Verification of stationary action trajectories via optimal control

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***Abstract*— A new optimal control based representation for stationary action trajectories is constructed by exploiting connections between semiconvexity, semiconcavity, and stationarity. This new representation is used to verify a known two-point boundary value problem characterization of stationary action.**

I. Introduction

The principle of stationary action, or *action principle*, is a fundamental variational postulate that underpins conservation laws in modern physics [7], [8], [10], [11]. A corollary of this principle states that *any trajectory of a conservative system must render the corresponding action stationary in the calculus of variation sense*, in which the *action* is the time integral of the corresponding Lagrangian.

When dynamical evolution is restricted to sufficiently short time horizons, the action involved is typically a convex function of the generalized velocity trajectory, at least where the generalized position space is finite dimensional, see for example [11], [6]. Consequently, on such short time horizons, stationary action is achieved as *least action*, and the trajectories involved can be characterized using tools from classical optimal control. In particular, for a specific conservation law, an optimal control problem can be formulated with respect to a cost function defined as the sum of the integrated Lagrangian and an artificial terminal cost, with the latter is used to capture terminal data. Dynamic programming may then be applied to characterize optimal trajectories, which necessarily correspond to trajectories of the underlying conservative system, subject to the imposed boundary conditions.

Recent efforts by the authors have successfully exploited this connection between least action and optimal control on short time horizons to develop a variety of fundamental solutions for conservative systems, including for the gravitational N-body problem [11]. However, on longer time horizons, or for systems evolving in infinite dimensions, this connection breaks down, typically due to a loss of convexity of the action. Indeed, the value functions associated with optimal control problems posed on these longer time horizons are typically afflicted by finite escape phenomena. As a minimum cannot be achieved in these cases, stationarity must explicitly be considered [6], [5].

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In this paper, connections between stationarity and *stationary control* [11], [4], [6], [5], [12], [13] are summarized and further explored, with a view to expanding the applicability of generalized optimal control tools to the evolution of conservative systems over longer time horizons. In Section II, the aforementioned connections between least action and optimal control are reviewed and formalized, and the indicated short horizon constraint elucidated. Subsequently, Section III briefly summarizes the relaxation of optimal control to stationary control that is required to deal with longer horizons, and provides the expected two-point boundary value problem (TPBVP) characterization of the stationary trajectory (i.e. along which the action is stationary). Finally, Section IV exploits connections between semiconvexity, semiconcavity, and stationarity in order to formulate two auxiliary optimal control problems that can be used to characterize the *staticizing* velocity input that yields upon integration the aforementioned stationary trajectory. The obtained characterization is used to verify the expected TPBVP formulation of Section III.

Throughout, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ denote the real, integer, and natural numbers respectively, with extended reals defined as $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{\pm\infty\}$. The space of continuous mappings between Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $C(\mathscr{X};\mathscr{Y})$. The set of bounded linear operators between the same spaces is denoted by $\mathcal{L}(\mathscr{X};\mathscr{Y})$, or $\mathcal{L}(\mathscr{X})$ if $\mathscr{X}$ and $\mathscr{Y}$ coincide. A function $f \in C(\mathscr{X};\mathscr{Y})$ is Fréchet differentiable at $x \in \mathscr{X}$, with derivative $Df(x) \in \mathcal{L}(\mathscr{X};\mathscr{Y})$, if $0 = \lim_{\|h\|_{\mathscr{X}}\to 0} \|df_x(h)\|_{\mathscr{Y}}$, with $df_x : \mathscr{X} \to \mathscr{X}$ defined by

$$df_x(h) \doteq \begin{cases} 0, & \|h\|_{\mathscr{X}} = 0, \\ \frac{f(x+h)-f(x)-Df(x)\,h}{\|h\|_{\mathscr{X}}}, & \|h\|_{\mathscr{X}} > 0. \end{cases}$$

By definition, the map $h \mapsto df_x(h)$ is continuous at 0.

II. Least action and optimal control

For a conservative system with generalized position evolving in a real Hilbert space $\mathscr{X}$, the action is formalized as a function defined with respect to a coercive inertia operator $\mathcal{M} \in \mathcal{L}(\mathscr{X})$, a potential field $V : \mathscr{X} \to \mathbb{R}$, and an artificial convex terminal cost $\Psi : \mathscr{X} \to \mathbb{R}$ that is included to encode terminal data [11], [6]. Given $t, T \in \mathbb{R}$, $t < T$, and $\mathscr{U}[t,T] \doteq \mathscr{L}_2([t,T];\mathscr{X})$, it is explicitly defined by $J_t[\Psi] : \mathscr{X} \times \mathscr{U}[t,T] \to \mathbb{R}$, with

$$J_t[\Psi](x,u) \doteq \int_t^T \tfrac{1}{2}\,\langle u_s, \mathcal{M}\,u_s\rangle - V(\xi_s)\,ds + \Psi(\xi_T), \quad (1)$$

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for all $x \in \mathscr{X}$, $u \in \mathscr{U}[t,T]$, in which $s \mapsto \xi_s$ is the generalized position trajectory

$$\xi_s \doteq x + \int_t^s u_\sigma \, d\sigma, \quad s \in [t,T], \tag{2}$$

defined with respect to a corresponding generalized velocity trajectory $s \mapsto u_s$ for $s \in [t,T]$. For convenience, in addition to coercivity of $\mathcal{M}$, it is assumed throughout that V, Ψ are three times continuously Fréchet differentiable with uniformly bounded Hessian as per [4], i.e.

$$\begin{aligned} V, \Psi &\in C^3(\mathscr{X};\mathbb{R}), \\ m &\doteq \inf_{h\in\mathscr{X}} \left\{ \langle h,\, \mathcal{M}\, h\rangle / \|h\|^2 \right\} > 0, \\ \kappa &\doteq 2 \sup_{h\in\mathscr{X}} \max\left\{ \|\nabla^2 V\|_{\mathcal{L}(\mathscr{X})},\, \|\nabla^2 \Psi(x)\|_{\mathcal{L}(\mathscr{X})} \right\} < \infty. \end{aligned} \tag{3}$$

For sufficiently short time horizons $T - t > 0$, and in the company of (3), the action $J_t[\Psi](x,\cdot) : \mathscr{U}[t,T] \to \mathbb{R}$, $x \in \mathscr{X}$, can be shown to be strictly convex and coercive for finite dimensional $\mathscr{X}$ [11], [6], see Theorem 2.1 below. In that case, an optimal control problem can be formulated to describe stationary action as *least* action. The value function involved is defined by $\overline{W}_t : \mathscr{X} \to \overline{\mathbb{R}}$, with

$$\overline{W}_t(x) \doteq \inf_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u) \tag{4}$$

for all $x \in \mathscr{X}$. The Hamiltonian $H : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ involved is subsequently defined by

$$\begin{aligned} H(x,p) &\doteq \tfrac{1}{2}\langle p,\, \mathcal{M}^{-1}\, p\rangle + V(x) \\ &= \sup_{u\in\mathscr{X}} \{ -\langle p, u\rangle - \tfrac{1}{2}\langle u,\, \mathcal{M}\, u\rangle \} + V(x) \end{aligned} \tag{5}$$

for all $x, p \in \mathscr{X}$, in which the second equality follows by completion of squares.

Theorem 2.1: Given m, κ as per (3) and $t_0 \in \mathbb{R}_{<T}$ satisfying $\max(T - t_0, 1)\,(T - t_0) < \frac{m}{\kappa}$, the following properties concerning the value function (4) hold:

(i) Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, the action (cost) $J_t[\Psi](x,\cdot) : \mathscr{U}[t,T] \to \mathbb{R}$ is strictly convex and coercive, and there exists an optimal input $\bar{u}^* \in \mathscr{U}[t,T]$ such that $\overline{W}_t(x) = J_t(x, \bar{u}^*) \in \mathbb{R}$;

(ii) Given the Hamiltonian H of (5), the function $(t,x) \mapsto \overline{W}_t(x)$ is the unique viscosity solution of the HJB PDE

$$\begin{cases} 0 = -\dfrac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)), \\ W_T(x) = \Psi(x), \end{cases} \tag{6}$$

for all $t \in [t_0, T]$, $x \in \mathscr{X}$; and

(iii) There exists a classical solution of the two-point boundary value problem (TPBVP)

$$\begin{cases} \dot{\bar{x}}_s = -\nabla_p H(\bar{x}_s, \bar{p}_s) = -\mathcal{M}^{-1}\, \bar{p}_s, & \bar{x}_t = x, \\ \dot{\bar{p}}_s = \nabla_x H(\bar{x}_s, \bar{p}_s) = \nabla V(\bar{x}_s), & \bar{p}_T = \nabla \Psi(\bar{x}_T), \end{cases} \tag{7}$$

for all $s \in [t,T]$, in which $\nabla_x H$ and $\nabla_p H$ denote Riesz representations of the Fréchet derivatives of the Hamiltonian (5), and the optimal input satisfies

$$\bar{u}^*_s = -\mathcal{M}^{-1}\, \bar{p}_s, \quad s \in [t,T]. \tag{8}$$

Proof: *(i)*: Fix arbitrary $t_0 < T$ such that $\max(T - t_0, 1)\,(T - t_0) < \frac{m}{\kappa}$, and fix any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u, \hat{u}, h \in \mathscr{U}[t,T]$. [4, Theorem 3.6, Assertion 2] states that the Fréchet derivative of the Riesz representation of the first Fréchet derivative of $J_t[\Psi](x,\cdot)$ is given by

$$\begin{aligned} &[D_u \nabla_u J_t[\Psi](x,u)\, h]_r = \mathcal{M}\, h_r \\ &\quad + \int_t^T \left[\int_{r\vee\rho}^T -\nabla^2 V(\xi_\sigma)\, d\sigma + \nabla^2 \psi(\xi_T) \right] h_\rho \, d\rho, \end{aligned} \tag{9}$$

for all $r \in [t,T]$, in which the trajectory $\sigma \mapsto \xi_\sigma$, $\sigma \in [t,T]$, is as per (2). Recalling (3), note that

$$\begin{aligned} &\left| \int_t^T \left[\int_{r\vee\rho}^T -\nabla^2 V(\xi_\sigma)\, d\sigma + \nabla^2 \psi(\xi_T) \right] h_\rho \, d\rho \right| \\ &\leqslant \int_t^T \left[\int_t^T \tfrac{\kappa}{2}\, d\sigma + \tfrac{\kappa}{2} \right] |h_\rho|\, d\rho \\ &\leqslant \kappa\, \max(T - t, 1) \int_t^T |h_\rho|\, d\rho. \end{aligned} \tag{10}$$

Hence, applying Cauchy-Schwartz and Hölder on $\mathscr{U}[t,T]$, along with coercivity of $\mathcal{M}$ as per (3), yields

$$\begin{aligned} &\langle h,\, D_u \nabla_u J_t[\Psi](x,u)\, h\rangle \\ &\geqslant \langle h,\, \mathcal{M}\, h\rangle_{\mathscr{U}[t,T]} - \kappa\, \max(T - t, 1) \left[\int_t^T |h_r|\, dr \right]^2 \\ &\geqslant \kappa \left[\tfrac{m}{\kappa} - \max(T - t_0, 1)\,(T - t_0) \right] \|h\|^2_{\mathscr{U}[t,T]} \\ &= \epsilon\, \|h\|^2_{\mathscr{U}[t,T]}, \end{aligned}$$

in which $\epsilon > 0$ by choice of t_0. Hence, $D_u \nabla_u J_t[\Psi](x,u) \in \mathcal{L}(\mathscr{U}[t,T))$ is coercive, so that $J_t[\Psi](x,\cdot)$ is strictly convex. Consequently, there exists a unique optimal control $\bar{u}^* \in \mathscr{U}[t,T]$ that is the minimizer of $J_t[\Psi](x,\cdot)$, i.e. $\overline{W}_t(x) = J_t[\Psi](x, \bar{u}^*) \in \mathbb{R}$. *(ii)*: Standard dynamic programming arguments yield the viscosity solution assertion, see for example [2, Theorem 7.4.14, p.223]. *(iii)*: Pontryagin's minimum principle [17], [1] and *(i)* yield existence of a solution to (7) and the minimum condition (8). The fact that this is a classical solution follows as a consequence of (3), see Remark 2.2 below. ■

Remark 2.2: Classical solutions for the characteristic system (7) in the statement of Theorem 2.1 *(iii)* can be asserted via a global Lipschitz property. Given $f : \mathscr{X}^2 \mapsto \mathscr{X}^2$,

$$f(X) \doteq \begin{pmatrix} -\mathcal{M}^{-1}\, p \\ \nabla V(x) \end{pmatrix}, \quad X \doteq \begin{pmatrix} x \\ p \end{pmatrix},$$

note that any mild solution of (7) is also a mild solution of the corresponding final value problem with compatible terminal condition, i.e.

$$\dot{\overline{X}}_s = f(\overline{X}_s), \qquad \overline{X}_T = \begin{pmatrix} y \\ \nabla\Psi(y) \end{pmatrix}, \tag{11}$$

for all $s \in [t,T]$, with $y \doteq \bar{x}_T \in \mathscr{X}$. By (3), observe that f

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is continuously differentiable. By the mean value theorem,

$$|f(X+h)-f(X)| \leqslant \left|\left(\int_0^1 Df(X+\eta\, h)\, d\eta\right)(h)\right| \leqslant \int_0^1 \|Df(X+\eta\, h)\|_{\mathcal{L}(\mathscr{X}^2)}\, d\eta\, |h| \leqslant \alpha\, |h|,$$

for all $X, h \in \mathscr{X}$, in which (3) implies that

$$\alpha \doteq \sup_{\zeta\in\mathscr{X}} \left\| \begin{pmatrix} 0 & -\mathcal{M}^{-1} \\ \nabla^2 V(\zeta) & 0 \end{pmatrix} \right\|_{\mathcal{L}(\mathscr{X}^2)} < \infty.$$

That is, f is globally Lipschitz, with Lipschitz constant $\alpha \in \mathbb{R}$. Hence, (11) has a unique classical solution, see for example [14, Theorem 5.1, p.127]. Consequently, any mild solution of TPBVP (7) must correspond to the unique classical solution of (11) defined by the compatible choice of $y \in \mathscr{X}$. □

Remark 2.3: As any solution of TPBVP (7) must be a classical solution by Remark 2.2, the map $s \mapsto \bar{p}_s$, $s \in (t,T)$, is differentiable. Hence, the first equation in (7) may be differentiated, yielding

$$\ddot{\bar{x}}_s = -\mathcal{M}^{-1}\, \dot{\bar{p}}_s = -\mathcal{M}^{-1}\, \nabla V(\bar{x}_s), \quad s \in (t,T),$$

which is a generalized form of Newton's second law. Moreover, applying (5), (7), and the chain rule,

$$\begin{aligned} &\tfrac{d}{ds} H(\bar{x}_s, \bar{p}_s) = \langle \nabla_x H(\bar{x}_s, \bar{p}_s), \dot{\bar{x}}_s \rangle + \langle \nabla_p H(\bar{x}_s, \bar{p}_s), \dot{\bar{p}}_s \rangle \\ &= \langle \nabla V(\bar{x}_s), -\mathcal{M}^{-1}\, \bar{p}_s \rangle + \langle \mathcal{M}^{-1}\, \bar{p}_s, \nabla V(\bar{x}_s) \rangle = 0, \end{aligned}$$

for all $s \in (t,T)$. Consequently, the Hamiltonian is the conserved quantity, along the characteristic flow, as expected by the minimum principle underlying (8). □

Theorem 2.1 demonstrates that solutions of the TPBVP (7) describe those trajectories that render the action stationary in the statement of the action principle. As expected, it is also possible to equivalently characterize these trajectories via a verification theorem for HJB PDE (6).

Theorem 2.4: Under the conditions of Theorem 2.1, suppose there exists $(t,x) \mapsto W_t(x) \in C^1((t_0,T)\times\mathscr{X};\mathbb{R})$ such that (6) holds, with $(\frac{\partial}{\partial t} W_t(x), \nabla W_t(x)) \in \mathbb{R}\times\mathscr{X}$ denoting its Fréchet derivative at $(t,x) \in (t_0,T)\times\mathscr{X}$. Then, $W_t(x) \leqslant J_t[\Psi](x,u)$ for all $u \in \mathscr{U}[t,T]$. Furthermore, if there exists a mild solution $s \mapsto \bar{x}^*_s$, $s \in (t,T)$, of (2) satisfying

$$\bar{x}^*_s = x + \int_t^s \bar{u}^*_\sigma\, d\sigma, \quad \bar{u}^*_\sigma = -\mathcal{M}^{-1}\, \nabla W_\sigma(\bar{x}^*_\sigma), \tag{12}$$

such that $\bar{x}^*_s \in \mathscr{X}$ for all $s \in (t,T)$, then $W_t(x) = J_t[\Psi](x, \bar{u}^*_t) = \overline{W}_t(x)$ for all $x \in \mathscr{X}$.

Proof: Fix $T \in \mathbb{R}$. Suppose the conditions of Theorem 2.1 hold, defining $t_0 \in \mathbb{R}$, $t_0 < T$. Suppose $(t,x) \mapsto W_t(x) \in C^1((t_0,T)\times\mathscr{X};\mathbb{R})$ is a (classical) solution of HJB PDE (6). Fix $t \in (t_0,T)$, $x \in \mathscr{X}$, $\bar{u} \in \mathscr{U}[t,T]$, and let $s \mapsto \bar{x}_s$ denote the corresponding mild solution of (2) with $\bar{x}_t = x$ and $u = \bar{u}$. Define $\bar{p}_s \doteq \nabla W_s(\bar{x}_s)$ and note that $\bar{p}_s \in \mathscr{X}$ for all $s \in (t,T)$. Fix $s \in (t,T)$ and observe by coercivity of $\mathcal{M}$ and completion of squares as per (5) that

$$\begin{aligned} &-\langle \bar{p}_s,\, \bar{u}_s \rangle - \tfrac{1}{2} \langle \bar{u}_s,\, \mathcal{M}\, \bar{u}_s \rangle \\ &= \tfrac{1}{2} \langle \bar{p}_s,\, \mathcal{M}^{-1}\, \bar{p}_s \rangle - \tfrac{1}{2} \| \mathcal{M}^{\frac{1}{2}}\, (\bar{u}_s + \mathcal{M}^{-1}\, \bar{p}_s) \|^2 \\ &\leqslant \tfrac{1}{2} \langle \bar{p}_s,\, \mathcal{M}^{-1}\, \bar{p}_s \rangle. \end{aligned} \tag{13}$$

Consequently, by the chain rule, (5), (6), and the definition of $\bar{p}_s$ above,

$$\begin{aligned} &\tfrac{d}{ds} W_s(\bar{x}_s) = \tfrac{\partial}{\partial s} W_s(\bar{x}_s) + \langle \nabla W_s(\bar{x}_s), \bar{u}_s \rangle \\ &= -[-\tfrac{\partial}{\partial s} W_s(\bar{x}_s) + H(x, \nabla W_s(\bar{x}_s))] \\ &\quad + H(x, \nabla W_s(\bar{x}_s)) + \langle \nabla W_s(\bar{x}_s), \bar{u}_s \rangle \\ &= \tfrac{1}{2} \langle \bar{p}_s,\, \mathcal{M}^{-1} \bar{p}_s \rangle + V(\bar{x}_s) + \langle \bar{p}_s, \bar{u}_s \rangle \\ &\geqslant -\tfrac{1}{2} \langle \bar{u}_s,\, \mathcal{M}\, \bar{u}_s \rangle + V(\bar{x}_s). \end{aligned}$$

Integrating with respect to $s \in (t,T)$, and recalling the terminal condition in (6),

$$\begin{aligned} &\Psi(\bar{x}_T) - W_t(x) \geqslant -\int_t^T \tfrac{1}{2} \langle \bar{u}_s,\, \mathcal{M}\, \bar{u}_s \rangle - V(\bar{x}_s)\, ds \\ &\implies W_t(x) \leqslant J_t[\Psi](x, \bar{u}). \end{aligned} \tag{14}$$

As $x \in \mathscr{X}$ and $\bar{u} \in \mathscr{U}[t,T]$ are arbitrary, the first assertion follows. Moreover, if $\bar{u}^*$ exists as per (12), then (13), (14) hold with equality, and the second assertion follows. ■

III. STATIONARY ACTION AND STATIONARY CONTROL

Theorem 2.1 guarantees that stationarity of the action (1) is achieved at a minimum, provided that the maximal time horizon $T - t_0$ is sufficiently short. For longer horizons, Theorem 2.1 is no longer applicable, typically due to a loss of convexity of (1). This is manifested in the optimal control problem (4) as finite escape phenomena exhibited by the value function $t \mapsto W_t$ as $T - t > 0$ increases.

As the connection between stationary (least) action and optimal control breaks down for longer time horizons, stationarity of the action is instead formalized by replacing the *inf* operation in (4) with a *stat* operation [12], [13]. This *stat* operation, along with the corresponding *argstat* operation, can be defined for Fréchet differentiable functions $F \in C^1(\mathscr{W};\mathbb{R})$ on any real Hilbert space $\mathscr{W}$ by

$$\begin{aligned} &\operatorname*{stat}_{w\in\mathscr{W}} F(w) \doteq \left\{ F(\overline{w}) \,\middle|\, \overline{w} \in \operatorname*{arg\,stat}_{w\in\mathscr{W}} F(w) \right\}, \\ &\operatorname*{arg\,stat}_{w\in\mathscr{W}} F(w) \doteq \{ \overline{w} \in \mathscr{W} \mid \nabla F(\overline{w}) = 0 \}, \end{aligned} \tag{15}$$

in which $\nabla F : \mathscr{W} \to \mathcal{L}(\mathscr{W};\mathbb{R})$ denotes the Riesz representation of the derivative. As the action $J_t[\Psi](x,\cdot) : \mathscr{U}[t,T] \to \mathbb{R}$ is continuously Fréchet differentiable, see [4, Theorem 3.6], and $\mathscr{U}[t,T]$ is a real Hilbert space, it is possible to select $\mathscr{W} \doteq \mathscr{U}[t,T]$ and $F \doteq J_t[\Psi](x,\cdot)$ in (15).

The ensuing *stationary control problem* is defined for *any* time horizon $T - t > 0$, $t, T \in \mathbb{R}$, by a (possibly set-valued) *stat* value function $\widetilde{W}_t : \mathscr{X} \to \overline{\mathbb{R}}$, with

$$\widetilde{W}_t(x) \doteq \operatorname*{stat}_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u), \tag{16}$$

for all $x \in \mathscr{X}$. Given a specific $x \in \mathscr{X}$, trajectories that render the action $J_t[\Psi](x, \cdot)$ stationary as per the action principle can be characterized as follows [4, Theorem 3.9].

Theorem 3.1: Suppose (3) holds. Given $t, T \in \mathbb{R}$, $t \leqslant T$, $x \in \mathscr{X}$, an input $\bar{u} \in \mathscr{U}[t,T]$ is *staticizing*, i.e.

$$\bar{u} \in \operatorname*{arg\,stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x, u) \tag{17}$$

if and only if there exists a classical solution of the TPBVP

$$\begin{cases} \dot{\bar{x}}_s = -\mathcal{M}^{-1}\, \bar{p}_s, & \bar{x}_t = x, \\ \dot{\bar{p}}_s = \nabla V(\bar{x}_s), & \bar{p}_T = \nabla \Psi(\bar{x}_T), \end{cases} \tag{18}$$

for all $s \in [t,T]$. Furthermore, $\bar{u} \in \mathscr{U}[t,T]$ satisfies

$$\bar{u}_s = -\mathcal{M}^{-1}\, \bar{p}_s, \quad s \in [t,T]. \tag{19}$$

Proof: [Sketch] The argument used here is a minor generalization of [4, Theorem 3.9], involving the replacement of a scalar inertia with the coercive inertia operator $\mathcal{M} \in \mathcal{L}(\mathscr{X})$, and strengthening the TPBVP solutions from mild to classical as per Remark 2.2. ∎

By inspection, TPBVPs (7), (18) are identical except for the time horizon on which solutions are sought. For short time horizons, as required in (4), (7), the input $\bar{u} \in \mathscr{U}[t,T]$ defined by (8) is a minimizer for the action (1). Theorems 2.1 and 2.4 provide a means for synthesizing this via solution of HJB PDE (6) and the application of Theorem 2.4.

For longer horizons, as allowed in (16), (18), the input $\bar{u} \in \mathscr{U}[t,T]$ defined by (17) need only render the action (1) stationary. As Theorems 2.1 and 2.4 are unavailable on these longer horizons, it is not possible to construct $\bar{u}$ via HJB PDE (6). However, verification of the stationary control is possible using an alternative approach that again appeals to a pair of optimal control and corresponding HJB PDEs, via semiconvex and semiconcave duality.

IV. Verification of the staticizing control

The aim is to develop a verification argument for the staticizing input (17), applicable to longer time horizons. Crucial to this development is a new characterization of the argstat operation (15) using semiconvex and semiconcave duality. This characterization is applicable for any real Hilbert space $\mathscr{W}$, although its application here will be restricted to the case $\mathscr{W} \doteq \mathscr{U}[t,T]$, given $t, T \in \mathbb{R}$, $t < T$. Unlike [12], this development will make use of a pair of optimal control problems, rather than a single stationary control problem.

A. Duality based characterization of argstat

Some preliminary definitions are required. A function $\psi : \mathscr{W} \to \overline{\mathbb{R}}$ is convex if its epigraph $\{(w,\alpha) \in \mathscr{W} \times \mathbb{R} \,|\, \psi(w) \leqslant \alpha\}$ is convex [15]. It is lower closed if $\psi = \text{cl}^- \psi$, in which $\text{cl}^- \psi$ is the lower closure of ψ, defined with respect to the corresponding lower semicontinuous envelope $\text{lsc}\,\psi$ by

$$\text{cl}^- \psi(w) \doteq \begin{cases} \text{lsc}\,\psi(w), & \text{lsc}\,\psi(w) > -\infty \ \forall\, w \in \mathscr{W}, \\ -\infty & \text{otherwise}, \end{cases}$$

for all $w \in \mathscr{W}$. A function ψ is concave if $-\psi$ is convex, and upper closed if $-\psi$ is lower closed, see [15, pp.15-17]. Recall that a proper lsc function is convex if and only if it is the pointwise supremum of its affine support functions, see for example [16, Theorem 8.13, p.309].

Semiconvexity and semiconcavity, and subsequent relaxed notions of duality, are defined with respect to a bivariate quadratic support or basis function $\varphi : \mathscr{W} \times \mathscr{W} \to \mathbb{R}$ that has a fixed coercive Hessian $\mathcal{C} \in \mathcal{L}(\mathscr{W})$. Explicitly,

$$\varphi(v,w) \doteq -\tfrac{1}{2}\langle v - w,\, \mathcal{C}\,(v-w)\rangle \tag{20}$$

for all $v, w \in \mathscr{W}$. Using (20), the spaces $\mathscr{S}_\varphi^+$ and $\mathscr{S}_\varphi^-$ of (uniformly) semiconvex and semiconcave functions are defined respectively by

$$\begin{aligned} \mathscr{S}_\varphi^+ &\doteq \left\{ \psi : \mathscr{W} \to \overline{\mathbb{R}} \,\middle|\, \begin{array}{c} v \mapsto \psi(v) - \varphi(v,0) \\ \text{convex, lower closed} \end{array} \right\}, \\ \mathscr{S}_\varphi^- &\doteq \left\{ \phi : \mathscr{W} \to \overline{\mathbb{R}} \,\middle|\, -\phi \in \mathscr{S}_\varphi^+ \right\}. \end{aligned} \tag{21}$$

These spaces are in duality, via either the semiconvex transform $\mathcal{D}_\varphi^+$, see for example [9], [3], or the analogously defined semiconcave transform $\mathcal{D}_\varphi^-$, i.e.

$$\mathscr{S}_\varphi^+ \underset{\mathcal{D}_\varphi^- \,\equiv\, [\mathcal{D}_\varphi^+]^{-1}}{\overset{\mathcal{D}_\varphi^+ \,\equiv\, [\mathcal{D}_\varphi^-]^{-1}}{\rightleftarrows}} \mathscr{S}_\varphi^-$$

The semiconvex transform and its inverse are given by

$$\begin{aligned} (\mathcal{D}_\varphi^+\, \psi)(w) &\doteq -\sup_{v \in \mathscr{W}} \{\varphi(v,w) - \psi(v)\}, \quad \psi \in \mathscr{S}_\varphi^+, \\ ([\mathcal{D}_\varphi^+]^{-1}\, \phi)(v) &= \sup_{w \in \mathscr{W}} \{\varphi(v,w) + \phi(w)\}, \quad \phi \in \mathscr{S}_\varphi^-, \end{aligned} \tag{22}$$

for all $v, w \in \mathscr{W}$, while for the semiconcave transform,

$$\begin{aligned} \mathcal{D}_\varphi^-\, \phi &\doteq -\mathcal{D}_\varphi^+[-\phi] = [\mathcal{D}_\varphi^+]^{-1}\phi, \quad \phi \in \mathscr{S}_\varphi^-, \\ [\mathcal{D}_\varphi^-]^{-1}\, \psi &= -[\mathcal{D}_\varphi^+]^{-1}\,[-\psi] = \mathcal{D}_\varphi^+\, \psi, \quad \psi \in \mathscr{S}_\varphi^+, \end{aligned} \tag{23}$$

in which the symmetry of φ with respect to its arguments is used to obtain the right-hand equivalences in (23).

Remark 4.1: Given any $\psi \in \mathscr{S}_\varphi^+$, by inspection of (22), the semiconvex transform provides a supremum of quadratics representation for ψ using φ, i.e.

$$\psi(v) = ([\mathcal{D}_\varphi^+]^{-1}\mathcal{D}_\varphi^+\, \psi)(v) = \sup_{w \in \mathscr{W}} \{\varphi(v,w) + (\mathcal{D}_\varphi^+\, \psi)(w)\}$$

for all $v \in \mathscr{W}$. Similarly, given any $\phi \in \mathscr{S}_\varphi^-$, by inspection of (23), the semiconcave transform provides an infimum of quadratics representation for ϕ using $-\varphi$, i.e.

$$\phi(w) = ([\mathcal{D}_\varphi^-]^{-1}\mathcal{D}_\varphi^-\, \phi)(w) = \inf_{v \in \mathscr{W}} \{-\varphi(w,v) + (\mathcal{D}_\varphi^-\, \psi)(v)\}$$

for all $w \in \mathscr{W}$, noting again the symmetry of φ. □

A new characterization of the argstat operation (15), using the spaces of semiconvex and semiconcave functions (21) and their respective transforms (22), (23), is as follows.

Theorem 4.2: Suppose $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. Then,

$$(\mathcal{D}_\varphi^+ F)(w) \leqslant F(w) \leqslant (\mathcal{D}_\varphi^- F)(w) \tag{24}$$

for all $w \in \mathscr{W}$, and

$$\overline{w} \in \operatorname*{arg\,stat}_{w \in \mathscr{W}} F(w) \iff (\mathcal{D}_\varphi^+ F)(\overline{w}) = (\mathcal{D}_\varphi^- F)(\overline{w}). \tag{25}$$

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Proof: Fix $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. With $w \in \mathscr{W}$, recalling (22), (23), and noting the symmetry of φ of (20), define

$$\begin{aligned} a(w) &\doteq (\mathcal{D}_\varphi^+ F)(w) = \inf_{v\in\mathscr{W}} \{F(v) - \varphi(v,w)\}, \\ b(w) &\doteq (\mathcal{D}_\varphi^- F)(w) = \sup_{v\in\mathscr{W}} \{F(v) + \varphi(v,w)\}. \end{aligned} \tag{26}$$

As $w \in \mathscr{W}$ is suboptimal in both right-hand sides in (26), and $\varphi(w,w) = 0$, by inspection,

$$a(w) \leqslant F(w) \leqslant b(w). \tag{27}$$

That is, (24) holds.

[Necessity]. Suppose the right-hand statement in (25) holds. That is, recalling (26), there exists $\overline{w} \in \mathscr{W}$ such that

$$a(\overline{w}) = b(\overline{w}). \tag{28}$$

Together, (27), (28), yield

$$a(\overline{w}) = F(\overline{w}) = b(\overline{w}). \tag{29}$$

Fix any $h \in \mathscr{W}$. As $\overline{w} + h \in \mathscr{W}$ is also suboptimal in the definitions of a and b, (26), (29) imply that

$$\begin{aligned} F(\overline{w}) = a(\overline{w}) &\leqslant F(\overline{w}+h) - \varphi(\overline{w}+h, \overline{w}), \\ F(\overline{w}) = b(\overline{w}) &\geqslant F(\overline{w}+h) + \varphi(\overline{w}+h, \overline{w}). \end{aligned}$$

These inequalities and (20) together yield

$$\begin{aligned} |F(\overline{w}+h) - F(\overline{w})| &\leqslant -\varphi(\overline{w}+h, h) = \tfrac{1}{2}\langle h, \mathcal{C}\, h\rangle \\ &\leqslant \|\mathcal{C}\|_{\mathcal{L}(\mathscr{W})}\, |h|^2. \end{aligned}$$

As $h \in \mathscr{W}$ is arbitrary, it follows that F is Fréchet differentiable at $\overline{w}$, with derivative and its Riesz representation given by $DF(\overline{w}) = 0 \in \mathcal{L}(\mathscr{W})$ and $\nabla F(\overline{w}) = 0 \in \mathscr{W}$ respectively. Hence, recalling (15), $\overline{w} \in \arg\operatorname{stat}_{w\in\mathscr{W}} F(w)$, as required.

[Sufficiency]. Suppose the left-hand statement in (25) holds, i.e. there exists a $\overline{w} \in \arg\operatorname{stat}_{w\in\mathscr{W}} F(w)$. By definition (15), $\nabla F(\overline{w}) = 0$. Note further by (20) that $\nabla_v \varphi(v, \overline{w})|_{v=\overline{w}} = 0$, so that

$$\nabla_v [F(v) - \varphi(v, \overline{w})]|_{v=\overline{w}} = 0. \tag{30}$$

Recall that $v \mapsto F(v) - \varphi(v, 0)$ is convex, as $F \in \mathscr{S}_\varphi^+$, and $\varphi(v,0) - \varphi(v, \overline{w})$ is affine. Hence, the map $v \mapsto F(v) - \varphi(v, \overline{w})$ must also be convex, while simultaneously satisfying (30). Hence, it has a global minimum at $v = \overline{w}$, so that

$$a(\overline{w}) = \inf_{v\in\mathscr{W}} \{F(v) - \varphi(v, \overline{w})\} = F(\overline{w}) - \varphi(\overline{w}, \overline{w}) = F(\overline{w}),$$

by (26). Similarly, as $F \in \mathscr{S}_\varphi^-$, the map $v \mapsto F(v) + \varphi(v, \overline{w})$ is concave, simultaneously satisfying (30). Hence, it has a global maximum at $v = \overline{w}$, so that

$$b(\overline{w}) = \sup_{v\in\mathscr{W}} \{F(v) + \varphi(v, \overline{w})\} = F(\overline{w}) + \varphi(\overline{w}, \overline{w}) = F(\overline{w}),$$

by (26). Hence, combining these two conclusions yields

$$(\mathcal{D}_\varphi^+ F)(\overline{w}) = a(\overline{w}) = F(\overline{w}) = b(\overline{w}) = (\mathcal{D}_\varphi^- F)(\overline{w}),$$

which completes the proof. ■

The following lemma is useful in the subsequent application of Theorem 4.2.

Lemma 4.3: With $F \in C^2(\mathscr{W}; \mathbb{R})$, suppose that the first Fréchet derivative of the Riesz representation of its first Fréchet derivative $D\nabla F : \mathscr{W} \to \mathcal{L}(\mathscr{W})$ is uniformly bounded, i.e. $\bar{c} \doteq \sup_{w\in\mathscr{W}} \|D\nabla F(w)\|_{\mathcal{L}(\mathscr{W})} < \infty$. Then, for any $\mathcal{C} \in \mathcal{L}(\mathscr{W})$, $\epsilon \geqslant 0$, satisfying $\langle h, \mathcal{C}\, h\rangle \geqslant (\bar{c} + \epsilon)\, |h|^2$ for all $h \in \mathscr{W}$, the support φ defined by (20) is such that *(i)* $w \mapsto F(w) - \varphi(w, 0)$ is (strictly) convex and $w \mapsto F(w) + \varphi(w, 0)$ is (strictly) concave for ($\epsilon > 0$) $\epsilon \geqslant 0$; and *(ii)* for any $\epsilon \geqslant 0$,

$$F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-. \tag{31}$$

Proof: Fix $F \in C^2(\mathscr{W}; \mathbb{R})$ and $\bar{c} < \infty$ as per the lemma statement. Fix $\epsilon \in \mathbb{R}_{\geqslant 0}$. Select any coercive $\mathcal{C} \in \mathcal{L}(\mathscr{W})$ such that $\langle h, \mathcal{C}\, h\rangle \geqslant (\bar{c} + \epsilon)\, |h|^2$ for all $h \in \mathscr{W}$, e,g, $\mathcal{C} \doteq c\mathcal{I}$, $c \geqslant \bar{c} + \epsilon$. Using this $\mathcal{C}$, define φ as per (20).

(i) Fix any $w, h \in \mathscr{W}$. As $F \in C^2(\mathscr{W}; \mathbb{R})$, its first Fréchet derivative at w satisfies $DF(w)\, h = \langle \nabla F(w), h\rangle$, in which $\nabla F(w) \in \mathscr{W}$ is the corresponding Riesz representation. Moreover,

$$D\,\nabla F(w) \in \mathcal{L}(\mathscr{W}), \quad D^2 F(w)\, h\, h = \langle h,\, D\,\nabla F(w)\, h\rangle,$$

see for example [4, Appendix]. Hence, for $\mu \doteq \pm 1$,

$$\begin{aligned} D^2[\,\mu\, F(w) - \varphi(w, 0)]\, h\, h &= \mu\langle h,\, D\,\nabla F(w)\, h\rangle + \langle h,\, \mathcal{C}\, h\rangle \\ &\geqslant -\bar{c}\, |h|^2 + \langle h,\, \mathcal{C}\, h\rangle \geqslant \epsilon\, |h|^2. \end{aligned}$$

As $w, h \in \mathscr{W}$ are arbitrary, it follows that $w \mapsto \pm F(w) - \varphi(w, 0)$ is (strictly) convex, as ($\epsilon > 0$) $\epsilon \geqslant 0$.

(ii) The maps $w \mapsto \pm F(w) - \varphi(w, 0)$ are continuous by definition, and hence lower closed. Hence, applying *(i)* and (21), $F \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-$. ■

B. Application to stationary control

Given fixed $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), the intention is to apply Theorem 4.2 to the stationary control problem (16), (17) for any $t \in [t_0, T)$, with $\mathscr{W} \doteq \mathscr{U}[t, T]$ and $F \doteq J_t[\Psi](x, \cdot)$. In order to explicitly define the quadratic support function $\varphi : \mathscr{U}[t,T] \times \mathscr{U}[t,T] \to \mathbb{R}$ as per (20), let

$$\begin{aligned} \mathcal{C} &\doteq c\mathcal{I} \in \bigcup_{t\in[t_0,T)} \mathcal{L}(\mathscr{U}[t,T]), \quad c \in \mathbb{R},\ c \geqslant c_{t_0}, \\ c_{t_0} &\doteq 1 + \|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})} + \kappa\, \max(T - t_0, 1)\, (T - t_0) < \infty. \end{aligned} \tag{32}$$

Lemma 4.4: Suppose (3) holds. Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, and support φ as per (20), (32), the following properties concerning the action (1) hold: *(i)* $u \mapsto J_t[\Psi](x, u) - \varphi(u, 0)$ is strictly convex and $u \mapsto J_t[\Psi](x, u) + \varphi(u, 0)$ is strictly concave; and *(ii)* $J_t[\Psi](x, \cdot)$ is simultaneously semiconvex and semiconcave, i.e.

$$J_t[\Psi](x, \cdot) \in \mathscr{S}_\varphi^+ \cap \mathscr{S}_\varphi^-. \tag{33}$$

Proof: Fix any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u, h, \tilde{h} \in \mathscr{U}[t, T]$. Recall by [4, Theorem 3.6] that the action (1) is three times Fréchet differentiable, i.e. $J_t[\Psi](x, \cdot) \in C^3(\mathscr{U}[t,T]; \mathbb{R})$. *(i)* Recalling (9), (10), and applying Cauchy-Schwartz and Hölder on $\mathscr{U}[t, T]$, the Fréchet derivative of the Riesz representation of the first Fréchet derivative

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of $J_t[\Psi](x,\cdot)$, i.e. $D_u \nabla_u J_t[\Psi](x,u) \in \mathcal{L}(\mathscr{U}[t,T])$, satisfies

$$\begin{aligned}\langle \tilde{h},\, D_u \nabla_u J_t[\Psi](x,u)\, h\rangle &\leqslant \|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})}\, |\tilde{h}|\, |h| \\ &+ \kappa\, \max(T-t,1) \int_t^T |\tilde{h}_r|\, dr \int_t^T |h_\rho|\, d\rho \\ &\leqslant \left[\|\mathcal{M}\|_{\mathcal{L}(\mathscr{X})} + \kappa\, \max(T-t,1)\, (T-t)\right] |\tilde{h}|\, |h|.\end{aligned}$$

As $h, \tilde{h} \in \mathscr{U}[t,T]$ are arbitrary, it follows immediately by definition (32) of c_{t_0} that $c_{t_0} \geqslant \bar{c}_{t_0} + \epsilon$, with $\epsilon \doteq 1$ and

$$\bar{c}_{t_0} \doteq \|D_u \nabla_u J_t[\Psi](x,u)\|_{\mathcal{L}(\mathscr{U}[t,T])}.$$

Hence, by definitions (20), (32) of φ, $\mathcal{C}$, Lemma 4.3 *(i)* implies that assertion *(i)* holds. Assertion *(ii)*, i.e. (33), is subsequently immediate by Lemma 4.3 *(ii)* ∎

Lemma 4.5: Suppose (3) holds. Given any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t,T]$,

$$(\mathcal{D}_\varphi^+ J_t[\Psi](x,\cdot))(u) \leqslant (\mathcal{D}_\varphi^- J_t[\Psi](x,\cdot))(u). \tag{34}$$

Moreover, the argstat condition (17) holds, i.e. $\bar{u} \in \arg\operatorname{stat}_{u\in\mathscr{U}[t,T]} J_t[\Psi](x,u)$, if and only if

$$(\mathcal{D}_\varphi^+ J_t[\Psi](x,\cdot))(\bar{u}) = (\mathcal{D}_\varphi^- J_t[\Psi](x,\cdot))(\bar{u}), \tag{35}$$

in which $\mathcal{D}_\varphi^\pm$ and φ are as per (23) and (20), (32).

Proof: Fix any $t \in [t_0, T)$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t,T]$. Observe by Lemma 4.4 that (33) holds. Hence, applying Theorem 4.2 with $\mathscr{W} \doteq \mathscr{U}[t,T]$, $F \doteq J_t[\Psi](x,\cdot)$, and φ defined via (20), (32), yields inequality (34) via (24), and the stated equivalence between (17) and (35) via (25). ∎

C. Auxiliary optimal control problems

In order to apply (35), it is useful to first rewrite both sides in a more familiar form. In particular, given $t \in [t_0, T)$ and $x \in \mathscr{X}$, and recalling the definitions (23) and (1) of $\mathcal{D}_\varphi^\pm$ and $J_t[\Psi](x,\cdot)$, observe that

$$\begin{aligned}(\mathcal{D}_\varphi^+ J_t[\Psi](x,\cdot))(\bar{u}) &= \inf_{u\in\mathscr{U}[t,T]} \{J_t[\Psi](x,u) - \varphi(u,\bar{u})\}, \\ (\mathcal{D}_\varphi^- J_t[\Psi](x,\cdot))(\bar{u}) &= \sup_{u\in\mathscr{U}[t,T]} \{J_t[\Psi](x,u) + \varphi(u,\bar{u})\}.\end{aligned} \tag{36}$$

That is, the two sides of (35) define a pair of auxiliary optimal control problems, parameterized by $\bar{u} \in \mathscr{U}[t,T]$. In view of (36), given any $v \in \mathscr{U}[t_0,T]$, explicitly define the auxiliary cost functions

$$J_t^v[\Psi],\ \widehat{J}_t^v[\Psi] : \mathscr{X} \times \mathscr{U}[t,T] \to \mathbb{R}$$

via (1) and (20), (32) by

$$\begin{aligned}J_t^v[\Psi](x,u) &\doteq J_t[\Psi](x,u) - \varphi(u,v) \\ &= \int_t^T \tfrac{1}{2}\langle u_s,\, \mathcal{E}\, u_s\rangle - \langle u_s,\, \mathcal{C}\, v_s\rangle + \tfrac{1}{2}\langle v_s,\, \mathcal{C}\, v_s\rangle - V(\xi_s)\, ds \\ &\qquad + \Psi(\xi_T),\end{aligned} \tag{37}$$

$$\begin{aligned}\widehat{J}_t^v[\Psi](x,u) &\doteq J_t[\Psi](x,u) + \varphi(u,v) \\ &= \int_t^T -\tfrac{1}{2}\langle u_s,\, \widehat{\mathcal{E}}\, u_s\rangle + \langle u_s,\, \mathcal{C}\, v_s\rangle - \tfrac{1}{2}\langle v_s,\, \mathcal{C}\, v_s\rangle - V(\xi_s)\, ds \\ &\qquad + \Psi(\xi_T),\end{aligned} \tag{38}$$

for all $t \in [t_0, T]$, $x \in \mathscr{X}$, and $u \in \mathscr{U}[t,T]$, in which

$$\mathcal{E}, \widehat{\mathcal{E}} \in \mathcal{L}(\mathscr{X}), \quad \mathcal{E} \doteq \mathcal{C} + \mathcal{M}, \quad \widehat{\mathcal{E}} \doteq \mathcal{C} - \mathcal{M}, \tag{39}$$

are coercive by (32). The aforementioned auxiliary optimal control problems are defined via their respective value functions $W_t^v, \widehat{W}_t^v : \mathscr{X} \times \mathscr{U}[t,T] \to \mathbb{R}$, with

$$W_t^v(x) \doteq \inf_{u\in\mathscr{U}[t,T]} J_t^v[\Psi](x,u), \tag{40}$$

$$\widehat{W}_t^v(x) \doteq \sup_{u\in\mathscr{U}[t,T]} \widehat{J}_t^v[\Psi](x,u), \tag{41}$$

for all $x \in \mathscr{X}$, $v \in \mathscr{U}[t,T]$. Analogously to the short horizon case of (4), (6), and Theorem 2.1, the relevant Hamiltonians are defined with respect to (5) by

$$H^v(t,x,p) \doteq H(x,p) - \tfrac{1}{2}\langle p + \mathcal{M}\, v_t,\, \mathcal{G}(p + \mathcal{M}\, v_t)\rangle, \tag{42}$$

$$\widehat{H}^v(t,x,p) \doteq H(x,p) - \tfrac{1}{2}\langle p + \mathcal{M}\, v_t,\, \widehat{\mathcal{G}}(p + \mathcal{M}\, v_t)\rangle, \tag{43}$$

for all $v \in \mathscr{U}[t,T]$, $t \in [t_0,T]$, $x, p \in \mathscr{X}$, in which

$$\mathcal{G}, \widehat{\mathcal{G}} \in \mathcal{L}(\mathscr{X}),\ \mathcal{G} \doteq \mathcal{M}^{-1} - \mathcal{E}^{-1},\ \widehat{\mathcal{G}} \doteq \mathcal{M}^{-1} + \widehat{\mathcal{E}}^{-1}, \tag{44}$$

are coercive, by coercivity of $\mathcal{E}$, $\widehat{\mathcal{E}}$ of (39). By (5), (44), the maps $p \mapsto H^v(t,x,p)$ and $p \mapsto \widehat{H}^v(t,x,p)$ are respectively convex and concave. Properties of these auxiliary optimal control problems follow analogously to Theorem 2.1, while being applicable to longer time horizons.

Theorem 4.6: Suppose (3) holds. Given arbitrary $v \in C([t_0,T];\mathscr{X})$, the following properties of (40) hold:

(i) Given $t \in [t_0,T]$, $x \in \mathscr{X}$, there exists an optimal input $u_v^* \in \mathscr{U}[t,T]$ such that $W_t^v(x) = J_t^v[\Psi](x,u_v^*) \in \mathbb{R}$;

(ii) Given Hamiltonian H^v of (42), the function $(t,x) \mapsto W_t^v(x)$ is the unique viscosity solution of the HJB PDE

$$\begin{cases} 0 = -\dfrac{\partial W_t}{\partial t} + H^v(t,x,\nabla_x W_t(x)), \\ W_T(x) = \Psi(x), \end{cases} \tag{45}$$

for all $t \in [t_0,T]$, $x \in \mathscr{X}$; and

(iii) There exists a classical solution of the TPBVP

$$\begin{cases} \dot{x}_s = v_s - \mathcal{E}^{-1}(\mathcal{M}\, v_s + p_s), & x_t = x, \\ \dot{p}_s = \nabla V(x_s), & p_T = \nabla\Psi(x_T) \end{cases} \tag{46}$$

for all $s \in [t,T]$, and the optimal input satisfies

$$[u_v^*]_s = v_s - \mathcal{E}^{-1}(\mathcal{M}\, v_s + p_s), \quad s \in [t,T]. \tag{47}$$

Proof: (i): Fix $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), and let $\mathcal{C} \in \mathcal{L}(\mathscr{X})$ be as per (32). Fix $t \in [t_0,T]$. Observe by Lemma 4.4 that $J_t^v[\Psi](x,\cdot) : \mathscr{U}[t,T] \to \mathbb{R}$ is strictly convex and coercive. Hence, there exists a unique optimal control $u_v^* \in \mathscr{U}[t,T]$ that is the minimizer of $J_t^v[\Psi](x,\cdot)$, i.e. $W_t^v(x) = J_t^v[\Psi](x,u_v^*) \in \mathbb{R}$. *(ii):* Standard dynamic programming arguments yield the viscosity solution assertion, see for example [2, Theorem 7.4.14, p.223]. *(iii):* The characteristic system (46) follows by inspection of (43). Existence of a solution to (46) follows by Pontryagin's minimum principle and *(i)*. Letting

$$f^v(X) \doteq \begin{pmatrix} -\mathcal{E}^{-1}\, p \\ \nabla V(x) \end{pmatrix} + \begin{pmatrix} \mathcal{I} - \mathcal{E}^{-1}\, \mathcal{M} \\ 0 \end{pmatrix} v,\ X \doteq \begin{pmatrix} x \\ p \end{pmatrix},$$

note as per Remark 2.2 that f^v is globally Lipschitz. Consequently, any mild solution of TPBVP (46) must correspond to the unique classical solution of the corresponding final (or initial) value problem, see for example [14, Theorem 5.1, p.127]. ■

Theorem 4.7: Suppose (3) holds. Given arbitrary $v \in C([t_0,T];\mathscr{X})$, the following properties concerning the value function (41) hold:

(i) Given $t \in [t_0,T]$, $x \in \mathscr{X}$, there exists an optimal input $\hat{u}_v^* \in \mathscr{U}[t,T]$ such that $\widehat{W}_t^v(x) = \widehat{J}_t^v[\Psi](x,\hat{u}_v^*) \in \mathbb{R}$;

(ii) Given Hamiltonian $\widehat{H}^v$ of (43), the function $(t,x) \mapsto \widehat{W}_t^v(x)$ is the unique viscosity solution of the HJB PDE

$$\begin{cases} 0 = -\dfrac{\partial W_t}{\partial t} + \widehat{H}^v(t,x,\nabla_x W_t(x)), \\ W_T(x) = \Psi(x), \end{cases} \tag{48}$$

for all $t \in [t_0,T]$, $x \in \mathscr{X}$; and

(iii) There exists a classical solution of the TPBVP

$$\begin{cases} \dot{\hat{x}}_s = v_s + \widehat{\mathcal{E}}^{-1}(\mathcal{M}\, v_s + \hat{p}_s), & \hat{x}_t = x, \\ \dot{\hat{p}}_s = \nabla V(\hat{x}_s), & \hat{p}_T = \nabla\Psi(\hat{x}_T) \end{cases} \tag{49}$$

for all $s \in [t,T]$, and the optimal input satisfies

$$[\hat{u}_v^*]_s = v_s + \widehat{\mathcal{E}}^{-1}(\mathcal{M}\, v_s + \hat{p}_s), \quad s \in [t,T]. \tag{50}$$

Proof: (i): Fix $t_0, T \in \mathbb{R}$ with $t_0 < T$ via (3), and let $\mathcal{C} \in \mathcal{L}(\mathscr{X})$ be as per (32). Fix $t \in [t_0,T]$. Observe by Lemma 4.4 that $-\widehat{J}_t^v[\Psi](x,\cdot) : \mathscr{U}[t,T] \to \mathbb{R}$ is strictly convex and coercive. Hence, there exists a unique optimal control $\hat{u}_v^* \in \mathscr{U}[t,T]$ that is the maximizer of $\widehat{J}_t^v[\Psi](x,\cdot)$, i.e. $\widehat{W}_t^v(x) = J_t^v[\Psi](x,\hat{u}_v^*) \in \mathbb{R}$. The remaining assertions *(ii)* and *(iii)* follow analogously as per Theorem 4.7. ■

Verification theorems analogous to Theorem 2.4 likewise follow.

Theorem 4.8: Under the conditions of Theorem 4.6, with $v \in C([t,T];\mathscr{X})$ fixed, suppose there exists $(t,x) \mapsto W_t(x) \in C^1((t_0,T)\times\mathscr{X};\mathbb{R})$ such that (45) holds, with $(\frac{\partial}{\partial t}W_t(x), \nabla W_t(x)) \in \mathbb{R}\times\mathscr{X}$ denoting the Fréchet derivative at $(t,x) \in (t_0,T)\times\mathscr{X}$. Then, $W_t^v(x) \leqslant J_t^v[\Psi](x,u)$ for all $u \in \mathscr{U}[t,T]$. Furthermore, if there exists a mild solution $s \mapsto (x_v^*)_s$, $s \in (t,T)$ of (2) satisfying

$$\begin{aligned} [x_v^*]_s &= x + \int_t^s [u_v^*]_\sigma \, d\sigma, \\ [u_v^*]_\sigma &= v_\sigma - \mathcal{E}^{-1}(\mathcal{M}\, v_\sigma + \nabla W_t^v([x_v^*]_\sigma)), \end{aligned} \tag{51}$$

such that $[x_v^*]_s \in \mathscr{X}$ for all $s \in (t,T)$, then $W_t^v(x) = J_t^v[\Psi](x,u_v^*)$ for all $x \in \mathscr{X}$.

Proof: The proof follows that of Theorem 2.4. Under the conditions of Theorem 4.6, with $v \in C([t,T];\mathscr{X})$ fixed, let $(t,x) \mapsto W_t(x) \in C^1((t_0,T)\times\mathscr{X};\mathbb{R})$ be such that (45) holds. Fix $t \in (t_0,T)$, $x \in \mathscr{X}$, $\bar{u} \in \mathscr{U}[t,T]$, and let $s \mapsto \bar{x}_s$ denote the corresponding mild solution of (2) with $\bar{x}_t = x$ and $u = \bar{u}$. Define $\bar{p}_s \doteq \nabla W_s(\bar{x}_s)$ and note that $\bar{p}_s \in \mathscr{X}$ for all $s \in (t,T)$. Fix $s \in (t,T)$ and observe by coercivity of $\mathcal{E}$ of (39) and completion of squares that

$$\begin{aligned} &\tfrac{1}{2}\langle \bar{p}_s, \mathcal{E}^{-1}\bar{p}_s\rangle + \langle \bar{p}_s, \bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s\rangle \\ &= \tfrac{1}{2}\|\mathcal{E}^{\frac{1}{2}}(\mathcal{E}^{-1}\bar{p}_s + \bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s)\|^2 - \tfrac{1}{2}\|\mathcal{E}^{\frac{1}{2}}(\bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s)\|^2 \\ &\geqslant -\tfrac{1}{2}\|\mathcal{E}^{\frac{1}{2}}(\bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s)\|^2, \end{aligned} \tag{52}$$

which holds with equality when

$$\bar{u}_s = \mathcal{G}\,\mathcal{M}\, v_s - \mathcal{E}^{-1}\bar{p}_s = v_s - \mathcal{E}^{-1}(\mathcal{M}\, v_s + \bar{p}_s). \tag{53}$$

Consequently, by the chain rule, (5), (39), (42), (44), (45), and the definition of $\bar{p}_s$ above,

$$\begin{aligned} \tfrac{d}{ds}W_s(\bar{x}_s) &= -[-\tfrac{\partial}{\partial s}W_s(\bar{x}_s) + H^v(s,\bar{x}_s,\nabla W_s^v(\bar{x}_s)] \\ &\qquad + H^v(s,\bar{x}_s,\nabla W_s(\bar{x}_s)) + \langle \nabla W_s(\bar{x}_s), \bar{u}_s\rangle \\ &= H(\bar{x}_s,\bar{p}_s) - \tfrac{1}{2}\langle \bar{p}_s + \mathcal{M}\, v_s, \mathcal{G}\,(\bar{p}_s + \mathcal{M}\, v_s)\rangle + \langle \bar{p}_s, \bar{u}_s\rangle \\ &= \tfrac{1}{2}\langle \bar{p}_s, \mathcal{E}^{-1}\bar{p}_s\rangle + \langle \bar{p}_s, \bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s\rangle \\ &\qquad - \tfrac{1}{2}\langle v_s, \mathcal{M}\,\mathcal{G}\,\mathcal{M}\, v_s\rangle + V(\bar{x}_s) \\ &\geqslant -\tfrac{1}{2}\|\mathcal{E}^{\frac{1}{2}}(\bar{u}_s - \mathcal{G}\,\mathcal{M}\, v_s)\|^2 - \tfrac{1}{2}\langle v_s, \mathcal{M}\,\mathcal{G}\,\mathcal{M}\, v_s\rangle + V(\bar{x}_s) \\ &= -\tfrac{1}{2}\langle \bar{u}_s, \mathcal{E}\,\bar{u}_s\rangle + \langle \bar{u}_s, \mathcal{E}\,\mathcal{G}\,\mathcal{M}\, v_s\rangle \\ &\qquad - \tfrac{1}{2}\langle v_s\,\mathcal{M}\,(\mathcal{G}\,\mathcal{E}\,\mathcal{G} + \mathcal{G})\,\mathcal{M}\, v_s\rangle + V(\bar{x}_s) \\ &= -\tfrac{1}{2}\langle \bar{u}_s, \mathcal{E}\,\bar{u}_s\rangle + \langle \bar{u}_s, \mathcal{C}\, v_s\rangle - \tfrac{1}{2}\langle v_s\,\mathcal{C}\, v_s\rangle + V(\bar{x}_s). \end{aligned} \tag{54}$$

Integrating with respect to $s \in [t,T]$, and recalling the terminal condition in (45),

$$\begin{aligned} &\Psi(\bar{x}_T) - W_t(x) \\ &\geqslant -\int_t^T \tfrac{1}{2}\langle \bar{u}_s, \mathcal{E}\,\bar{u}_s\rangle - \langle \bar{u}_s, \mathcal{C}\, v_s\rangle + \tfrac{1}{2}\langle v_s\,\mathcal{C}\, v_s\rangle - V(\bar{x}_s)\, ds \\ &\Longrightarrow\; W_t(x) \leqslant J_t^v[\Psi](x,\bar{u}). \end{aligned} \tag{55}$$

As $x \in \mathscr{X}$ and $\bar{u} \in \mathscr{U}[t,T]$ are arbitrary, the first assertion follows. Moreover, if u_v^* exists as per (51), (53), then (54), (57) hold with equality, and the second assertion follows. ■

Theorem 4.9: Under the conditions of Theorem 4.7, with $v \in C([t,T];\mathscr{X})$ fixed, suppose there exists $(t,x) \mapsto W_t(x) \in C^1((t_0,T)\times\mathscr{X};\mathbb{R})$ such that (48) holds, with $(\frac{\partial}{\partial t}W_t(x), \nabla W_t(x)) \in \mathbb{R}\times\mathscr{X}$ denoting the Fréchet derivative at $(t,x) \in (t_0,T)\times\mathscr{X}$. Then, $W_t^v(x) \geqslant \widehat{J}_t^v[\Psi](x,u)$ for all $u \in \mathscr{U}[t,T]$. Furthermore, if there exists a mild solution $s \mapsto (x_v^*)_s$, $s \in (t,T)$ of (2) satisfying

$$\begin{aligned} [\hat{x}_v^*]_s &= x + \int_t^s [\hat{u}_v^*]_\sigma \, d\sigma, \\ [\hat{u}_v^*]_\sigma &= v_\sigma + \widehat{\mathcal{E}}^{-1}(\mathcal{M}\, v_\sigma + \nabla W_t^v([\hat{x}_v^*]_\sigma)), \end{aligned} \tag{56}$$

such that $[\hat{x}_v^*]_s \in \mathscr{X}$ for all $s \in (t,T)$, then $W_t^v(x) = \widehat{J}_t^v[\Psi](x,\hat{u}_v^*)$ for all $x \in \mathscr{X}$.

Proof: Following the proof of Theorem 4.8, the key steps are completion of squares and application of the chain rule. In particular, using the analogous notation, observe by coercivity of $\widehat{\mathcal{E}}$ of (39) and completion of squares that

$$\begin{aligned} &-\tfrac{1}{2}\langle \bar{p}_s, \widehat{\mathcal{E}}^{-1}\bar{p}_s\rangle + \langle \bar{p}_s, \bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\, v_s\rangle \\ &\quad = -\tfrac{1}{2}\|\widehat{\mathcal{E}}^{\frac{1}{2}}(\widehat{\mathcal{E}}^{-1}\bar{p}_s - (\bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\, v_s))\|^2 \\ &\qquad\qquad + \tfrac{1}{2}\|\widehat{\mathcal{E}}^{\frac{1}{2}}(\bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\, v_s)\|^2 \\ &\quad \leqslant \tfrac{1}{2}\|\widehat{\mathcal{E}}^{\frac{1}{2}}(\bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\, v_s)\|^2, \end{aligned}$$

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which holds with equality when

$$\bar{u}_s = \widehat{\mathcal{G}}\,\mathcal{M}\,v_s + \widehat{\mathcal{E}}^{-1}\,\bar{p}_s = v_s + \widehat{\mathcal{E}}^{-1}(\mathcal{M}\,v_s + \bar{p}_s).$$

Consequently, by the chain rule, (5), (39), (43), (44), (48),

$$\begin{aligned}
\tfrac{d}{ds} W_s(\bar{x}_s) &= -[-\tfrac{\partial}{\partial s} W_s(\bar{x}_s) + \widehat{H}^v(s, \bar{x}_s, \nabla W_s(\bar{x}_s))] \\
&\qquad + \widehat{H}^v(s, \bar{x}_s, \bar{p}_s) + \langle \nabla W_s(\bar{x}_s), \bar{u}_s \rangle \\
&= H(\bar{x}_s, \bar{p}_s) - \tfrac{1}{2}\langle \bar{p}_s + \mathcal{M}\,v_s,\, \widehat{\mathcal{G}}\,(\bar{p}_s + \mathcal{M}\,v_s)\rangle + \langle \bar{p}_s, \bar{u}_s\rangle \\
&= -\tfrac{1}{2}\langle \bar{p}_s, \widehat{\mathcal{E}}^{-1}\,\bar{p}_s\rangle + \langle \bar{p}_s, \bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\,v_s\rangle \\
&\qquad - \tfrac{1}{2}\langle v_s,\, \mathcal{M}\,\widehat{\mathcal{G}}\,\mathcal{M}\,v_s\rangle + V(\bar{x}_s) \\
&\leqslant \tfrac{1}{2}\|\widehat{\mathcal{E}}^{\frac{1}{2}}(\bar{u}_s - \widehat{\mathcal{G}}\,\mathcal{M}\,v_s)\|^2 - \tfrac{1}{2}\langle v_s,\, \mathcal{M}\,\widehat{\mathcal{G}}\,\mathcal{M}\,v_s\rangle + V(\bar{x}_s) \\
&= \tfrac{1}{2}\langle \bar{u}_s, \widehat{\mathcal{E}}\,\bar{u}_s\rangle - \langle \bar{u}_s, \widehat{\mathcal{E}}\,\widehat{\mathcal{G}}\,\mathcal{M}\,v_s\rangle \\
&\qquad + \tfrac{1}{2}\langle v_s\,\mathcal{M}\,(\widehat{\mathcal{G}}\,\widehat{\mathcal{E}}\,\widehat{\mathcal{G}} - \widehat{\mathcal{G}})\,\mathcal{M}\,v_s\rangle + V(\bar{x}_s) \\
&= \tfrac{1}{2}\langle \bar{u}_s, \widehat{\mathcal{E}}\,\bar{u}_s\rangle - \langle \bar{u}_s, \mathcal{C}\,v_s\rangle + \tfrac{1}{2}\langle v_s\,\mathcal{C}\,v_s\rangle + V(\bar{x}_s).
\end{aligned}$$

Integrating with respect to $s \in [t,T]$, and recalling the terminal condition in (48),

$$\begin{aligned}
&\Psi(\bar{x}_T) - W_t(x) \\
&\leqslant \int_t^T \tfrac{1}{2}\langle \bar{u}_s, \widehat{\mathcal{E}}\,\bar{u}_s\rangle - \langle \bar{u}_s, \mathcal{C}\,v_s\rangle + \tfrac{1}{2}\langle v_s\,\mathcal{C}\,v_s\rangle - V(\bar{x}_s)\,ds \\
&\Longrightarrow\ W_t(x) \geqslant \widehat{J}_t^v[\Psi](x, \bar{u}), \qquad (57)
\end{aligned}$$

and the remainder of the proof proceeds analogously with that of Theorem 4.8. ■

Theorem 4.10: Suppose (3) holds. Then,

$$W_t^v(x) \leqslant \widehat{W}_t^v(x). \qquad (58)$$

for all $t \in [t_0, T)$, $x \in \mathscr{X}$, and $v \in \mathscr{U}[t,T]$. Moreover, (17) holds, i.e. $\bar{u} \in \arg\mathrm{stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$, if and only if

$$W_t^{\bar{u}}(x) = \widehat{W}_t^{\bar{u}}(x). \qquad (59)$$

for all $t \in [t_0, T)$, $x \in \mathscr{X}$.

Proof: Immediate by Lemma 4.5, and (36), (37), (38), (40), (41). ■

D. Verification of Theorem 3.1 via optimal control

Theorems 4.6, 4.7, and 4.10 may be applied to directly verify the long time horizon $\arg\mathrm{stat}$ characterization provided by Theorem 3.1. In particular, by application of Theorems 4.6 and 4.7, assertions *(i)* and *(iii)*, it is evident that given $t \in [t_0, T]$, $x \in \mathscr{X}$, and $v \in \mathscr{U}[t,T]$ defined via the TPBVP

$$\begin{cases} \dot{x}_s = v_s \doteq -\mathcal{M}^{-1}\,p_s, & x_t = x, \\ \dot{p}_s = \nabla V(x_s), & p_T = \nabla\Psi(x_T) \end{cases}$$

for $s \in [t,T]$, that

$$J_t^v[\Psi](x, u_v^*) = W_t^v(x) = \widehat{W}_t^v(x) = \widehat{J}_t^v(x, \hat{u}_v^*),$$

and $u_v^* = v = \hat{u}_v^*$. Hence, Theorem 4.10 immediately yields that $v \in \arg\mathrm{stat}_{u \in \mathscr{U}[t,T]} J_t[\Psi](x,u)$, as per Theorem 3.1.

V. Conclusions

The stationary action principle is a fundamental physical postulate that underpins the temporal evolution of dynamical systems that obey conservation laws. Where this evolution involves a finite dimensional generalized position space, and is over a sufficiently short time horizon, this action principle can be encapsulated within an optimal control problem, and tools from classical optimal control can be brought to bear in the computation of system trajectories. However, on longer time horizons, this encapsulation is known to break down, typically due to a loss of convexity of the integrated Lagrangian. In this paper, a new characterization of the stationary action principle is developed that exploits connections between stationarity, semiconvexity, semiconcavity, and optimal control. In particular, it is shown that the stationary action principle can be characterized by a pair of related optimal control problems that are well-defined on arbitrarily long finite horizons.

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