An Optimization Approach to the Witsenhausen Counterexample

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I. Introduction

We examine the structure of the Witsenhausen counterexample/problem and its solution. In particular, we find it useful to work with the associated quantile function, rather than the controller itself or its distribution. With this transformation, the problem is reduced to minimization of a certain criterion over a particular function space. The optimization criterion is the sum of two functionals. The first, representing the control cost, is a simple quadratic. The second, representing the expected squared estimation error, has a more complex structure over this space. Nonetheless, it has a unique minimum (i.e., no other local minima). The problem of determining the parameter region over which the total cost criterion has a unique minimum remains open, although numerical experimentation suggests that this may "typically" be the case. Numerical results also indicate the form of the solution.

II. Background and Definition

The Witsenhausen counterexample [14] gained substantial notoriety for several reasons. The first is that it is a simply formulated problem, which may appear to a casual observer to be of linear/quadratic form, but is in fact far from such, and certainly does not have a linear controller as the optimal solution. Second, it is a problem where there is incomplete communication between the controller and the estimator. As such, it can be viewed as a benchmark problem in the area of networked control problems, which is of course an area of great current interest. Moreover, the controller is attempting not only to minimize its own effort, but is also attempting to aid the estimator through its control action. In fact, it is this latter role which drives the controller to act at all.

The problem formulation is quite simple, and one might place it in the arena of optimization rather than control, as one could argue that the problem does not have the time-structure which separates control from optimization. The problem is as follows. The first input is a scalar normal random variable, $W \sim \mathcal{N}(0,c)$, and we let its range be denoted as $\mathcal{W} \doteq I\!\!R$. A "controller", $\zeta : \mathcal{W} \to I\!\!R$ acts additively on the first input generating output $X = W + \zeta(W)$. We assume that $\zeta(\cdot)$ is measurable, of course. An observation, $Y = X + \nu$ is made, where $\nu \sim \mathcal{N}(0,d)$, and we let the range of Y be denoted by $\mathcal{Y} = I\!\!R$. The estimator generates estimate e_Y, knowing Y, but not $\zeta(W)$. Note that we assume the estimator does know the control strategy to be followed, $\zeta(\cdot)$, but not the actual control applied. The payoff to be minimized is

$$\hat{J}(\zeta(\cdot), e.) \doteq \mathbf{E}\left\{k_0|\zeta(W)|^2 + |X - e_Y|^2\right\}, \tag{1}$$

where $k_0 \in [0,\infty)$. Due to the squared-error form of the second term on the right, the optimal estimate is the conditional expectation, which will be denoted by $\hat{e}_Y$, and given that, we let

$$J(\zeta) \doteq \mathbf{E}\left\{k_0|\zeta(W)|^2 + |X - \hat{e}_Y|^2\right\}. \tag{2}$$

Clearly, the solution depends only on the three parameters, c, d, and k_0. Upon examining (2), we see that an optimal control must not only be measurable, but must have finite variance. Consequently, we take the control space to be

$$\mathcal{Z} \doteq \left\{\zeta : \mathcal{W} \to I\!\!R \,|\, \text{ measurable and } \mathbf{E}[\zeta^2(W)] < \infty\right\}. \tag{3}$$

We let

$$V = V(c,d,k_0) \doteq \inf_{\zeta\in\mathcal{Z}} J(\zeta) = \inf_{\zeta\in\mathcal{Z}} J(\zeta; c, d, k_0). \tag{4}$$

In this form, we see that the problem reduces to an (infinite-dimensional) optimization problem.

A good deal of quite interesting work has used this problem as a basis for development (c.f., [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [13]). Of particular relevance to the analysis here is [7]. In [7], the authors assume a signaling structure (originally suggested in [14]) for the controller, where the controller acts to make X take on one of a small finite set of possible values, the selection of which is based on input W. This, allows the estimator to correctly identify X with high probability, particularly if the gap between possible X values is relatively large compared with $\sqrt{d}$. Using a more general approach here, we find that such solutions emerge naturally in an interesting region of parameter space, while solutions similar to normal random variables, corresponding to nearly linear controllers, occur in another region. Hybrids of these appear in intervening regions.

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III. Quantile Representation

We will find it helpful to optimize not over controller u, but instead over the resulting distribution of X, F_X. Further, we will find it helpful to perform the bulk of the analysis not with distribution function F_X, but with the corresponding quantile function, which we denote as G. Consequently, it is helpful to review the transformation between the two representations. Let $\mathcal{F}$ denote the space of probability distribution functions on $I\!R$ with finite second moments. Let

$$\mathcal{G} \doteq \Big\{ G : (0,1) \to I\!R \,\Big|\, \int_{(0,1)} G^2(u)\,du < \infty, \text{ mono. inc.,} \\ \text{continuous on the left, limits on the right} \Big\}. \tag{5}$$

Given $F \in \mathcal{F}$, let

$$\mathcal{I}[F](u) \doteq \inf\{x \mid F(x) \geq u\}, \tag{6}$$

for all $u \in (0,1)$.

Theorem 3.1: $\mathcal{I}$ is a bijection from $\mathcal{F}$ to $\mathcal{G}$.

Proof: This is somewhat classical. We mention some useful points in the proof. Fix $F \in \mathcal{F}$, and let $G \doteq \mathcal{I}[F]$. First note that by the definitions of $\mathcal{I}$ and $\mathcal{F}$, $G(u) \in I\!R$ for all $u \in (0,1)$. For $u \in (0,1)$, let $A_u \doteq \{y \in I\!R \,|\, F(y) \geq u\}$. It is both useful and not difficult to show that $A_u = [G(u), \infty)$. In particular, one immediately sees that $v \geq u$ implies that $A_v \subseteq A_u$, and consequently, G is monotonically increasing. Of course, this implies that G has limits on the right.

To see the left continuity, let $\{u_n\}_{n\in\mathbb{N}} \subset (0,1)$ be monotonically increasing, and in particular, let $u_n \to \hat{u} < 1$. Let $x_n = G(u_n)$ $\forall n \in \mathbb{N}$ and $\hat{x} = G(\hat{u})$. Since G is monotonically increasing, $\{x_n\}$ is monotonically increasing, and there exists $\bar{x} \in I\!R$ such that $x_n \uparrow \bar{x} \leq \hat{x}$. Suppose $\bar{x} < \hat{x}$. By the definitions of x_n and $\mathcal{I}$,

$$u_n \leq F(x_n) \quad \forall\, n. \tag{7}$$

Let $\epsilon = (\hat{x} - x_n)/2 > 0$. Then, using (7) and the monotonicity of F,

$$u_n \leq F(x_n + \epsilon) = F(\hat{x} - \epsilon). \tag{8}$$

On the other hand, by definition, $\hat{x} = G(\hat{u})$ implies that $\inf\{x \mid F(x) \geq \hat{u}\} = \hat{x}$, and consequently, there exists $\delta > 0$ such that

$$F(\hat{x} - \varepsilon) \leq \hat{u} - \delta. \tag{9}$$

Combining (8) and (9), we see that $u_n \leq \hat{u} - \delta$ for all n, which is a contradiction. Therefore, $x_n \uparrow \hat{x}$, and we have left continuity.

We refer the reader to standard texts (c.f., [3]) for the remaining assertions. ■

Next, given $G \in \mathcal{G}$, let

$$\mathcal{J}[G](x) \doteq \begin{cases} \sup\{u \,|\, G(u) \leq x\} & \text{if } \{u \,|\, G(u) \leq x\} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

for all $x \in I\!R$.

Theorem 3.2: $\mathcal{J} = \mathcal{I}^{-1}$.

Proof: We refer the reader to standard texts (c.f., [3]) for more detail. ■

Remark 3.3: For purposes of intuition, it is helpful to consider the simple smooth, strictly increasing case. In this case, with $G \doteq \mathcal{I}[F]$, we may write $G = F^{-1}$ where the inverse function is interpreted in the classical sense, and one has, formally,

$$du = \frac{du}{dx}\,dx = \frac{dF}{dx}\,dx = f(x)\,dx$$

with f denoting the corresponding density.

Remark 3.4: Further, for square-integrable H,

$$\mathbf{E}[H(X)] = \int_{I\!R} H(x)\,dF(x) = \int_{(0,1)} H(G(u))\,du, \tag{11}$$

where this equivalence holds in the general case, and we do not include the proof.

IV. Cost Structure

We now examine the structure of the cost criterion, J, using a quantile functional representation. First, let

$$\hat{A}(\zeta) \doteq \mathbf{E}\left\{k_0|\zeta(W)|^2\right\} \text{ and } \hat{B}(\zeta) \doteq \mathbf{E}\left\{|Y - \hat{e}_Y|^2\right\}. \tag{12}$$

First, we look at $\hat{B}$. Note that the conditional expectation of X given $Y = y$ is

$$\hat{e}_y \doteq \mathbf{E}\{X \,|\, Y = y\} = \frac{k_1}{k_0} = \frac{\int_{I\!R} x h_d(x,y)\,dF_X(x)}{\int_{I\!R} h_d(x,y)\,dF_X(x)} \tag{13}$$

where

$$h_d(x,y) \doteq \frac{1}{\sqrt{2\pi d}} \exp\left[\frac{-(x-y)^2}{2d}\right].$$

Employing change of variables (11), this becomes

$$\hat{e}_y = \frac{\int_{(0,1)} G(u) h_d(G(u),y)\,du}{\int_{(0,1)} h_d(G(u),y)\,du}, \tag{14}$$

where $G = \mathcal{I}[F_X]$ and du indicates integration with respect to Lebesgue measure. Further, noting that for measurable $\mathcal{C} \subseteq I\!R$, $P(Y \in \mathcal{C}) = \int_{\mathcal{C}} \int_{I\!R} h_d(x,y) dF_X(x)\,dy$, we see that

$$\hat{B}(\zeta) = \int_{I\!R} \int_{I\!R} |x - \hat{e}_y|^2 h_d(x,y)\,dF_X(x)\,dy,$$

which again by the change of variables,

$$= \int_{I\!R} \int_{(0,1)} |G(u) - \hat{e}_y|^2 h_d(G(u),y)\,du\,dy \doteq B(G). \tag{15}$$

Next, we look to represent $\hat{A}$ in terms of the quantile function corresponding to X. Due to the explicit presence of ζ, one should examine the transformation carefully. Note that ζ wishes to transform input W into some form (presumably more useful to the estimator), X. Considering (15), we see that the expected estimator error, B, depends not on ζ, but only on the resulting distribution, F_X, or equivalently, the quantile function G. Consequently, ζ would like to generate any given F_X, with the minimum squared effort given by $\hat{A}$. It seems intuitively clear that in order to minimize cost, for any given F_X, one would choose a monotonically increasing

$\zeta(\cdot)$. However, to be completely rigorous, we do not presume this form, but find this form, along with the representation of the problem, in terms of the quantile function.

Let F_W denote the distribution corresponding to (normally distributed) input W, and let $G_W \doteq \mathcal{I}[F_W]$, which is of course, C^∞.

Theorem 4.1: For any $G \in \mathbb{G}$,

$$\min_\zeta \{\mathbf{E}[\zeta^2] \,|\, \mathcal{I}[F_X](\cdot) = G(\cdot)\} = \int_{(0,1)} [G(u) - G_W(u)]^2 du.$$

Proof: We first demonstrate that there is a ζ such that

$$\mathbf{E}[\zeta^2] = \int_{(0,1)} [G(u) - G_W(u)]^2 du.$$

Equivalently, we construct a function, $\hat{X}(w)$, with distribution F_X such that

$$\int_{\mathbb{R}} [\hat{X}(w) - w]^2 P(dw) = \int_{(0,1)} [G(u) - G_W(u)]^2 du. \quad (16)$$

For each $w \in \mathbb{R}$, define

$$\hat{X}(w) \doteq G(F_W(w)). \quad (17)$$

Note that F_W belongs to the class indicated in Remark 3.3, and consequently, $G_W = F_W^{-1}$ in the classical sense. Thus, it suffices to show

$$\int_{\mathbb{R}} [\hat{X}(w) - w]^2 P(dw) = \int_{(0,1)} [\hat{X}(G_W(u)) - G_W(u)]^2 du. \quad (18)$$

The proof proceeds by approximating the arguments of the integrals in (18) by simple functions.

Denote the Lebesgue measure on $(0,1)$ as $\lambda(\cdot)$ and define the indicator function $I_A(x)$, with argument x, to equal 1 when $x \in A$ and 0 otherwise. Given a Borel set $B \subset (0,1)$, a useful fact (see, for instance, [12]) is that

$$P(B) = \lambda(\{u | G_W(u) \in B\}). \quad (19)$$

Since $\hat{X}$ and W have finite second moments, the integrand of the left-hand side of (18) is dominated by an integrable function, specifically $[\hat{X}(w) - w]^2 \leq 2[\hat{X}^2(w) + w^2]$.

Now we are in a position to define a sequence of dominated simple functions, $\{s_n(w)\}_{n\in\mathbb{N}}$,

$$s_n(w) \triangleq \sum_k \beta_n(k) I_{B_n(k)}(w) \leq 2[\hat{X}^2(w) + w^2], \text{ a.e.,}$$

which converges as $\lim_{n\to\infty} s_n(w) = [\hat{X}(w) - w]^2$ for almost every $w \in \mathbb{R}$. By the Lebesgue Convergence Theorem,

$$\mathbf{E}[s_n] = \int_{\mathbb{R}} s_n(w) P(dw) = \sum_k \beta_n(k) P(B_n(k)) \quad (20)$$

$$\xrightarrow[n\to\infty]{} \int_{\mathbb{R}} [\hat{X}(w) - w]^2 P(dw). \quad (21)$$

Also, because the mapping between almost every $u \in (0,1)$ and almost every $w \in \mathbb{R}$ is bijective and continuous,

$$S_n(u) \triangleq s_n \circ G_W(u) = \sum_k \beta_n(k) I_{\{u|G_W(u)\in B_n(k)\}}(u) \xrightarrow[n\to\infty]{} [\hat{X}(G_W(u)) - G_W(u)]^2$$

almost everywhere. Furthermore, $S_n(F_W(w))$ is dominated by $2[\hat{X}(w)^2 + w^2]$; thus, $S_n(u)$ is dominated almost everywhere by an integrable function. Using (19) and (20),

$$\mathbf{E}[s_n] = \sum_k \beta_n(k) P(B_n(k)) = \sum_k \beta_n(k) \lambda(\{u | G_W(u) \in B_n(k)\}) = \int_{(0,1)} S_n(u) du. \quad (22)$$

Again using the Lebesgue Convergence Theorem,

$$\lim_{n\to\infty} \int_{(0,1)} S_n(u) du = \int_{(0,1)} [\hat{X}(G_W(u)) - G_W(u)]^2 du. \quad (23)$$

Combining (21)–(23), we arrive at

$$\int_{\mathbb{R}} [\hat{X}(w) - w]^2 P(dw) = \int_{(0,1)} [\hat{X}(G_W(u)) - G_W(u)]^2 du.$$

It remains to prove the reverse inequality. This proof is substantially more technical, and we do not include it here, although we do note that if one assumes ζ to be monotonically increasing, the proof is substantially less demanding. ■

Given Theorem 4.1, it is natural to define

$$A(G) \doteq \int_{(0,1)} [G(r) - G_W(r)]^2 dr. \quad (24)$$

Then, combining the above, and with a bit more work, one finally finds that problem (4) is equivalently

$$V = V(c, d, k_0) = \inf_{G\in\mathcal{G}} \bar{J}(G) = \inf_{G\in\mathcal{G}} \bar{J}(G; c, d, k_0), \quad (25)$$

where

$$\bar{J}(G) = k_0 A(G) + B(G). \quad (26)$$

Further, given an optimal G, one could construct the corresponding controller from $\zeta(w) = \hat{X}(w) - w$ where $\hat{X}$ is given by (17).

V. Solution Form

We make some remarks on the form of the solution of our optimization problem. First, of course, in this quantile representation, the $A(G)$ functional is simply a quadratic, with minimum of zero at $G = G_W$.

The $B(G)$ functional is more complex, and is the source of the difficulties. Fix some $G \in \mathcal{G}$. We will consider certain L_2 variations around G. Let $\gamma \in L_2(0,1)$ (with specific form to follow), and $\delta > 0$. Recall from (13) that $\hat{e}_y = k_1/k_0$ where k_1, k_0 are given there. By standard computations,

$$\begin{aligned} &k_1(G + \delta\gamma) - k_1(G) \\ &= \delta \int_{(0,1)} h_d(y, G(v)) \left[1 + G(v)\left(\frac{y - G(v)}{d}\right)\right] \gamma(v)\, dv \\ &\quad + \mathcal{O}(\delta^2), \end{aligned} \quad (27)$$

$$\begin{aligned} &k_0(G + \delta\gamma) - k_0(G) \\ &= \delta \int_{(0,1)} h_d(y, G(v)) \left(\frac{y - G(v)}{d}\right) \gamma(v)\, dv + \mathcal{O}(\delta^2). \end{aligned} \quad (28)$$

Taking a similar differential in (13) , and employing (27) and (28), one obtains

$$\begin{aligned}
\hat{e}_y(G+\delta\gamma) - \hat{e}_y(G) &= \delta \int_{(0,1)} \frac{h_d(y,G(v))}{k_0(y,G)} \\
&\quad \cdot \left\{ 1 + \left[G(v) - \frac{k_1(y,G)}{k_0(y,G)} \right] \left(\frac{y - G(u)}{d} \right) \right\} \gamma(v)\, dv \\
&\quad + \mathcal{O}(\delta^2) \\
&\doteq \delta \int_{(0,1)} \Delta_e^0(y,G,G(v))\gamma(v)\, dv + \mathcal{O}(\delta^2), \qquad (29)
\end{aligned}$$

where, for clarity, we remark that when $G(v)$ appears as an argument, it indicates dependence on G evaluated at v, and when G appears as an argument with no argument of its own, this indicates dependence on the entire function.

Continuing with this process, and suppressing dependence of $\hat{e}_y$ on y, G, we find

$$\begin{aligned}
B(G+\delta\gamma) - B(G) &= \delta \int_{(0,1)} \int_{I\!R} h_d(y,G(u)) \\
&\quad \cdot \left[2(G(u) - \hat{e}_y) + |G(u) - \hat{e}_y|^2 \left(\frac{y - G(u)}{d} \right) \right] \\
&\quad + \Delta_e^1(y,G)\Delta_e^0(y,G,G(u))\, dy\, \gamma(u)\, du + \mathcal{O}(\delta^2) \\
&\doteq \delta \int_{(0,1)} b(G,G(u))\gamma(u)\, du + \mathcal{O}(\delta^2), \qquad (30)
\end{aligned}$$

where

$$\Delta_e^1(y,G) \doteq \int_{(0,1)} 2(\hat{e}_y - G(v)) h_d(y,G(v))\, dv. \qquad (31)$$

It is worthwhile explicitly noting that for $\alpha \in I\!R$,

$$\begin{aligned}
b(G,\alpha) &= \int_{I\!R} \Bigg\{ h_d(y,\alpha) \\
&\quad \cdot \left[2(\alpha - \hat{e}_y(G)) + |\alpha - \hat{e}_y(G)|^2 \left(\frac{y-\alpha}{d} \right) \right] \qquad (32) \\
&\quad + \Delta_e^1(y,G) \frac{h_d(y,\alpha)}{k_0(y,G)} \left[1 + \left(\alpha - \frac{k_1(y,G)}{k_0(y,G)} \right) \left(\frac{y-\alpha}{d} \right) \right] \Bigg\}\, dy
\end{aligned}$$

which is clearly C^∞ in α.

Now, recalling that G is monotonically increasing, there exists at most a countably infinite number of discontinuities. Consequently, there exists a finite or countably infinite set of open intervals $\{(\beta_k, \beta_{k+1})\}_{k\in\mathcal{K}}$ such that G is continuous on each open interval and such that $(0,1) \setminus \bigcup_{k\in\mathcal{K}} (\beta_k, \beta_{k+1})$ consists of at most a countably infinite number of points. Suppose there exist $k \in \mathcal{K}$, $\hat{\beta}_k \in (\beta_k, \beta_{k+1})$, $\hat{\varepsilon}_k > 0$ and $\tilde{\delta}_k > 0$ such that $B_{\hat{\varepsilon}_k}(\hat{\beta}_k) \subseteq (\beta_k, \beta_{k+1})$ and $G(v) - G(u) \geq \tilde{\delta}_k (v - u)$ for all $v \geq u$ in $B_{\hat{\varepsilon}_k}(\hat{\beta}_k)$. (Otherwise, G is piecewise constant.) Suppose $b(G, G(\hat{\beta}_k)) \neq 0$, and in particular, suppose $b(G, G(\hat{\beta}_k)) > 0$. Since $b(G,\alpha)$ is C^∞ in α, there exists $\bar{\delta}_k > 0$ and $\bar{b}_k > 0$ such that

$$b(G,\alpha) > \bar{b}_k \quad \forall\, \alpha \in B_{\bar{\delta}_k}(G(\hat{\beta}_k)). \qquad (33)$$

Also, since $G \in C((\beta_k, \beta_{k+1}))$, there exists $\bar{\varepsilon}_k \in (0, \hat{\varepsilon}_k)$ such that

$$G(u) \in B_{\bar{\delta}_k}(G(\hat{\beta}_k)) \quad \forall\, u \in B_{\bar{\varepsilon}_k}(\hat{\beta}_k). \qquad (34)$$

Again, recalling that $I_\mathcal{A}$ denotes the indicator function for measurable set $\mathcal{A}$, let

$$\gamma(u) \doteq I_{[\hat{\beta}_k, \hat{\beta}_k + \bar{\varepsilon}_k)}(u)(\hat{\beta}_k + \bar{\varepsilon}_k - u),$$

and let $\delta \in (0, \tilde{\delta}_k)$. Then,

$$(G(v) + \delta\gamma(v)) - (G(u) + \delta\gamma(v)) \geq (\tilde{\delta}_k - \delta)(v-u) \geq 0$$

for all $v \geq u$ in $(\hat{\beta}_k, \hat{\beta}_k + \bar{\varepsilon}_k)$. Consequently, $G + \delta\gamma \in \mathcal{G}$ for all $\delta \in [0, \tilde{\delta}_k)$.

Further, with this choice of δ and γ, and using (31),

$$\begin{aligned}
B(G+\delta\gamma) - B(G) &= \delta \int_{(0,1)} b(G,G(u))\gamma(u)\, du + \mathcal{O}(\delta^2) \\
&= \delta \int_{(\hat{\beta}_k, \hat{\beta}_k + \bar{\varepsilon}_k)} b(G,G(u))(\hat{\beta}_k + \bar{\varepsilon}_k - u)\, du + \mathcal{O}(\delta^2),
\end{aligned}$$

which by (33) and (34)

$$\begin{aligned}
&> \delta \bar{b}_k \int_{(\hat{\beta}_k, \hat{\beta}_k + \bar{\varepsilon}_k)} (\hat{\beta}_k + \bar{\varepsilon}_k - u)\, du + \mathcal{O}(\delta^2) \\
&= \delta \bar{b}_k \bar{\varepsilon}_k + \mathcal{O}(\delta^2) > 0
\end{aligned}$$

for $\delta > 0$ sufficiently small. Consequently, G cannot be optimal. The case where $b(G, G(\hat{\beta}_k)) < 0$ is similar. We see that if G is optimal and not constant on some $(\hat{\beta}_k, \hat{\beta}_{k+1})$, then $b(G, G(u)) = 0$ for all $u \in (\hat{\beta}_k, \hat{\beta}_{k+1})$. As it appears technically demanding to prove, for the present, we assume:

$b(G,\alpha)$ has only isolated zeros as a function of α for any $G \in \mathcal{G}$. $\qquad (A.1)$

The reader may choose to examine (32) for an understanding of the motivation behind this assumption. If G is not constant on $(\hat{\beta}_k, \hat{\beta}_{k+1})$, then by $(A.1)$ and the continuity of G over this interval, there exists $u \in (\hat{\beta}_k, \hat{\beta}_{k+1})$ such that $b(G, G(u)) \neq 0$, and so G cannot be optimal. We have:

Lemma 5.1: Assume $(A.1)$. Suppose B has a local minimum at $\bar{G} \in \mathcal{G}$. Then, $\bar{G}$ is piecewise constant.

We say $G \in \mathcal{G}$ is antisymmetric (around $1/2$) if $G(u) = -G(1-u)$ for almost every $u \in (0,1)$. (Alternatively, $G(1/2 - \delta) = -G(1/2 + \delta)$ for almost every $\delta \in (0, 1/2)$.) Similarly, $F \in \mathcal{F}$ is antisymmetric (around range value $1/2$) if $F(-x) = 1 - F(x)$ for all $x \in I\!R$. Both of these correspond to a symmetric density function when such exists. Let

$$\mathcal{G}^a \doteq \{ G \in \mathcal{G} \,|\, G \text{ is antisymmetric} \}.$$

The following is obivous.

Lemma 5.2: The minimum of B (as well as the minimum of A) over $\mathcal{G}$ is attained on $\mathcal{G}^a$.

Of the piecewise constant quantile functions, the entirely constant function is important. Suppose there exists $\bar{g} \in I\!R$ such that $G(u) = \bar{g}$ for all $u \in (0,1)$. Then,

$$\hat{e}_y(G) = \frac{k_1(y,G)}{k_0(y,G)} = \frac{\bar{g} k_0(y,G)}{k_0(y,G)} = \bar{g}.$$

Consequently, $|\hat{e}_y(G) - G(u)|^2 = 0$ for all y, u, and we see $B(G) = 0$.

Now, suppose G is not constant. Then, noting the monotonicity, there exist $\varepsilon > 0$ and $0 < \bar{u} < \bar{v} < 1$ such that $G(\bar{v}) - G(\bar{u}) = \varepsilon$. This implies

$$\begin{aligned} G(u) &\leq G(\bar{u}) \quad \forall\, u \in (0, \bar{u}] \\ G(u) &\geq G(\bar{v}) \quad \forall\, u \in [\bar{v}, 1). \end{aligned}$$

Therefore, since $\hat{e}_y(G)$ is independent of u, for any $y \in I\!\!R$,

$$\begin{aligned} \text{either} \quad & |G(u) - \hat{e}_y(G)| \geq \frac{\varepsilon}{2} \quad \forall\, u \in [\bar{v}, 1), \\ \text{or} \quad & |G(u) - \hat{e}_y(G)| \geq \frac{\varepsilon}{2} \quad \forall\, u \in (0, \bar{u}]. \end{aligned}$$

Employing this in (15), one finds

$$\begin{aligned} B(G) \geq \int_{I\!\!R} \min\Bigg\{ & \int_{[\bar{v},1)} \frac{\varepsilon^2}{4} h_d(y, G(u))\, du, \\ & \int_{(0,\bar{u}]} \frac{\varepsilon^2}{4} h_d(y, G(u))\, du \Bigg\}\, dy > 0. \end{aligned}$$

Consequently, we have

Lemma 5.3: If G is constant, then $B(G) = 0$; otherwise, $B(G) > 0$.

At this point, one knows that any G that minimizes B is piecewise constant, and that one may restrict the search for minima to $\mathcal{G}^a$. One also knows that constant functions yield the minimum, with the constant function, $\bar{G}_0(u) \equiv 0$ being the minimizer within $\mathcal{G}^a$. We have not yet shown that there do not exist other local minima. We briefly indicate this result.

Lemma 5.4: Neglecting the absolute minimizer, $\bar{G}_0(u) \equiv 0$, there are no other local minimizers of B over $\mathcal{G}^a$.

Proof: (Sketch of proof.) Here we find it convenient to work with $F_X \in \mathcal{F}$ and the corresponding density "function" represented in terms of Dirac δ functions. Consider a piecewise constant $G^p \in \mathcal{G}^a$. Associated to this is a density function. Without loss of generality, we let this density be

$$f^p(x) = \lambda_0 \delta_0(x) + \sum_{i=1}^{N} \frac{\lambda_i}{2} \left[\delta_{\xi_i}(x) + \delta_{-\xi_i}(x)\right],$$

where $\lambda_0, \lambda_i \in [0,1]$ and $\lambda_0 + \sum_{i=1}^{N} \lambda_i = 1$. We will show that one can construct a path from G^p to $\bar{G}_0$ such that $B(G)$ monotonically decreases along the path. As our proof is quite technical, the details cannot be included here. However, the main points will be indicated. We will take $\hat{\xi} > 0$ sufficiently large such that several terms in B become quite small. Let $\mu \in [0, \lambda_N]$, and let $\tilde{e}_y(\mu) \doteq \frac{\tilde{k}_1(\mu,y)}{\tilde{k}_0(\mu,y)}$ where

$$\begin{aligned} \tilde{k}_0(\mu, y) =& \lambda_0 h_d(0, y) + \sum_{i=1}^{N-1} \frac{\lambda_i}{2} \left[h_d(\xi_i, y) + h_d(-\xi_i, y)\right] \\ & + \frac{\lambda_N - \mu}{2} \left[h_d(\xi_N, y) + h_d(-\xi_N, y)\right] \\ & + \frac{\mu}{2} \left[h_d(\hat{\xi}, y) + h_d(-\hat{\xi}, y)\right], \end{aligned}$$

$$\begin{aligned} \tilde{k}_1(\mu, y) =& \sum_{i=1}^{N-1} \frac{\lambda_i}{2} \xi_i \left[h_d(\xi_i, y) - h_d(-\xi_i, y)\right] \\ & + \frac{\lambda_N - \mu}{2} \xi_N \left[h_d(\xi_N, y) - h_d(-\xi_N, y)\right] \\ & + \frac{\mu}{2} \hat{\xi} \left[h_d(\hat{\xi}, y) - h_d(-\hat{\xi}, y)\right]. \end{aligned}$$

Letting $\tilde{B}(\mu)$ denote the corresponding cost, one has $\tilde{B}(\mu) = B(\tilde{G}^\mu)$ where $\tilde{G}^\mu$ is the quantile function corresponding to density

$$\begin{aligned} \tilde{f}^\mu(x) =& \lambda_0 \delta_0(x) + \sum_{i=1}^{N-1} \frac{\lambda_i}{2} \left[\delta_{\xi_i}(x) + \delta_{-\xi_i}(x)\right] \\ & + \frac{\lambda_N - \mu}{2} \left[\delta_{\xi_N}(x) + \delta_{-\xi_N}(x)\right] + \frac{\mu}{2} \left[\delta_{\hat{\xi}}(x) + \delta_{-\hat{\xi}}(x)\right]. \end{aligned}$$

Note that

$$\begin{aligned} \tilde{B}(\mu) =& \int_{I\!\!R} \Bigg\{ \lambda_0 \tilde{e}_y^2(\mu) h_d(0, y) \\ & + \sum_{i=1}^{N-1} \frac{\lambda_i}{2} \left[|\xi_i - \tilde{e}_y(\mu)|^2 h_d(\xi_i, y) + |\xi_i + \tilde{e}_y(\mu)|^2 h_d(-\xi_i, y)\right] \\ & + \frac{\lambda_N - \mu}{2} \Big[|\xi_N - \tilde{e}_y(\mu)|^2 h_d(\xi_N, y) \\ & \quad + |\xi_N + \tilde{e}_y(\mu)|^2 h_d(-\xi_N, y)\Big] + \frac{\mu}{2} \Big[|\hat{\xi} - \tilde{e}_y(\mu)|^2 h_d(\xi_N, y) \\ & \quad + |\hat{\xi} + \tilde{e}_y(\mu)|^2 h_d(-\xi_N, y)\Big] \Bigg\}\, dy. \end{aligned}$$

One shows that for $\hat{\xi}$ sufficiently large, $\tilde{B}(\mu)$ is monotonically decreasing in μ. For intuition, note that as one may choose $\hat{\xi}$ quite "far" from the ξ_i, when y is large positive, the estimator predicts $\tilde{e}_y(\mu)$ to be quite close to $\hat{\xi}$, and thus the contribution from this term is lower than that from the ξ_N term. Thus, as one increases μ from 0 to λ_N, the cost decreases. Next, one proceeds to apply the same method to the $N-1$ term in the sum, with exactly the same (sufficiently large) $\hat{\xi}$. By induction, one finally obtains G^2 which is piecewise constant, with at most three segments, and a corresponding density

$$f^2(x) = \lambda_0 \delta_0(x) + \left(\frac{1-\lambda_0}{2}\right) \left[\delta_{\hat{\xi}}(x) + \delta_{-\hat{\xi}}(x)\right],$$

where $B(G^2) \leq B(G^p)$. Lastly, one shows that there is a path from G^2 to $\bar{G}_0$ along which $B(G)$ is monotonically decreasing. ■

The above simple results indicate something of the structure of the optimization problem. One desires to minimize the sum of a quadratic, A, with minimum at $G = G_W$, and a functional, B, with the somewhat odd structure indicated here. This interplay is what leads to the variety of solutions one finds over the parameter space, where the "signaling" optima are those where the B component plays a more significant role than the cases where the solution looks closer to a normal random variable. Note that although both A and B have unique minima, this does not imply that $\bar{J} = k_0 A + B$ does not possess extraneous local minima. Sufficient conditions guaranteeing such will be the subject of a longer paper.

VI. Numerical Results and Comments

Although a complete analysis indicating under exactly what parameter sets we are able to guarantee that the total cost, $\bar{J}(G) \doteq k_0 A(G) + B(G)$, has no extraneous local minima is not included, we nonetheless begin experimentation to verify the structure of the optimal distribution. In particular, a gradient-descent algorithm has been constructed to search for the optimum under various sets of parameter values. We note that this algorithm obtains the same numerical values (to the published number of digits) as obtained in [2] for the cases given there. The algorithm optimizes over the quantile representation of the control-induced distribution of X. Roughly, at each iteration, it proceeds by moving in the direction opposite the gradient, until the cost begins to rise. At that point another gradient calculation is done, and the process is repeated. It is not fruitful to indicate the finer details such as stopping criteria at this point, and in the limited space.

In some parameter regions, the algorithm finds (approximate) optima which are of the signaling form discussed in [7], [14], while in other regions, the optima are generated by controllers that are roughly linear. Intuitively, the distinction flows from the relative importance of $k_0 A$ and B in the total cost, where consideration of A alone would lead to linear optima, while B (with some impetus from A) pushes the solution toward signaling forms. We include three figures below. Perhaps the most interesting is the second figure, where the parameters are in an area between these two regions, and the optimal solution appears to have a mix of the two forms.

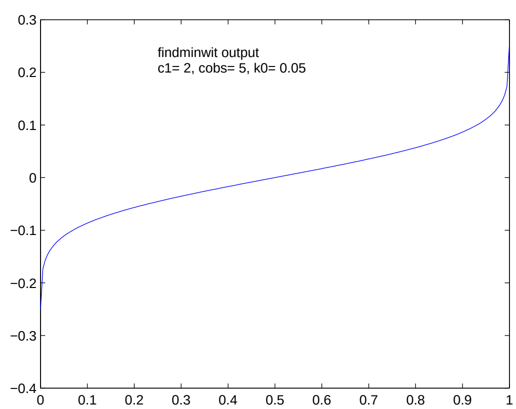

Fig. 1. Approximate solution: $c = 2$, $d = 5$, $k_0 = 0.05$.

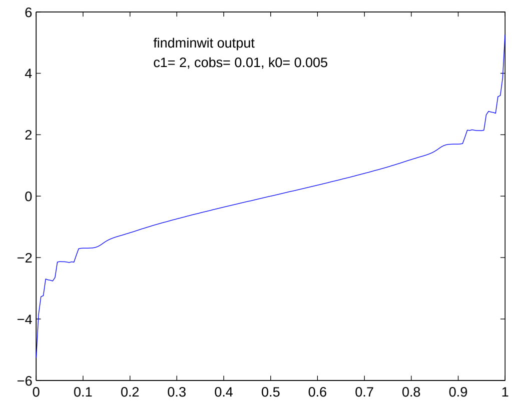

Fig. 2. Approximate solution: $c = 2$, $d = 0.01$, $k_0 = 0.005$.

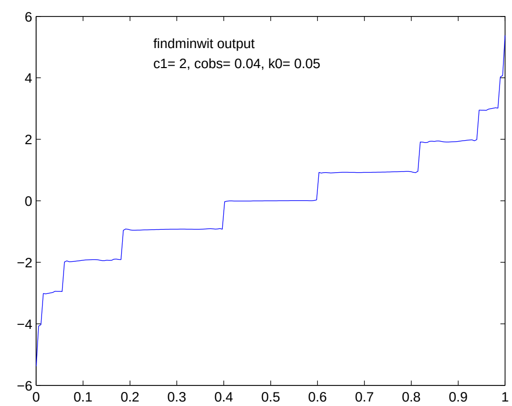

Fig. 3. Approximate solution: $c = 2$, $d = 0.04$, $k_0 = 0.05$.

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