Connections Between Risk–Sensitive Stochastic Control, Differential Games, and H$^\infty$ Control: the Nonlinear Case

by

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Vita

The author was born on May 6, 1957 in Schenectady, New York. In 1960, his family moved to Colonie, New York where he spent his youth. He attended St. Clare's Grade School and Bishop Gibbons High School in the same area.

After graduating from high school, the author enrolled directly at the Rensselaer Polytechnic Institute. He attended Rensselaer from September 1975 to December 1976, and again from September 1977 to December 1979. He then spent two years working at PAR Technology Corporation in Rome, New York (and later New Hartford, New York), a defense aerospace firm.

In January 1982, he returned to Rensselaer to pursue an advanced degree. He received an M.S. degree in Mathematics in May 1983. He then continued on in pursuit of a doctoral degree for another year, working with Dr. Andre Manitius in the field of Optimization and Control. In May 1984, he left Rensselaer for personal reasons to work at the Jet Propulsion Laboratory in Pasadena, California. There he worked in the Navigation Systems Section for five years. He contributed in several areas including aeromaneuvering guidance studies for proposed Mars missions. While at the Jet Propulsion Laboratory, he co–authored two journal articles and a conference paper: Error Analysis for a Guided Mars Landing, Optimal Aeroassisted Guidance Using Loh's Term Approximations, and Gravity Gradient Measurements, respectively.

In September 1989, he enrolled at Brown University in the Division of Applied Mathematics. He was awarded a University Fellowship during his first year, and was subsequently supported by teaching and research assistantships. He received the

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Table of Contents

To the web-reader:

In the years between the original writing of this thesis and TeX-ing of the current version for the web, it appears that some lengths or font sizes may have changed slightly. This has produced some changes in the formatting of the document including some minor errors in the page-numbering. Rather than re-examing the entire document, I put it on the web without significant modification of the TeX fonts and sizes. Your understanding is appreciated. - Bill McEneaney, Aug. 1999

1.0 INTRODUCTION

There are various approaches to treating disturbances in control systems. In stochastic control, disturbances are modelled as stochastic processes. On the other hand, in robust H^∞ control, disturbances are modelled deterministically. The theory of risk–sensitive optimal stochastic control provides a link between these two approaches. In the linear/quadratic case, this connection has been known for some time. In this thesis, the connection is made for a variety of nonlinear systems as well.

Work on H^∞–based control theory began in the early 1980's with the seminal work of Zames [47]. The early work was aimed at linear systems and formulated in the frequency domain. The symbol H^∞ refers to the Hardy space H^∞. The standard problem in that work was to minimize the H^∞–norm of some transfer matrix associated with the system [20]. The associated state–space formulation was found via some results from Fourier Analysis; see Francis [20] and the references contained therein, particularly Dym and McKean [11]. This state–space formulation can naturally be viewed in terms of two–player, zero–sum games; see, for instance

Basar and Bernhard [2] and Limebeer et al. [33]. This formulation also allows one to extend the definition of H$^\infty$ control to nonlinear systems. This connection with differential games is also utilized in [23], [24], [4], and [27]. A complementary way of viewing the nonlinear H$^\infty$ control problem is via the concept of dissipative systems. This approach is utilized in [41], [3], and [28]. With this approach, one has a partial differential inequality (PDI) associated with the dissipative system. Much of the work to date has assumed a C^1 solution to this PDI. James [28] avoids this assumption via the concept of viscosity solutions.

On the other hand, the dynamic programming approach to stochastic control has long been an fruitful avenue of research. In the case where the noise entering the system is modelled via Brownian motion, this leads to a dynamic programming equation (DPE) which takes the form of a partial differential equation (PDE), often called the Hamilton–Jacobi (or Hamilton–Jacobi–Bellman) equation. If the system is sufficiently regular and the noise enters in a nondegenerate form, the DPE is a nonlinear, nondegenerate, second order PDE. Notable references are [16] and [30]. More recently, a theory of weak solutions known as viscosity solutions was developed for certain PDE's which do not possess smooth (or even continuous) solutions [8], [9]. Interestingly, viscosity solutions have turned out to be the correct weak solutions (i.e. the value function) to DPE's which do not possess classical solutions. This subject is fully discussed in [17]. Viscosity solutions are also the correct weak solutions to the Isaacs equations representing deterministic and stochastic differential games [13], [18].

The stochastic control problem analogous to the original H$^\infty$ problem is the linear quadratic stochastic regulator. In this formulation, the control applied is insensitive to the size of the disturbance noise. In 1973, Jacobson [26] introduced the linear exponential quadratic stochastic regulator, the significance of which was not fully realized until recently. In this formulation, the choice of control is dependent upon the intensity of the noise. Whittle introduces a risk–sensitivity parameter, $\frac{1}{\epsilon}$, into this formulation [43], [44], [45], [46], leading to the name risk–sensitive control. When $\frac{1}{\epsilon}$ is positive, we have a "risk–averse" controller, and when it is negative we have a "risk–seeking" controller. As $\frac{1}{\epsilon}$ approaches zero, one obtains the linear quadratic regulator. As $\frac{1}{\epsilon} \to \infty$, the control becomes "infinitely risk–averse." We are concerned with the risk–averse case. In this case, the risk–sensitive control problem has an equivalent formulation as a differential game. In particular, as $\frac{1}{\epsilon} \to \infty$, the differential game becomes the differential game associated with H^∞ control [21], [40].

This thesis is concerned with making the same link in the nonlinear context. It will be shown that as $\frac{1}{\epsilon} \to \infty$, the risk–sensitive control problem leads to a differential game, and in particular, the differential game associated with an H^∞ controller. The approach is as follows. We consider the value function associated with the risk–averse control problem. The value function is the solution of the associated DPE. The logarithmic transform of Fleming (see [17]) is applied to this function yielding a solution to the transformed PDE. As $\frac{1}{\epsilon} \to \infty$, this solution converges to the viscosity solution of the (first–order) limiting PDE. We then demonstrate the relation between this solution and the value of the differential game associated with H^∞ control. Related work via similar techniques is being undertaken by James [27], [28], [29]. A different approach (Operator Theory) to a very similar problem is being taken by Runolfsson [37], [38]. Another related result appears in [5].

The remainder of the thesis can be divided into two parts: the finite time horizon result (Chapter 2), and the infinite time horizon result (Chapter 3). Each chapter is nearly self–contained.

For the finite time horizon problem, we assume that the dynamics are Lipschitz, and that the control takes values in a compact set. The stochastic control problem has a risk–averse cost criterion with a running cost and terminal cost, both of which must satisfy a quadratic growth condition. It is shown that the logarithm of the value function is the solution to the associated PDE. As $\frac{1}{\epsilon} \to \infty$, a viscosity solution to a first–order PDE is obtained. In parallel, it is shown that the value of a certain differential game is also a viscosity solution of the same first–order PDE. A uniqueness result is then proved which implies that the solutions are identical. This differential game is precisely the game that would be associated with a finite time horizon, nonlinear, H^∞ control problem.

In Chapter 3, the infinite time horizon case is considered. A similar result is obtained via similar techniques although with visible differences. In this case, the dynamics are assumed to be endowed with a certain ergodicity property, and the running cost must satisfy a linear growth condition. First, Girsanov's Theorem is used to show that if a certain DPE has a solution, then this solution is associated with the value of the infinite time horizon, risk–averse control problem. Then a series of stochastic games with increasingly relaxed assumptions are employed to demonstrate the existence of a solution to this DPE. Coincidentally, we also obtain a stochastic differential game which is related to the same DPE. We then let $\frac{1}{\epsilon} \to \infty$ (total risk aversion), and obtain the viscosity solution to a first–order PDE. This limit PDE is related to a somewhat deterministic differential game (this will be made clear later). More importantly, this PDE is the dissipation equation discussed above, which is, in turn, related to a nonlinear, infinite time horizon, H^∞ control.

It should be noted that many of the results which appear here were discussed in [14], [15].

A final remark on notation is in order. Generic constants will be used throughout the text, and will be denoted by A_i, B_i, C_i, etc... They may be assumed to have the same value throughout any given section, but due to the large number of such constants, no attempt has been made to have their values remain constant throughout the thesis.

2.0 FINITE TIME HORIZON CASE

2.1 Introduction and Assumptions

For the finite time horizon case, we consider the risk–averse stochastic control problem with dynamics

$$\begin{aligned} dX_t^\epsilon &= f(t, X_t^\epsilon, u_t)\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_s^\epsilon &= x \end{aligned} \tag{1.1}$$

and cost criterion

$$\begin{aligned} I^\epsilon(s,x,u) &= \epsilon \log J^\epsilon(s,x,u) \\ &= \epsilon \log \mathrm{E}_{s,x} \exp\left\{ \frac{1}{\epsilon} \left[\int_s^T L(t, X_t^\epsilon, u_t)\, dt + \Psi(X_T^\epsilon) \right] \right\} \end{aligned}$$

where $X_t^\epsilon \in \Re^n$ is the state, u is the control, and B is an n–dimensional Brownian motion relative to some reference probability system, ν (see Section 2.2). We let $\epsilon > 0$ be a small parameter which is both a scale factor for the Brownian motion and a measure of the risk–sensitivity in the cost criterion. The constant γ must be large enough so that the expectation exists. The role of γ will become clearer later. We assume that the control is progressively measurable (cf. [31]), and takes values in the compact set U. We make the following assumptions on f, L, and Ψ.

$$\begin{aligned} &f \in C([0,T] \times \Re^n \times U) \\ &|f(t,x,v) - f(t,y,v)| \le K|x-y| \qquad \forall\, t \in [0,T],\ \forall\, x,y \in \Re^n,\ \forall\, v \in U \\ &|f(t,x,v)| \le K(1+|x|) \qquad \forall\, t \in [0,T],\ \forall\, x \in \Re^n,\ \forall\, v \in U. \end{aligned} \tag{AF1}$$

(Although an inequality of the second type above follows from the first, we include it explicitly to indicate that we will be using the same constant, K, for both bounds.)

$$
\begin{aligned}
&L \in C([0,T] \times \Re^n \times U) \\
&L(t,x,v) \geq 0 \qquad \forall\, t \in [0,T],\ \forall\, x \in \Re^n,\ \forall\, v \in U \\
&|L(t,x,v) - L(t,y,v)| \leq C(1+|x|+|y|)|x-y| \quad \forall\, t \in [0,T],\ \forall\, x,y \in \Re^n,\ \forall\, v \in U \\
&|L(t,x,v)| \leq C(1+|x|^2) \qquad \forall\, t \in [0,T],\ \forall\, x \in \Re^n,\ \forall\, v \in U.
\end{aligned}
\tag{AF2}
$$

$$
\begin{aligned}
&\Psi \in C(\Re^n) \\
&\Psi(x) \geq 0 \qquad \forall\, x \in \Re^n \\
&|\Psi(x) - \Psi(y)| \leq C(1+|x|+|y|)|x-y| \qquad \forall\, x,y \in \Re^n \\
&|\Psi(x)| \leq C(1+|x|^2) \qquad \forall\, x \in \Re^n
\end{aligned}
\tag{AF3}
$$

Assumptions (AF2)–(AF3) are nearly optimal in the sense that one cannot in general exceed quadratic growth in the cost function without losing existence of I^ϵ. The conditions that L and Ψ be bounded below by zero could be replaced by a condition that they be bounded below by some finite (possibly negative) constant with only very minor changes. The assumption of continuity of L and Ψ is essential in many of the proofs. The Lipschitz continuity of f could not easily be weakened without producing difficulties in the existence and uniqueness of the stochastic differential equation (1.1), as well as in many of the other proofs.

It might be desirable to consider the addition of a state dependent multiplier on the Brownian motion term in (1.1). However, we have not considered this here.

The controller, u, is a minimizing controller, that is, we consider a value function of the form

$$V^\epsilon(s,x) = \inf_{u \in A_{s,\nu}} I^\epsilon(s,x,u)$$

where $A_{s,\nu}$ is the set of progressively measurable controls with values in U. Note that the system (1.1) possesses a strong solution, so there is no need to take an infimum over reference probability systems, ν (defined in Section 2.2); the value is the same for all reference probability systems.

We consider the case of infinite risk–aversion with corresponding scaling down of the Brownian motion term. This is achieved by letting $\epsilon \downarrow 0$. It will be shown that $V^\epsilon \to V$ as $\epsilon \downarrow 0$ where V is the value of the deterministic differential game with dynamics

$$\begin{aligned} \frac{dX}{dt} &= f(t, X_t, u_t) + w_t \\ X_s &= x \end{aligned} \tag{1.3}$$

and payoff

$$P_{s,x}(u,w) = \int_s^T \left[L(t, X_t, u_t) - \gamma^2 |w_t|^2 \right] \, dt + \Psi(X_T).$$

Here the system, including the controls u and w, is deterministic. Regarding the controls, u is the minimizing control which still takes values in U, and w is the maximizing control which takes values in $\Re^n$. The Elliott–Kalton [12] definition of the value of the game will be used throughout. One might note that the Isaacs condition holds since the controls $u.$ and $w.$ appear separately in the game dynamics and payoff. The game will be defined more completely in Section 2.4.

Since H^∞ control was originally formulated in the frequency domain, most of the results which have been generated have been for the infinite time horizon problem. However, as discussed in Chapter 1, the differential game/state–space formulation has more recently been developed. The above game is the finite time horizon analogue of the H^∞ differential game. It is the author's opinion that the finite time horizon case is of interest in itself, and not merely as a stepping stone to the infinite time horizon results. For instance, in aerospace trajectory guidance, control and navigation applications, one would often most naturally be interested in the finite time horizon result.

The results of this chapter are somewhat similar to those of James [27], whose research has been moving along a parallel track.

2.2 Existence and Uniqueness

Here we remark on the existence and uniqueness of the solutions to the ordinary differential equation (ODE) describing the dynamics of the game problem and the stochastic differential equation (SDE) describing the dynamics of the risk–sensitive control problem.

We first prove the deterministic existence and uniqueness results. Without loss of generality, we take $s = 0$, and consider the ODE

$$\begin{aligned} \frac{dX}{dt} &= f(t, X_t, u_t) + w_t \\ X_0 &= x \end{aligned} \tag{2.1}$$

(Note: Although the dynamics are deterministic, we will use the common stochastic notation X_t to indicate $X(t)$ for convenience and uniformity of presentation.)

Here $u.$ is Lebesgue measurable with values in the compact set $U \subset \Re^m$ and $w. \in L^2([0,T];\Re^n)$, hereafter denoted simply by $L^2[0,T]$. As is standard throughout this chapter, we assume f satisfies (AF1).

Theorem 2.2.1: There exists a unique solution to (2.1) on [0,T].

proof: Let $\xi_t = X_t - \int_0^t w_r\,dr$. Then $\xi.$ satisfies

$$\begin{aligned} \frac{d\xi}{dt} &= f(t, \xi_t + \int_0^t w_r\,dr, u_t) \\ \xi_0 &= x. \end{aligned} \tag{2.2}$$

By Miller and Michel [36], Theorem 2.4.6, (2.2) has a unique solution, ξ^*. Letting $X_t^* = \xi_t^* + \int_0^t w_r\,dr$, we see that X^* is the unique solution to (2.1). ∎

We now turn to the stochastic case. As in [17], Section 4.2, we define a reference probability system ν as a 4–tuple

$$\nu = (\Omega, \{\mathcal{F}_t\}, P, B_\cdot)$$

where $(\Omega, \mathcal{F}_T, P)$ is a probability space. That is, Ω is a sample space, $\mathcal{F}_T$ a σ–algebra on Ω, and P a probability measure on $\mathcal{F}_T$. Then $\{\mathcal{F}_t\}$ is an increasing family of σ–algebras and $B_\cdot$ is an n–dimensional Brownian motion adapted to $\mathcal{F}_\cdot$ on $[0, T]$.

Consider the SDE

$$\begin{aligned} dX_t &= f(t, X_t, u_t)\, dt + \sigma\, dB_t \qquad t \in [0, T] \\ X_0 &= x \end{aligned} \tag{2.3}$$

where $x \in \Re^n$, $u_\cdot : \Omega \times \Re \to \Re^m$ is $\mathcal{F}_t$–progressively measurable ($\mathcal{F}_t$-p.m.) with $u_t(\omega) \in U$ for all $t \in [0, T]$, $\omega \in \Omega$. (Here we assume, as in Section 2.1, that U is compact.) We let σ be a positive constant.

Recall that, as in [31], a strong solution to (2.3) exists if there exists a continuous, $\mathcal{F}_t$–p.m. process X with the following properties (see [31], definition 5.2.1):

$$\begin{aligned} &(i) \quad X \text{ is adapted to } \{\mathcal{F}_t\} \\ &(ii) \quad P(X_0 = x) = 1 \\ &(iii) \quad P\left\{ \int_0^t |f_i(r, X_r, u_r)|\, dr < \infty \right\} = 1 \quad \forall\, i = 1...n,\ t \in [0, T] \\ &(iv) \quad (2.3) \text{ is satisfied } \forall\, t \in [0, T], \text{ a.s.} \end{aligned} \tag{2.4}$$

Theorem 2.2.2: There exists a unique strong solution to (2.3) for any reference probability system ν. This solution is $\mathcal{F}_t$-p.m., and

$$\mathrm{E}|X_t|^2 \le 2\left(1 + |x|^2 + \sigma^2\right) e^{(4K^2+1)t} \qquad \forall t \in [0, T].$$

proof: Existence and uniqueness follows from [30], Theorem 2.5.7. To obtain the bound, let $\xi_t = X_t - x$ so that

$$\xi_t = \int_0^t f(r, \xi_r + x, u_r)\, dr + \sigma B_t$$
$$\xi_0 = 0.$$

Then employ [30], Lemma 2.5.2 to obtain a bound on ξ_t. ∎

2.3 The Value Function for the Stochastic Control Problem

In this section, we consider the risk–averse stochastic control problem with dynamics

$$\begin{aligned} dX_t^\epsilon &= f(t, X_t^\epsilon, u_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t \\ X_s^\epsilon &= x \end{aligned} \tag{3.1}$$

and cost criterion

$$I^\epsilon(s,x,u) = \epsilon \log \mathrm{E}_{s,x} \exp\left\{ \frac{1}{\epsilon}\left[\int_s^T L(t, X_t^\epsilon, u_t)\,dt + \Psi(X_T^\epsilon) \right] \right\}.$$

In general, we will drop the ϵ superscript on $X.$ in this section so that the notation does not become too cumbersome.

To simplify the expressions we define

$$A^\epsilon(s,x,u,\omega) = \left[\int_s^T L(t, X_t, u_t)\,dt + \Psi(X_T) \right]$$

and

$$J^\epsilon(s,x,u) = \mathrm{E}_{s,x} \exp\left[\frac{1}{\epsilon} A^\epsilon(s,x,u,\omega) \right]$$

so that

$$I^\epsilon(s,x,u) = \epsilon \log J^\epsilon(s,x,u) = \epsilon \log \mathrm{E}_{s,x} \exp\left[\frac{1}{\epsilon} A^\epsilon(s,x,u,\omega) \right].$$

We consider the corresponding value functions

$$V^\epsilon(s,x) = \inf_{u \in A_{s,\nu}} I^\epsilon(s,x,u) \tag{3.2}$$

where $A_{s,\nu}$ was given in Section 2.1, and

$$\Phi^\epsilon(s,x) = \inf_{u \in A_{s,\nu}} J^\epsilon(s,x,u)$$

so that $V^\epsilon = \epsilon \log \Phi^\epsilon$.

We will show that V^ϵ is a viscosity solution to the dynamic programming equation

$$\begin{aligned} 0 &= V^\epsilon_s + \frac{\epsilon}{4\gamma^2}\Delta V^\epsilon + \frac{1}{4\gamma^2}|\nabla V^\epsilon|^2 + \min_{v\in U}[f(s,x,v)\cdot\nabla V^\epsilon + L(s,x,v)] \\ V^\epsilon(T,x) &= \Psi(x). \end{aligned} \tag{3.3}$$

Note that Φ^ϵ will also be a solution to

$$\begin{aligned} 0 &= \Phi^\epsilon_s + \frac{\epsilon}{4\gamma^2}\Delta\Phi^\epsilon + \min_{v\in U}\left[f(s,x,v)\cdot\nabla\Phi^\epsilon + \frac{1}{\epsilon}L(s,x,v)\Phi^\epsilon\right] \\ \Phi^\epsilon(T,x) &= \exp\frac{1}{\epsilon}\Psi(x). \end{aligned} \tag{3.4}$$

We will also bound the value function by a quadratic independent of $\epsilon \le 1$, and prove equicontinuity of the value functions for values of $\epsilon \le 1$. These estimates require that γ not be too small. They will be used in an application of Ascoli–Arzela to follow in a later section. Note that one could use the method of Barles and Perthame [6] rather than Ascoli–Arzela. However, since the equicontinuity estimate with respect to x will be necessary for the uniqueness proof (to follow in Sec. 2.5), it is unlikely that this would result in a great simplification of the computations.

In order to obtain this result, we must first consider the "cutoff" problem, where the possibly unbounded functions f, L, and Ψ are replaced by bounded counterparts. We obtain the analogous result for this cutoff problem, and then take a limit to obtain the desired result.

The cutoff functions, indexed by k, are given by

$$f^k(t,x,v) = \begin{cases} f(t,x,v) & \text{if } |f(t,x,v)| \le k \\ k\frac{f(t,x,v)}{|f(t,x,v)|} & \text{if } |f(t,x,v)| > k \end{cases}$$

$$L^k(t,x,v) = \begin{cases} k & \text{if } L(t,x,v) > k \\ L(t,x,v) & \text{if } L(t,x,v) \le k \end{cases}$$

$$\Psi^k(x) = \begin{cases} k & \text{if } \Psi(x) > k \\ \Psi(x) & \text{if } \Psi(x) \le k. \end{cases}$$

Let N_0 be the smallest integer greater than

$$\max\left\{\Psi(0), \max_{t\in[0,T],v\in U} L(t,0,v), \max_{t\in[0,T],v\in U}|f(t,0,v)|\right\}.$$

By (AF1)–(AF3), $N_0 < \infty$. Let

$$B_k = \sup\{b: \quad f(t,x,v) = f^k(t,x,v), \quad L(t,x,v) = L^k(t,x,v),$$
$$\Psi(x) = \Psi^k(x), \qquad \forall |x| \le b, \forall t \in [0,T], \forall v \in U\}.$$

Note that $B_k > 0$ for all $k \ge N_0$ and $B_k \to \infty$ monotonically as $k \to \infty$.

We also define X^k by

$$\begin{aligned} dX_t^k &= f^k(t, X_t^k, u_t)\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_s^k &= x \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} A^{\epsilon,k}(s,x,u,\omega) &= \left[\int_s^T L(t, X_t^k, u_t)\, dt + \Psi(X_T^k) \right] \\ J^{\epsilon,k}(s,x,u) &= \mathrm{E}_{s,x} \exp\left[\frac{1}{\epsilon} A^{\epsilon,k}(s,x,u,\omega) \right] \\ I^{\epsilon,k}(s,x,u) &= \epsilon \log J^{\epsilon,k}(s,x,u) \\ \Phi^{\epsilon,k}(s,x) &= \inf_{u \in A_{s,\nu}} J^{\epsilon,k}(s,x,u) \\ V^{\epsilon,k}(s,x) &= \inf_{u \in A_{s,\nu}} I^{\epsilon,k}(s,x,u) = \epsilon \log \Phi^{\epsilon,k}(s,x). \end{aligned}$$

The dynamic programming equations for $V^{\epsilon,k}$ and $\Phi^{\epsilon,k}$ are

$$\begin{aligned} 0 &= V_s + \frac{\epsilon}{4\gamma^2} \Delta V + \frac{1}{4\gamma^2} |\nabla V|^2 + \min_{v \in U} [f^k(s,x,v) \cdot \nabla V + L^k(s,x,v)] \\ V(T,x) &= \Psi^k(x) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} 0 &= \Phi_s + \frac{\epsilon}{4\gamma^2} \Delta \Phi + \min_{v \in U} [f^k(s,x,v) \cdot \nabla \Phi + \frac{1}{\epsilon} L^k(s,x,v) \Phi] \\ \Phi(T,x) &= \exp \frac{1}{\epsilon} \Psi^k(x). \end{aligned} \tag{3.7}$$

Lemma 2.3.1: There exists a bounded, classical solution $V \in C^{1,2}([0,T) \times \Re^n)$ to (3.6).

proof: This follows directly from Ladyženskaja et al. [34], Theorem 5.8.1 and Remark 5.8.1 which directly follows it. The conditions necessary to apply [34] Theorem 5.8.1 are easily verified for the problem at hand. ∎

For clarity, we note that we are not claiming a uniform bound for all k in Lemma 2.3.1, but only that for each k, the solution is bounded.

Remark 2.3.2: Since the lemma states that the solution to (3.6) is bounded, we have from Ladyženskaja et al. [34], Theorem 5.3.1 that $|\nabla V|$ is also bounded. Consequently the solution to (3.7) satisfies $|\Phi|, |\nabla\Phi|$ bounded also.

Lemma 2.3.3: The solution to (3.6) is the value function $V^{\epsilon,k}$, and the solution to (3.7) is the value function $\Phi^{\epsilon,k}$. An admissible feedback solution exists which yields the minimum. Furthermore, $V^{\epsilon,k}$ is the unique, bounded, classical solution to (3.6).

proof: We prove the result for Φ. The result for V follows.

Let Φ be the solution to (3.7). First we show $\Phi(s,x) \leq J^{\epsilon,k}(s,x,u)$ for all $u \in A_{s,\nu}$ and $(s,x) \in Q_T = [0,T] \times \Re^n$.

For a fixed $\mathcal{F}_t$–progressively measurable conttrol, the solution to the stochastic differential equation, X^k (which we shall denote simply as X throughout the remainder of this proof), is a continuous semi-martingale, with, in fact, square–integrable martingale part. Thus, since $\Phi \in C^{1,2}$, we can apply Ito's rule to yield

$$\begin{aligned}\Phi(t,X_t) = \Phi(s,x) + \int_s^t \Big[& \Phi_r(r,X_r) + \nabla\Phi(r,X_r)\cdot f^k(r,X_r,u_r) \\ & + \frac{\epsilon}{4\gamma^2}\Delta\Phi(r,X_r)\Big]\,dr \\ & + \frac{\epsilon}{2\gamma^2}\int_s^t \nabla\Phi(r,X_r)\cdot dB_r\end{aligned}$$

where the notation $\int g \cdot dB_r$ indicates $\sum_{i=1}^n \int g^{(i)}\,dB_r^{(i)}$. By Remark 2.3.2, there exists a such that $|\nabla\Phi| \leq a$ for all $(s,x) \in Q_T$ which implies $\mathrm{E}\int_s^T |\nabla\Phi|^2\, dt \leq a^2 T$, and consequently (see for instance [31], Section 3.2), $\int_s^t \nabla\Phi(r, X_r) \cdot dB_r$ is a square–integrable martingale. Thus $\Phi(t, X_t)$ is also a continuous semi–martingale.

Note that since L^k is bounded, $\mathcal{E}_t \equiv \exp[\frac{1}{\epsilon}\int_s^t L^k(e, X_r, u_r)\, dr]$ is a continuous semi–martingale with zero martingale part. Therefore we may apply the stochastic integration by parts formula (see [31], Section 3.3) to the product $\mathcal{E}_t\Phi(t, X_t)$ to yield

$$\begin{aligned}
\mathcal{E}_t\Phi(t,X_t) - \Phi(s,x) = &\int_s^t \Phi(r,X_r)\frac{1}{\epsilon}L^k(r,X_r,u_r)\mathcal{E}_r\, dr \\
&+ \int_s^t \mathcal{E}_r \bigg[\Phi_r(r,X_r) + \nabla\Phi(r,X_r)\cdot f^k(r,X_r,u_r) \\
&\qquad + \frac{\epsilon}{4\gamma^2}\Delta\Phi(r,X_r)\bigg]\, dr \\
&+ \frac{\epsilon}{2\gamma^2}\int_s^t \mathcal{E}_r\nabla\Phi(r,X_r)\cdot dB_r.
\end{aligned}$$

Using DPE (3.7) to eliminate the first two terms on the right, we have

$$\mathcal{E}_t\Phi(t,X_t) - \Phi(s,x) \geq \frac{\epsilon}{2\gamma^2}\int_s^t \mathcal{E}_r\nabla\Phi(r,X_r)\cdot dB_r.$$

Since L^k and $\nabla\Phi$ are bounded, so is $\mathcal{E}_r\nabla\Phi(r,X_r)$. Thus, by the same argument as above, $\int_s^t \mathcal{E}_r\nabla\Phi(r,X_r)\cdot dB_r$ is a square–integrable martingale. Therefore, taking $t = T$

$$\Phi(s,x) \leq \mathrm{E}_{s,x}[\mathcal{E}_T\Phi(T,X_T)]$$

which by definition of $\mathcal{E}$ and the terminal condition in (3.7)

$$\begin{aligned}
&= \mathrm{E}_{s,x}\exp\left\{\frac{1}{\epsilon}\left[\int_s^T L^k(t,X_t,u_t)\, dt + \Psi(X_T)\right]\right\} \\
&= J^{\epsilon,k}(s,x,u).
\end{aligned}$$

Now suppose there exists $u^* \in A_{s,\nu}$ such that

$$u_t^* \in \mathrm{argmin}_{v\in U}[f^k(t,X_t^*,v)\cdot\nabla\Phi(t,X_t^*) + \frac{1}{\epsilon}L^k(t,X_t^*,v)\Phi(t,X_t^*)]$$

for all $t \in [s,t]$. Then we would have equality in in the above, and consequently,

$$\Phi(s,x) = J^{\epsilon,k}(s,x,u^*).$$

However, we have not shown that such a $u^* \in A_{s,\nu}$ exists for any $A_{s,\nu}$.

It is easily seen from [16], Appendix B, that there exists a Borel measurable function, $g(t,x)$ such that

$$g(t,x) \in \text{argmin}_{v \in U}[f^k(t,x,v) \cdot \nabla\Phi(t,x) + \frac{1}{\epsilon} L^k(t,x,v)\Phi(t,x)]$$

for all $(t,x) \in Q_T$. Consider the SDE

$$dX_t = f^k(t, X_t, g(t, X_t))\, dt + \frac{\epsilon}{2\gamma^2}\, dB_t.$$

By Veretennikov [42], Theorem 1, it has a unique strong solution for any reference probability system, ν. Letting $u_t^* = g(t, X_t)$ for the strong solution yields $u^* \in A_{s,\nu}$. Therefore,

$$\Phi(s,x) = \min_{u \in A_{s,\nu}} J^{\epsilon,k}(s,x,u) = \Phi^{\epsilon,k}(s,x).$$

To prove the uniqueness claim, suppose there exists another bounded, classical solution, $\hat{\Phi}$. Then, by the same proof as above, it is the value function $\Phi^{\epsilon,k}$. ∎

Now, given $R < \infty$, let $Z_\cdot$ be the solution of the integral equation

$$Z_t = R + K(t-s) + K\int_s^t Z_r\, dr + \sqrt{\frac{\epsilon}{2\gamma^2}}|B_t|.$$

Lemma 2.3.4: For solutions $X_\cdot$ and $X_\cdot^k$ with initial conditions $|x| \le R$, we have

$$|X_t| \le Z_t \qquad \forall t \in [0,T], \omega \in \Omega$$

$$|X_t^k| \le Z_t \qquad \forall k,\ \forall t \in [0,T],\ \forall \omega \in \Omega$$

proof: By (AF1)

$$|X_t| \leq R + K(t-s) + K\int_s^t |X_r|\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}}|B_t|.$$

So

$$|X_t| - Z_t \leq K\int_s^t (|X_r| - Z_r)\ dr$$

which, by Gronwall, implies

$$|X_t| \leq Z_t \qquad\qquad \forall t \in [0,T], \omega \in \Omega.$$

Now note that

$$|f^k(t,x,v)| \leq |f(t,x,v)| \leq K(1+|x|)$$

so we obtain the same result for X^k. ∎

It is useful here to prove a quadratic growth bound on the value function, V^ϵ. Note that this bound can then clearly be applied to $V^{\epsilon,k}$ as well.

Lemma 2.3.5: Let $\gamma > \gamma_1$ where γ_1 is defined in the proof. Then there exists C_2 (also given in the proof) such that for all ϵ, for all $u \in A_{s,\nu}$, for all $s \in [0,T]$, and for all $x \in \Re^n$ we have

$$I^\epsilon(s,x,u) \leq C_2(1+|x|^2)$$

and consequently

$$V^\epsilon(s,x) \leq C_2(1+|x|^2).$$

proof: For simplicity, we take $s=0$. By (AF1)

$$|X_t - x| \leq K(1+|x|)t + K\int_0^t |X_r - x|\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}}|B_t|.$$

Let

$$M_T = \max_{t\in[0,T]} |B_t|$$

$$C_1 = \left(1 + KTe^{KT}\right).$$

By Gronwall

$$|X_t - x| \le C_1 \left[K(1+|x|)T + \sqrt{\frac{\epsilon}{2\gamma^2}} M_T\right] \tag{3.9}$$

and consequently

$$|X_t| \le |x| + C_1 \left[K(1+|x|)T + \sqrt{\frac{\epsilon}{2\gamma^2}} M_T\right]. \tag{3.10}$$

By (AF2) and (AF3)

$$A^\epsilon(s,x,u,\omega) \le C(1+T) + C|X_T|^2 + C\int_0^t |X_t|^2\,dt.$$

Employing (3.10) and the fact that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have

$$A^\epsilon(s,x,u,\omega) \le C(1+T)\left[1 + 3|x|^2 + 3c_1^2K^2(1+|x|)^2T^2 + 3C_1^2\frac{\epsilon}{2\gamma^2}M_T^2\right].$$

Thus

$$\begin{aligned}
\mathrm{E}_{s,x}\exp\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega) &\le \exp\left\{\frac{1}{\epsilon}C(1+T)\left[1+3|x|^2+3C_1^2K^2(1+|x|)^2T^2\right]\right\} \\
&\quad\cdot \mathrm{E}_{s,x}\exp\left[3C_1^2\frac{\epsilon}{2\gamma^2}M_T^2\right]
\end{aligned}$$

which upon recalling (as in Appendix 4.1) that $P(M_T > a) \le 4nP(B_T^{(1)} > a)$

$$\begin{aligned}
&\le \exp\left\{\frac{1}{\epsilon}C(1+T)\left[1+3|x|^2+3C_1^2K^2(1+|x|)^2T^2\right]\right\} \\
&\quad\cdot 4n\int_0^\infty \frac{1}{\sqrt{2\pi T}} e^{\frac{3CC_1^2(1+T)}{2\gamma^2}z^2} e^{-\frac{z^2}{2T}}\,dz.
\end{aligned}$$

Clearly we need $\gamma > \gamma_1 \equiv [3CC_1^2T(1+T)]^{\frac{1}{2}}$ in order for this integral to exist.

Since $I^\epsilon = \epsilon\log\mathrm{E}_{s,x}\frac{1}{\epsilon}A^\epsilon$, we have

$$\begin{aligned}
I^\epsilon(s,x,u) &\le C(1+T)\left[1+3|x|^2+3C_1^2K^2(1+|x|)^2T^2\right] \\
&\quad + \epsilon\log 4n\int_0^\infty \frac{1}{\sqrt{2\pi T}} e^{\frac{3CC_1^2(1+T)}{2\gamma^2}z^2} e^{-\frac{z^2}{2T}}\,dz \\
&\le C_2(1+|x|^2)
\end{aligned}$$

for proper choice of C_2.

Since the bound is independent of u, it also holds for V^ϵ. ∎

Remark 2.3.6: Note that the above bound is actually of the form

$$I^\epsilon(s,x,u) \le C_2'(1+|x|^2) + \epsilon C_3'$$

where C_2', C_3' are independent of ϵ.

Theorem 2.3.7: Let $\gamma > \gamma_1$ and $R < \infty$. Then

$$J^{\epsilon,k}(s,x,u) \to J^\epsilon(s,x,u)$$

$$I^{\epsilon,k}(s,x,u) \to I^\epsilon(s,x,u)$$

uniformly for $|x| \le R$, $u \in A_{s,\nu}$ and $s \in [0,T]$.

proof: We show that $J^{\epsilon,k} \to J^\epsilon$ uniformly on compact sets. From Lemma 2.3.5, we know that J^ϵ, $J^{\epsilon,k}$ exist for $\gamma > \gamma_1$. Also

$$|J^\epsilon - J^{\epsilon,k}| \le \mathrm{E}_{s,x}|\exp(\frac{1}{\epsilon}A^\epsilon) - \exp(\frac{1}{\epsilon}A^{\epsilon,k})|. \tag{3.11}$$

Define

$$\mathcal{A}_k = \{\omega \in \Omega | \quad |X_t(\omega)| \le B_k \quad \forall t \in [0,T]\}.$$

Then (3.11) becomes

$$\begin{aligned}|J^\epsilon - J^{\epsilon,k}| \le &\int_{\mathcal{A}_k} \left|\exp\left(\frac{1}{\epsilon}A^\epsilon\right) - \exp\left(\frac{1}{\epsilon}A^{\epsilon,k}\right)\right| dP(\omega) \\ &+ \int_{\mathcal{A}_k^c} \left|\exp\left(\frac{1}{\epsilon}A^\epsilon\right) - \exp\left(\frac{1}{\epsilon}A^{\epsilon,k}\right)\right| dP(\omega).\end{aligned}$$

But $X_t^k(\omega) = X_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \mathcal{A}_k$. Therefore, the first integral is zero, and we have

$$\begin{aligned}|J^\epsilon - J^{\epsilon,k}| &\le \int_{\mathcal{A}_k^c} \left|\exp\left(\frac{1}{\epsilon}A^\epsilon\right) - \exp\left(\frac{1}{\epsilon}A^{\epsilon,k}\right)\right| dP(\omega) \\ &\le \int_{\mathcal{A}_k^c} \left[\exp\left(\frac{1}{\epsilon}A^\epsilon\right) + \exp\left(\frac{1}{\epsilon}A^{\epsilon,k}\right)\right] dP(\omega)\end{aligned}$$

which by (AF2), (AF3) and Lemma 2.3.4

$$\leq 2e^{\frac{C}{\epsilon}(1+T)} \int_{\mathcal{A}_k^c} \exp\left\{ \frac{C}{\epsilon} \left[\int_0^T Z_t^2 \, dt + Z_T^2 \right] \right\} dP(\omega).$$

(It is easily seen that this integral exists for $\gamma > \gamma_1$.)

Note that this is independent of $|x| \leq R$, $u \in A_{s,\nu}$ and $s \in [0,T]$. Also note that the integrand is independent of k; only the region $\mathcal{A}_k^c$ is changing. By a standard result, this integral goes to zero if $P(\mathcal{A}_k^c) \to 0$. But by Lemma 2.3.4,

$$\begin{aligned} P(\mathcal{A}_k^c) &\leq P(\max_{t\in[s,T]} |Z_t| > B_k) \\ &\leq P(\max_{t\in[0,T]} |Z_t| > B_k) \end{aligned}$$

(where the reference probability system is shifted by $-s$, which is acceptable here since we have strong solutions). But since $B_k \to \infty$, we have $P(\max_{t\in[0,T]} |Z_t| > B_k) \to 0$ as $k \to \infty$, (to see this, use Gronwall and the definition of Z.) and this convergence is uniform in $|x| \leq R$, $u \in A_{s,\nu}$ and $s \in [0,T]$. ∎

Theorem 2.3.8: Let $\gamma > \gamma_1$. Then

$$V^{\epsilon,k}(s,x) \to V^{\epsilon}(s,x)$$

and

$$\Phi^{\epsilon,k}(s,x) \to \Phi^{\epsilon}(s,x)$$

uniformly on compact sets.

proof: We use the standard proof. Let $(s,x) \in D$ where D is compact. Given $\beta > 0$, by Theorem 2.3.7, there exists N such that for all $k \geq N$

$$|I^{\epsilon,k}(s,x,u) - I^{\epsilon}(s,x,u)| \leq \frac{\beta}{2} \qquad \forall u \in A_{s,\nu}, \forall (s,x) \in D. \tag{3.12}$$

Now, there exists $u_{\beta,s,x}$ such that

$$I^{\epsilon}(s,x,u_{\beta,s,x}) \leq V^{\epsilon}(s,x) + \frac{\beta}{2}.$$

This implies that

$$V^{\epsilon,k}(s,x) - V^{\epsilon}(s,x) \leq I^{\epsilon,k}(s,x,u_{\beta,s,x}) - I^{\epsilon}(s,x,u_{\beta,s,x}) + \frac{\beta}{2}$$

which by (3.12)

$$\leq \beta$$

for $k \geq N$ uniformly in D. The reverse bound follows similarly. Since $\beta > 0$ was arbitrary, we have the desired result. ∎

Theorem 2.3.9: Let $\gamma > \gamma_1$. Then the value function, V^{ϵ}, of the stochastic control problem is a continuous viscosity solution of (3.3).

proof: We may rewrite equations (3.3) and (3.6) as

$$0 = V_s^{\epsilon} + \frac{\epsilon}{4\gamma^2}\Delta V^{\epsilon} + \frac{1}{4\gamma^2}|\nabla V^{\epsilon}|^2 + F^{\infty}(s,x,\nabla V^{\epsilon}) \qquad (3.13a)$$

and

$$0 = V_s + \frac{\epsilon}{4\gamma^2}\Delta V + \frac{1}{4\gamma^2}|\nabla V|^2 + F^{k}(s,x,\nabla V) \qquad (3.13b)$$

where

$$F^{\infty}(s,x,p) = \min_{v\in U}\left[f(s,x,v)\cdot p + L(s,x,v)\right]$$
$$F^{k}(s,x,p) = \min_{v\in U}\left[f^k(s,x,v)\cdot p + L^k(s,x,v)\right].$$

Note that $F^k \to F^{\infty}$ uniformly on compact sets. By Theorem 2.3.8, $V^{\epsilon,k} \to V^{\epsilon}$ uniformly on compact sets where $V^{\epsilon,k}$ is the classical solution to (3.13b). Therefore, by [17], Lemma 2.6.2, V^{ϵ} is a viscosity solution to (3.13a). (For a discussion of viscosity solutions, see [17], [8], or [9].) Clearly, it satisfies the terminal condition, and is continuous by Theorem 2.3.7. ∎

Actually, one could show that V^{ϵ} is a classical solution (via local existence theorems for classical solutions on cylindrical regions with the viscosity solution as boundary data), however that will not be needed here.

Now we proceed with obtaining the local Lipschitz condition in x. Note that the Lipschitz constant is linear in R; this is essential to the uniqueness proof later on. This linear requirement on the Lipschitz constant makes the proof somewhat tedious. In fact, there is a rather technical estimate which is relegated to the Appendix.

Lemma 2.3.10: Let $\gamma > \gamma_1$. Given $R < \infty$, there exists ϵ_0 such that for all $\epsilon \leq \epsilon_0$, for all $|x|, |y| \leq R$, for all $s \in [0, T]$, and for all $u \in A_{s,\nu}$ we have

$$|I^\epsilon(s, x, u) - I^\epsilon(s, y, u)| \leq [C_6 + C_7 R]|x - y|$$

and

$$|V^\epsilon(s, x) - V^\epsilon(s, y)| \leq [C_6 + C_7 R]|x - y|.$$

proof: Assume $|x - y| \leq 1$.

$$\begin{aligned}\frac{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)\right]}{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]} &= \mathrm{E}\Bigg\{\exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega) - \frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right] \\ &\qquad \cdot \frac{\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]}{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]}\Bigg\} \\ &= \int_\Omega \exp\Bigg[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega) \\ &\qquad - \frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\Bigg]\, d\hat{P}(\omega) \end{aligned} \tag{3.14}$$

where

$$\hat{P}(\mathcal{A}) = \int_{\mathcal{A}} \frac{\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]}{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]}\, dP. \tag{3.15}$$

By assumption (AF2)

$$|L(t, X_t, u_t) - L(t, Y_t, u_t)| \leq C(1 + |X_t| + |Y_t|)|X_t - Y_t| \tag{3.16}$$

where X_t and Y_t are solutions to (3.1) driven by u. with initial conditions (s,x) and (s,y), respectively. Similarly by (AF3)

$$|\Psi(X_T) - \Psi(Y_T)| \le C(1 + |X_T| + |Y_T|)|X_T - Y_T| \tag{3.17}$$

By (AF1) and Gronwall, we have

$$|X_t - Y_t| \le C_1|x - y| \qquad \forall t \in [s,T] \tag{3.18}$$

where $C_1 = 1 + KTe^{KT}$.

Also, as in (3.10), we have

$$|X_t| \le b_1 + b_2|x| + \sqrt{\frac{\epsilon}{2\gamma^2}}|B_t| + b_3\sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^t |B_r|\,dr \qquad \forall t \in [s,T] \tag{3.19}$$

where

$$b_1 = KTC_1$$

$$b_2 = 1 + b_1$$

and

$$b_3 = Ke^{KT}$$

and a similar bound for Y_t.

Combining (3.16)–(3.19), we have

$$\begin{aligned}\frac{1}{\epsilon}|A^\epsilon(s,x,u,\omega) - A^\epsilon(s,y,u,\omega)| \le \Big\{ & C_3(T+1)\left[2 + 2b_1 + b_2(|x| + |y|)\right] \\ & + C_3\Big[2\sqrt{\frac{\epsilon}{2\gamma^2}}(1+b_3)\int_s^T |B_t|\,dt \\ & \quad + 2b_3\sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^T\int_s^t |B_r|\,dr\,dt \\ & \quad + 2\sqrt{\frac{\epsilon}{2\gamma^2}}|B_T|\Big]\Big\}|x-y|\frac{1}{\epsilon} \\ \equiv \{ & C_3(T+1)\left[2b_1 + b_2(|x|+|y|)\right] + Q\}\,\frac{|x-y|}{\epsilon}\end{aligned} \tag{3.20}$$

where $C_3 = CC_1$. (Note (3.20) defines Q.)

Employing this in (3.14) yields

$$\frac{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)\right]}{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]} \le \exp\left\{\frac{1}{\epsilon}|x-y|C_3(T+1)[2+2b_1+b_2(|x|+|y|)]\right\}$$
$$\int_\Omega \exp\left[Q\frac{|x-y|}{\epsilon}\right]\, d\hat{P}(\omega)$$
$$\equiv \exp\left\{[C_4 + C_5(|x|+|y|)]\frac{|x-y|}{\epsilon}\right\}$$
$$\cdot \int_\Omega \left[\exp\left(\frac{Q}{\epsilon}\right)\right]^{|x-y|} d\hat{P}$$

which defines C_4 and C_5. By assumption $|x-y| \le 1$. So, by Hölder we have

$$\frac{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)\right]}{\mathrm{E}\exp\left[\frac{1}{\epsilon}A^\epsilon(s,y,u,\omega)\right]} \le \exp\left\{[C_4 + C_5(|x|+|y|)]\frac{|x-y|}{\epsilon}\right\}$$
$$\cdot \left[\int_\Omega \exp\left(\frac{Q}{\epsilon}\right)\, d\hat{P}\right]^{|x-y|}.$$

Therefore, using the definition of I^ϵ and symmetry

$$|I^\epsilon(s,x,u) - I^\epsilon(s,y,u)| \le |x-y|\left[C_4 + 2C_5R + \epsilon\log\int_\Omega \exp\left(\frac{Q}{\epsilon}\right)\, d\hat{P}\right].$$

By the bound in the Appendix, this yields

$$|I^\epsilon(s,x,u) - I^\epsilon(s,y,u)| \le [C_6 + C_7R]|x-y|$$

where $C_6 = C_4 + \alpha_1$ and $C_7 = 2C_5 + \alpha_2$.

Again, since this bound is independent of u, we have the bound for V^ϵ also. ∎

We now move to the equicontinuity with respect to time. The proof of this result will draw on the above equicontinuity result in the space variable, however some additional technicalities must be dealt with.

Lemma 2.3.11: Let $\gamma > \max\{\gamma_1, \gamma_2\}$ where γ_2 is given in the proof. Let $R < \infty$. Then there exists $\epsilon_0 \le 1$ such that for all $\epsilon \le \epsilon_0$, for all $|x| < R$, and for all $u \in A_{s,\nu}$

$$|I^\epsilon(s,x,u) - I^\epsilon(t,x,u)| \le g_R(|t-s|)$$

and

$$|V^\epsilon(s,x) - V^\epsilon(t,x)| \le g_R(|t-s|)$$

where $g_R(\rho) \downarrow 0$ as $\rho \downarrow 0$.

proof: Without loss of generality, let $0 \le s \le t \le T$. We will prove the two sides of the bound separately. We will first show

$$I^\epsilon(t,x,u) - I^\epsilon(s,x,u) \le g_R(|t-s|)$$

We have

$$I^\epsilon(t,x,u) - I^\epsilon(s,x,u) = \epsilon \log \frac{\mathrm{E}\exp\frac{1}{\epsilon}\left[\int_t^T L(r,Y_r,u_r)\,dr + \Psi(Y_T)\right]}{\mathrm{E}\exp\frac{1}{\epsilon}\left[\int_s^T L(r,X_r,u_r)\,dr + \Psi(X_T)\right]}$$

where

$$Y_r = x + \int_t^r f(\rho, Y_\rho, u_\rho)\,d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}}(B_r - B_t) \tag{3.21}$$

$$X_r = x + \int_s^r f(\rho, X_\rho, u_\rho)\,d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}}(B_r - B_s) \tag{3.22}$$

which for $r \ge t$

$$= X_t + \int_t^r f(\rho, X_\rho, u_\rho)\,d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}}(B_r - B_t). \tag{3.23}$$

Using the fact that $L \ge 0$ and flipping the numerator and denominator,

$$\begin{aligned} I^\epsilon(t,x,u) - I^\epsilon(s,x,u) &\le -\epsilon \log \frac{\mathrm{E}\exp\frac{1}{\epsilon}\left[\int_t^T L(r,X_r,u_r)\,dr + \Psi(X_T)\right]}{\mathrm{E}\exp\frac{1}{\epsilon}\left[\int_t^T L(r,Y_r,u_r)\,dr + \Psi(Y_T)\right]} \\ &= -\epsilon \log \frac{\mathrm{E}\left\{\mathrm{E}\left[\exp\frac{1}{\epsilon}\left(\int_t^T L(r,X_r,u_r)\,dr + \Psi(X_T)\right)\,\middle|\, X_t\right]\right\}}{\mathrm{E}\exp\frac{1}{\epsilon}\left[\int_t^T L(r,Y_r,u_r)\,dr + \Psi(Y_T)\right]}. \end{aligned}$$

By incorporating the running cost into the state vector and letting $\tilde{f}(r,X_r) \equiv f(r,X_r,u_r)$ where $\tilde{f}$ is now random, we may employ Krylov, Theorem 2.9.4 [30] directly to obtain

$$\begin{aligned}
I^\epsilon(t,x,u) - I^\epsilon(s,x,u) &\le -\epsilon \log \frac{\mathrm{E}J^\epsilon(t,X_t,u)}{J^\epsilon(t,x,u)} \\
&= -\epsilon \log \mathrm{E}\frac{J^\epsilon(t,X_t,u)}{J^\epsilon(t,x,u)}
\end{aligned}$$

which, by Jensen's inequality

$$\le \mathrm{E}\left\{-\epsilon \log \frac{J^\epsilon(t,X_t,u)}{J^\epsilon(t,x,u)}\right\}.$$

Let $R_1 = R + \delta$ where $\delta > 0$ and

$$A_1 = \int_{|X_t| \le R_1} \epsilon \log \frac{J^\epsilon(t,x,u)}{J^\epsilon(t,X_t,u)}\, dP(\omega)$$

and

$$A_2 = \int_{|X_t| > R_1} \epsilon \log \frac{J^\epsilon(t,x,u)}{J^\epsilon(t,X_t,u)}\, dP(\omega)$$

so that

$$I^\epsilon(t,x,u) - I^\epsilon(s,x,u) \le A_1 + A_2. \tag{3.24}$$

We consider each term separately. By Lemma 2.3.10, we have for $\epsilon \le \epsilon_0$

$$A_1 \le \int_{|X_t| \le R_1} [C_6 + C_7 R_1] |X_t - x|\, dP(\omega) \le [C_6 + C_7 R_1] \mathrm{E}|X_t - x|. \tag{3.25}$$

Employing a minor modification of (3.9), we have

$$\begin{aligned}
A_1 &\le [C_6 + C_7 R_1] C_1 \left[K(1+|x|)(t-s) + \sqrt{\frac{\epsilon}{2\gamma^2}} \mathrm{E}M_t \right] \\
&\le [C_6 + C_7 R_1] C_1 \left[K(1+R)(t-s) + \frac{1}{\sqrt{2\gamma^2}} \mathrm{E}M_t \right]
\end{aligned}$$

for $\epsilon \le 1$. This clearly goes to zero as $|t-s| \downarrow 0$ independently of $|x| \le R$ and $u \in A_{s,\nu}$.

Now we consider A_2. Since $L, \Psi \ge 0$, $J^\epsilon \ge 1$. This implies

$$A_2 \le \int_{|X_t| > R_1} \epsilon \log J^\epsilon(t,x,u)\, dP(\omega) = I^\epsilon(t,x,u) P(|X_t| > R_1)$$

which by Lemma 2.3.5

$$\leq C_2(1+R_1^2)P(|X_t| > R_1).$$

But $P(|X_t| > R_1) \leq P(|X_t - x| > \delta)$ which clearly goes to zero as $|t-s|$ goes to zero.

Since both A_1 and A_2 have the proper behavior, we have

$$I^\epsilon(t,x,u) - I^\epsilon(s,x,u) \leq g_R(|t-s|) \tag{3.26}$$

for the proper choice of g_R.

Now we turn to the reverse bound. We have

$$I^\epsilon(s,x,u) - I^\epsilon(t,x,u) = \epsilon \log \frac{J^\epsilon(s,x,u)}{J^\epsilon(t,x,u)}$$

which again by Krylov, Theorem 2.9.4 [30]

$$\begin{aligned}
&= \epsilon \log \mathrm{E}_{s,x}\Big\{\mathrm{E}_{t,X_t}\Big[\exp\Big(\int_t^T L(r,X_r,u_r)\,dr + \Psi(X_T)\Big) \\
&\qquad \cdot \exp\Big(\int_s^t L(r,X_r,u_r)\,dr\Big)\Big]/J^\epsilon(t,x,u)\Big\} \\
&= \epsilon \log \mathrm{E}_{s,x} \frac{J^\epsilon(t,X_t,u)\exp\left(\int_s^t L(r,X_r,u_r)\,dr\right)}{J^\epsilon(t,x,u)}
\end{aligned}$$

which by Lemma 2.3.10 and (AF2)

$$\begin{aligned}
&\leq \epsilon \log \mathrm{E}_{s,x} \exp \frac{1}{\epsilon}\Big\{[C_5 + C_6 R + C_6|X_t|]\,|X_t - x| \\
&\qquad + C(t-s) + C\int_s^t |X_r|^2\,dr\Big\}.
\end{aligned}$$

Now employing (3.9) and (3.10) (with t replacing T on the right hand side), and letting $C_8(R,t) = (1+R)Kt(1+Kte^{Kt})$, we have

$$\begin{aligned}
I^\epsilon(s,x,u) - I^\epsilon(t,x,u) &\leq \epsilon \log \mathrm{E}_{s,x} \exp \frac{1}{\epsilon}\Big\{\Big[C_5 + C_6 R + C_6\Big(R + C_8(R,t-s) \\
&\qquad + \sqrt{\frac{\epsilon}{2\gamma^2}}(1+Kt)M_t\Big)\Big] \\
&\qquad \cdot \Big[C_8(R,t-s) + \sqrt{\frac{\epsilon}{2\gamma^2}}(1+Kt)M_t\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ C(t-s) + 2C(R + C_8^2(R,T)(t-s) \\
&+ 2C \int_s^t \frac{\epsilon}{2\gamma^2} \left[|B_r| + K \int_s^r |B_\rho| \, d\rho \right]^2 dr \Big\} \\
\le \epsilon \log \mathrm{E}_{s,x} \exp \frac{1}{\epsilon} &\left[a_1 + a_2 \sqrt{\frac{\epsilon}{2\gamma^2}} M_t + a_3 \frac{\epsilon}{2\gamma^2} M_t^2 \right]
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= C_8(R,t-s)(C_5 + 2C_6 R + C_6 C_8(R,T)) + C(1 + 2R + C_8^2(R,T))(t-s) \\
a_2 &= C_6(R + 2C_8(R,t-s))(1+Kt) \\
a_3 &= (1+Kt)^2 + 2C(t + 2Kt^2 + K^2t^2).
\end{aligned}
$$

Extracting the deterministic part, and employing Hölder to separate the remainder, we have

$$
\begin{aligned}
I^\epsilon(s,x,u) - I^\epsilon(t,x,u) \le a_1 &+ \frac{\epsilon}{2} \log \mathrm{E}_{s,x} e^{\sqrt{\frac{2}{\gamma^2 \epsilon}} a_2 M_t} \\
&+ \frac{\epsilon}{2} \log \mathrm{E}_{s,x} e^{\frac{1}{\gamma^2} a_3 M_t^2}.
\end{aligned}
$$

Clearly the first term, a_1 goes to zero as $(t-s) \downarrow 0$ independent of $\epsilon \le \epsilon_0$, $|x| \le R$ and $u \in A_{s,\nu}$. Employing (A.4) from the Appendix, we see that the second term is bounded above by

$$
\log(2n) \frac{4a_2^2(t-s)}{\gamma^2}
$$

which also goes to zero independent of $\epsilon \le \epsilon_0$, $|x| \le R$ and $u \in A_{s,\nu}$. Finally, the third term exists for

$$
\gamma \ge \gamma_2 \equiv \left\{ 2T \left[(1+KT)^2 + 2C(T + 2KT^2 + K^2T^2) \right] \right\}^{\frac{1}{2}}
$$

and goes to zero independent of $\epsilon \le \epsilon_0$, $|x| \le R$ and $u \in A_{s,\nu}$. This completes the proof of the bound on the second side. ∎

At this point, we have demonstrated that the value function, V^ϵ, for the risk–averse stochastic control problem (3.1), (3.2) is a viscosity solution to the corresponding

DPE (3.3). We have also shown that V^ϵ is bounded on compact sets independent of ϵ and is equicontinuous. These results will be used later in an application of Ascoli–Arzela to show that the V^ϵ converge to a limit which is a viscosity solution of the limiting DPE.

2.4 Game Theory Results

In this section, we will demonstrate that the value function associated with the game discussed in Section 2.1 is the viscosity solution of a certain partial differential equation (PDE), i.e. the Isaacs equation. A similar result was obtained in [13]. However, in that case the assumptions were much more strict. In particular, the dynamics were bounded as well as Lipschitz. The cost functions were also bounded and Lipschitz. In addition, the controls for both players took values in a compact space. From our assumptions in Section 2.1, and the definition of the game below, it is clear that the assumptions are greatly relaxed here. Therefore, it is necessary to first bound the behavior of the value function, and show that any player's control which is near optimal must also be bounded in some sense. Once these results are established, we proceed in a manner similar to that in [13], however the calculations from that point on are somewhat simplified since the Isaacs condition is satisfied.

First let us define the game and the notion of value which we will be using more explicitly. As given in Section 2.2, the (deterministic) dynamics of the game will be

$$\begin{aligned} \frac{dX}{dt} &= f(t, X_t, u_t) + w_t \qquad t \in [s, T] \\ X_s &= x \end{aligned} \tag{4.1}$$

where X_t is the state at time t, and s is the initial time. Here, $u.$ is the control for the minimizing player. We assume that it is measurable with values in the compact set $U \subset \Re^m$. On the other hand, $w.$ is the control for the maximizing player which we assume is in $L^2([s,T]; \Re^n)$, hereafter denoted simply by $L^2[s,T]$. We will use the notation $\|w\|$ to indicate the $L^2[s,T]$ norm of w. The payoff will contain a running cost, L, and a terminal cost, Ψ. We let it be given by

$$P_{s,x}(u, w) = \int_s^T \left[L(t, X_t, u_t) - \gamma^2 |w_t|^2 \right] \, dt + \Psi(X_T) \tag{4.2}$$

Here γ is a positive constant which determines the size of the penalty which the maximizing player must pay for using control. This constant will be discussed further below. The feasible control sets for the minimizing and maximizing players, respectively, are

$$N(s) = \{u : [s,T] \to U, \text{measurable}\}$$

$$G(s) = L^2[s,T].$$

We also need to define the concept of strategy. A strategy for the minimizing player is an element from the set of mappings from $G(s)$ to $N(s)$ which satisfies the strategy condition. This condition is simply that the player cannot have foreknowledge of the opposing player's moves. Stating this precisely, the set of strategies for the minimizing player is given by

$$\Delta(s) = \{\beta : G(s) \to N(s) \quad \text{such that given any} \quad \tau \in [s,T], \quad w_t = \hat{w}_t \quad \forall t \in [s,\tau]$$
$$\text{implies} \quad \beta[w]_t = \beta[\hat{w}]_t \quad \forall t \in [s,\tau]\}.$$

Similarly, the strategies for the maximizing player are

$$\Gamma(s) = \{\alpha : N(s) \to G(s) \quad \text{such that given any} \quad \tau \in [s,T], \quad u_t = \hat{u}_t \quad \forall t \in [s,\tau]$$
$$\text{implies} \quad \alpha[u]_t = \alpha[\hat{u}]_t \quad \forall t \in [s,\tau]\}.$$

We use the Elliott–Kalton value of the game (see [12],[13]). In particular, the lower and upper values of the game are given by

$$W(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G(s)} P_{s,x}(\beta[w], w) \tag{4.3a}$$

and

$$U(s,x) = \sup_{\alpha \in \Gamma(s)} \inf_{u \in N(s)} P_{s,x}(u, \alpha[u]), \tag{4.3b}$$

respectively. If $U = W$, then we say that the game has a value.

We will be working mainly with the lower value. First, we get a bound on the behavior of this value. Let $Q_T = [0,T] \times \Re^n$.

Lemma 2.4.1:

$$W(s,x) \geq 0 \qquad \forall (s,x) \in Q_T. \tag{4.4}$$

If $\gamma \geq \gamma_1 \equiv \left[3CT(T+1)(1+K^2T^2e^{2KT})\right]^{\frac{1}{2}}$, then

$$W(s,x) < \infty \qquad \forall (s,x) \in Q_T. \tag{4.5}$$

In particular, for $\gamma \geq \gamma_1$,

$$W(s,x) < b_1 + b_2|x|^2 \qquad \forall (s,x) \in Q_T \tag{4.6}$$

where b_1 and b_2 are given in the proof of the theorem.

Remark 2.4.2: This is the same γ_1 as given in the stochastic control section.

proof: Taking $w \equiv 0$ and noting that $L, \Psi \geq 0$, implies $W(s,x) \geq 0$.

Now we proceed to the second assertion. By (4.1) and assumptions (AF1)

$$|X_t| \leq |x| + K\int_s^t (1+|X_r|)\,dr + \int_s^t |w_r|\,dr$$

which, by Gronwall, implies

$$|X_t| \leq \left(1+K(t-s)e^{KT}\right)|x| + K(t-s) + \frac{K^2(t-s)^2}{2}e^{KT} + \int_s^t |w_r|\,dr$$
$$+ Ke^{KT}\int_s^t\int_s^r |w_p|\,dp\,dr.$$

Noting that if $r \leq t$

$$\int_s^r |w_p|\,dp \leq \int_s^t |w_p|\,dp \tag{4.7}$$

and using $t \leq T$ to obtain a crude bound, we have

$$|X_t| \leq a_1 + (1+a_2)\int_s^t |w_r|\,dr$$

where

$$a_1 = \left(1 + KTe^{KT}\right)|x| + KT + \frac{K^2(t-s)^2}{2}e^{KT}$$

and

$$a_2 = KTe^{KT}.$$

Noting that $(\sum_{i=1}^n c_i)^2 \leq n\sum_{i=1}^n c_i^2$, and using Hölder's inequality yields

$$|X_t|^2 \leq 3a_1^2 + 3(1+a_2^2)(t-s)\int_s^t |w_r|^2\,dr. \tag{4.8}$$

Now by assumptions (AF2) and (AF3),

$$\int_s^T [L(r,X_r,u_r) - \gamma^2|w_r|^2]\,dr + \Psi(X_T) \leq \int_s^T [C(1+|X_r|^2) - \gamma^2|w_r|^2]\,dr + C(1+|X_T|^2)$$

which by (4.7) (with $|w_r|^2$ replacing $|w_r|$) and (4.8)

$$\begin{aligned} &\leq C(1+T)(1+3a_1^2) + \left[3CT(1+T)(1+a_2^2) - \gamma^2\right]\int_s^T |w_r|^2\,dr \\ &= C(1+T)(1+3a_1^2) + (\gamma_1^2 - \gamma^2)\int_s^T |w_r|^2\,dr. \end{aligned} \tag{4.9}$$

From (4.9), we see that if $\gamma \geq \gamma_1$, then

$$\sup_{w\in G(s)} \int_s^T \left[L(t,X_t,u_t) - \gamma^2|w_2|^2\right]\,dt + \Psi(X_T) \leq C(1+T)(1+3a_1^2) < \infty$$

which proves our second assertion.

Finally,

$$C(1+T)(1+3a_1^2) \leq b_1 + b_2|x|^2$$

where

$$b_1 = C(1+T)\left[1 + 6\left(KT + \frac{K^2T^2}{2}e^{2KT}\right)^2\right]$$

and

$$b_2 = 6C(1+T)\left(1 + KTe^{KT}\right)^2$$

which proves our third assertion. ∎

Remark 2.4.3: We made very loose estimates in calculating γ_1. Clearly a more careful treatment could produce a smaller γ_1.

Lemma 2.4.4: Let $\gamma > \gamma_1$ and $\epsilon \leq 1$. Given $\beta \in \Delta(s)$, let $\tilde{w}$ be such that

$$P_{s,x}(\beta[\tilde{w}], \tilde{w}) \geq \sup_{w \in G(s)} P_{s,x}(\beta[w], w) - \epsilon.$$

Then

$$\int_{t_1}^{t_2} |\tilde{w}_r| \, dr \leq (C_1 + C_2|x|)|t_2 - t_1|^{1/2} \qquad \forall \, t_1, t_2 \in [s, T].$$

proof: Since $\gamma > \gamma_1$, $\sup_{w \in G(s)} P_{s,x}(\beta[w], w) < \infty$. By assumption and the first assertion of Lemma 2.4.1, $P_{s,x}(\beta[\tilde{w}], \tilde{w}) \geq -\epsilon$ which implies

$$\gamma^2 \int_s^T |\tilde{w}_t|^2 \, dt \leq \left[\int_s^T L(t, X_t, \beta[\tilde{w}]_t) \, dt + \Psi(X_T) + \epsilon \right]$$

which as in the proof of Lemma 2.4.1 yields (noting $\epsilon \leq 1$)

$$\int_{t_1}^{t_2} |\tilde{w}_t|^2 \, dt \leq \frac{1}{\gamma^2 - \gamma_1^2} [b_1' + b_2'|x|^2] \qquad \forall \, s \leq t_1 \leq t_2 \leq T \tag{4.10}$$

for proper choice of b_1', b_2' depending on (C, K, T).

By Cauchy–Schwarz then

$$\int_{t_1}^{t_2} |\tilde{w}_t| \, dt \leq \left[\frac{b_1' + b_2'|x|^2}{\gamma^2 - \gamma_1^2} \right]^{1/2} |t_2 - t_1|^{1/2}$$

which yields

$$\int_{t_1}^{t_2} |\tilde{w}_t| \, dt \leq (C_1 + C_2|x|)|t_2 - t_1|^{1/2}$$

for proper choice of C_1 and C_2. ∎

Remark 2.4.5: Let $M_x = C_1 + C_2|x|$, $M \in [M_x, \infty)$ and define

$$G_M(s) = \left\{ w \in G(s) \ : \ \int_{t_1}^{t_2} |w_t|\, dt \leq M|t_2 - t_1|^{1/2} \quad \forall\, t_1, t_2 \in [s, T] \right\}.$$

We have shown

$$W(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_M(s)} P_{s,x}(\beta[w], w). \tag{4.11}$$

Remark 2.4.6: Alternatively, one might choose to note that we have shown that we may assume

$$\|w\|_{L^2}^2 \leq \frac{1}{\gamma^2 - \gamma_1^2}[b_1' + b_2'|x|^2].$$

Lemma 2.4.7: Let X_r be a solution of (4.1) on $[s,T]$ driven by $\tilde{w} \in G_M(s)$ and β. Then there exist B_3, B_4, B_5 depending on K, T, M such that

$$|X_r - x| \leq B_3(r-s) + B_4\sqrt{r-s} + B_5|x|(r-s) \qquad \forall r \in [s,T].$$

proof: By (4.1), (AF1) and the assumption on $\tilde{w}$, we have

$$|X_r - x| \leq K(1+|x|)(r-s) + M\sqrt{r-s} + K\int_s^r |X_\rho - x|\, d\rho$$

which by Gronwall implies

$$|X_r - x| \leq K(1+|x|)(r-s) + M\sqrt{r-s} + [K(1+|x|)T + M\sqrt{T}]Ke^{KT}(r-s).$$

This yields the desired result for the proper choice of B_3, B_4, B_5. ∎

Lemma 2.4.8: Let $\gamma > \gamma_1$ and $\epsilon \leq 1$. Given $\beta \in \Delta(s)$, let $\tilde{w}$ be such that

$$P_{s,x}(\beta[\tilde{w}], \tilde{w}) \geq \sup_{w \in G(s)} P_{s,x}(\beta[w], w) - \epsilon$$

where X_t is the solution of (4.1) driven by $\tilde{w}, \beta$. Then

$$|X_t - x| \leq (B_3 + B_5|x|)T + B_4\sqrt{T} \qquad \forall t \in [s, T].$$

proof: Combine Lemmas 2.4.4 and 2.4.7. ∎

The next proposition and the two remarks which follow it are not essential to the development, but are somewhat interesting in their own right. The proposition demonstrates that if one follows a near optimal control for a portion of the trajectory, then any near optimal control for the remainder will satisfy the same bound as for the initial trajectory.

Lemma 2.4.9: Let $\gamma > \gamma_1$ and $\epsilon < 1$. Given $\beta \in \Delta(s)$, let $\tilde{w}$ be such that

$$P_{s,x}(\beta[\tilde{w}], \tilde{w}) \geq \sup_{w \in G(s)} P_{s,x}(\beta[w], w) - \epsilon/2. \tag{4.12}$$

Let $X_t = x + \int_s^t f(r, X_r, \beta[\tilde{w}]_r) + \tilde{w}_r \, dr$. Let $\hat{w} \in G(t)$ be such that

$$P_{t,X_t}(\beta[\hat{w}], \hat{w}) \geq \sup_{w \in G(t)} P_{t,X_t}(\beta[w], w) - \epsilon/2. \tag{4.13}$$

Then $\hat{w} \in G_{M_x}(t)$.

proof: Define $\hat{w}' \in G(s)$ by

$$\hat{w}'_\tau = \begin{cases} \tilde{w}_\tau & \text{if } \tau \leq t \\ \hat{w}_\tau & \text{if } \tau > t. \end{cases}$$

Then

$$\begin{aligned} P_{s,x}(\beta[\hat{w}'], \hat{w}') = & \int_s^t [L(r, X_r, \beta[\hat{w}']_r) - \gamma^2|\tilde{w}_r|^2] \, dr \\ & + \int_t^T [L(r, X_r, \beta[\hat{w}']_r) - \gamma^2|\hat{w}_r|^2] \, dr + \Psi(X_T) \end{aligned} \tag{4.14}$$

where

$$X_r = x + \int_s^r f(p, X_p, \beta[\hat{w}']_p) + \hat{w}'_p \, dp.$$

By (4.13) and (4.14), we have

$$P_{s,x}(\beta[\hat{w}'], \hat{w}') \geq \int_s^t [L(r, X_r, \beta[\hat{w}']_r) - \gamma^2 |\tilde{w}_r|^2] \, dr + \sup_{w \in G(t)} P_{t,X_t}(\beta[w], w) - \epsilon/2$$

which by the definition of supremum

$$\geq P_{s,x}(\beta[\tilde{w}], \tilde{w}) - \epsilon/2$$

which by (4.12)

$$\geq \sup_{w \in G(s)} P_{s,x}(\beta[w], w) - \epsilon.$$

Therefore, by Lemma 2.4.4

$$\int_{t_1}^{t_2} |\hat{w}_r| \, dr \leq M_x |t_2 - t_1|^{1/2} \qquad \forall t_1, t_2 \in [t, T]. \blacksquare$$

Remark 2.4.10: We say that $(t, y) \in R^{\epsilon/2}(s, x)$ if there exist β and $\tilde{w}$ satisfying (4.12) such that $y = X_t$ where $X.$ is the solution of (4.1) driven by β and $\tilde{w}$. We have shown that

$$W(t, y) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(t)} P_{t,y}(\beta[w], w) \qquad \forall y \in R^{\epsilon/2}(s, x). \tag{4.15}$$

Remark 2.4.11: Let $y \in R^{\epsilon/2}(s, x)$. Then by the lemma

$$\begin{aligned} \sup_{G_{M_x}(t)} P_{t,y}(\beta[w], w) &= \sup_{w \in G(t)} P_{t,y}(\beta[w], w) \\ &= \sup_{w \in G_M(t)} P_{t,y}(\beta[w], w) \end{aligned}$$

for any $M \in [M_x, \infty)$. By (4.15) this implies

$$\begin{aligned} W(t, y) &= \inf_{\beta \in \Delta(t)} \sup_{w \in G(t)} P_{t,y}(\beta[w], w) \\ &= \inf_{\beta \in \Delta(t)} \sup_{w \in G_M(t)} P_{t,y}(\beta[w], w) \end{aligned} \tag{4.16}$$

for any $M \geq M_x$.

Let $h(s,x,u,w) = L(s,x,u) - \gamma^2|w|^2$. The next theorem is our main dynamic programming principle.

Theorem 2.4.12: Let $\gamma > \gamma_1$ and $0 \leq s \leq s+\sigma \leq T$. Then

$$W(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + W(s+\sigma, X_{s+\sigma}) \right].$$

proof: Let

$$R(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + W(s+\sigma, X_{s+\sigma}) \right]. \quad (4.17)$$

Given $\epsilon > 0$ (we take this ≤ 1), there exists $\tilde{\beta} \in \Delta(s)$ such that

$$R(s,x) \geq \sup_{w \in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \tilde{\beta}[w]_t, w_t)\, dt + W(s+\sigma, X_{s+\sigma}) \right] - \epsilon. \quad (4.18)$$

Now, for any $y \in \Re^n$

$$W(s+\sigma, y) = \inf_{\beta \in \Delta(s+\sigma)} \sup_{w \in G(s+\sigma)} \left[\int_{s+\sigma}^{T} h(t, X_t, \beta[w]_t, w_t)\, dt + \Psi(X_T) \right]$$

which implies there exists $\tilde{\beta}'_y \in \Delta(s+\sigma)$ such that

$$W(s+\sigma, y) \geq \sup_{w \in G(s+\sigma)} \left[\int_{s+\sigma}^{T} h(t, X_t, \tilde{\beta}'_y[w]_t, w_t)\, dt + \Psi(X_T) \right] - \epsilon. \quad (4.19)$$

Define $\hat{\beta} \in \Delta(s)$ as follows. For each $w \in G(s)$, let

$$\hat{\beta}[w]_t = \begin{cases} \tilde{\beta}[w]_t & \text{if } t \leq s+\sigma \\ \tilde{\beta}'_{X_{s+\sigma}}[w]_t & \text{if } t > s+\sigma. \end{cases}$$

(One may note that I am abusing notation slightly in this last line so as to avoid excessively complex expressions.) Also, given $\tilde{w} \in G_{M_x}(s)$ and $\tilde{w}' \in G(s+\sigma)$, define

$$\hat{w} = \begin{cases} \tilde{w} & \text{if } t \leq s+\sigma \\ \tilde{w}' & \text{if } t > s+\sigma. \end{cases}$$

Then (4.18), (4.19) imply

$$R(s,x) \geq \sup_{\tilde{w}\in G_{M_x}(s)} \sup_{\tilde{w}'\in G(s+\sigma)} \left\{ \int_s^{s+\sigma} h(t,X_t,\hat{\beta}[\hat{w}]_t,\hat{w}_t)\,dt + \left[\int_{s+\sigma}^T h(t,X_t,\hat{\beta}[\hat{w}]_t,\hat{w}_t)\,dt + \Psi(X_T)\right]\right\} - 2\epsilon$$

which since $G_{M_x} \subset \{\hat{w} \,:\, \tilde{w} \in G_{M_x}(s),\ \tilde{w}' \in G(s+\sigma)\}$

$$\geq \sup_{w\in G_{M_x}(s)} \left[\int_s^T h(t,X_t,\hat{\beta}[w]_t,w_t)\,dt + +\Psi(X_T)\right] - 2\epsilon.$$

This implies

$$R(s,x) \geq \inf_{\beta\in\Delta(s)} \sup_{w\in G_{M_x}(s)} P_{s,x}(\beta[w],w) - 2\epsilon$$

which implies (since ϵ is arbitrary) that

$$R(s,x) \geq W(s,x).$$

Now we only need to prove the reverse inequality. Given $\epsilon > 0$, there exists $\tilde{\beta} \in \Delta(s)$ such that

$$W(s,x) \geq \sup_{w\in G(s)} \left[\int_s^T h(t,X_t,\tilde{\beta}[w]_t,w_t)\,dt + \Psi(X_T)\right] - \epsilon. \tag{4.20}$$

By (4.17)

$$R(s,x) \leq \sup_{w\in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t,X_t,\tilde{\beta}[w]_t,w_t)\,dt + W(s+\sigma,X_{s+\sigma})\right].$$

Consequently there exists $\tilde{w} \in G_{M_x}(s)$ such that

$$R(s,x) \leq \int_s^{s+\sigma} h(t,X_t,\tilde{\beta}[\tilde{w}]_t,\tilde{w}_t)\,dt + W(s+\sigma,X_{s+\sigma}) + \epsilon. \tag{4.21}$$

As before

$$W(s+\sigma,X_{s+\sigma}) \leq \sup_{w\in G(s+\sigma)} \left[\int_{s+\sigma}^T h(t,X_t,\hat{\beta}[w]_t,w_t)\,dt + \Psi(X_T)\right]$$

where, given $w \in G(s+\sigma)$, we define $\hat{w} \in G(s)$ by

$$\hat{w}_t = \begin{cases} \tilde{w}_t & \text{if } t \leq s+\sigma \\ w_t & \text{if } t > s+\sigma \end{cases}$$

and $\hat{\beta} \in \Delta(s+\sigma)$ by $\hat{\beta}[w] = \tilde{\beta}[\hat{w}]$. This implies there exists $\hat{w}' \in G(s+\sigma)$ such that

$$W(s+\sigma, X_{s+\sigma}) \leq \int_{s+\sigma}^{T} h(t, X_t, \hat{\beta}[\hat{w}']_t, \hat{w}'_t)\, dt + \Psi(X_T) + \epsilon. \tag{4.22}$$

Define

$$\hat{w}_t = \begin{cases} \tilde{w}_t & \text{if } t \leq s+\sigma \\ \hat{w}'_t & \text{if } t > s+\sigma. \end{cases}$$

Then (4.21), (4.22) imply

$$R(s,x) \leq \int_{s}^{T} h(t, X_t, \tilde{\beta}[\hat{w}]_t, \hat{w}_t)\, dt + \Psi(X_T) + 2\epsilon$$

which by (4.20) implies

$$R(s,x) \leq W(s,x) + 3\epsilon.$$

Since ϵ was arbitrary, we have our reverse inequality. ∎

Remark 2.4.13: Minor modifications of the proof also yield

$$W(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G(s)} \left[\int_{s}^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + W(s+\sigma, X_{s+\sigma}) \right].$$

and

$$W(s,x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_M(s)} \left[\int_{s}^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + W(s+\sigma, X_{s+\sigma}) \right].$$

for any $M \geq M_x$.

Now that we have our dynamic programming principles, we proceed to prove that the value function is the viscosity solution of the corresponding Isaacs equation. Two short lemmas precede the main Theorem.

Let

$$H(t,x,p) = \max_{w\in\Re^n} \min_{v\in U} \left\{ L(t,x,v) - \gamma^2|w|^2 + [f(t,x,v)+w]\cdot p \right\}$$

and note that

$$\begin{aligned} H(t,x,p) &= \max_{w\in\Re^n} \{w\cdot p - \gamma^2|w|^2\} + \min_{v\in U}\{L(t,x,v) + f(t,x,v)\cdot p\} \\ &= \frac{|p|^2}{4\gamma^2} + \min_{v\in U}\{L(t,x,v) + f(t,x,v)\cdot p\}, \end{aligned}$$

that is, the Isaacs condition holds.

Lemma 2.4.14: Let $g \in C^1([0,T]\times\Re^n)$ satisfy

$$g_t(t_0,x_0) + H(t_0,x_0,\nabla g(t_0,x_0)) \le -\theta < 0. \tag{4.23}$$

Then for any $M \ge M_{x_0}$, for $\sigma > 0$ sufficiently small (depending on M, x_0), there exists $\beta \in \Delta(t_0)$ such that for all $w \in G_M(t_0)$

$$\int_{t_0}^{t_0+\sigma} h(r,X_r,\beta[w]_r,w_r) + [f(r,X_r,\beta[w]_r) + w_r]\cdot\nabla g(r,X_r) + g_t(r,X_r)\,dr \le -\frac{\sigma\theta}{2}.$$

proof: Let

$$\Lambda(t,x,v,w) = h(t,x,v,w) + [f(t,x,v)+w]\cdot\nabla g(t,x) + g_t(t,x).$$

By (4.23) $\max_{w\in\Re^n}\min_{v\in U}\Lambda(t,x,v,w) \le -\theta$. Let $v_0 = \operatorname{argmin}_{v\in U}\Lambda(t_0,x_0,v,w)$, and note that v_0 is independent of w (a result of the Isaacs condition holding), so that

$$\Lambda(t_0,x_0,v_0,w) \le -\theta \qquad \forall\, w\in\Re^n.$$

Now let $w \in G_M(t_0)$ and $\beta[w] \equiv v_0$. By Lemma 2.4.7,

$$|X_{t_0+r} - x_0| \le B_3 r + B_4\sqrt{r} + B_5|x_0|r$$

where

$$X_{t_0+r} = x_0 + \int_{t_0}^{t_0+r} f(\rho, x_\rho, \beta[w]_\rho) + w_\rho \, d\rho.$$

Therefore, given $\delta > 0$, there exists $\sigma > 0$ (depending on M, x_0) such that

$$|X_{t_0+r} - x_0| \leq \delta \qquad \forall r \leq \sigma, \, \forall w \in G_M(t_0), \, \forall \beta \in \Delta(t_0).$$

Then by continuity (taking $\delta > 0$ sufficiently small),

$$\Lambda(t_0 + r, X_{t_0+r}, \beta[w]_{t_0+r}, w_{t_0+r}) \leq -\frac{\theta}{2} \qquad \forall r \leq \sigma, \, \forall w \in G_M(t_0).$$

Integrating from t_0 to $t_0 + \sigma$, we obtain the desired result. ∎

Lemma 2.4.15: Let $g \in C^1([0,T] \times \Re^n)$ satisfy

$$g_t(t_0, x_0) + H(t_0, x_0, \nabla g(t_0, x_0)) \geq \theta > 0.$$

Then there exists M' such that for all $M \geq M'$, for $\sigma > 0$ sufficiently small (depending on M, x_0), there exists $w \in G_M(t_0)$ such that for all $\beta \in \Delta(t_0)$

$$\int_{t_0}^{t_0+\sigma} h(r, X_r, \beta[w]_r, w_r) + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) + g_t(r, X_r) \, dr \geq \frac{\sigma\theta}{2}.$$

proof: We have

$$\max_{w \in \Re^n} \min_{v \in U} \Lambda(t_0, x_0, v, w) \geq \theta.$$

This implies that there exists w^* such that

$$\Lambda(t_0, x_0, v, w^*) \geq \theta \qquad \forall v \in U.$$

Let $w_s = w^*$ for all s (and note $w. \in G_M(t_0)$ for all $M \geq M' \equiv |w^*|\sqrt{T}$).

As before, by Lemma 2.4.7, for any u.

$$|X_{t_0+r} - x_0| \leq B_3 r + B_4 \sqrt{r} + B_5 |x_0| r$$

which implies that given $\delta > 0$, there exists $\sigma > 0$ (depending on M, x_0) such that

$$|X_{t_0+r} - x_0| \leq \delta \qquad \forall r \leq \sigma, \, \forall u \in N(t_0).$$

Therefore by continuity (taking $\delta > 0$ sufficiently small),

$$\Lambda(T_0 + r, X_{t_0+r}, \beta[w]_{t_0+r}, w_{t_0+r}) \geq \frac{\theta}{2} \qquad \forall r \leq \sigma, \, \forall \beta \in \Delta(t_0)$$

Integrating from t_0 to $t_0 + \sigma$, we obtain the desired result. ∎

Theorem 2.4.16: W is a viscosity solution to

$$\begin{aligned} &W_s + H(s, x, \nabla W) = 0 \\ &W(T, x) = \Psi(x). \end{aligned} \tag{4.24}$$

proof: It will be shown that W satisfies the conditions of a viscosity subsolution and supersolution of (4.24) according to Definition 2.4.1 of [17]. It is easily seen that solutions of (4.1) are continuous with respect to initial conditions, and consequently W is continuous. Thus W belongs to the proper class of functions.

Now, let $g \in C^1([0, T] \times \Re^n)$ and suppose that $W - g$ has a local maximum at (t_0, x_0). We must show

$$g_t(t_0, x_0) + H(t_0, x_0, \nabla g(t_0, x_0)) \geq 0. \tag{4.25}$$

Suppose not. Then there exists $\theta > 0$ such that

$$g_t(t_0, x_0) + H(t_0, x_0, \nabla g(t_0, x_0)) \leq -\theta < 0.$$

Let $M \geq M_{x_0}$. By Lemma 2.4.14, there exists $\sigma > 0$ (depending on M, x_0) and $\beta \in \Delta(t_0)$ such that for all $w \in G_M(t_0)$,

$$\int_{t_0}^{t_0+\sigma} h(r, X_r, \beta[w]_r, w_r) + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) + g_t(r, X_r) \, dr \leq -\frac{\sigma\theta}{2}. \tag{4.26}$$

Since $W - g$ has a local maximum at (t_0, x_0), using Lemma 2.4.7 yields

$$W(t_0, x_0) - g(t_0, x_0) \geq W(t_0 + r, X_{t_0+r}) - g(t_o + r, X_{t_0+r}) \qquad \forall r \leq \sigma \tag{4.27}$$

(where we may have to reduce the size of σ from above).

From Theorem 2.4.12,

$$W(s, x) = \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + W(s + \sigma, X_{s+\sigma}) \right]. \tag{4.28}$$

Substituting (4.27) into (4.28), we have

$$\inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + g(s + \sigma, X_{s+\sigma}) - g(t_0, x_0) \right] \geq 0$$

which implies

$$\begin{aligned} \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \bigg\{ \int_s^{s+\sigma} & h(t, X_t, \beta[w]_t, w_t)\, dt + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) \\ & + g_t(r, X_r)\, dr \bigg\} \geq 0. \end{aligned} \tag{4.29}$$

But (4.26) implies

$$\begin{aligned} \inf_{\beta \in \Delta(s)} \sup_{w \in G_{M_x}(s)} \bigg\{ \int_s^{s+\sigma} & h(t, X_t, \beta[w]_t, w_t)\, dt + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) \\ & + g_t(r, X_r)\, dr \bigg\} \leq -\frac{\sigma\theta}{2}. \end{aligned} \tag{4.30}$$

Since (4.30) contradicts (4.29), (4.25) holds. Thus W is a viscosity supersolution.

Now suppose that $W - g$ has a local minimum at (t_0, x_0). We must show

$$g_t(t_0, x_0) + H(t_0, x_0, \nabla g(t_0, x_0)) \leq 0. \tag{4.31}$$

Suppose not. Then there exists $\theta > 0$ such that

$$g_t(t_0, x_0) + H(t_0, x_0, \nabla g(t_0, x_0)) \geq \theta < 0.$$

By Lemma 2.4.15, there exists M, $\sigma > 0$ (depending on M, x_0) and $w \in G_M(t_0)$ such that for all $\beta \in \Delta(t_0)$

$$\int_{t_0}^{t_0+\sigma} h(r, X_r, \beta[w]_r, w_r) + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) + g_t(r, X_r)\, dr \geq \frac{\sigma\theta}{2}$$

which implies

$$\begin{aligned} \inf_{\beta\in\Delta(s)} \sup_{w\in G_{M_x}(s)} \Big\{ \int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) \\ + g_t(r, X_r)\, dr \Big\} \geq \frac{\sigma\theta}{2}. \end{aligned} \tag{4.32}$$

Since $W - g$ has a local minimum at (t_0, x_0), using Lemma 2.4.7 yields

$$W(t_0, x_0) - g(t_0, x_0) \leq W(t_0 + r, X_{t_0+r}) - g(t_o + r, X_{t_0+r}) \qquad \forall r \leq \sigma$$

(where we may have to reduce the size of σ from above). Substituting this into (4.28), we have

$$\inf_{\beta\in\Delta(s)} \sup_{w\in G_{M_x}(s)} \left[\int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + g(s+\sigma, X_{s+\sigma}) - g(t_0, x_0) \right] \leq 0$$

which implies

$$\begin{aligned} \inf_{\beta\in\Delta(s)} \sup_{w\in G_{M_x}(s)} \Big\{ \int_s^{s+\sigma} h(t, X_t, \beta[w]_t, w_t)\, dt + [f(r, X_r, \beta[w]_r) + w_r] \cdot \nabla g(r, X_r) \\ + g_t(r, X_r)\, dr \Big\} \leq 0. \end{aligned}$$

But this contradicts (4.32), so (4.31) must hold. This implies that W is a viscosity subsolution.

Therefore, W is a viscosity solution of (4.24). ∎

Note that, since the Isaacs condition is satisfied by (4.24) (see the expressions just prior to Lemma 2.4.14), we can similarly prove that the upper value is also a viscosity solution to (4.24). Then by the uniqueness result to follow in Section 2.5, $U = W$.

We now prove an additional minor result which will be needed later on. Namely, we prove a specific local Lipschitz bound on the variation of W in x.

Lemma 2.4.17: Let $\gamma > \gamma_1$. Let $R < \infty$ and M_R defined as in Remark 2.4.5 (i.e. $M_R = C_1 + C_2 R$). Then for all x, y such that $|x|, |y| \leq R$ and for all $s \in [0, T]$ we have

$$|W(s,x) - W(s,y)| \leq [B_6 + B_7 R]|x - y|$$

where B_6,B_7 are given in the proof.

proof: Let $w \in G_{M_R}(s)$, $\beta \in \Delta(s)$ and $u = \beta[w]$. Let X_t and Y_t be solutions of (4.1) with initial conditions $X_s = x$ and $Y_s = y$, respectively. By (AF1) and Gronwall

$$|X_t - Y_t| \leq (1 + KTe^{KT})|x - y| \equiv a_3|x - y|. \tag{4.33}$$

Also, by Lemma 2.4.7 (with the value of B_4 used explicitly),

$$|X_t| \leq B_3 T + a_3 M_R \sqrt{T} + B_5 T|x| \qquad \forall\, t \in [s, T] \tag{4.34}$$

with a similar bound for Y_t.

By assumption (AF2)

$$|h(t, X_t, u_t, w_t) - h(t, Y_t, u_t, w_t)| \leq C(1 + |X_t| + |Y_t|)|X_t - Y_t| \tag{4.35}$$

while by (AF3)

$$|\Psi(X_T) - \Psi(Y_T)| \leq C(1 + |X_T| + |Y_T|)|X_T - Y_T| \tag{4.36}$$

Combining (4.33) – (4.36), we have

$$|P_{s,x}(\beta[w], w) - P_{s,y}(\beta[w], w)| \leq 2Ca_3(1+T)\left[1 + B_3 T + a_3\sqrt{T} M_R + B_5 T R\right]|x - y|$$

which, recalling the definition of M_R from Remark 2.4.5, is

$$
\begin{aligned}
&= 2Ca_3(1+T)\Big[1 + B_3T + a_3C_1\sqrt{T} \\
&\qquad\qquad + (a_3C_2\sqrt{T} + B_5T)R\Big]|x-y| \\
&\equiv (B_6 + B_7R)|x-y|. \qquad (4.37)
\end{aligned}
$$

We now follow the standard approach to obtain the desired result. For instance, given $\delta > 0$, choose $w_\delta \in G_{M_r(s)}$ such that

$$
P_{s,x}(\beta[w_\delta], w_\delta) \geq \sup_{w \in G_{M_R}(s)} P_{s,x}(\beta[w], w) - \delta
$$

which implies

$$
\begin{aligned}
\sup_{w \in G_{M_R}(s)} P_{s,y}(\beta[w], w) - \sup_{w \in G_{M_R}(s)} P_{s,x}(\beta[w], w) &\leq P_{s,y}(\beta[w_\delta], w_\delta) \\
&\quad - P_{s,x}(\beta[w_\delta], w_\delta) + \delta
\end{aligned}
$$

which by (4.37)

$$
\leq (B_6 + B_7R)|x-y| + \delta.
$$

Since $\delta > 0$ was arbitrary, we have

$$
\sup_{w \in G_{M_R}(s)} P_{s,y}(\beta[w], w) - \sup_{w \in G_{M_R}(s)} P_{s,x}(\beta[w], w) \leq (B_6 + B_7R)|x-y|.
$$

By symmetry we obtain

$$
|\sup_{w \in G_{M_R}(s)} P_{s,y}(\beta[w], w) - \sup_{w \in G_{M_R}(s)} P_{s,x}(\beta[w], w)| \leq (B_6 + B_7R)|x-y|.
$$

Repeating this procedure for the $\inf_{\beta \in \Delta(s)}$ operation yields the desired result. ∎

2.5 Uniqueness

Recall that the Isaacs equation for the dynamic game of Section 2.4 was given by

$$\begin{aligned} &W_s + H(s, x, \nabla W) = 0 \\ &W(T, x) = \Psi(x) \end{aligned} \tag{5.1}$$

where

$$H(s, x, p) = \frac{1}{4\gamma^2}|p|^2 + \min_{v \in U}\left[f(s, x, v) \cdot p + L(s, x, v)\right].$$

We showed that the value function for the game was a viscosity solution to (5.1). Note also that in Section 2.3, we showed that the value function for the risk–averse stochastic control problem was a viscosity solution to

$$\begin{aligned} &V_s + H^\epsilon(s, x, \nabla V, \Delta V) = 0 \\ &V(T, x) = \Psi(x) \end{aligned} \tag{5.2}$$

where

$$H^\epsilon(s, x, p, r) = H(s, x, p) + \frac{\epsilon}{4\gamma^2}r.$$

We also showed in Section 2.3 that the $\{V^\epsilon\}$ were uniformly bounded on compact sets and equicontinuous. Thus, if we use Ascoli–Arzela, we will have a subsequence (which we will denote here also by $\{V^\epsilon\}$) converging to some continuous V as $\epsilon \to 0$. From [17], Lemma 2.6.2, we know that V will also be a viscosity solution to the limiting version of (5.2), which is, in fact, (5.1). If we can prove that (5.1) has a unique viscosity solution, then we will know that $V = W$. We will prove a slightly weaker result which will, however, be sufficient. We will prove uniqueness among the class of solutions which are continuous and locally Lipschitz in x with a Lipschitz constant which is at most affine in $|x|$. Note that by Lemmas 2.3.10 and 2.4.17, V and W are in this class.

Some comments should be made regarding this uniqueness result. There are three peculiarities of this problem, the combination of which makes the result subtle. They are:

1) the domain is unbounded
2) the solution may grow quadratically
3) the Hamiltonian is quadratic in p.

The saving grace is that the form of the time dependence is such that we can employ a cone of dependence argument.

Uniqueness for this problem has not previously been discussed in the literature ([22], [7], [8]). Much of the uniqueness results are derived first for equations without time dependence (eg. stationary problems), and then these methods are extended to produce the uniqueness results for equations with time dependence (eg. Cauchy problems). However, a uniqueness result cannot be proved for the time independent analogue of (5.1). This is noted in some generality in [22], Remark 1.6.

In order to obtain the uniqueness result for (5.1), we return, as suggested by [35], to an earlier approach from [9], that is, a cone of dependence argument. In fact, Lemmas 2.5.2 and 2.5.3 below are directly modelled on Lemmas 5.3 and 5.4 of [9].

So that our uniqueness result will have the standard form, we reverse the time variable in (5.1). That is, we consider the PDE

$$W_s + \hat{H}(s, x, \nabla W) = 0 \tag{5.3a}$$

$$W(0, x) = \Psi(x) \tag{5.3b}$$

where

$$\hat{H}(s, x, p) = -\frac{1}{4\gamma^2}|p|^2 - \min_{v \in U}\left[f(T - s, x, v) \cdot p + L(T - s, x, v)\right].$$

Let W,V be continuous viscosity solutions of (5.3) that satisfy a local Lipschitz condition with a Lipschitz constant that is affine in $|x|$. We consider the bounded, globally Lipschitz modifications of W, V given by

$$\widetilde{W}(s,x) = \begin{cases} W(s,x) & \text{if } |x| \le R \\ W(s, \frac{x}{|x|}R) & \text{if } |x| > R \end{cases}$$

and similarly for $\widetilde{V}$.

Note that since $\widetilde{W}$ and $\widetilde{V}$ are Lipschitz, we can define

$$c = \max\{\|\nabla\widetilde{W}(s,\cdot)\|_{L^\infty(\Re^n)},\ \|\nabla\widetilde{V}(s,\cdot)\|_{L^\infty(\Re^n)}\} \tag{5.4}$$

where we note that c is independent of s and $c < \infty$ by the results of Sections 2.3 and 2.4 and Rademacher's Theorem (see, for instance, Ziemer [48]). We can also define

$$m = \max\{\|\widetilde{W}\|_{L^\infty([0,T]\times\Re^n)},\ \|\widetilde{V}\|_{L^\infty([0,T]\times\Re^n)}\}$$

and note that $m < \infty$ by the results of Sections 2.3 and 2.4. Clearly $\widetilde{W}$,$\widetilde{V}$ satisfy (5.3a) in the viscosity sense on $[0,T] \times \{|x| < R\}$ and the initial data (5.3b) on $|x| \le R$. Note also that

$$\begin{aligned} |\hat{H}(s,x,p) - \hat{H}(s,x,q)| &\le \left[\frac{c}{2\gamma^2} + K(1+R)\right] |p-q| \\ &\equiv a|p-q| \end{aligned} \tag{5.5}$$

on $[0,T] \times \{|x| \le R\} \times \{|p|, |q| \le c\}$.

Let $B(x,R)$ denote the closed ball of radius R centered at x. The following result, which we state without proof, may be found in [9].

Lemma 2.5.1: Let $w \in C(\Re^n)$, $\phi \in C^1(\Re^n)$, $\sup_{\Re^n} w\phi \in (0,\infty)$, and $x_0 \in \{x \in \Re^n : \quad w(x)\phi(x) = \max_{\Re^n}(w\phi)\}$. If $\nabla w \in L^\infty(B(x_0,R))$ for some $R > 0$, then

$$w(x_0)\frac{|\nabla\phi(x_0)|}{\phi(x_0)} \le \|\nabla w\|_{L^\infty(B(x_0,R))}.$$

Lemma 2.5.2: Let $\widetilde{W}$, $\widetilde{V}$ be as above. Let $\Lambda \in C^1([0,T] \times \Re^n)$, $\Lambda \ge 0$, $\Lambda = 0$ if $|x| \ge R$, and

$$-\Lambda_t > a|\nabla\Lambda|$$

in $(\mathrm{supp}\Lambda)^\circ$ where the $\circ$ superscript indicates interior. (An example of such a Λ will appear on p. 55.) Then

$$\widetilde{W} = \widetilde{V}$$

on $(\mathrm{supp}\Lambda)^\circ$.

proof: We will prove $\widetilde{W} \leq \widetilde{V}$. The proof of the reverse inequality is identical with the roles reversed.

Let $\phi \in C^\infty(\Re^n)$, $\mathrm{supp}(\phi) \subset B(0,1)$, $\phi(0) = 1$, $0 \leq \phi \leq 1$, and $\eta \in C^\infty(\Re)$, $\mathrm{supp}(\eta) \subset B(0,1)$, $\eta(0) = 1$, $0 \leq \eta \leq 1$. Let $\phi_\alpha(x) = \phi(x/\alpha)$ and $\eta_\alpha(s) = \eta(s/\alpha)$. Finally, as usual, let Q_T denote $[0,T] \times \Re^n$.

Assume

$$M \equiv \max_{\mathrm{supp}\Lambda, t\in[0,T]} \left[(\widetilde{W} - \widetilde{V})\Lambda^2\right] > 0. \tag{5.6}$$

(Otherwise, there is nothing to prove.) Let

$$M_\alpha \equiv \max_{Q_T \times Q_T} \left\{\phi_\alpha(x-y)\eta_\alpha(t-s)\Lambda(t,x)\Lambda(s,y)[\widetilde{W}(t,x) - \widetilde{V}(s,y)]\right\}$$

where we note that the max is obtained since $(\mathrm{supp}(\Lambda) \cap [0,T]) \subset [0,T] \times \{|x| \leq R\}$. (Also note that $M_\alpha \geq M$.) Suppose the maximum is obtained at $\{(t_\alpha, x_\alpha), (s_\alpha, y_\alpha)\}$ so that

$$M_\alpha = \phi_\alpha(x_\alpha - y_\alpha)\eta_\alpha(t_\alpha - s_\alpha)\Lambda(t_\alpha, x_\alpha)\Lambda(s_\alpha, y_\alpha)[\widetilde{W}(t_\alpha, x_\alpha) - \widetilde{V}(s_\alpha, y_\alpha)]$$

Note that $M_\alpha \to M > 0$ as $\alpha \downarrow 0$, $(t_\alpha, x_\alpha), (s_\alpha, y_\alpha) \in (\mathrm{supp}\Lambda)^\circ$, and $|t_\alpha - s_\alpha| \to 0$, $|x_\alpha - y_\alpha| \to 0$ as $\alpha \downarrow 0$. Without loss of generality, let there exist (t_0, x_0) such that $s_\alpha, t_\alpha \to t_0$ and $x_\alpha, y_\alpha \to x_0$ as $\alpha \downarrow 0$ (where we may have to take subsequences). Note that $t_0 > 0$, $|x_0| < R$ and $(t_0, x_0) \in (\mathrm{supp}\Lambda)^\circ$ (since $M > 0$) which implies that given $\delta > 0$ sufficiently small, there exists α_0 such that

$$
\begin{aligned}
&t_\alpha, s_\alpha \geq \delta \qquad \forall \alpha \leq \alpha_0 \\
&|x_\alpha|, |y_\alpha| \leq R - \delta \qquad \forall \alpha \leq \alpha_0 \\
&d\left((t_0, x_0), (\mathrm{supp}\Lambda)^c\right) \geq \delta \qquad \forall \alpha \leq \alpha_0
\end{aligned}
\tag{5.7}
$$

where $d(z, A)$ represents the distance from z to the set A and a superscript c indicates the complement of a set.

This implies that for sufficiently large α,

$$\phi_\alpha(x_\alpha - y_\alpha)\eta_\alpha(t_\alpha - s_\alpha)\Lambda(t_\alpha, x_\alpha)\Lambda(s_\alpha, y_\alpha) > 0$$

Then we may use the alternative definition of viscosity solution , [9] definition 5.3, (see [9], Section 1.5 for the proof of equivalence) to assert that at (t_α, x_α),(s_α, y_α)

$$-\left(\frac{\eta_\alpha{}'}{\eta_\alpha} + \frac{\Lambda_t}{\Lambda}\right)(\widetilde{W} - \widetilde{V}) + \hat{H}\left(t_\alpha, x_\alpha, -\left[\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla_x \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right) \leq 0$$

and

$$\left(-\frac{\eta_\alpha{}'}{\eta_\alpha} + \frac{\Lambda_s}{\Lambda}\right)(\widetilde{W} - \widetilde{V}) + \hat{H}\left(s_\alpha, y_\alpha, \left[-\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla_y \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right) \geq 0$$

where we have used the fact that $\widetilde{W}$ is independent of (s, y) and $\widetilde{V}$ is independent of (t, x) so that they may be viewed as constants in the relevant inequalities. (We have also used the notation ∇_x and ∇_y to indicate that we are taking the gradient of $\Lambda(t, x)$ and $\Lambda(s, y)$, respectively.) Combining these last two inequalities we have at (t_α, x_α),(s_α, y_α)

$$
\begin{aligned}
-\left(\frac{\Lambda_t}{\Lambda} + \frac{\Lambda_s}{\Lambda}\right)(\widetilde{W} - \widetilde{V}) + \hat{H}\left(t_\alpha, x_\alpha, -\left[\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla_x \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right) \\
- \hat{H}\left(s_\alpha, y_\alpha, \left[-\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla_y \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right) \leq 0.
\end{aligned}
\tag{5.8}
$$

Using the value of c from (5.4) and Lemma 2.5.1

$$
\begin{aligned}
\left|\left[\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right| \leq c \\
\left|\left[-\frac{\phi_\alpha{}'}{\phi_\alpha} + \frac{\nabla_y \Lambda}{\Lambda}\right](\widetilde{W} - \widetilde{V})\right| \leq c,
\end{aligned}
\tag{5.9}
$$

and by (5.7), $\Lambda(t_\alpha, x_\alpha)$, $\Lambda(s_\alpha, y_\alpha)$ are bounded away from zero for sufficiently large α. Thus we may take the (possibly subsequential) limit in (5.8) as $\alpha \downarrow 0$ to find

$$\begin{aligned}
&-2\frac{\Lambda_t(t_0,x_0)}{\Lambda(t_0,x_0)}(\widetilde{W}-\widetilde{V})(t_0,x_0)\\
&+\hat{H}\left(t_0,x_0,-\left[\frac{{\phi_\alpha}'(0)}{\phi_\alpha(0)}+\frac{\nabla\Lambda(t_0,x_0)}{\Lambda(t_0,x_0)}\right](\widetilde{W}-\widetilde{V})(t_0,x_0)\right)\\
&-\hat{H}\left(t_0,x_0,\left[-\frac{{\phi_\alpha}'(0)}{\phi_\alpha(0)}+\frac{\nabla\Lambda(t_0,x_0)}{\Lambda(t_0,x_0)}\right](\widetilde{W}-\widetilde{V})(t_0,x_0)\right)\le 0
\end{aligned}$$

which by (5.9) and (5.5) yields

$$-2\Lambda_t(t_0,x_0)(\widetilde{W}-\widetilde{V})(t_0,x_0)-2a|(\widetilde{W}-\widetilde{V})(t_0,x_0)|\nabla\Lambda(t_0,x_0)\le 0.$$

Using (5.6) and the fact that (t_0, x_0) is a maximum point to assert the positivity of $(\widetilde{W}-\widetilde{V})$, we have

$$-\Lambda_t(t_0,x_0)\le a|\nabla\Lambda(t_0,x_0)|$$

with $(t_0, x_0) \in (\mathrm{supp}\Lambda)^\circ$. But this contradicts the assumption of the lemma. Therefore, $M \le 0$ which implies

$$\widetilde{W}\le\widetilde{V}$$

on $(\mathrm{supp}\Lambda)^\circ$.

Again, the reverse inequality is proved similarly. ∎

Lemma 2.5.3: Let $\widetilde{W}$, $\widetilde{V}$ be as above and a as in (5.5). Then

$$\widetilde{W}=\widetilde{V}\quad\text{on}\quad |x|\le R-at,\, t\in[0,\min\{T,R/a\}].$$

proof: Let $g \in C^\infty(\Re)$, $g(r) = 0$ if $r \le 0$, and $g'(r) > 0$ if $r > 0$. Let $\Lambda(t, x) = g(R_0 - at - \lambda|x|^{1+\alpha})$ for some constants R_0, λ, α. Then

$$\mathrm{supp}\Lambda=\left\{(t,x):\, t\in[0,R_0/a],|x|\le[\lambda^{-1}(R_0-at)]^{\frac{1}{1+\alpha}}\right\}$$

and

$$\{x : \Lambda(x,0) > 0\} = \left\{x : |x| < (\lambda^{-1}R_0)^{\frac{1}{1+\alpha}}\right\}.$$

So we want

$$(\lambda^{-1}R_0)^{\frac{1}{1+\alpha}} \leq R. \tag{5.10}$$

Now, by definition

$$g'(R_0 - at - \lambda|x|^{1+\alpha}) > 0 \quad \text{on} \quad (\text{supp}\Lambda)^\circ$$

and

$$a|\nabla\Lambda| = a\lambda(1+\alpha)|x|^\alpha g'(R_0 - at - \lambda|x|^{1+\alpha})$$

$$-\Lambda_t = ag'(R_0 - at - \lambda|x|^{1+\alpha}).$$

Therefore, if

$$\lambda(1+\alpha)(\lambda^{-1}R_0)^{\frac{\alpha}{1+\alpha}} < 1 \tag{5.11}$$

then $-\Lambda_t > a|\nabla\Lambda|$ on $(supp\Lambda)^\circ$, and so by Lemma 2.5.2

$$\widetilde{W} = \widetilde{V} \quad \text{on} \quad (\text{supp}\Lambda)^\circ.$$

We will be done if we can show that we can choose $\lambda(\alpha)$,$R_0(\alpha)$ such that (5.10) and (5.11) hold and $\lambda(\alpha) \to 1$, $R_0(\alpha) \to R$ as $\alpha \downarrow 0$. Let R_0 satisfy

$$(1+2\alpha)R_0^{1+\alpha} = R^{1+\alpha}.$$

Then (5.10), (5.11) become

$$\lambda \geq \frac{1}{(1+2\alpha)^{\frac{1}{1+\alpha}}R^\alpha}$$

and

$$\lambda < \frac{(1+2\alpha)^{\frac{\alpha}{1+\alpha}}}{(1+\alpha)R^\alpha}.$$

Considering these last two inequalities, we choose

$$\lambda(\alpha) = \frac{1}{(1+2\alpha)^{\frac{1}{1+\alpha}}R^\alpha}.$$

Then (5.10), (5.11) are satisfied and $\lambda(\alpha) \to 1$, $R_0(\alpha) \to R$ as $\alpha \downarrow 0$. ∎

We have now shown that $\widetilde{W} = \widetilde{V}$ on cones of the form above, and consequently, $W = V$ on these cones as well. Now let us consider the slope of these cones more carefully. By (5.4), Lemma 2.4.17, and Lemma 2.3.10 (and the uniform convergence of V^ϵ to V on compact sets) there exists a constant $D < \infty$ such that

$$c \leq D(1 + R),$$

which implies that

$$a \leq \left(\frac{D}{2\gamma^2} + K\right)(1 + R) \equiv \tilde{a}(R).$$

For simplicity, assume that $T \geq R/\tilde{a}(R)$, and let

$$A_{R,a} = \{(t, x) : \, |x| \leq R - at, \, t \in [0, R/a]\}.$$

Then we know that $W(t, x) = V(t, x)$ for all $(t, x) \in A_{R,\tilde{a}(R)}$ for any $R < \infty$. Let τ_R denote the height of these cones. Then (since $R - \tilde{a}(R)\tau_R = 0$), we have

$$\tau_R = \frac{R}{\left(\frac{D}{2\gamma^2} + K\right)(1 + R)}.$$

Note that $\tau_r \uparrow \tau_\infty \equiv \frac{1}{\frac{D}{2\gamma^2} + K} > 0$ as $R \to \infty$.

Let $t < \tau_\infty$. Then for any $x \in \Re^n$, $(t, x) \in A_{R,\tilde{a}(R)}$ for some sufficiently large R. Therefore, letting τ be any value in $(0, \tau_\infty)$, we have $W(t, x) = V(t, x)$ for all $(t, x) \in [0, \tau] \times \Re^n$.

We can then restart the problem at $t = \tau$ to show that $W(t, x) = V(t, x)$ for all $(t, x) \in [0, 2\tau] \times \Re^n$. Repeating this process, we obtain the following.

Theorem 2.5.4: Let W, V be two continuous viscosity solutions of (5.3) which are locally Lipschitz in x with a Lipschitz constant which grows no faster than an affine function in $|x|$. Then $W(t, x) = V(t, x)$ for all $(t, x) \in [0, T] \times \Re^n$.

2.6 Convergence

Theorem 2.6.1: Suppose assumptions (AF1)–(AF3) are satisfied, and that $\gamma > \max\{\gamma_1, \gamma_2\}$. Then $V^\epsilon \to V$ uniformly on compact sets where V is a continuous viscosity solution to

$$\begin{aligned} &0 = V_s + \frac{1}{4\gamma^2} V_x^2 + \min_{v \in U} \left[f(s,x,v) \cdot V_x + L(s,x,v) \right] \\ &V(T,x) = \Psi(x), \end{aligned} \tag{6.1}$$

and consequently the value of the game (2.4.1)–(2.4.3). This solution is the unique viscosity solution in the class of continuous functions which meet a quadratic growth condition in x (i.e. locally Lipschitz in x with a Lipschitz constant which is at most affine in x).

proof: On any compact set, we know that the V^ϵ are bounded independent of $\epsilon \leq 1$ and equicontinuous by Lemmas 2.3.5, 2.3.10 and 2.3.11. Therefore, given any sequence $\epsilon_n \to 0$, by Ascoli–Arzela there exists a subsequence converging uniformly on compact sets to a continuous V. By [17], Lemma 2.6.2, this V is a viscosity solution to (6.1). Note that, since the V^ϵ were uniformly locally Lipschitz in x with a constant which grew at most linearly with $|x|$ (Lemma 2.3.10), V also satisfies this condition.

Now suppose that $V^\epsilon \not\to V$. This implies that there exists some subsequence ϵ_m such that $\epsilon_m \to 0$, but $V^{\epsilon_m} \not\to V$. Again, by Ascoli–Arzela, there exists a further subsequence of this that converges uniformly on compact sets to some continuous $\widehat{V}$. By [17], Lemma 2.6.2, this $\widehat{V}$ is also a viscosity solution to (6.1). Of course it satisfies the quadratic growth condition. But then by Theorem 2.5.4, we must have $\widehat{V} = V$. Therefore, $V^\epsilon \to V$ as $\epsilon \to 0$ where V is a continuous viscosity solution to

(6.1).

The uniqueness follows from Theorem 2.5.4.

By Theorem 2.4.16, the lower value of the game (2.4.1)–(2.4.3a) is a viscosity solution to (6.1). By symmetry, the upper value (2.4.1),(2.4.2), and (2.4.3b) is also a viscosity solution to (6.1). By Lemma 2.4.17, both satisfy the quadratic growth condition, and consequently, by Theorem 2.5.4, they are equal. Therefore the game has a value, say W. Once again applying Theorem 2.5.4, we see that $W = V$. ∎

3.0 INFINITE TIME HORIZON CASE

3.1 Introduction and Assumptions

For the infinite time horizon case, we consider the risk–averse stochastic control problem with dynamics

$$\begin{aligned} dX_t^\epsilon &= f(X_t^\epsilon, u_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t \\ X_0^\epsilon &= x \end{aligned} \tag{1.1}$$

and cost criterion

$$\liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^\epsilon, u_t)\,dt\right] \tag{1.2}$$

where $X_t^\epsilon \in \Re^n$ is the state, u is the control, and B is an n–dimensional Brownian motion relative to some reference probability system, ν (see Section 2.2). We let $\epsilon > 0$ be a small parameter which is both a scale factor for the Brownian motion and a measure of the risk–sensitivity in the cost criterion. The role of γ will become clearer later. We assume that the control is progressively measurable, and takes values in the compact set U. We make the following assumptions on f and L.

$$f \in C(\Re^n \times U; \Re^n)$$

$$\exists\, a > 0 \,\text{such that} \quad (x-y)\cdot[f(x,v)-f(y,v)] \le -a|x-y|^2 \qquad \forall\, x,y \in \Re^n,\ v \in U \tag{AI1}$$

$$\begin{aligned} &L \in C(\Re^n \times U; \Re) \\ &|L(x,v) - L(y,v)| \le K|x-y| \qquad \forall\, x,y \in \Re^n,\ v \in U \\ &0 \le L(x,v) \le K(1+|x|) \qquad \forall\, x \in \Re^n,\ v \in U \end{aligned} \tag{AI2}$$

(Although the third inequality above follows from the first, we include it explicitly to indicate that we will be using the same constant K for both.)

Letting

$$f^k(x,v) = \begin{cases} f(x,v) & \text{if } |f(x,v)| \le k \\ k\frac{f(x,v)}{|f(x,v)|} & \text{if } |f(x,v)| > k, \end{cases}$$

we also assume that either

$$\exists\ C_L \text{ such that } |f(x,v) - f(y,v)| \le C_L|x-y| \qquad \forall\, x,y \in \Re^n,\ v \in U \tag{AI3a}$$

or

$$\begin{aligned} &(x-y)\cdot\left[f^k(x,v) - f^k(y,v)\right] \le 0 \qquad \forall\, x,y \in \Re^n,\ v \in U \\ &\exists\ C_B \text{ such that } |f(x,v)| \le e^{C_B\sqrt{1+|x|^2}} \qquad \forall\, x \in \Re^n,\ v \in U. \end{aligned} \tag{AI3b}$$

It will also be useful to define

$$\mathcal{M} = \max_{v \in U} |f(0,v)|. \tag{1.3}$$

The first two inequality assumptions in (AI1) ensure the existence of a solution to (1.1) for all $t < \infty$. Either assumption (AI3a) or (AI3b) is needed for some lemmas

(3.4.3 and 3.4.10) which involve an intermediary cutoff problem. It may be possible to obtain the result without either of them, however that has not been achieved here. The class of functions satisfying both (AI3a) and (AI1) is clearly nontrivial. An example from the class which satisfies both (AI1) and (AI3b) is given by

$$f(x,v) = -g(|x|,v)x \tag{1.4}$$

where $g \in C^2(\,[0,\infty) \times U;\ [a,\infty)\,)$ (where a is as in (AI1)) and $g_\rho(\rho,v) \geq 0$ for all $(\rho,v) \in [0,\infty) \times U$.

It is easily seen that (AI1) is satisfied by an application of the Mean Value Theorem to the first term of the dot product

$$[g(|x|,v)x - g(|y|,v)y] \cdot (x-y).$$

To show that (AI3b) is satisfied, let

$$A \equiv \left[f^k(x,v) - f^k(y,v)\right] \cdot (x-y).$$

The case where $|f(x,v)|, |f(y,v)| \leq k$ is covered by the same argument as for (AI1). In the case where $|f(x,v)|, |f(y,v)| > k$ we have

$$A = -k\left(\frac{x}{|x|} - \frac{y}{|y|}\right) \cdot (x-y).$$

In the case where $|f(x,v)| > k \geq |f(y,v)|$, we have

$$A = -\left(k\frac{x}{|x|} - g(|y|,v)y\right) \cdot (x-y).$$

The last case will be covered by symmetry. Thus, it is sufficient to show that

$$(\alpha x - \beta y) \cdot (x-y) \geq 0$$

for all $x, y \in \Re^n$ and $\alpha, \beta \in [0, \infty)$ such that $|x| \geq |y|$ implies $\alpha|x| \geq \beta|y|$ and vice versa (i.e. $|y| \geq |x|$ implies $\beta|y| \geq \alpha|x|$). By symmetry, it is sufficient to show that

$$C \equiv (x - \rho y) \cdot (x - y) \geq 0$$

for $|x| \geq |y|$, $\rho \in \left[0, \frac{|x|}{|y|}\right]$. But then

$$C \geq (|x| - \rho|y|)(|x| - |y|) \geq 0.$$

Thus, functions of the form (1.4) are in the class which satisfy (AI3b) and (AI1). Finally, one might note that in the scalar case ($n = 1$), any f which satisfies (AI1) automatically satisfies the first inequality of (AI3b).

The Lipschitz growth bound on L in (AI2) is not a sharp condition. It is expected that the following results could be obtained with only a quadratic growth bound on L, as long as the quadratic growth rate were not too large relative to the size of a in assumption (AI1). However the estimates would be considerably more delicate in that case. It might also be desirable to consider the addition of a state dependent multiplier on the Brownian motion term in (1.1). However, we have not considered this here.

We will first demonstrate a connection between the value of the risk–averse control problem (1.1)–(1.2) and the number $\lambda^\epsilon \in \Re$ such that $\lambda^\epsilon, W^\epsilon$ are a solution to

$$\lambda^\epsilon = \frac{\epsilon}{4\gamma^2}\Delta W^\epsilon + \frac{1}{4\gamma^2}|\nabla W^\epsilon|^2 + \min_{v \in U}\left[f(x,v) \cdot \nabla W^\epsilon + L(x,v)\right]. \tag{1.5}$$

It will also be shown that this λ^ϵ is related to the value of a certain stochastic differential game with long–run average–cost–per–unit–time criterion. This game has dynamics

$$dX_t^\epsilon = \left[f(X_t^\epsilon, u_t) + w_t\right] dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \tag{1.6}$$

and payoff

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{E}_x \int_0^T \left[L(X_t^\epsilon, u_t) - \gamma^2 |w_t|^2 \right] \, dt \tag{1.7}$$

where $u.$ and $w.$ are the minimizing and maximizing controls, respectively.

We then consider the case of infinite risk–aversion with corresponding scaling down of the Brownian motion term. This is achieved by letting $\epsilon \downarrow 0$. As $\epsilon \downarrow 0$, λ^ϵ converges along subsequences to values $\{\lambda^0\}$ where each λ^0 and a corresponding $W^0 \in C(\Re^n)$ are a viscosity solution to the limiting version of (1.5), that is

$$\lambda^0 = \frac{1}{4\gamma^2} |\nabla W^0|^2 + \min_{v\in U} \left[f(x,v) \cdot \nabla W^0 + L(x,v) \right]. \tag{1.8}$$

It is shown that λ^0 is also related to the value of the game given by (1.6), (1.7) with $\epsilon = 0$. However, it has not been shown that the controls for this limiting game are deterministic.

On the other hand, (1.8) has another interpretation. If there is a $\gamma_0 < \infty$ large enough so that $\lambda^0 \leq 0$ (note that $\lambda^0 \geq 0$ since $L \geq 0$, so we may as well say $\lambda^0 = 0$), then we may view (1.8) as a dissipation relation (see Secs. 1.0, 3.7) for the deterministic system

$$dX_t^0 = f(X_t^0, u_t) + w_t$$

with cost criterion $\int_0^T L(X_t^0, u_t)\, dt$. In particular, under certain conditions this will imply that

$$\inf_u \int_0^T L(X_t^0, u_t)\, dt \leq \gamma^2 \int_0^T |w_t|^2 \, dt$$

for initial data at a reference point (say $x = 0$) for all $T < \infty$ and all w in some set, where the infimum will be taken over some appropriate set. By definition, this implies that the system satisfies an H^∞ bound of γ_0. Here we are using the H^∞ bound for nonlinear systems as discussed in Section 3.7. Thus we see that under some conditions, if there exists $\gamma_0 < \infty$ such that the value of the risk–averse stochastic control problem is going to zero, then the associated deterministic system will be able to achieve an H^∞ bound of γ_0.

3.2 Existence and Uniqueness

In this section, we prove existence and uniqueness of the solution to the differential equation associated with the risk–averse stochastic control problem and the stochastic differential game under consideration. This system is

$$\begin{aligned} dX_t &= [f(X_t, u_t) + w_t]\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_0 &= x. \end{aligned} \tag{2.1}$$

In particular, we assume throughout the section that we have a reference probability system, $\nu = (\Omega, \{\mathcal{F}_t\}, P, B)$ (see Section 2.2), $u \in A_\nu$ (the set of progressively measurable processes with values in U), and that w is an $\mathcal{F}_t$–progressively measurable process with values in $\Re^n$ bounded by some constant $m < \infty$ for all $\omega \in \Omega$. In the stochastic control case, w will be identically zero, and in the game case, it will be the control for the maximizing player. We also prove existence results for the cases where either u or w, or both, are replaced by feedback controls of certain classes. We do not assume either condition (AI3a) or (AI3b). If we had assumed (AI3a), then the existence and uniqueness results of Section 2.2 would have been immediately applicable, however we do not want to restrict ourselves to that case alone.

These existence and uniqueness results will differ from those of Section 2.2 in that we are no longer assuming that f is globally Lipschitz, or bounded for that matter, which are the typical assumptions for existence and uniqueness proofs. Instead, we are using a stability type of assumption, (AI1). This obligates us to use a different method of proof than that used in Section 2.2. In particular, we use a Lyapunov approach along the lines of Kushner [32].

In order to encompass the feedback cases and the case with $\mathcal{F}_t$–progressively measurable controls, (2.1), consider the more general system given by

$$
\begin{aligned}
dX_t &= [f(X_t, u_t, g(t, X_t)) + w_t + h(t, X_t)]\ dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\
X_0 &= x
\end{aligned}
\tag{2.2}
$$

where g, h are measurable mappings into U and $\{x \in \Re^n : |x| \leq m\}$ respectively, and this f still satisfies (AI1) with the extra variable going along for the ride (i.e. $(x-y) \cdot [f(x, v_1, v_2) - f(y, v_1, v_2)] \leq -a|x-y|^2$ for all $x, y \in \Re^n$ and $v_1, v_2 \in U$). We will need a cutoff version of (2.2). Let

$$
f^k(x, u, v) = \begin{cases} f(x, u, v) & \text{if } |f(x, u, v)| \leq k \\ k\frac{f(x,u,v)}{|f(x,u,v)|} & \text{if } |f(x, u, v)| > k \end{cases}
\tag{2.3}
$$

so that we have the cutoff system

$$
\begin{aligned}
dX_t^k &= \left[f^k(X_t^k, u_t, g(t, X_t^k)) + w_t + h(t, X_t^k)\right]\ dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\
X_0 &= x
\end{aligned}
\tag{2.4}
$$

Lemma 3.2.1: Assume the above. Suppose there exist unique strong solutions to (2.4) on $[0, T]$ for all $T < \infty$ and $k < \infty$. Then there exists a strong solution to (2.2) on $[0, T]$ for any $T < \infty$.

proof: Fix $T < \infty$. Let

$$
G(x) = \sqrt{1 + |x|^2}
$$

so that

$$
\nabla G(x) = \frac{x}{\sqrt{1 + |x|^2}} \tag{2.5a}
$$

$$
\Delta G(x) \leq n. \tag{2.5b}
$$

By Ito's rule,

$$\begin{aligned}G(X_t^k) = G(x) + \int_0^t \Big\{ &\frac{\epsilon}{4\gamma^2}\Delta G(X_r^k) \\ &+ \left[f^k(X_r^k, u_r, g(r, X_r^k)) + w_r + h(r, X_r^k)\right] \cdot \nabla G(X_r^k)\Big\}\, dr \\ + \sqrt{\frac{\epsilon}{2\gamma^2}} &\int_0^t \nabla G(X_r^k) \cdot\, dB_r\end{aligned}$$

where the notation $\int \rho \cdot dB$ indicates $\sum_{i=1}^n \int \rho^{(i)}\, dB^{(i)}$. By (2.5) and the boundedness of w, h

$$\begin{aligned}G(X_t^k) \le G(x) + \int_0^t &\left\{ \frac{\epsilon}{4\gamma^2} n + 2m + f^k(X_r^k, u_r, h(r, X_r^k)) \cdot \frac{X_r^k}{\sqrt{1+|X_r^k|^2}} \right\} dr \\ + \sqrt{\frac{\epsilon}{2\gamma^2}} &\int_0^t \nabla G(X_r^k) \cdot\, dB_r. \end{aligned} \tag{2.6}$$

But note that

$$f^k(x,u,v) \cdot x = \left[f^k(x,u,v) - f^k(0,u,v)\right] \cdot (x - 0) + f^k(0,u,v) \cdot x$$

which by (1.3), (2.3), (AI1) and a few computations ((AI3) is not used)

$$\le 2\mathcal{M}|x|.$$

To see this, consider for instance the case where the cutoff is active at x but not at the origin. Then from the above, we have

$$\begin{aligned}f^k(x,u,v) \cdot x \le &\frac{k}{f(x,u,v)}[f(x,u,v) - f(0,u,v)] \cdot (x - 0) \\ &\left(\frac{k}{f(x,u,v)} - 1\right) f(0,u,v) \cdot x + f^k(0,u,v) \cdot x \\ \le &\ 0 + \mathcal{M}|x| + \mathcal{M}|x|.\end{aligned}$$

The other cases follow similarly. Substituting this bound into (2.6) yields

$$\begin{aligned}G(X_t^k) \le G(x) + &\left\{ \frac{\epsilon}{4\gamma^2} n + 2m + 2\mathcal{M} \right\} T \\ &+ \sqrt{\frac{\epsilon}{2\gamma^2}} T \sum_{i=1}^n \max_{t \in [0,T]} |B_t^{(i)}|.\end{aligned}$$

This implies

$$\max_{t\in[0,T]}|X_t^k| \le \sqrt{1+|x|^2} + \left(\frac{\epsilon}{4\gamma^2}n + 2m + \mathcal{M}\right)T$$
$$+\sqrt{\frac{\epsilon}{2\gamma^2}}T\sum_{i=1}^{n}\max_{t\in[0,T]}|B_t^{(i)}|$$
$$\equiv C_1 + C_2T + \sqrt{\frac{\epsilon}{2\gamma^2}}T\sum_{i=1}^{n}\max_{t\in[0,T]}|B_t^{(i)}|$$

which defines C_1, C_2. Therefore

$$\mathrm{E}_{s,x}\max_{t\in[0,T]}|X_t^k| \le C_1 + C_2T + \sqrt{\frac{\epsilon}{2\gamma^2}}Tn\mathrm{E}_{s,x}\max_{t\in[0,T]}|B_t^{(1)}|,$$

and so by (A.4) (from the Appendix)

$$\mathrm{E}_{s,x}\max_{t\in[0,T]}|X_t^k| \le C_1 + C_2T + C_3T^{\frac{3}{2}} \equiv C_4 \tag{2.7}$$

where $C_3 = \sqrt{\frac{\epsilon}{2\gamma^2}}\frac{4}{\sqrt{2\pi}}n^2$. Let

$$D_k = \left\{x : \max_{u,v\in U}|f(x,u,v)| \le k\right\}$$

and $d_k = \inf_{x\notin D_k}|x|$ so that $d_k \to \infty$ as $k \to \infty$ by continuity. Then let

$$B_k = \{\omega : \max_{t\in[0,T]}|X_t^k| > d_k\}.$$

From (2.7) and Chebyshev, we see that

$$P(B_k) \le \frac{C_4}{d_k}.$$

Also note that if $\omega \notin B_k$ then $\omega \notin B_m$ for all $m \ge k$ which implies

$$B_k \supset B_{k+1} \supset \cdots.$$

Thus, letting $N = \bigcap_{k=1}^{\infty} B_k$, we have $P(N) = 0$. Let $\Omega^* = \Omega - N$. If $\omega \in \Omega^*$, there exists k_ω such that $\omega \notin B_k$ for all $k \ge k_\omega$ which implies

$$X_t^k = X_t \qquad \forall\, k \ge k_\omega,\ \forall t \in [0,T]$$

and, consequently,

$$X_\cdot^k(\omega) \to X_\cdot(\omega) \qquad \text{uniformly on} \qquad [0, T],$$

and $X_\cdot$ is continuous. Since $X_t^k(\omega) \to X_t(\omega)$ on $\Omega^* \times [0, t]$ for all $t \in [0, T]$, X is $\mathcal{F}_t$–progressively measurable . (More precisely, we can take $\tilde{X}_t^k(\omega) = X_t^k(\omega)$ on $\Omega^* \times [0, T]$ and $\tilde{X}_t^k(\omega) = 0$ on $N \times [0, T]$. Then $\tilde{X}^k \to X$ on $\Omega \times [0, t]$ for all $t \in [0, T]$. By, for instance, [39] Theorem 3.20, $X : \Omega \times [0, t] \to \Re^n$ is $\mathcal{F}_t \times \mathcal{B}$ measurable for all $t \in [0, T]$.)

Thus, referring to the definition of a strong solution, (2.2.4), we see that condition (i) is satisfied by $X_\cdot$. Conditions (ii) and (iii) (by (2.7)) are also clearly satisfied. Also, since $X_{[0,T]}^k = X_{[0,T]}$ for all $k \geq k_\omega$ almost surely and X^k satisfies (2.1) for $\omega \in \Omega^*$, $k \geq k_\omega$, condition (iv) is satisfied. Therefore X is a strong solution on $[0, T]$. Since T was arbitrary, we are done. ∎

Now we consider specific cases. First we prove existence and uniqueness for system (2.1) with $\mathcal{F}_t$–progressively measurable control.

Theorem 3.2.2: Under the above assumptions there exists a strong solution to (2.1).

proof: Consider the cutoff version of (2.1), that is

$$\begin{aligned} dX_t^k &= \left[f^k(X_t^k, u_t) + w_t \right] \, dt + \sqrt{\frac{\epsilon}{2\gamma^2}} \, dB_t \\ X_0^k &= x \end{aligned}$$

where f^k is the cutoff given by (2.3). From Theorem 2.2.2, there exists a strong solution for any k. Then by Lemma 3.2.1, there exists a strong solution to (2.1). ∎

Theorem 3.2.3: The strong solution to (2.1) is strongly unique.

proof: Let X and Y be strong solutions to (2.1), and define the vector $\begin{pmatrix} X \\ Y \end{pmatrix}$. We will treat X and Y as scalars for simplicity. Then $\begin{pmatrix} X \\ Y \end{pmatrix}$ satisfies

$$\begin{pmatrix} X \\ Y \end{pmatrix}_t = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} \int_0^t f(X_r, u_r) + w_r \, dr \\ \int_0^t f(Y_r, u_r) + w_r \, dr \end{pmatrix} + \sqrt{\frac{\epsilon}{2\gamma^2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_t.$$

Now define

$$G(x, y) = |x - y|^2$$

which implies

$$\nabla G(x, y) = 2(x - y) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the matrix of second derivatives is given by

$$H = 2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then, by Ito's rule

$$\begin{aligned} G\left(\begin{pmatrix} X \\ Y \end{pmatrix}_t\right) = G\left(\begin{pmatrix} x \\ x \end{pmatrix}\right) &+ \int_0^t \Big\{ 2(X_r - Y_r) \cdot [f(X_r, u_r) + w_r] \\ &\qquad - 2(X_r - Y_r) \cdot [f(Y_r, u_r) + w_r] \Big\} \, dr \\ &+ \sum_{i=1}^{2} \sum_{i=1}^{2} \int_0^t \frac{\epsilon}{4\gamma^2} H_{i,j} \, d\langle B, B \rangle_r \\ &+ \sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^t 2(X_r - Y_r) \, dB_r - \sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^t 2(X_r - Y_r) \, dB_r \\ = 2 \int_0^t (X_r - Y_r) \cdot [f(X_r, u_r) - f(Y_r, u_r)] \, dr & \end{aligned} \tag{2.8}$$

which by (AI1)

$$\leq -2a \int_0^t |X_r - Y_r|^2 \, dr. \tag{2.9}$$

Therefore, by Gronwall's inequality

$$|X_t - Y_t|^2 = 0 \qquad \forall\, t \in [0, \infty), \ \forall\, \omega \in \Omega. \quad \blacksquare$$

Inequality (2.9) is interesting in itself as it implies a contraction property due to (AI1); that is $|X_t - Y_t| \leq e^{-at}|x - y|$.

Now we consider the case where one or both of the controls is a feedback control. Recall that

$$g : [0, \infty) \times \Re^n \to U$$
$$h : [0, \infty) \times \Re^n \to \{x \in \Re^n : \ |x| \leq m\}.$$

We assume both are measurable.

Remark 3.2.4: Assume the above. System (2.1) where one or both of the controls is a feedback of the above form has a weak solution by [31], Remark 5.3.8 and arguments like those above.

Theorem 3.2.5: Assume the above. Consider system (2.1) where u is replaced by $g(\cdot, X_\cdot)$, and w is replaced by $h(\cdot, X_\cdot)$. This system has a strong solution.

proof: By [42], Theorem 1, the cutoff version has a unique strong solution. Then by Lemma 3.2.1, we have existence for the system without the cutoff. ∎

Lastly, we consider the systems

$$\begin{aligned} dX_t &= [f(X_t, u_t) + h(t, X_t)]\ dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_0 &= x \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} dX_t &= [f(X_t, g(t, X_t)) + w_t]\ dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_0 &= x. \end{aligned} \tag{2.11}$$

Existence proofs for these systems require additional assumptions.

Theorem 3.2.6: Assume the above. Suppose that h is Lipschitz in its second variable. Then (2.10) has a strong solution.

proof: Put a cutoff on f as usual, and let $\tilde{f}^k(t,x,v) = f^k(x,v) + h(t,x)$. Then by Theorem 2.2.2, the system

$$dX_t^k = \tilde{f}^k(t, X_t^k, u_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t$$
$$X_0^k = x$$

has a strong solution for any k. By Lemma 3.2.1, we are done. ∎

Theorem 3.2.7: Assume the above. Suppose that g is Lipschitz in its second variable. Then (2.11) has a strong solution.

proof: Put a cutoff on f as usual, and let $\tilde{f}^k(t,x,w) = f^k(x, g(t,x)) + w$. Note that $\tilde{f}^k$ is Lipschitz in x. Then by Theorem 2.2.2, the system

$$dX_t^k = \tilde{f}^k(t, X_t^k, w_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t$$
$$X_0^k = x$$

has a strong solution for any k. By Lemma 3.2.1, we are done. ∎

3.3 Dynamic Programming Equation

In this section, we assume existence of a solution to a dynamic programming equation (DPE) which takes the form (3.2). The value λ is then shown to be the minimum cost for an associated risk–averse control problem with long–run expected rate of exponential growth cost criterion.

Let $\nu^0 = \left(\Omega, \{\mathcal{F}_t\}, P^0, B^0_\cdot\right)$ be any reference probability system (see Section 2.2). As usual, the dynamics are of the form

$$\begin{aligned} dX_t^0 &= f(X_t^0, u_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t^0 \\ X_0^0 &= x. \end{aligned} \tag{3.1}$$

For simplicity, we let the initial time be $s = 0$ here since, for this cost criterion, the result is independent of the initial time.

Theorem 3.3.1: Suppose that there exist $\lambda \in \Re$ and $W \in C^2(\Re^n)$ such that $|\nabla W(x)| \le c < \infty$ for all $x \in \Re^n$ and

$$\lambda = \frac{\epsilon}{4\gamma^2}\Delta W + \frac{1}{4\gamma^2}|\nabla W|^2 + \min_{v\in U}[f(x,v)\cdot \nabla W + L(x,v)]. \tag{3.2}$$

Then

$$\lambda \le \liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0, u_t)\,dt\right]$$

for any $u \in A_{\nu^0}$ (see Section 3.2) where X^0 is the solution of (3.1) corresponding to u, and E_x^0 represents the expectation with respect to P^0.

Furthermore, there exists $u^* \in A_{\nu^0}$ given by

$$u_t^* = g(X_t^*) \in \arg\min_{v\in U}\left[f(X_t^*, v)\cdot \nabla W(X_t^*) + L(X_t^*, v)\right]$$

where $g(\cdot)$ is bounded and Borel measurable, X^* is the corresponding solution to (3.1), and

$$\lambda = \lim_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^*, u_t^*)\, dt\right].$$

proof: First we note that it is sufficient to prove the result for $\lambda \geq 0$. Otherwise, let M be such that $\lambda + M \geq 0$. Then let $\tilde{\lambda} = \lambda + M$ and $\tilde{L} = L + M$, and suppose we prove the result for $\tilde{\lambda}$, $\tilde{L}$. Then we have for instance,

$$\lambda + M \leq \liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T \left(L(X_t^0, u_t) + M\right)\, dt\right]$$

which, upon cancellation of the M terms, yields the result for λ, L.

Now, let $w_t \equiv \frac{1}{2\gamma^2}\nabla W(X_t^0)$ which is $\mathcal{F}_t$–progressively measurable and bounded by assumption. Let $u \in A_{\nu^0}$. From (3.1),

$$\begin{aligned} X^0 &= x + \int_0^t [f(X_r^0, u_r) + w_r]\, dr + \sqrt{\frac{\epsilon}{2\gamma^2}} B_t^0 - \int_0^t w_r\, dr \\ &\equiv x + \int_0^t [f(X_r^0, u_r) + w_r]\, dr + \sqrt{\frac{\epsilon}{2\gamma^2}} B_t \end{aligned} \tag{3.3}$$

which defines B.

We would like a probability measure under which B is a Brownian motion. By the boundedness of w, the Novikov condition is met, and so we can apply Girsanov's Theorem (cf. [31]) to assert the existence of a probability measure, P, under which B is a Brownian motion, and further,

$$\begin{aligned} P(d\omega) &= \exp\left[\sqrt{\frac{2\gamma^2}{\epsilon}}\int_0^t w_r \cdot dB_r^0 - \frac{\gamma^2}{\epsilon}\int_0^t |w_r|^2\, dr\right] P^0(d\omega) \\ &= \exp\left[\sqrt{\frac{2\gamma^2}{\epsilon}}\int_0^t w_r \cdot dB_r + \frac{\gamma^2}{\epsilon}\int_0^t |w_r|^2\, dr\right] P^0(d\omega) \end{aligned} \tag{3.4}$$

where we write $\int w \cdot dB$ to indicate $\sum_{i=1}^n \int w^{(i)}\, dB^{(i)}$.

Under this change of measure

$$\mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0,u_t)\,dt\right] = \mathrm{E}_x \exp\Bigg\{\frac{1}{\epsilon}\Bigg[\int_0^T \left(L(X_t^0,u_t)-\gamma^2|w_t|^2\right)\,dt \\ -\sqrt{2\gamma^2\epsilon}\int_0^T w_t\cdot\,dB_t\Bigg]\Bigg\} \tag{3.5}$$

where E_x represents expectation with respect to P.

On the other hand, by (3.2), (3.3) and Ito's rule

$$W(X_T^0)-W(x) \geq \int_0^T \left[\nabla W(X_t^0)\cdot w_t + \lambda - L(X_t^0,u_t) - \frac{1}{4\gamma^2}|\nabla W(X_t^0)|^2\right]\,dt \\ +\sqrt{\frac{\epsilon}{2\gamma^2}}\int_0^T \nabla W(X_t^0)\cdot\,dB_t$$

which upon recalling the definition of w_t becomes

$$W(X_T^0)-W(x) \geq \int_0^T \left[\lambda - L(X_t^0,u_t)+\gamma^2|w_t|^2\right]\,dt \\ +\sqrt{2\gamma^2\epsilon}\int_0^T w_t\,dB_t. \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$\mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0,u_t)\,dt\right] \geq \exp\left[\frac{\lambda T+W(x)}{\epsilon}\right]\mathrm{E}_x\exp\left[-\frac{1}{\epsilon}W(X_T^0)\right]$$

which implies

$$\liminf_{T\to\infty}\frac{\epsilon}{T}\log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0,u_t)\,dt\right] \geq \lambda + \liminf_{T\to\infty}\frac{\epsilon}{T}\log\mathrm{E}_x\exp\left[-\frac{1}{\epsilon}W(X_T^0)\right] \tag{3.7}$$

By the assumptions on W, (AI1), and Lemma 3.3.5 (to follow), there exists $C_1 < \infty$ such that

$$\frac{1}{C_1} \leq \mathrm{E}_x\exp\left[-\frac{1}{\epsilon}W(X_T^0)\right] \leq C_1 \qquad \forall\, T. \tag{3.8}$$

Therefore, (3.7) implies

$$\liminf_{T\to\infty}\frac{\epsilon}{T}\log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0,u_t)\,dt\right] \geq \lambda$$

which was our first claim.

For our second claim, note that there exists a bounded measurable g as given in the theorem statement by [16], Appendix B. By Theorem 3.2.5, there exists a strong solution to (3.1) with this feedback control. By the choice of u^*, we obtain equality in (3.6) which yields

$$\mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^*, u_t^*)\, dt\right] = \exp\left[\frac{\lambda T + W(x)}{\epsilon}\right] \mathrm{E}_x \exp\left[-\frac{1}{\epsilon} W(X_T^*)\right]$$

which, in turn, yields both

$$\liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^*, u_t^*)\, dt\right] = \lambda + \liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x \exp\left[-\frac{1}{\epsilon} W(X_T^*)\right]$$

and

$$\limsup_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^*, u_t^*)\, dt\right] = \lambda + \limsup_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x \exp\left[-\frac{1}{\epsilon} W(X_T^*)\right].$$

By (3.8) again, we have

$$\lim_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0, u_t)\, dt\right] = \lambda. \quad \blacksquare$$

If one fixes the control in Theorem 3.3.1, one can obtain the following result similarly. We suppose that $\tilde{f}(x)$ and $\tilde{L}(x)$ satisfy conditions of the form (AI1) and (AI2), and consider the system

$$\begin{aligned} d\tilde{X}_t &= \tilde{f}(\tilde{X}_t)\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t^0 \\ \tilde{X}_0 &= x. \end{aligned} \tag{3.9}$$

Theorem 3.3.2: Suppose there exist $\lambda \in \Re$ and $W \in C^2(\Re^n)$ such that $|\nabla W(x)| \leq c < \infty$ for all $x \in \Re^n$ and

$$\lambda = \frac{\epsilon}{4\gamma^2}\Delta W + \frac{1}{4\gamma^2}|\nabla W|^2 + [\tilde{f}(x)\cdot\nabla W + \tilde{L}(x)].$$

Then

$$\lambda = \lim_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}^0_x \exp\left[\frac{1}{\epsilon}\int_0^T \tilde{L}(\tilde{X}_t)\,dt\right].$$

This result bears some similarity to results from the Theory of Large Deviations. In fact, if one fixes the control for the results that follow as well, there is a similarity to the optimal control approach to Large Deviations [10].

The required bound on $\mathrm{E}_x \exp\left[-\frac{1}{\epsilon}W(X^0_T)\right]$ is typical of many bounds which will be required in this chapter. Thus the following analysis will be referenced and/or reproduced in an altered fashion often.

Given a solution, $X.$ to

$$dX_t = [f(X_t, u_t) + w_t]\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t$$
$$X_0 = x,$$

define the stopped process, $X^k_\cdot$ by

$$\tau_k = \inf\{t : |X_t| \geq k\}$$

(τ_k is a stopping time)

$$X^k_t = \begin{cases} X_t & \text{if } t \leq \tau_k \\ X_{\tau_k} & \text{if } t > \tau_k. \end{cases} \tag{3.10}$$

Then X^k is a bounded semi–martingale, i.e.

$$X^k_t = x + \int_0^{t\wedge\tau_k} [f(X_r, u_r) + w_r]\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}} B_{t\wedge\tau_k}$$

where $\wedge$ represents the minimum operator.

Let $G \in C^2(\Re^n)$. Since X^k is bounded,

$$\mathrm{E}_x \int_0^t |\nabla G(X^k_r)|^2\,dr < \infty \qquad \forall t$$

which implies (see, for instance [31], Definition 3.2.9) that

$$\int_0^t \sqrt{\frac{\epsilon}{2\gamma^2}} \nabla G(X_r^k) \cdot dB_r$$

is a square–integrable martingale.

By Ito's rule, we have

$$G(X_t^k) = G(x) + \int_0^{t\wedge\tau_k} AG(X_r^k)\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^{t\wedge\tau_k} \nabla G(X_r^k) \cdot dB_r \tag{3.11}$$

where A is the generator of X, that is

$$Ag = \frac{\epsilon}{4\gamma^2}\Delta g + (f+w)\cdot\nabla g. \tag{3.12}$$

Assuming $|x| \le k$, we have by the martingale property of the last term in (3.11)

$$\mathrm{E}_x G(X_t^k) = G(x) + \mathrm{E}_x\left[\int_0^{t\wedge\tau_k} AG(X_r^k)\,dr\right]. \tag{3.13}$$

Lemma 3.3.3: Let

$$G(x) = e^{K\sqrt{1+|x|^2}} \tag{3.14}$$

for any $K < \infty$. There exists $C_2 < \infty$ such that

$$\mathrm{E}_x G(X_t) \le G(x) + C_2 t \qquad \forall\, t < \infty.$$

proof: Let the stopped process be defined as above. Note that

$$\begin{aligned} \nabla G(x) &= \frac{KG(x)}{\sqrt{1+|x|^2}} x \\ \Delta G(x) &\le K(n+K)G(x). \end{aligned} \tag{3.16}$$

By (3.12) and (3.16) there exists $R_0 < \infty$ such that for $t < \tau_k$

$$\begin{aligned} AG(X_t^k) &= \frac{KG(X_t^k)}{\sqrt{1+|X_t^k|^2}}\left\{2K\left[f(X_t^k,u_t)+w_t\right]\cdot X_t^k + \frac{\epsilon}{4\gamma^2}(n+K)\sqrt{1+|X_t^k|^2}\right\} \\ &\le \frac{KG(X_t^k)}{\sqrt{1+|X_t^k|^2}}\left\{2Km|X_t^k| - 2aK|X_t^k|^2 + \frac{\epsilon}{4\gamma^2}(n+K)(1+|X_t^k|)\right\} \\ &\equiv F(X_t^k) \\ &\le 0 \qquad \text{if } |X_t^k| \ge R_0. \end{aligned} \tag{3.17}$$

Thus, letting $C_2 = \max_{|x| \le R_0} F(x)$, we have

$$AG(X_t^k) \le C_2.$$

Therefore, by (3.13)

$$\mathrm{E}_x G(X_t^k) \le G(x) + C_2 t \qquad \forall t < \infty$$

where C_2 is independent of k. Thus, by Fatou's Lemma,

$$\mathrm{E}_x G(X_t) \le G(x) + C_2 t \qquad \forall t < \infty. \quad \blacksquare$$

The previous lemma implies that a certain stochastic integral is a square–integrable martingale. This, in turn, allows us to prove an even better result as follows.

Lemma 3.3.4: Given G and X as above, there exists $C_1 < \infty$ (dependent upon x) such that

$$\mathrm{E}_x G(X_t) \le C_1 \qquad \forall\, t < \infty.$$

proof: Applying Ito's rule to the original system, we have

$$\begin{aligned} G(X_t) = G(x) &+ \int_0^t AG(X_r)\, dr \\ &+ \sqrt{\frac{\epsilon}{2\gamma^2}} K \int_0^t \frac{G(X_r)}{\sqrt{1+|X_r|^2}} X_t \cdot dB_r. \end{aligned} \tag{3.18}$$

But, by Lemma 3.3.3,

$$\mathrm{E}_x G^2(X_t) \le G^2(x) + C_2' t \qquad \forall\, t < \infty$$

for some $C_2' < \infty$. Therefore, the last term in (3.18) is a square–integrable martingale. Thus, taking expectations in (3.18), we obtain

$$\mathrm{E}_x G(X_t) \le G(x) + \mathrm{E}_x \int_0^t AG(X_r)\, dr. \tag{3.19}$$

Now note that

$$AG \le \left\{ \frac{\epsilon}{4\gamma^2} K(n+K) + [f(x,u)+w] \cdot x \frac{K}{\sqrt{1+|x|^2}} \right\} G.$$

Recall that

$$\begin{aligned} x \cdot f(x,u) &= (x-0) \cdot [f(x,u) - f(0,u)] + x \cdot f(0,u) \\ &\le -a|x|^2 + \mathcal{M}|x| \end{aligned}$$

where $\mathcal{M}$ is given by (1.3). This implies

$$\begin{aligned} AG &\le \left\{ \frac{\epsilon}{4\gamma^2} K(n+K) + (m+\mathcal{M})K - aK \frac{|x|^2}{\sqrt{1+|x|^2}} \right\} G \\ &\le \left\{ \frac{\epsilon}{4\gamma^2} K(n+K) + (m+\mathcal{M})K + aK - a \log G \right\} G \\ &\equiv \{C_3 - a \log G\} G. \end{aligned} \tag{3.20}$$

Employing this in (3.19), we have

$$\mathrm{E}_x G(X_t) \le G(x) + \mathrm{E}_x \int_0^t \{C_3 - a \log G(X_r)\} G(X_r)\, dr$$

which by Lemma 3.3.3, Tonelli's Theorem, and Fubini's Theorem,

$$= G(x) + \int_0^t \mathrm{E}_x \left[\{C_3 - a \log G(X_r)\} G(X_r) \right] dr.$$

Noting that the expression $(C_3 - a \log z)z$ is concave in z, we have, by Jensen's inequality,

$$\mathrm{E}_x G(X_t) \le G(x) + \int_0^t \{C_3 - a \log \mathrm{E}_x G(X_r)\} \mathrm{E}_x G(X_r)\, dr.$$

Since $\mathrm{E}_x G(X_r) \ge e^{\frac{C_3}{a}}$ implies that the integrand is nonpositive, we have

$$\mathrm{E}_x G(X_t) \le \max \left\{ G(x), e^{\frac{C_3}{a}} \right\} \equiv C_1 \qquad \forall\, t \in [0, \infty). \quad \blacksquare$$

Lemma 3.3.5: Let X be as above. Let $W(\cdot)$ satisfy

$$-K\sqrt{1+|x|^2} \le W(x) \le K\sqrt{1+|x|^2} \qquad \forall x \in \Re^n.$$

Then there exists $C_1 < \infty$ such that

$$\frac{1}{C_1} \le \mathrm{E}_x \exp\left(-W(X_t)/\epsilon\right) \le C_1 \qquad \forall t \in [0,\infty).$$

proof: The right hand side follows from Lemma 3.3.4. For the other side, note

$$\begin{aligned} 1 &= \left\{\mathrm{E}\left[\exp\left(K\sqrt{1+|X_t|^2}/2\right)\exp\left(-K\sqrt{1+|X_t|^2}/2\right)\right]\right\}^2 \\ &\le \mathrm{E}_x \exp\left(K\sqrt{1+|X_t|^2}\right)\mathrm{E}_x \exp\left(-K\sqrt{1+|X_t|^2}\right). \end{aligned}$$

Then, by Lemma 3.3.4, for some $K < \infty$

$$\mathrm{E}_x \exp\left(-W(X_t)/\epsilon\right) \ge \mathrm{E}_x \exp\left(-K\sqrt{1+|X_t|^2}\right) \ge \frac{1}{C_1}. \quad \blacksquare$$

This last lemma was the result required in the proof of Theorem 3.3.1.

3.4 The Finite Time Horizon Stochastic Game

In order to employ Theorem 3.3.1, we need to prove that a classical solution to (3.2) exists. We will prove this by using the representation of the solution of (3.2) as the value of a long–run average–cost–per-unit–time stochastic differential game. However, in order to obtain the result for that infinite time horizon game, we first consider a finite time horizon stochastic game with discounted cost criterion. We start with a simple case first, and then relax the assumptions.

Consider the following game. The state is $X_\cdot$, the control for the minimizing player is $u_\cdot$, and the control for the maximizing player is $w_\cdot$. The dynamics are given by

$$\begin{aligned} dX_t &= \left[f^k(X_t, u_t) + w_t\right]\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, dB_t \\ X_s &= x \end{aligned} \tag{4.1}$$

where

$$f^k(x,v) = \begin{cases} f(x,v) & \text{if } |f(x,v)| \le k \\ k\frac{f(x,v)}{|f(x,v)|} & \text{if } |f(x,v)| > k. \end{cases} \tag{4.2}$$

The payoff is

$$J^k(s,x,u,w) = \mathrm{E}_{s,x} \int_s^T e^{-ct}\left[L^k(X_t,u_t) - \gamma^2|w_t|^2\right]\, dt \tag{4.3}$$

where

$$L^k(x,v) = \begin{cases} L(x,v) & \text{if } |L(x,v)| \le k \\ k & \text{if } |L(x,v)| > k. \end{cases} \tag{4.4}$$

The controls u are in A_ν, and $w \in \mathcal{W}_m$ where $\mathcal{W}_m$ is the set of $\mathcal{F}_t$–progressively measurable controls with values in $\Re^n$ satisfying $|w_t(\omega)| \le m$ for all $t \in [0,\infty)$ and $\omega \in \Omega$. As in [18], we define the strategies for the players as follows. A strategy for the minimizing player is a mapping $\beta : \mathcal{W}_m \to A_\nu$ such that if for any $t \in [s,T]$, one has $w_r = \hat{w}_r$ for all $r \in [s,t]$ a.s., then $\beta[w]_r = \beta[\hat{w}]_r$ for all $r \in [s,t]$ a.s. The set of strategies for the minimizing player is denoted by Δ. The strategies for the maximizing player are defined analogously, and denoted by $\alpha \in \Lambda_m$. One might note that we do not specify the control domain endpoints (eg. $[s, T]$) in the definitions, since nothing is lost here by assuming the controls exist on a larger domain (eg. $[0, \infty)$), and that will simplify the notation. Finally, note that the discount factor c which appears in the cost will be necessary when we take the limit as the time horizon goes to infinity.

Take ν to be the canonical sample space for the Brownian motion; that is the case where

$$\Omega = \{\omega \in C([s,T]; \Re^n) \text{ such that } \omega_s = 0\},$$

the filtration is generated by the sample paths of the Brownian motion, and P is Wiener measure. The lower value is defined to be

$$V_l^{k,m} = \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}_m} J^k(s, x, \beta[w], w), \tag{4.5a}$$

and the upper value is

$$V_u^{k,m} = \sup_{\alpha \in \Lambda_m} \inf_{u \in A_\nu} J^k(s, x, u, \alpha[u]). \tag{4.5b}$$

By [18] Theorem 2.6, $V_l^{k,m} = V_u^{k,m}$ (we denote the common value by $V^{k,m}$), and $V^{k,m}$ is the unique viscosity solution to

$$\begin{aligned} 0 = V_s &+ \frac{\epsilon}{4\gamma^2}\Delta V + \min_{v \in U}\left[f^k(x,v)\cdot \nabla V + e^{-cs}L^k(x,v)\right] \\ &+ \max_{|w|\le m}\left[w \cdot \nabla V - \gamma^2 e^{-cs}|w|^2\right] \qquad (4.6) \\ V(T,x) &= 0. \end{aligned}$$

(Note that the Isaacs condition holds for (4.6).) Furthermore, by [18], Lemma 1.5, V is bounded, globally Lipschitz in x, and Hölder continuous in s. In fact, more can be said. By the proofs of Lemmas 3.4.3 and 3.4.6 to follow, $V^{k,m}$ is Lipschitz on bounded sets, and consequently a generalized solution as well [9].

Now, we can rewrite (4.6) as

$$0 = V_s + \frac{\epsilon}{4\gamma^2}\Delta V + \Gamma(s,x)$$
$$V(T,x) = 0$$

where

$$\Gamma(s,x) = \min_{v\in U}\left[f^k(x,v)\cdot\nabla V + e^{-cs}L^k(x,v)\right] + \max_{|w|\le m}\left[w\cdot\nabla V - \gamma^2 e^{-cs}|w|^2\right].$$

By the above Lipschitz property, it is easily seen that $\|\Gamma(s,x)\|_{L^2}$ is bounded on bounded sets (in fact, $\|\Gamma(s,x)\|_{L^\infty}$ is bounded). Therefore, by [16] Appendix E, $V^{k,m} \in C^{1,2}([0,T]\times\Re^n)$; $V^{k,m}$ is a classical solution. Now suppose there exists another classical solution $\hat{V}$. Then $\hat{V}$ is a viscosity solution, and since there is a unique viscosity solution, $\hat{V} = V^{k,m}$. (Actually, one could prove this without reference to the uniqueness of viscosity solutions. Let $W = \hat{V} - V^{k,m}$. Then W satisfies $W_s + \frac{\epsilon}{4\gamma^2}\Delta W + G(s,x)\cdot\nabla W = 0$ with terminal data $W(T,x) = 0$ where $G(s,x) = \int_0^1 H(x, \nabla\hat{V} + \lambda(\nabla V^{k,m} - \nabla\hat{V}))\,d\lambda$. Then one can show $W \equiv 0$ by considering the SDE $d\xi_t = G(t,\xi_t)\,dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\,dB_t$ with $\xi_s = x$ and applying Ito's rule to $W(s,x)$.) We summarize this result.

Theorem 3.4.1: There exists a unique, classical solution (in $C^{1,2}$) to (4.6), and it is the value, $V^{k,m}$, of the stochastic differential game (4.1)–(4.5).

We would like to weaken the assumptions that we made in obtaining this result. With this in mind, we prove some boundedness results regarding the behavior of $V^{k,m}$, where these bounds will be independent of k.

Lemma 3.4.2:

$$|J^k(s,x,u,w)| \le \frac{M_1}{c} \qquad \forall\, u\in A_\nu,\ w\in\mathcal{W}_m,\ s\in[0,T],\ x\in\Re^n,\ k<\infty$$

and

$$0 \leq V^{k,m}(s,x) \leq \frac{M_1}{c} \qquad \forall\, s \in [0,T],\ x \in \Re^n,\ k < \infty$$

where

$$M_1 = K\left[\frac{\epsilon n}{4\gamma^2} + m\right] T + K\sqrt{1+|x|^2}.$$

proof: We use the standard technique. One might note that we do not need a cutoff as in Section 3.3 since here we know that the relevant expectations exist. Let

$$G(x) = \sqrt{1+|x|^2}$$

which implies

$$\nabla G(x) = \frac{x}{\sqrt{1+|x|^2}}$$
$$\Delta G(x) \leq n.$$

As in (3.3.13),

$$\mathrm{E}_{s,x} G(X_t) = G(x) + \int_0^t \mathrm{E}_{s,x} AG(X_r)\, dr \tag{4.7}$$

where

$$Ag = \frac{\epsilon}{4\gamma^2}\Delta g + (f^k + w)\cdot \nabla g. \tag{4.8}$$

From (4.8) and the above derivatives of G,

$$AG(X_t) \leq \frac{\epsilon n}{4\gamma^2} + m + f^k(X_t,u_t)\cdot \frac{X_t}{\sqrt{1+|X_t|^2}}.$$

However, by (AI1), (4.2) and (1.3), $f^k(X_t,u_t)\cdot X_t \leq 2\mathcal{M}|X_t|$ (see p. 67) which implies

$$AG(X_t) \leq \frac{\epsilon n}{4\gamma^2} + m + 2\mathcal{M}.$$

Substituting this into (4.7) yields

$$\mathrm{E}_{s,x} G(X_t) \leq G(x) + \left[\frac{\epsilon n}{4\gamma^2} + m + 2\mathcal{M}\right] T$$

which implies

$$\mathrm{E}_{s,x}\sqrt{1+|X_t|^2} \le \sqrt{1+|x|^2} + \left[\frac{\epsilon n}{4\gamma^2} + m + 2\mathcal{M}\right] T \equiv \sqrt{1+|x|^2} + C_5 T. \qquad (4.9)$$

Combining (4.9), (4.3), and (AI2), we have

$$\begin{aligned}|J^k(s,x,u,w)| &\le K\left[\sqrt{1+|x|^2} + C_5 T\right] \int_0^T e^{-ct}\, dt \\ &\le \frac{K\left[\sqrt{1+|x|^2} + C_5 T\right]}{c}.\end{aligned}$$

This yields the first assertion and the second inequality of the second assertion. To prove the first inequality of the second assertion, simply consider the case where $w \equiv 0$. ∎

In the next two lemmas we obtain a bound independent on $\nabla V^{k,m}$ independent of k. It is necessary here to invoke either (AI3a) or (AI3b). In Lemma 3.4.3a, we prove the result under assumptions (AI1), (AI2) and (AI3a). In Lemma 3.4.3b, we prove the same result, but under assumptions (AI1), (AI2) and (AI3b) instead.

Lemma 3.4.3a: Assume (AI1), (AI2) and (AI3a). Then

$$|\nabla V^{k,m}(s,x)| \le \frac{K}{c}\left(1 + C_L T e^{C_L T}\right) \qquad \forall\, s \in [0,T],\ x \in \Re^n$$

proof: By (AI3a) and Gronwall,

$$|X_t - Y_t| \le \left(1 + C_L T e^{C_L T}\right)|x-y| \qquad \forall\, t \in [s,T]$$

where X_t, Y_t satisfy (4.1) with initial conditions x, y, respectively. Then by (AI2), we have

$$\begin{aligned}|J^k(s,x,u,w) - J^k(s,y,u,w)| &\le K\left(1 + C_L T e^{C_L T}\right)|x-y| \int_s^T e^{-ct}\, dt \\ &\le \frac{K}{c}\left(1 + C_L T e^{C_L T}\right)|x-y|\end{aligned}$$

for all $s \in [0,T]$, $x, y \in \Re^n$, $u \in A_{s,\nu}$, and $w \in \mathcal{W}_m$.

Then by the standard technique (see, for instance, the proof of Lemma 2.4.17, or the proof of Theorem 2.3.8), we have the same Lipschitz bound for $V^{k,m}$. Since $V^{k,m} \in C^{1,2}$, we have the result. ∎

Lemma 3.4.3b: Assume (AI1), (AI2) and (AI3b) (see p. 61 for notation). Then

$$|\nabla V^{k,m}(s,x)| \leq \frac{K}{c} \qquad \forall\, s \in [0,T],\ \forall\, x \in \Re^n$$

proof: We again use the standard procedure. Let X_t and Y_t satisfy (4.1) with initial conditions x and y, respectively. As in the proof of Theorem 3.2.3, we form the vector process $\begin{pmatrix} X \\ Y \end{pmatrix}_t$, and, for simplicity, deal with X and Y as scalars. Fix $\delta_2 > 0$ and consider x, y such that $|x-y| \geq \delta_2$. Let $\delta_1 > 0$ and define

$$G_{\delta_1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \sqrt{\delta_1^2 + |x-y|^2}.$$

It could be noted here that we would really like to look at $G_0 = |x-y|$, but since this is not differentiable, we look at G_{δ_1} instead. Then

$$\nabla G_{\delta_1} = \frac{x-y}{G_{\delta_1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and letting H denote the matrix of second derivatives, we have

$$H_{1,1} = \frac{\delta_1^2}{G_{\delta_1}^3}$$

which for $|x-y| \geq \delta_2$

$$\leq \frac{\delta_1^2}{\delta_2^3}$$

as well as $H_{1,2} \leq 0$, $H_{2,1} \leq 0$, and $H_{2,2} \leq \frac{\delta_1^2}{\delta_2^3}$.

Then, by Ito's rule applied to $G_{\delta_1}\left(\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right)$

$$\begin{aligned}
G_{\delta_1}\left(\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right) \leq G_{\delta_1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &+ \int_s^t \frac{X_r - Y_r}{G_{\delta_1}} \cdot \left[f^k(X_r, u_r) - f^k(Y_r, u_r)\right] \, dr \\
&+ \sum_{i=1}^{2}\sum_{j=1}^{2} \int_s^t \frac{\epsilon}{4\gamma^2} H_{i,j} \, d\langle B \rangle_r \\
&+ 2\sqrt{\frac{\epsilon}{2\gamma^2}} \int_s^t \frac{X_r - Y_r}{G_{\delta_1}} \, dB_r - 2\sqrt{\frac{\epsilon}{2\gamma^2}} \int_s^t \frac{X_r - Y_r}{G_{\delta_1}} \, dB_r \\
\leq G_{\delta_1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &+ \int_s^t \frac{X_r - Y_r}{G_{\delta_1}} \cdot \left[f^k(X_r, u_r) - f^k(Y_r, u_r)\right] \, dr \\
&+ \frac{\epsilon}{2\gamma^2} \frac{\delta_1^2}{\delta_2^3}(t - s).
\end{aligned}$$

Noting, by (AI3b), that $(x-y) \cdot [f^k(x,v) - f^k(y,v)] \leq 0$ for all $x, y \in \Re^n$ and $v \in U$, we have

$$G_{\delta_1}\left(\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right) \leq G_{\delta_1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) + \frac{\epsilon}{2\gamma^2} \frac{\delta_1^2}{\delta_2^3} T$$

or,

$$\sqrt{\delta_1^2 + |X_t - Y_t|^2} \leq \sqrt{\delta_1^2 + |x - y|^2} + \frac{\epsilon}{2\gamma^2} \frac{\delta_1^2}{\delta_2^3} T.$$

Since this is true for all $\delta_1 > 0$, we take the limit as $\delta_1 \downarrow 0$ to yield

$$|X_t - Y_t| \leq |x - y|$$

for all $t \in [0, \infty)$ and for all $x, y \in \Re^n$ such that $|x - y| \geq \delta_2$. Since $\delta_2 > 0$ was arbitrary, we have

$$|X_t - Y_t| \leq |x - y|$$

for all $t \in [0, \infty)$ and $x, y \in \Re^n$.

By (AI2), we then have

$$|J^k(s, x, u, w) - J^k(s, y, u, w)| \leq K|x - y| \int_0^T e^{-ct} \, dt \leq \frac{K|x - y|}{c}$$

for all $s \in [0,T]$, $x \in \Re^n$, $u \in A_\nu$, $w \in \mathcal{W}_m$. Then by the standard technique (see, for instance, the proof of Lemma 2.4.17, or the proof of Theorem 2.3.8), we have the same Lipschitz bound for $V^{k,m}$. Since $V^{k,m} \in C^{1,2}$, we have the result. ∎

Remark 3.4.4: Note that the $\nabla V^{k,m}$ bound is independent of k and m.

Remark 3.4.5: In both Lemmas 3.4.3a and 3.4.3b, assumptions AI3a and AI3b were used to get a bound on $|X_t - Y_t|$ as a function of $|x-y|$. Recall from (2.9) that such a bound could be obtained directly from (AI1) if it were not for the cutoff.

Lemma 3.4.6: Given $B < \infty$

$$|V_s^{k,m}(s,x)| \le M_2 \qquad \forall\,(s,x) \in [0,T] \times \{|x| \le B\}$$

where

$$M_2 = 2\left\{K\sqrt{1+B^2} + \left[\frac{\epsilon n}{4\gamma^2} + m + \mathcal{M}\right] KT + \gamma^2 m^2\right\}.$$

proof: Let $0 \le s \le t \le T$ and $|x| \le B < \infty$. Suppose the reference probability system is denoted by $\nu = (\Omega, \{\mathcal{F}_t\}, P, B)$, and let $u \in A_\nu$, $w \in \mathcal{W}_m$. Then

$$J^k(s,x,u,w) = \mathrm{E}_{s,x} \int_s^T e^{-cr} \tilde{L}^k(X_r, u_r, w_r)\, dr \tag{4.11}$$

where

$$\tilde{L}^k(x,u,w) = L^k(x,u) - \gamma^2 |w|^2$$

and

$$X_r = x + \int_s^r \left[f^k(X_\rho, u_\rho) + w_\rho\right]\, d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}} B_r.$$

Now we consider $J^k(t,x,\hat{u},\hat{w})$. Recalling that J^k is independent of the reference probability system (implied by strong uniqueness), we choose

$$\hat{\nu} = (\Omega, \{\hat{\mathcal{F}}_t\}, P, \hat{B})$$

where $\hat{\mathcal{F}}_r = \mathcal{F}_{r-(t-s)}$ and $\hat{B}_r = B_{r-(t-s)}$. Also let $\hat{u}_r = u_{r-(t-s)}$ and $\hat{w}_r = w_{r-(t-s)}$, and note that $\hat{u} \in A_{\hat{\nu}}$ and $\hat{w} \in \mathcal{W}_m(\hat{\nu})$ where we include the argument in $\mathcal{W}_m$ for clarity.

Thus letting $\hat{X}$ be the state for this time–shifted system, we have

$$\begin{aligned}\hat{X}_r &= x + \int_s^r \left[f^k(\hat{X}_\rho, \hat{u}_\rho) + \hat{w}_\rho\right]\, d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}}\hat{B}_r \\ &= X_{r-(t-s)}\end{aligned}$$

which implies

$$\begin{aligned}J^k(t,x,\hat{u},\hat{w}) &= \mathrm{E}_{s,x}\int_t^T e^{-cr}\tilde{L}^k(X_{r-(t-s)}, u_{r-(t-s)}, w_{r-(t-s)})\, dr \\ &= \mathrm{E}_{s,x}e^{-c(t-s)}\int_s^{T-(t-s)} e^{-c\rho}\tilde{L}^k(X_\rho, u_\rho, w_\rho)\, d\rho. \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we have

$$\begin{aligned}|J^k(s,x,u,w) - J^k(t,x,u,w)| = \Big|(1-e^{-c(t-s)})\mathrm{E}_{s,x}&\int_s^{T-(t-s)} e^{-cr}\tilde{L}^k(X_r,u_r,w_r)\, dr \\ &+ \mathrm{E}_{s,x}\int_{T-(t-s)}^T e^{-cr}\tilde{L}^k(X_r,u_r,w_r)\, dr\Big| \end{aligned} \tag{4.13}$$

Now, by (4.9) and (AI2)

$$\mathrm{E}_{s,x}|\tilde{L}^k(X_r,u_r,w_r)| \le M_1 \tag{4.14}$$

where

$$M_1 = K\sqrt{1+B^2} + \left[\frac{\epsilon n}{4\gamma^2} + m + \mathcal{M}\right]KT + \gamma^2 m^2.$$

Combining (4.13) and (4.14), we have

$$\begin{aligned}|J^k(s,x,u,w) - J^k(t,x,u,w)| &\le M_1\Big|(1-e^{-c(t-s)})\int_s^{T-(t-s)} e^{-cr}\, dr \\ &\qquad + \int_{T-(t-s)}^T e^{-cr}\, dr\Big| \\ &\le 2M_1(t-s). \end{aligned} \tag{4.15}$$

We complete the proof by the usual argument with a minor variation introduced by the $\hat{u}$ and $\hat{w}$. For instance, given β, define $\hat{\beta}$ by $\hat{\beta}[\hat{w}]_r = \beta[w]_{r-(t-s)}$ and note $\hat{\beta} \in \Delta$. Also, let

$$I^{k,m}(s,x,\beta) \equiv \sup_{w \in \mathcal{W}_m} J^k(s,x,\beta[w],w).$$

Then, given $\delta > 0$, there exists $w_\delta \in \mathcal{W}_m$ such that

$$I^{k,m}(s,x,\beta) - I^{k,m}(t,x,\hat{\beta}) \leq J^k(s,x,\beta[w_\delta],w_\delta) - J^k(t,x,\hat{\beta}[\hat{w}_\delta],\hat{w}_\delta) + \delta$$

which by (4.15)

$$\leq 2M_1|t-s| + \delta.$$

Letting $\delta \downarrow 0$, and then using symmetry, we have

$$|I^{k,m}(s,x,\beta) - I^{k,m}(t,x,\hat{\beta})| \leq 2M_1|t-s|$$

for all $0 \leq s \leq t \leq T$, $|x| \leq B$, and $\beta \in \Delta$. Repeating this process for the $\inf_{\beta \in \Delta}$ operation yields the desired result. ∎

Lemma 3.4.7: Let $B < \infty$. Then

$$|\Delta V^{k,m}(s,x)| \leq M_3 \qquad \forall\, s \in [0,T],\ |x| \leq B$$

where M_3 depends on $K, B, \epsilon, n, \gamma, m$, C_L (if Assumption (AI3a) is used), and $\max_{\{|x| \leq B\} \times U} |f|$, but not k.

proof: Combine (4.6), Lemmas 3.4.3, Lemma 3.4.6, and the fact that $|f^k| \leq |f|$, $|L^k| \leq |L|$. ∎

Having bounded the behavior of $V^{k,m}$ independent of k, we will prove a convergence lemma which will, in turn, be applied to the $V^{k,m}$.

Lemma 3.4.8: Let $W^k \in C^{1,2}([0,T] \times \Re^n)$ satisfy

$$\begin{aligned} 0 &= W_s^k + \frac{\sigma^2}{2}\Delta W^k + H_k(s,x,\nabla W^k) \qquad \text{on} \quad [0,T]\times\Re^n \\ W^k(T,x) &= 0. \end{aligned} \tag{4.16}$$

Assume that given $B \in (0,\infty)$, there exists M_B independent of k such that

$$|W^k| + |\nabla W^k| + |W_s^k| + |\Delta W^k| \le M_B \qquad \forall\,(s,x) \in [0,T]\times\{|x|\le B\}. \tag{4.17}$$

Assume that the H_k are locally Lipschitz uniformly in k and that $H_k \to H$ uniformly on compact sets (i.e. $H_k(s,x,p) \to H(s,x,p)$ uniformly on compact sets of (s,x,p)). Then there exists a subsequence (which we also index by k) such that

$$\begin{aligned} W^k &\to W^0 \\ \nabla W^k &\to \nabla W^0 \\ W_s^k &\to W_s^0 \\ \Delta W^k &\to \Delta W^0 \end{aligned} \tag{4.18}$$

uniformly on compact sets (which implies $W^0 \in C^{1,2}$), and

$$\begin{aligned} 0 &= W_s^0 + \frac{\sigma^2}{2}\Delta W^0 + H(s,x,\nabla W^0) \qquad \text{on} \quad [0,T]\times\Re^n \\ W^0(T,x) &= 0. \end{aligned} \tag{4.19}$$

proof: Fix $B < \infty$, and let $Q_B = [0,T]\times\{|x|\le B\}$. By (4.17),

$$\{W^k\} \quad \text{are equicontinuous on } Q_B. \tag{4.20}$$

Also, by [16], (E.9) (or, equivalently, Friedman [19] or [34]),

$$\{\nabla W^k\} \quad \text{satisfy a Hölder condition} \tag{4.21}$$

with some exponential coefficient $\mu > 0$ on Q_B uniformly in k. (To use Friedman, let $\widetilde{W}^k = \tilde{\alpha}(x)W^k$ where $\tilde{\alpha}$ is a test function which is 1 on $\{|x| \le B-\delta\}$, and 0 on $\{|x|\ge B\}$. Then consider the PDE for $\widetilde{W}^k$, and use Friedman, Theorem 7.4.)

Now, by assumption, the H_k are locally Lipschitz uniformly in k, so there exists $C_5 < \infty$ such that

$$\begin{aligned} |H_k(s,x,\nabla W^k(s,x)) - H_k(t,y,\nabla W^k(t,y))| &\le C_5|x-y| + C_5|t-s| \\ &\quad + C_5|\nabla W^k(s,x) - \nabla W^k(s,y)| \\ &\quad + C_5|\nabla W^k(s,y) - \nabla W^k(t,y)| \end{aligned}$$

which by (4.17) and (4.21)

$$\begin{aligned} &\le C_5(1+M_B)|x-y|^{\mu} \\ &\quad + C_5(1+C_6)|t-s|^{\mu/2} \end{aligned}$$

for some $C_6 < \infty$ on Q_B (with $|x-y|, |t-s| \le 1$).

Therefore, by [16], (E.10),

$$\{W_s^k\}, \{\Delta W^k\} \quad \text{satisfy a Hölder condition} \tag{4.22}$$

on Q_B uniformly in k.

By (4.17), (4.20), (4.21), and (4.22), the W^k, ∇W^k, W_s^k, and ΔW^k are equicontinuous and bounded uniformly in k. Then, by Ascoli–Arzela and a standard result relating the derivatives of limits to the limits of derivatives, there exists $W^0 \in C^{1,1}([0,T] \times \Re^n)$ and a subsequence (which we also denote by k) such that

$$\begin{aligned} W^k &\to W^0 \\ \nabla W^k &\to \nabla W^0 \\ W_s^k &\to W_s^0 \\ \Delta W^k &\to \Delta W^0 \end{aligned} \tag{4.23}$$

uniformly on compact sets. Note that by standard Schauder estimates (cf. [16], (E.10) or [34], Theorem 4.10.1) $W^0 \in C^{1,2}([0,T] \times \Re^n)$. Thus we have proved our first assertion. To prove the second, simply note that

$$
\begin{aligned}
|W_s^0 + \frac{\sigma^2}{2}\Delta W^0 + H(s,x,\nabla W^0)| \le |W_s^k &+ \frac{\sigma^2}{2}\Delta W^k + H_k(s,x,\nabla W^k)| \\
&+ |W_s^k - W_s^0| + \frac{\sigma^2}{2}|\Delta W^k - \Delta W^0| \\
&+ |H(s,x,\nabla W^0) - H_k(s,x,\nabla W^0)| \\
&+ |H_k(s,x,\nabla W^0) - H_k(s,x,\nabla W^k)|
\end{aligned}
$$

and employ (4.16), (4.23), and the assumptions on H_k. ∎

Lemma 3.4.9: Let the $V^{k,m}$ be as above. Then there exists $V^m \in C^{1,2}$ and a subsequence (also denoted by $V^{k,m}$) such that $V^{k,m} \to V^m$, $\nabla V^{k,m} \to \nabla V^m$, $V_s^{k,m} \to V_s^m$, and $\Delta V^{k,m} \to \Delta V^m$ uniformly on compact sets, and

$$
\begin{aligned}
0 &= V_s^m + \frac{\epsilon}{4\gamma^2}\Delta V^m + H_m(s,x,\nabla V^m) \\
V^m(T,x) &= 0
\end{aligned}
\tag{4.24}
$$

where

$$
\begin{aligned}
H_m(s,x,\nabla V^m) = &\max_{|w|\le m}\left[w\cdot\nabla V^m - e^{-cs}\gamma^2|w|^2\right] \\
&+ \min_{v\in U}\left[f(x,v)\cdot\nabla V^m + e^{-cs}L(x,v)\right].
\end{aligned}
$$

proof: It is only necessary to verify that the conditions of Lemma 3.4.8 are met. First, (4.16) is obvious from (4.6) with

$$
H_{k,m}(s,x,p) = \max_{|w|\le m}\left[w\cdot p - e^{-cs}\gamma^2|w|^2\right] + \min_{v\in U}\left[f^k(x,v)\cdot p + e^{-cs}L^k(x,v)\right].
$$

Also, (4.17) is clearly met by Lemmas 3.4.2, 3.4.3, 3.4.6, and 3.4.7. It is also clear by definition that the $H_{k,m} \to H_m$ uniformly on compact sets. We need only show that the $H_{k,m}$ are locally Lipschitz uniformly in k. But this is also clear by the definition of the $H_{k,m}$, the local Lipschitz nature of f, L, and the min, max over compact sets. ∎

We will now obtain some verification results which will provide representations for V^m. A preliminary result is needed prior to the first verification. This result requires either assumption (AI3a) or (AI3b). Consequently we present two lemmas, each producing the same result, but with the first requiring (AI3a) and the second requiring (AI3b). This will be the last place where either (AI3a) or (AI3b) will be needed.

Lemma 3.4.10a: Assume (AI1), (AI2) and (AI3a). Let $B, T < \infty$.

$$|J^k(s,x,u,w) - J(s,x,u,w)| \to 0 \qquad \text{as } k \to \infty$$

uniformly for all $|x| \leq B$, $s \in [0,T]$, $u \in A_{s,\nu}$, and $w \in \mathcal{W}_m$ where J^k is given by (4.3) and

$$\begin{aligned} J(s,x,u,w) &= \mathrm{E}_{s,x} \int_s^T e^{-ct} \left[L(X_t,u_t) - \gamma^2 |w_t|^2 \right] dt \\ X_t &= x + \int_s^t \left[f(X_r,u_r) + w_r \right] dr + \sqrt{\frac{\epsilon}{2\gamma^2}} B_t. \end{aligned} \tag{4.25}$$

proof: We have

$$\begin{aligned} |J^k(s,x,u,w) - J(s,x,u,w)| &\leq \mathrm{E}_{s,x} \int_s^T e^{-ct} |L^k(X_t^k,u_t) - L^k(X_t,u_t)|\, dt \\ &\quad + \mathrm{E}_{s,x} \int_s^T e^{-ct} |L^k(X_t,u_t) - L(X_t,u_t)|\, dt \\ &\equiv A_1 + A_2. \end{aligned} \tag{4.26}$$

We prove the result for A_1 and A_2 separately. First consider A_1. By (AI2), (4.4), and Tonelli's Theorem

$$A_1 \leq K \int_s^T e^{-ct} \mathrm{E}_{s,x} |X_t^k - X_t|\, dt. \tag{4.27}$$

Now,

$$\mathrm{E}_{s,x} |X_t^k - X_t| \leq \int_{\{\max_{r \in [s,t]} |X_r| > R(k)\}} \left[|X_t^k| + |X_t| \right] dP. \tag{4.28}$$

where $R(k) \equiv \sup\{R : f(x,v) = f^k(x,v) \; \forall \, |x| \le R, \, v \in U\}$. But by (AI3a), and Lemma 2.3.4,

$$|X_t^k| \le Z_t \quad \text{and} \quad |X_t| \le Z_t \tag{4.29}$$

where Z_t is the solution to the integral equation

$$Z_t = B + C_L T + C_L \int_s^t Z_r \, dr + \sqrt{\frac{\epsilon}{2\gamma^2}} |B_t|.$$

Thus, by (4.28), we have

$$\mathrm{E}_{s,x}|X_t^k - X_t| \le 2 \int_{\{\max_{r \in [s,t]} |X_r| > R(k)\}} Z_t \, dP.$$

But by a standard result of Real Analysis, this integral goes to zero as $k \to \infty$ since $P\left(\max_{r \in [s,t]} |X_r| > R(k)\right) \to 0$ as $k \to \infty$. Thus, by (4.27), $A_1 \to 0$ as $k \to \infty$ uniformly for all $s \in [0,T]$, $|x| \le B$, $u \in \mathcal{A}_{s,\nu}$, and $w \in \mathcal{W}$.

Now consider A_2. By Tonelli's Theorem

$$A_2 \le \int_s^T e^{-ct} \mathrm{E}_{s,x} |L^k(X_t, u_t) - L(X_t, u_t)| \, dt$$

and by (AI2) and (4.4)

$$\le 2K \int_s^T e^{-ct} \int_{\{\max_{t \in [s,T]} |X_t| > \frac{k}{K}\}} (1 + |X_t|) \, dP \, dt$$

which by (4.29)

$$\le 2K \int_s^T e^{-ct} \int_{\{\max_{t \in [s,T]} |X_t| > \frac{k}{K}\}} (1 + Z_t) \, dP \, dt.$$

Thus, by the same reasoning as for A_1, $A_2 \to 0$ as $k \to \infty$ uniformly for all $s \in [0,T]$, $|x| \le B$, $u \in \mathcal{A}_{s,\nu}$, and $w \in \mathcal{W}$. ∎

Lemma 3.4.10b: Assume (AI1), (AI2) and (AI3b). Let $B, T < \infty$. Then the same implications as given in Lemma 3.4.10a hold.

proof: The proof is identical to that for Lemma 3.4.10a up to and including (4.27). Then let $G(x,y) = |x-y|^2$. By (2.8) with X^k, f^k replacing Y, f, we have

$$\begin{aligned}
\mathrm{E}_{s,x}|X_t^k - X_t|^2 &\le 2\mathrm{E}_{s,x}\int_s^t (X_r^k - X_r)\cdot \left[f^k(X_r^k,u_r) - f(X_r,u_r)\right]\,dr \\
&= 2\mathrm{E}_{s,x}\int_s^t (X_r^k - X_r)\cdot \Big[f^k(X_r^k,u_r) - f^k(X_r,u_r) \\
&\qquad + f^k(X_r,u_r) - f(X_r,u_r)\Big]\,dr
\end{aligned}$$

which by (AI3b)

$$\le 0 + 2\mathrm{E}_{s,x}\int_s^t \left(|X_r^k| + |X_r|\right) e^{C_B\sqrt{1+|X_r|^2}}\,dr$$

and by using Tonelli's Theorem and Hölder's inequality

$$\begin{aligned}
&\le 2\int_s^t \Bigg\{ \left[\mathrm{E}_{s,x}|X_r|^2\right]^{\frac{1}{2}} \left[\int_{\{\max_{r\in[s,t]}|X_r|>R(k)\}} e^{(C_B+1)\sqrt{1+|X_r|^2}}\,dP\right]^{\frac{1}{2}} \\
&\qquad + \int_{\{\max_{r\in[s,t]}|X_r|>R(k)\}} e^{2C_B\sqrt{1+|X_r^k|^2}}\,dP \Bigg\}\,dr.
\end{aligned}$$

Now, by a modification of the reasoning that led to (4.9), but with $|x|^2$ replacing $\sqrt{1+|x|^2}$, it is easily shown that there exists $C_1 < \infty$ such that $\mathrm{E}_{s,x}|X_t^k|^2 \le C_1$ for all $t \in [0,T]$. Thus,

$$\begin{aligned}
\mathrm{E}_{s,x}|X_t^k - X_t|^2 \le 2\int_s^t \Bigg\{ &C_1 \left[\int_{\{\max_{r\in[s,t]}|X_r|>R(k)\}} e^{2C_B\sqrt{1+|X_r|^2}}\,dP\right]^{\frac{1}{2}} \\
&+ \int_{\{\max_{r\in[s,t]}|X_r|>R(k)\}} e^{(C_B+1)\sqrt{1+|X_r|^2}}\,dP \Bigg\}\,dr.
\end{aligned}$$

But by Lemma 3.3.3 (or Lemma 3.3.4), given $C_2 < \infty$ there exists $C_3 < \infty$ such that $\mathrm{E}_{s,x}e^{C_2\sqrt{1+|X_r|^2}} \le C_3$ for all $t \in [0,\infty)$. Thus, since $P\left(\max_{r\in[s,t]}|X_r| > R(k)\right) \to 0$ as $k \to \infty$,

$$\int_{\{\max_{r\in[s,t]}|X_r|>R(k)\}} e^{C_2\sqrt{1+|X_r|^2}}\,dP \to 0$$

as $k \to \infty$ uniformly for all $s \in [0,T]$, $|x| \le B$, $u \in A_{s,\nu}$, and $w \in \mathcal{W}$.

For A_2, one may use the same proof as used for Lemma 3.4.10a. ∎

Lemma 3.4.11:

$$V^m(s,x) = \inf_{\beta\in\Delta} \sup_{w\in\mathcal{W}_m} J(s,x,\beta[w],w).$$

proof: By the uniform convergence of Lemmas 3.4.10, and the standard technique (see the proofs of Lemmas 2.4.17 or 2.3.8), we have the desired result. ∎

The next Lemma shows that we also have a pair of feedback controls which yield V^m.

Lemma 3.4.12: Let g, h be the bounded Borel measurable functions (actually h is continuous) such that

$$g(t,x) \in \arg\min_{v\in U}\left[f(x,v)\cdot\nabla V^m(t,x) + e^{-ct}L(x,v)\right] \tag{4.28}$$

and

$$h(t,x) \in \arg\max_{w\in\Re^n}\left[w\cdot\nabla V^m(t,x) - \gamma^2 e^{-ct}|w|^2\right]. \tag{4.29}$$

Then

$$V^m(s,x) = J(s,x,g(\cdot,X.),h(\cdot,X.))$$

where

$$X_t = x + \int_s^t \left[f(X_r, g(r,X_r)) + h(r,X_r)\right] dr + \sqrt{\frac{\epsilon}{2\gamma^2}}B_t. \tag{4.30}$$

proof: The existence of measurable g, h as above follows from [16], Appendix B. The existence of the solution to (4.30) follows from Theorem 3.2.5.

By Ito's rule and the fact that $V^m(T,x) \equiv 0$

$$\begin{aligned} V^m(s,x) = &-\int_s^T \left\{V_s^m + \frac{\epsilon}{4\gamma^2}\Delta V^m + \left[f(X_t, g(t,X_t)) + h(t,X_t)\right]\cdot\nabla V^m\right\} dt \\ &- \sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^T \Delta V^m \cdot dB_t. \end{aligned} \tag{4.31}$$

But by (4.24), (4.28), and (4.29)

$$0 = V_s^m + \frac{\epsilon}{4\gamma^2}\Delta V^m + [f(x, g(s,x)) + h(s,x)] \cdot \nabla V^m$$
$$+ e^{-cs}\left[L(x, g(s,x)) - \gamma^2 |h(s,x)|^2\right]. \tag{4.32}$$

Substituting (4.32) into (4.31) yields

$$V^m(s,x) = \int_s^T e^{-ct}\left[L(X_t, g(t,X_t)) - \gamma^2 |h(t,X_t)|^2\right] dt$$
$$+ \sqrt{\frac{\epsilon}{2\gamma^2}} \int_s^T \nabla V^m \cdot dB_t.$$

But by Lemma 3.4.3, this last term is a square–integrable martingale. Therefore

$$V^m(s,x) = \mathrm{E}_{s,x} \int_s^T e^{-ct}\left[L(X_t, g(t,X_t)) - \gamma^2 |h(t,X_t)|^2\right] dt. \blacksquare$$

Remark 3.4.13: By a similar proof as for Lemma 3.4.12 and Remark 3.2.4, we also have

$$V^m(s,x) \le \inf_{u \in A_{s,\nu}} J(s,x,u,h(\cdot, X.)).$$

Now note that, by Lemmas 3.4.3 and 3.4.9,

$$|\nabla V^m(s,x)| \le \frac{K}{c} C_4 \qquad \forall\, (s,x) \in [0,T] \times \{|x| \le B\}$$

where C_4 is 1 under assumption (AI3b), and $1 + C_L T e^{C_L T}$ under assumption (AI3a). Therefore, letting

$$m^* = \frac{K}{2c\gamma^2} C_4 e^{cT} \tag{4.33}$$

we have for all $m \ge m^*$

$$\max_{|w| \le m}\left[w \cdot \nabla V^m - \gamma^2 e^{-cs}|w|^2\right]$$
$$= \max_{|w| \le m^*}\left[w \cdot \nabla V^m - \gamma^2 e^{-cs}|w|^2\right] \tag{4.34}$$
$$= \max_{w \in \Re^n}\left[w \cdot \nabla V^m - \gamma^2 e^{-cs}|w|^2\right].$$

Thus, for any $m \geq m^*$, $V^m = V^{m^*} = V$ where $V \in C^{1,2}$ and satisfies

$$\begin{aligned} 0 &= V_s + \frac{\epsilon}{4\gamma^2}\Delta V + H(s,x,\nabla V) \\ V(T,x) &= 0 \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} H(s,x,\nabla V) &= \max_{w\in\Re^n}\left[w\cdot\nabla V - \gamma^2 e^{-cs}|w|^2\right] + \min_{v\in U}\left[f(s,x)\cdot\nabla V + e^{-cs}L(s,x)\right] \\ &= H_{m^*}(s,x,\nabla V). \end{aligned} \tag{4.36}$$

We also have

Lemma 3.4.14:

$$V(s,x) = \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J(s,x,\beta[w],w) \tag{4.37}$$

$$= J(s,x,g(\cdot,X.),h(\cdot,X.)) \tag{4.38}$$

where J, g, and h are given by (4.25), (4.28) and (4.29) and

$$\mathcal{W} = \{w:\ [0,T]\times\Omega\to\Re^n \text{ which are bounded, } \mathcal{F}_t\text{–progressively measurable }\}.$$

proof: For the first assertion, note that given $\beta \in \Delta$ and $\delta > 0$ there exists $w_\delta \in \mathcal{W}$ such that

$$\sup_{w\in\mathcal{W}} J(s,x,\beta[w],w) \leq J(s,x,\beta[w_\delta],w_\delta) + \delta$$

But by the definition of $\mathcal{W}$, $w_\delta \in \mathcal{W}_{m_0}$ for some $m_0 \geq m^\star$, so

$$\sup_{w\in\mathcal{W}} J(s,x,\beta[w],w) \leq \sup_{w\in\mathcal{W}_{m_0}} J(s,x,\beta[w],w) + \delta.$$

Taking the infimum, we have

$$\begin{aligned} \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J(s,x,\beta[w],w) &\leq \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}_{m_0}} J(s,x,\beta[w],w) + \delta \\ &= V^{m_0}(s,x) + \delta = V(s,x) + \delta. \end{aligned}$$

Since δ was arbitrary,

$$\inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}} J(s, x, \beta[w], w) \leq V(s, x).$$

The reverse is obvious since $\mathcal{W} \supset \mathcal{W}_{m^\star}$.

The second assertion is clear from the fact that $V = V^{m^*}$ and Lemma 3.4.12. ∎

Remark 3.4.15: By (4.34) we also have the representation

$$V(s, x) = \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}_{m^*}} J(s, x, \beta[w], w).$$

Remark 3.4.16: All the above results hold for the upper value as well as the lower by analogous proofs. In particular,

$$V(s, x) = \sup_{\alpha \in \Lambda} \inf_{u \in A_{s,\nu}} J(s, x, u, \alpha[u]).$$

We have now proved an Isaacs equation result for a finite time horizon game satisfying (AI1), (AI2) with compact controls for the minimizing player. In the next section, we use this result to obtain results for infinite time horizon games.

3.5 The Infinite Time Horizon Stochastic Game

In the last section, we showed that a certain finite time horizon stochastic differential game had value, and that that value function was a classical solution to the associated Isaacs equation. In particular, the payoff function for that game contained a discount factor. Here we will let the time horizon go to infinity so as to obtain an analogous result for the infinite time horizon, discounted payoff, stochastic differential game. After obtaining that result, we will let the discount factor go to zero, and scale the value such that we obtain a result for a long–run average–cost–per–unit–time, stochastic differential game. The Isaacs equation for this game will be equation (3.2). Thus we will obtain the promised result necessary for the application of Theorem 3.3.1.

First, in order to let the time horizon go to infinity for the game of the last section, we need to obtain some bounds on the behavior of the value function for that game. In particular, we want to bound the behavior independent of the time horizon, T, by using the ergodic properties of the problem.

Let us modify the notation to emphasize the parameters of interest in this section. Let $J^{T,c}$ be the payoff of (4.25), and $V^{T,c}$ be the value function of (4.37). We showed that $V^{T,c}$ was a classical solution of (4.35), (4.36).

First we will obtain a new bound for $\nabla V^{T,c}$ which decays exponentially with s. This will then be used to obtain a bound on the values of w (to be denoted by $m^\star$ as opposed to the previous bound m^*) which will be independent of T.

Lemma 3.5.1:

$$|\nabla V^{T,c}(s,x)| \leq \frac{K}{a+c} e^{-cs} \qquad \forall\, 0 \leq s \leq T < \infty,\ x \in \Re^n.$$

proof: By (2.9) and Gronwall, we have

$$|X_t - Y_t|^2 \le e^{-2at}|x-y|^2 \qquad \forall\, t \in [s,\infty),\ \ x,y \in \Re^n$$

or,

$$|X_t - Y_t| \le e^{-at}|x-y| \qquad \forall\, t \in [s,\infty),\ \ x,y \in \Re^n.$$

Thus, by (AI2)

$$\begin{aligned}|J^{T,c}(s,x,u,w) - J^{T,c}(s,y,u,w)| &\le K|x-y| \int_s^T e^{-a(r-s)} e^{-cr}\, dr \\ &= \frac{K}{a+c}|x-y| e^{as} e^{-(a+c)s}\left(1 - e^{-(a+c)(T-s)}\right) \\ &\le \frac{K}{a+c} e^{-cs}|x-y|.\end{aligned}$$

Since this is true for all u, w, by the usual argument we have

$$|V^{T,c}(s,x) - V^{T,c}(s,y)| \le \frac{K}{a+c} e^{-cs}|x-y| \qquad \forall\, 0 \le s \le T < \infty,\ \ x,y \in \Re^n.$$

Since $V^{T,c} \in C^{1,2}$, we are done. ∎

With this new bound, we proceed as at the end of Section 3.4. That is, letting

$$m^\star = \frac{1}{2\gamma^2}\frac{K}{a+c} \tag{5.1}$$

we have for all $m \ge m^\star$

$$\begin{aligned}&\max_{|w|\le m}\left[w \cdot \nabla V^{T,c} - \gamma^2 e^{-cs}|w|^2\right] \\ &= \max_{|w|\le m^\star}\left[w \cdot \nabla V^{T,c} - \gamma^2 e^{-cs}|w|^2\right] \\ &= \max_{w\in\Re^n}\left[w \cdot \nabla V^{T,c} - \gamma^2 e^{-cs}|w|^2\right].\end{aligned}$$

Thus $V^{T,c}$ is the unique classical solution to (4.31) where we now have

$$H(s,x,\nabla V^{T,c}) = H_{m^\star}(s,x,\nabla V^{T,c}),$$

and it is sufficient to consider $w \in \mathcal{W}_{m^\star}$ in calculating the value.

Lemma 3.5.2: Let $(s,x) \in [0,T] \times \{|x| \leq B\}$ where $B < \infty$. Then

$$0 \leq V^{T,c}(s,x) \leq \frac{C_2}{c} e^{-cs} \leq \frac{C_2}{c}$$

where C_2 depends on a, n, γ, B, K.

proof: The proof is a modification of the proof of Lemma 3.4.2. Refer to the earlier proof. Again we use $G(x) = \sqrt{1+|x|^2}$. In this case, however

$$Ag = \frac{\epsilon}{4\gamma^2} \Delta g + (f + w) \cdot \nabla g.$$

By (AI1) and Remark 3.4.15, Let $w \in \mathcal{W}_{m^\star}$, and note that by (AI1),

$$\begin{aligned} AG(X_t) &\leq \frac{\epsilon n}{4\gamma^2} + m^\star - a \frac{|X_t|^2}{\sqrt{1+|X_t|^2}} \\ &\leq \frac{\epsilon n}{4\gamma^2} + m^\star + a - aG(X_t) \\ &= C_1 - aG(X_t) \end{aligned}$$

which defines C_1. Employing this in (4.7), and using the linearity to move the expectation operator yields

$$\mathrm{E}_{s,x} G(X_t) \leq G(x) + \int_s^t \left[C_1 - a \mathrm{E}_{s,x} G(X_r) \right] dr.$$

As usual, this implies

$$\mathrm{E}_{s,x} G(X_t) \leq \max \left\{ G(x), \frac{C_1}{a} \right\}.$$

By (AI2), this implies

$$\mathrm{E}_{s,x} \left[L(X_t, u_t) - \gamma^2 |w_t|^2 \right] \leq C_2 \tag{5.2}$$

where

$$C_2 = K \max\left\{\sqrt{1+B^2}, \frac{C_1}{a}\right\} + \gamma^2 m^{\star 2}.$$

Thus

$$V^{T,c}(s,x) \le C_2 \int_s^T e^{-ct}\, dt \le \frac{C_2}{c}. \quad \blacksquare$$

Lemma 3.5.3: Let $(s,x) \in [0,T] \times \{|x| \le B\}$ where $B < \infty$. Then

$$|V_s^{T,c}(s,x)| \le 2C_2$$

where C_2 depends on a, n, γ, B, K. (This is the same C_2 as in Lemma 3.5.2.)

proof: Refer to the proof of Lemma 3.4.6. All the steps up to and including (4.13) remain the same with $J^{T,c}$ replacing J^k, f replacing f^k, and L replacing L^k. Then, in place of (4.14), we use (5.2). Thus we obtain, in place of (4.15),

$$\begin{aligned} |J^{T,c}(s,x,u,w) - J^{T,c}(t,x,u,w)| &\le C_2 \left| \left(1 - e^{-c(t-s)}\right) \int_s^{T-(t-s)} e^{-cr}\, dr \right. \\ &\qquad \left. + \int_{T-(t-s)}^T e^{-cr}\, dr \right| \\ &\le 2C_2 |t-s|. \end{aligned}$$

The remainder of the proof mimics that of Lemma 3.4.6 with C_2 replacing M_1. $\blacksquare$

Lemma 3.5.4: Let $(s,x) \in [0,T] \times \{|x| \le B\}$ where $B < \infty$. Then

$$|\Delta V^{T,c}(s,x)| \le C_3$$

where C_3 depends on a, n, γ, B, K, and $\max_{\{|x| \le B\} \times U} |f|$.

proof: Combine Lemmas 3.5.1 and 3.5.3 with (4.31) and (4.32), and note that the new bound $m^\star$ replaces m^* in (4.32). $\blacksquare$

We now let the time horizon go to infinity. First we show that the value function for the finite time horizon game converges to a solution of the Isaacs equation for the infinite time horizon game with discounted cost. Then we prove a verification theorem which implies that this solution is the value function for the associated game.

Lemma 3.5.5: There exists $V^c \in C^{1,2}([0,\infty)\times\Re^n)$ and a sequence (also denoted by $V^{T,c}$) with $T\to\infty$ such that

$$\begin{aligned} &V^{T,c} \to V^c \\ &\nabla V^{T,c} \to \nabla V^c \\ &V_s^{T,c} \to V_s^c \\ &\Delta V^{T,c} \to \Delta V^c \end{aligned} \tag{5.3}$$

uniformly on compact sets, and V^c satisfies

$$0 = V_s + \frac{\epsilon}{4\gamma^2}\Delta V + H(s,x,\nabla V) \qquad \forall\,(s,x)\in[0,\infty)\times\Re^n. \tag{5.4}$$

proof: Fix a compact set $[0,T_0]\times\{|x|\le B\}$. Consider the set $\{V^{T,c}\}_{T\ge T_0}$. All the $V^{T,c}$ satisfy

$$0 = V_s + \frac{\epsilon}{4\gamma^2}\Delta V + H_{m^\star}(s,x,\nabla V)$$

on $[0,T_0]\times\{|x|\le B\}$. Now note that all the assumptions of Lemma 3.4.8 are met on $[0,T_0]\times\{|x|\le B\}$ for the set $\{V^{T,c}\}_{T\ge T_0}$ with the exception of the boundary condition in (4.16). However, this boundary condition is never used in the proof (except to produce the boundary condition in the result). Therefore, by this trivial modification of Lemma 3.4.8, we have the desired result. ∎

Lemma 3.5.6: Let $(s,x)\in[0,\infty)\times\Re^n$. Then

$$V^c(s,x) = \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J^c(s,x,\beta[w],w) \tag{5.5a}$$

$$= \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}_{m^\star}} J^c(s,x,\beta[w],w) \tag{5.5b}$$

$$= \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}_{m}} J^c(s,x,\beta[w],w) \qquad \forall\, m \geq m^\star \tag{5.5c}$$

$$= J^c(s,x,g(\cdot,X.),h(\cdot,X.)) \tag{5.5d}$$

where

$$J^c(s,x,u,w) = \mathrm{E}_{s,x} \int_s^\infty e^{-ct} \left[L(X_t,u_t) - \gamma^2 |w_t|^2\right] \, dt \tag{5.6}$$

$$X_t = x + \int_s^r [f(X_\rho,u_\rho) + w_\rho] \, d\rho + \sqrt{\frac{\epsilon}{2\gamma^2}} B_t. \tag{5.7}$$

and g, h are bounded, Borel measurable functions such that

$$g(t,x) \in \arg\min_{v \in U} \left[f(x,v) \cdot \nabla V^c(t,x) + e^{-ct} L(x,v)\right] \tag{5.8}$$

$$h(t,x) \in \arg\max_{w \in \Re^n} \left[w \cdot \nabla V^c(t,x) - e^{-ct}\gamma^2 |w|^2\right]. \tag{5.9}$$

proof: Let $B < \infty$, $|x| \leq B$, $u \in A_{s,\nu}$, and $w \in \mathcal{W}_m$. Then

$$|J^{T,c}(s,x,u,w) - J^c(s,x,u,w)| \leq \mathrm{E}_{s,x} \int_T^\infty e^{-ct} |L(X_t,u_t)| \, dt$$

which by Tonelli's Theorem and a minor modification of (5.2) yields

$$\leq C_2' \int_T^\infty e^{-ct} \, dt = \frac{C_2'}{c} e^{-cT}$$

which goes to zero uniformly for $u \in A_{s,\nu}$ and $w \in \mathcal{W}_m$. By the standard proof (see the proofs of Lemmas 2.4.17 or 2.3.8) this implies

$$V^c(s,x) = \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}_m} J^c(s,x,\beta[w],w)$$

for any $m \geq m^\star$. This of course implies (5.5b) and (5.5c).

Now note that given $\beta \in \Delta$ and $\delta > 0$ there exists $w_\delta \in \mathcal{W}$ such that

$$\sup_{w \in \mathcal{W}} J^c(s,x,\beta[w],w) \leq J^c(s,x,\beta[w_\delta],w_\delta) + \delta$$

But by the definition of $\mathcal{W}$, $w_\delta \in \mathcal{W}_{m_0}$ for some $m_0 \geq m^\star$, so

$$\sup_{w \in \mathcal{W}} J^c(s,x,\beta[w],w) \leq \sup_{w \in \mathcal{W}_{m_0}} J^c(s,x,\beta[w],w) + \delta.$$

Taking the infimum, we have

$$\begin{aligned}\inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J^c(s,x,\beta[w],w) &\le \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}_{m_0}} J^c(s,x,\beta[w],w)+\delta\\ &= V^c(s,x)+\delta.\end{aligned}$$

Since δ was arbitrary,

$$\inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J^c(s,x,\beta[w],w) \le V^c(s,x).$$

The reverse is obvious since $\mathcal{W}\supset\mathcal{W}_{m^\star}$. This yields (5.5a).

Finally, (5.5d) follows in a similar manner to the proof of Lemma 3.4.12. Refer to that proof. In particular, with g,h given by (5.8), (5.9), we have

$$V^c(s,x)-V^c(T,x)=\mathrm{E}_{s,x}\int_s^T e^{-ct}\left[L(X_t,g(t,X_t))-\gamma^2|h(t,X_t)|^2\right]\,dt \tag{5.10}$$

By Lemma 3.5.2

$$V^c(s,x)=\lim_{T\to\infty}\mathrm{E}_{s,x}\int_s^T e^{-ct}\left[L(X_t,g(t,X_t))-\gamma^2|h(t,X_t)|^2\right]\,dt$$

Then by (5.2) and the Lebesgue Convergence Theorem,

$$V^c(s,x)=\mathrm{E}_{s,x}\int_s^\infty e^{-ct}\left[L(X_t,g(t,X_t))-\gamma^2|h(t,X_t)|^2\right]\,dt \tag{5.11}$$

which is (5.5d). ∎

Now, from (5.5d), we have

$$V^c(s,x)=\mathrm{E}_{s,x}\int_s^\infty e^{-ct}\left[L(X_t^0,g(t,X_t))-\gamma^2|h(t,X_t)|^2\right]\,dt$$

where

$$X_t^0=x+\int_s^t\left[f(X_r^0,g(r,X_r))+h(r,X_r)\right]dr+\sqrt{\frac{\epsilon}{2\gamma^2}}B_t.$$

As in the proof of Lemma 3.4.6, we may time–shift both the reference probability system and the controls. That is, we let $\hat{\mathcal{F}}_t = \mathcal{F}_{t-s}$, $\hat{B}_t = B_{t-s}$, $\hat{u}_t = u^0_{t-s}$, $\hat{w}_t = w^0_{t-s}$, and find the resulting time–shifted state satisfies $\hat{X}_t = X^0_{t-s}$. Then, we easily see

$$V^c(s,x) = e^{-cs}V^c(0,x) \tag{5.12}$$

for all $s \in [0,\infty)$. This of course implies

$$V^c_s = -cV^c. \tag{5.13}$$

Substituting (5.13) into (5.4), we see that V^c satisfies

$$\begin{aligned} cV^c = \frac{\epsilon}{4\gamma^2}\Delta V^c &+ \min_{v\in U}[f(x,v)\cdot\nabla V^c + e^{-cs}L(x,v)] \\ &+ \max_{w\in\Re^n}[w\cdot\nabla V^c - \gamma^2 e^{-cs}|w|^2]. \end{aligned}$$

Defining $\mathcal{V}^c(x) = V^c(0,x) = e^{cs}V^c(s,x)$, we have

$$\begin{aligned} c\mathcal{V}^c &= \frac{\epsilon}{4\gamma^2}\Delta\mathcal{V}^c + \min_{v\in U}[f(x,v)\cdot\nabla\mathcal{V}^c + L(x,v)] \\ &\qquad \max_{w\in\Re^n}[w\cdot\nabla\mathcal{V}^c - \gamma^2|w|^2] \qquad (5.14) \\ &= \frac{\epsilon}{4\gamma^2}\nabla\mathcal{V}^c + H^0(x,\nabla\mathcal{V}^c) \end{aligned}$$

which defines H^0. Thus, we have obtained a time–independent Isaacs equation and solution.

By Lemma 3.5.6,

$$\begin{aligned} \mathcal{V}^c(x) &= \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J^c(0,x,\beta[w],w) && (5.15a) \\ &= \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}_{m^\star}} J^c(0,x,\beta[w],w) && (5.15b) \\ &= \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}_m} J^c(0,x,\beta[w],w) \qquad \forall\, m \geq m^\star && (5.15c) \\ &= J^c(0,x,g(\cdot,X_\cdot),h(\cdot,X_\cdot)) && (5.15d) \end{aligned}$$

for all $x \in \Re^n$.

Now we come to the last step of this arduous process; we let the discount factor go to zero. As usual, we first bound the behavior of $\mathcal{V}^c$ independent, in a sense, of c. By Lemmas 3.5.1, 3.5.2, 3.5.4, 3.5.5, and (5.4) and the definition of $\mathcal{V}^c$, we have

$$0 \le c\mathcal{V}^c(x) \le C_2 \tag{5.16a}$$

$$|\nabla \mathcal{V}^c(x)| \le \frac{K}{a} \tag{5.16b}$$

$$|\Delta \mathcal{V}^c(x)| \le C_3 \tag{5.16c}$$

for all $|x| \le B < \infty$ and $c > 0$ where

$$\begin{aligned} C_2 &= K \max\left\{\sqrt{1+B^2}, \frac{\epsilon n}{4a\gamma^2} + \frac{m^\star}{a} + 1\right\} + \gamma^2 m^{\star 2} \\ C_3 &= \frac{4\gamma^2}{\epsilon}\left(2C_2 + \|f\|_B \frac{K}{a} + K\sqrt{1+B^2} + m^\star \frac{K}{a} + \gamma^2 m^{\star 2}\right) \end{aligned}$$

and

$$\|f\|_B = \max_{\{|x|\le B\}\times U} |f(x,v)|.$$

(Note this is a more specific value for C_3 than we wrote down earlier.)

Now, recall that the bounds in Lemmas 3.5.1, 3.5.3 and 3.5.4 implied $|\nabla V^{T,c}| \le \frac{K}{a}$, $|V_s^{T,c}| \le 2C_2$, and $|\Delta V^{T,c}| \le C_3$ where these bounds were independent of c as well as T. Thus (see the proof of Lemma 3.4.8), the $V^{T,c}$, $\nabla V^{T,c}$, $V_s^{T,c}$, and $\Delta V^{T,c}$ (actually all second derivatives by [16], (E.10)) are equicontinuous both in T and c. Therefore, by (5.3) and the definition of $\mathcal{V}^c$, the $c\mathcal{V}^c$, $\nabla\mathcal{V}^c$, and $\Delta\mathcal{V}^c$ are equicontinuous in c. Thus we may apply Ascoli–Arzela to yield a sequence $c_n \to \infty$ such that

$$c_n \mathcal{V}^{c_n} \to \mathcal{V}_0^0$$

$$\nabla \mathcal{V}^{c_n} \to \mathcal{V}_1^0$$

$$\Delta \mathcal{V}^{c_n} \to \mathcal{V}_2^0$$

uniformly on compact sets as $c_n \downarrow 0$, and $\mathcal{V}_i^0 \in C(\Re^n)$ for all i. Since $|\nabla \mathcal{V}^c| \leq \frac{K}{a}$ for all $c > 0$, this implies $\nabla(c\mathcal{V}^c) \to 0$, so we may replace $\mathcal{V}_0^0$ by a constant λ. Also, a standard theorem of mathematical analysis implies that $\mathcal{V}_2^0 = \sum_{i=1}^n \left(\mathcal{V}_1^0\right)_{x_i}^{(i)}$. Thus, we may write

$$\begin{aligned} & c_n \mathcal{V}^{c_n} \to \lambda \\ & \nabla \mathcal{V}^{c_n} \to \nabla W \\ & \Delta \mathcal{V}^{c_n} \to \Delta W \end{aligned} \tag{5.17}$$

where λ is a constant and $W \in C^2(\Re^n)$. As usual, the uniform convergence and (5.14) combine to imply that λ, W are a solution to

$$\begin{aligned} \lambda = & \frac{\epsilon}{4\gamma^2} \Delta W + \min_{v \in U} [f(x,v) \cdot \nabla W + L(x,v)] \\ & \quad + \max_{w \in \Re^n} [w \cdot \nabla W - \gamma^2 |w|^2] \\ = & \frac{\epsilon}{4\gamma^2} \Delta W + \min_{v \in U} [f(x,v) \cdot \nabla W + L(x,v)] + \frac{1}{4\gamma^2} |\nabla W|^2. \end{aligned} \tag{5.18}$$

Recall that (5.18) is identical to (3.2). Thus we have obtained the sought after result that allows us to invoke Theorem 3.3.1.

Remark 3.5.7: We also have, for any $B < \infty$

$$\begin{aligned} & |\nabla W| \leq \frac{K}{a} \qquad \forall\, x \in \Re^n \\ & |\Delta W| \leq C_3 \qquad \forall\, |x| \leq B \\ & \lambda \in \left[0, K\left(\frac{\epsilon n}{4a\gamma^2} + \frac{m^\star}{a} + 1\right) + \gamma^2 m^{\star 2}\right] \end{aligned} \tag{5.19}$$

where we have used the fact that λ is a constant and the fact that the $c\mathcal{V}^c$ bound holds for all $B < \infty$ in obtaining the last bound.

Note that we also have the following verifications.

Lemma 3.5.8: $c\mathcal{V}^c$ converges to a unique λ (not just subsequentially) which is the value of the control problem of Theorem 3.3.1.

proof: Let λ, W be given by the subsequential limit of (5.17). By Theorem 3.3.1 and (5.18), λ is the value of the game given there. Suppose $c\mathcal{V}^c \not\to \lambda$. Then there exists a subsequence and some $\delta > 0$ such that $|c_n\mathcal{V}^{c_n} - \lambda| > \delta$ for all n. But by Ascoli–Arzela, this subsequence must also possess a further subsequence such that

$$c_n\mathcal{V}^{c_n} \to \lambda_2$$

$$\nabla\mathcal{V}^{c_n} \to \nabla W_2$$

$$\Delta\mathcal{V}^{c_n} \to \Delta W_2.$$

But then by Theorem 3.3.1 and (5.18), λ_2 is also the value of the control problem given there. Therefore $\lambda = \lambda_2$ which is a contradiction. ∎

Lemma 3.5.9: Let g^0, h^0 be bounded Borel measurable functions such that

$$g^0(x) \in \arg\min_{v\in U}\left[f(x,v)\cdot\nabla W + L(x,v)\right] \tag{5.20a}$$

$$h^0(x) \in \arg\max_{w\in\Re^n}\left[w\cdot\nabla W - \gamma^2|w|^2\right]. \tag{5.20b}$$

Then,

$$\lambda = \lim_{T\to\infty}\frac{1}{T}\mathrm{E}_x\int_0^T [L(X_t, g^0(X_t)) - \gamma^2|h^0(X_t)|^2]\,dt. \tag{5.21}$$

and if h^0 is Lipschitz (i.e. ∇W Lipschitz), then

$$\lambda \le J^{\epsilon-}(x,u,h^0(X_\cdot)) \qquad \forall\, u \in A_\nu \tag{5.22a}$$

$$\lambda \le \sup_{\alpha\in\Lambda}\inf_{u\in A_\nu} J^{\epsilon-}(x,u,\alpha[u]). \tag{5.22b}$$

If f, L, U are such that g^0 is Lipschitz (typical conditions are given in [16], Section 6.6),

$$\lambda \ge J^{\epsilon+}(x, g^0(X_\cdot), w) \qquad \forall\, w \in \mathcal{W} \tag{5.23a}$$

$$\lambda \ge \inf_{\beta\in\Delta}\sup_{w\in\mathcal{W}} J^{\epsilon+}(x,\beta[w],w) \tag{5.23b}$$

where

$$J^{\epsilon -}(x,u,w) = \liminf_{T\to\infty} \frac{1}{T}\mathrm{E}_x \int_0^T [L(X_t,u_t) - \gamma^2|w_t|^2]\, dt \tag{5.24a}$$

$$J^{\epsilon +}(x,u,w) = \limsup_{T\to\infty} \frac{1}{T}\mathrm{E}_x \int_0^T [L(X_t,u_t) - \gamma^2|w_t|^2]\, dt. \tag{5.24a}$$

proof: First we prove (5.21). Note that by Theorem 3.2.5, we have existence for the SDE with both feedback controls. By Ito's rule,

$$\begin{aligned} W(X_T^0) - W(x) = &\int_0^T \left\{ \frac{\epsilon}{4\gamma^2}\Delta W + \nabla W \cdot [f(X_t^0, g^0(X_t)) + h^0(X_t)] \right\} dt \\ &+ \sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^T \nabla W \cdot dB_t. \end{aligned} \tag{5.25}$$

Then, by combining this with (5.18) and (5.20),

$$\begin{aligned} \lambda T = &\int_0^T \left[L(X_t^0, g^0(X_t)) - \gamma^2 |h^0(X_t)|^2 \right] \, dt \\ &+ \sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^T \nabla W \cdot dB_t + W(X_T^0) - W(x). \end{aligned} \tag{5.26}$$

Taking expectations and noting that by (5.19) the stochastic integral term is a martingale, we have

$$\lambda T = \mathrm{E}_x \int_0^T \left[L(X_t^0, g^0(X_t)) - \gamma^2 |h^0(X_t)|^2 \right] \, dt + \mathrm{E}_x W(X_T^0) - W(x) \tag{5.27}$$

which implies

$$\lambda \le \liminf_{T\to\infty} \left\{ \frac{1}{T}\mathrm{E}_x \int_0^T \left[L(X_t^0, g^0(X_t)) - \gamma^2 |h^0(X_t)|^2 \right] \, dt + \frac{\mathrm{E}_x W(X_T^0)}{T} \right\} \tag{5.28a}$$

$$\lambda \ge \limsup_{T\to\infty} \left\{ \frac{1}{T}\mathrm{E}_x \int_0^T \left[L(X_t^0, g^0(X_t)) - \gamma^2 |h^0(X_t)|^2 \right] \, dt + \frac{\mathrm{E}_x W(X_T^0)}{T} \right\} \tag{5.28b}$$

But (5.19) implies that $W(\cdot)$ is at most linear, and so by the same proof as for Lemma 3.5.2, $\mathrm{E}_x W(X_T^0)$ is bounded. Therefore, (5.28a) implies (5.21).

Now we prove (5.22). Note that we have existence again for the SDE by Theorem 3.2.6. Proceeding as above, but with any u replacing $g^0(X_\cdot)$, we obtain in place of (5.28)

$$\lambda \le \liminf_{T\to\infty} \frac{1}{T}\mathrm{E}_x \int_0^T [L(X_t,u_t) - \gamma^2|h^0(X_t)|^2]\, dt. \tag{5.29}$$

Since this is true for all $u \in A_\nu$, we have (5.22a). Now, given u and h^0, we have a solution to the SDE, X^0. Let $\alpha^0[u]_t = h^0(X_t^0)$. This is a strategy. Therefore (5.29) implies (5.22b).

Now we prove (5.23). Under the assumption that g^0 is Lipschitz, we have existence for the SDE by Theorem 3.2.7. Then proceeding as in the proof of (5.22), we obtain (5.23). ∎

3.6 Equivalence and the Deterministic Limit

At this point, we have obtained a relation between the infinite time horizon risk–averse control problem and an associated stochastic differential game with long–run average–cost–per–unit–time criterion. In particular, we have the following.

Theorem 3.6.1: Let (AI1), (AI2) hold. Then there exist $\lambda \in [0, \infty)$ and $W \in C^2(\Re^n)$ which satisfy (5.19) and

$$\begin{aligned}\lambda &= \frac{\epsilon}{4\gamma^2}\Delta W + \min_{v \in U}[f(x,v)\cdot\nabla W + L(x,v)] + \frac{1}{4\gamma^2}|\nabla W|^2 \\ &\equiv F^\epsilon(x, \nabla W, \Delta W).\end{aligned} \tag{6.1}$$

(Since W is defined only up to an additive constant, we may take $W(0) = 0$.) Furthermore, let X, Y satisfy

$$X_t = x + \int_0^t f(X_r, u_r)\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}}B_t^0$$
$$Y_t = x + \int_0^t [f(X_r, u_r) + w_r]\,dr + \sqrt{\frac{\epsilon}{2\gamma^2}}B_t$$

where $B_\cdot^0$ and $B_\cdot$ are Brownian motions relative to ν^0 and ν, respectively. (Note that ν^0 and ν may be any reference probability systems, and in fact, we may take $\nu^0 = \nu$ if we so choose.) Then

$$\lambda = \inf_{u \in A_\nu} \limsup_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t, u_t)\,dt\right] \tag{6.2a}$$

$$\lambda = \inf_{u \in A_\nu} \liminf_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t, u_t)\,dt\right] \tag{6.2b}$$

$$\lambda = \lim_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E}_x^0 \exp\left[\frac{1}{\epsilon}\int_0^T L(X_t^0, g^0(X_t^0))\,dt\right] \tag{6.2c}$$

$$\lambda = \lim_{T\to\infty} \frac{1}{T}\mathrm{E}_x \int_0^T [L(Y_t^0, g^0(Y_t^0)) - \gamma^2|h^0(Y_t^0)|^2]\,dt \tag{6.2d}$$

and if ∇W is Lipschitz

$$\lambda \leq \sup_{\alpha \in \Lambda} \inf_{u \in A_\nu} \liminf_{T \to \infty} \left\{ \frac{1}{T} \mathrm{E}_x \int_0^T [L(Y_t, u_t) - \gamma^2 |\alpha[u]_t|^2] \, dt \right\} \tag{6.2e}$$

while if g^0 is Lipschitz

$$\lambda \geq \inf_{\beta \in \Delta} \sup_{w \in \mathcal{W}} \limsup_{T \to \infty} \left\{ \frac{1}{T} \mathrm{E}_x \int_0^T [L(Y_t, \beta[w]_t) - \gamma^2 |w_t|^2] \, dt \right\} \tag{6.2f}$$

where

$$g^0(Y_t^0) \in \arg\min_{v \in U} [f(Y_t^0, v) \cdot \nabla W(Y_t^0) + L(Y_t^0, v)]$$

$$h^0(Y_t^0) \in \arg\max_{w \in \Re^n} [w \cdot \nabla W(Y_t) - \gamma^2 |w|^2]$$

and Y^0, X^* are the states corresponding to these specific controls.

proof: (6.2a), (6.2b), and (6.2c) follow from Theorem 3.3.1. The remainder follow from Lemma 3.5.9. ∎

Now we will take the limit as $\epsilon \downarrow 0$. In the risk–averse stochastic control problem, we are taking the risk–averse limit while at the same time scaling down the noise as $\sqrt{\epsilon}$. In the stochastic game, the $\sqrt{\epsilon}$ multiplier on the Brownian motion is going to zero so that the game is, in a sense to be specified below, becoming a deterministic game.

Let us superscript λ and W by ϵ to indicate their dependence on the parameter. Thus, from (6.1), $\lambda^\epsilon, W^\epsilon$ are a solution to

$$\lambda^\epsilon = F^\epsilon(x, \nabla W^\epsilon, \Delta W^\epsilon).$$

Let us suppose $\epsilon \leq \epsilon_0$ where we take $\epsilon_0 = 1$ for simplicity. Then, from Remark 3.5.7 and (5.1), we have

$$\begin{aligned} &\lambda^\epsilon \in \left[0,\, K + \frac{Kn}{4\gamma^2 a} + \frac{3K^2}{4\gamma^2 a}\right] \\ &|\nabla W^\epsilon(x)| \leq \frac{K}{a} \qquad\qquad \forall\, x \in \Re^n \end{aligned} \tag{6.3}$$

for all $\epsilon \leq 1$. Recall that we may assume

$$W^\epsilon(0) = 0 \qquad \forall\, \epsilon. \tag{6.4}$$

From (6.3), (6.4), Ascoli–Arzela, and some standard results from Mathematical Analysis, we know there exists

$$\lambda^0 \in \left[0,\, K + \frac{Kn}{4\gamma^2 a} + \frac{3K^2}{4\gamma^2 a}\right], \tag{6.5}$$

$W^0 \in C(\Re^n)$ where $W^0(0) = 0$, and a subsequence, say $\{\epsilon_n\}$ such that $\epsilon_n \downarrow 0$,

$$\begin{aligned} \lambda^{\epsilon_n} &\to \lambda^0, \\ W^{\epsilon_n} &\to W^0 \end{aligned} \tag{6.6}$$

uniformly on compact sets and

$$W^0 \text{ satisfies a Lipschitz condition with constant } \tfrac{K}{a}. \tag{6.7}$$

Now let

$$F^0(x,p) = \min_{v \in U}[f(x,v) \cdot p + L(x,v)] + \frac{1}{4\gamma^2}|p|^2$$

and note that

$$F^\epsilon \to F^0 \tag{6.8}$$

uniformly on compact sets as $\epsilon \downarrow 0$. Therefore, by (6.6), (6.8) and [17] Lemma 2.6.2, W^0 is a viscosity solution to

$$\lambda^0 = F^0(x, \nabla W^0). \tag{6.9}$$

Unfortunately however, all we have are subsequences converging to solutions of (6.9). We do not have a uniqueness result which would imply that all subsequences are converging to the same λ^0. In particular, if we had a λ^0 which was the value of a deterministic game, we could not claim that it was the same value as a subsequential limit. We will now show that in fact, that there is a connection between the subsequential limits and the upper and lower values of a (somewhat) deterministic game.

Let

$$X_t^\epsilon = x + \int_0^t [f(X_r^\epsilon, u_r) + w_r]\, dr + \sqrt{\frac{\epsilon}{2\gamma^2}} B_t \tag{6.10}$$

and $J^{\epsilon+}, J^{\epsilon-}$ be given by (5.24) with $X = X^\epsilon$. Let

$$X_t^0 = x + \int_0^t [f(X_r^0, u_r) + w_r]\, dr \tag{6.11a}$$

$$J^{0+}(x,u,w) = \limsup_{T\to\infty} \frac{1}{T} \mathrm{E}_x \int_0^T [L(X_t^0, u_t) - \gamma^2 |w_t|^2]\, dt \tag{6.11b}$$

$$J^{0-}(x,u,w) = \liminf_{T\to\infty} \frac{1}{T} \mathrm{E}_x \int_0^T [L(X_t^0, u_t) - \gamma^2 |w_t|^2]\, dt \tag{6.11c}$$

for any $u \in A_\nu$, $w \in \mathcal{W}$.

Theorem 3.6.2: Then

$$\begin{aligned}\limsup_{\epsilon\downarrow 0} \sup_{\alpha\in\Lambda} \inf_{u\in A_\nu} J^{\epsilon-}(x,u,\alpha[u]) &= \sup_{\alpha\in\Lambda} \inf_{u\in A_\nu} J^{0-}(x,u,\alpha[u]) \\ &\equiv \lambda^{0+}\end{aligned} \tag{6.12a}$$

(which defines λ^{0+}). Moreover, if h^0 (see (5.20b)) is Lipschitz

$$\lambda^{0+} \geq \limsup_{\epsilon\downarrow 0} \lambda^\epsilon. \tag{6.12b}$$

Also,

$$\begin{aligned}\liminf_{\epsilon\downarrow 0} \inf_{\beta\in\Delta} \sup_{w\in\mathcal{W}} J^{\epsilon+}(x,\beta[w],w) &= \inf_{\beta\in\Delta} \sup_{w\in\mathcal{W}} J^{0+}(x,\beta[w],w) \\ &\equiv \lambda^{0-}\end{aligned} \tag{6.12c}$$

(which defines λ^{0-}).

If f, L, U are such that g^0 (given by (5.20a) is Lipschitz, then

$$\lambda^{0-} \leq \liminf_{\epsilon \downarrow 0} \lambda^{\epsilon}. \tag{6.12d}$$

Of course, if the limit game has value, this implies that λ^ϵ converges to that value.

proof: Let $u \in A_\nu$ and $w \in \mathcal{W}$. Let $\begin{pmatrix} X^\epsilon \\ X^0 \end{pmatrix}_t$ be the process created by combining X^ϵ and X^0 into a single vector process. We again deal with X^ϵ and X^0 as scalars for simplicity. The vector process satisfies

$$\begin{pmatrix} X^\epsilon \\ X^0 \end{pmatrix}_t = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_s^t \left[\begin{pmatrix} f(X_r^\epsilon, u_r) \\ f(X_r^0, u_r) \end{pmatrix} + w_r \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] dr$$
$$+ \begin{pmatrix} \sqrt{\frac{\epsilon}{2\gamma^2}} B_t \\ 0 \end{pmatrix}.$$

Let

$$G\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = |x - y|^2$$

so that

$$\nabla G = 2(x-y) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the matrix of second derivatives is

$$H = 2I.$$

By Ito's rule

$$G\left(\begin{pmatrix} X^\epsilon \\ X^0 \end{pmatrix}_t \right) = \int_0^t \left\{ \frac{\epsilon n}{2\gamma^2} + 2(X_r^\epsilon - X_r^0) \cdot [f(X_r^\epsilon, u_r) - f(X_r^0, u_r)] \right\} dr$$
$$+ 2\sqrt{\frac{\epsilon}{2\gamma^2}} \int_0^t (X_r^\epsilon - X_r^0) \cdot dB_r.$$

But from the proof of Lemma 3.2.1 (see (2.7)), the last term is a square–integrable martingale. Therefore, taking expectations and noting (AI1), we have

$$\mathrm{E}G\left(\begin{pmatrix} X^\epsilon \\ X^0 \end{pmatrix}_t \right) \leq \int_0^t \left[\frac{\epsilon n}{2\gamma^2} - 2a \mathrm{E}_x G\left(\begin{pmatrix} X^\epsilon \\ X^0 \end{pmatrix}_r \right) \right] dr$$

which, as usual, implies

$$\mathrm{E}_x|X_t^\epsilon - X_t^0|^2 \le \frac{\epsilon n}{4a\gamma^2} \qquad \forall\, t < \infty$$

which by Hölder yields

$$\mathrm{E}_x|X_t^\epsilon - X_t^0| \le \sqrt{\frac{\epsilon n}{4a\gamma^2}} \qquad \forall\, t < \infty.$$

Combining this with (AI2) yields

$$\left|\frac{1}{T}\mathrm{E}_x \int_0^T [L(X_t^\epsilon, u_t) - \gamma^2|w_t|^2]\, dt - \frac{1}{T}\mathrm{E}_x \int_0^T [L(X_t^0, u_t) - \gamma^2|w_t|^2]\, dt\right| \le K\sqrt{\frac{\epsilon n}{4a\gamma^2}}$$

for all $T < \infty$, $u \in A_\nu$, $w \in \mathcal{W}$, and $x \in \Re^n$. This implies

$$|J^{\epsilon+}(x,u,w) - J^{0+}(x,u,w)| \le K\sqrt{\frac{\epsilon n}{4a\gamma^2}} \tag{6.13}$$

and

$$|J^{\epsilon-}(x,u,w) - J^{0-}(x,u,w)| \le K\sqrt{\frac{\epsilon n}{4a\gamma^2}} \tag{6.14}$$

for all $u \in A_\nu$, $w \in \mathcal{W}$, and $x \in \Re^n$.

By the standard proof (see the proofs of Lemmas 2.4.17 or 2.3.8), (6.13), (6.14) yield (6.12a), (6.12c).

Combining this with Theorem 3.6.1 yields (6.12b), (6.12d). ∎

Thus we have two results. The first is that the lim inf and lim sup of λ^ϵ are converging to λ^{0-} and λ^{0+} where λ^{0-}, W^{0-} and λ^{0+}, W^{0+} are solutions, in the viscosity sense, of (6.9). (Here we have defined W^{0-}, W^{0+} to be subsequential limits of $W^{\epsilon_n^-}, W^{\epsilon_n^+}$ for subsequences $\epsilon_n^-, \epsilon_n^+$ such that $\lambda^{\epsilon_n^-} \to \lambda^{0-}$, $\lambda^{\epsilon_n^+} \to \lambda^{0+}$.) The ramifications of this will be explored in the next section. Secondly, the upper and lower values of the stochastic game converge to the upper and lower values of the game with deterministic dynamics and payoff (6.11) (and these values bound the subsequential limits of λ^ϵ). However these values are calculated via infimum and supremum over stochastic controls rather than deterministic controls.

3.7 Connections with H^∞ Control

We have demonstrated the connection between infinite time horizon stochastic control and a particular stochastic differential game with long–run average–cost–per–unit–time criterion. We have also showed that in the risk–averse limit (with proper scaling down of the Brownian motion) we obtain an infinite time horizon differential game with deterministic dynamics albeit with possibly stochastic controls. In the process, we found that in the limit as $\epsilon \downarrow 0$, we also obtained viscosity solutions to a certain first–order PDE, (6.9). In this section, we shall use this last representation to obtain a connection with state–space, non–linear H^∞ control.

First we recall some results from state–space H^∞ control. The H^∞ control problem was, of course, originally formulated in frequency space. The associated state–space formulation was found via some results from Fourier Analysis; see Francis [20] and the references contained therein, particularly Dym and McKean [11]. This state–space formulation can naturally be viewed in terms of two–player, zero–sum games; see, for instance Basar and Bernhard [2]. This formulation also allows one to extend the definition of H^∞ control to nonlinear systems. Some notable results in this area are Ball and Helton [3], Isidori [23], [24] and [25], James [28], and van der Schaft [41].

A typical system considered in the above is that of [23], [25]

$$\begin{aligned} \frac{dX}{dt} &= \bar{f}(X_t) + g_2(X_t)u_t + g_1(X_t)w_t \\ z_t &= h_1(X_t) + k_{12}(X_t)u_t. \end{aligned} \tag{7.1}$$

(Note that here the control is not constrained to lie in a compact set.) In our presentation, this would correspond to

$$\begin{aligned} f(x,v) &= \bar{f}(x) + g_2(x)v \\ L(x,v) &= |z|^2 \end{aligned} \tag{7.2}$$

Our assumptions require

$$g_1(x) \equiv 1, \tag{7.3}$$

and we will assume this henceforth.

If there exists $\gamma \in [0,\infty)$ and a control u_0 such that system (7.1) with initial condition $X_0 = 0$ ($x = 0$ is a special reference point since $\bar{f}(0) = h_1(0) = 0$ as noted below) satisfies

$$\int_0^T |z_t|^2\, dt \le \gamma^2 \int_0^T |w_t|^2\, dt \tag{7.4}$$

for all $T < \infty$ and for all $w \in \bar{\mathcal{W}}$ (where $\bar{\mathcal{W}}$ may be bounded piece–wise continuous functions [23], or L_2 [41]), then u_0 is a control for which the system has an H^∞ bound (or, equivalently, L_2–gain bound) of γ. Note that minimizing the H^∞ norm is equivalent to finding the smallest $\gamma > 0$ such that the game

$$\inf_{u\in\bar{\mathcal{U}}} \sup_{w\in\bar{\mathcal{W}}} \int_0^T \left[|z_t|^2 - \gamma^2|w_t|^2\right]\, dt \tag{7.5}$$

has value $\lambda \le 0$ for all $T < \infty$, where $\bar{\mathcal{U}}$ is some appropriate set of controls.

Isidori makes the assumptions that $k_{12}^T(x)k_{12}(x)$ is nonsingular (and therefore positive definite) for all $x \in \Re^n$, $\bar{f}(0) = 0$, $h_1(0) = 0$, and that the trajectory $\dot{X}_t = \bar{f}(X_t) + g_2(X_t)u_t$ with control given by $h_1(x) + k_{12}(x)u = 0$ satisfies $X_t \to 0$ as $t \to \infty$. Under these assumptions, [23], Proposition 2.2 states that if there exists $\widetilde{W} \in C^1(\Re^n)$ which is positive definite and satisfies

$$\begin{aligned} 0 \ge \min_{v\in\Re^n} &\left[(\bar{f} + g_2 v)^T \nabla\widetilde{W} + (h_1 + k_{12}v)^T(h_1 + k_{12}v)\right] \\ &+ \frac{1}{4\gamma^2}|\nabla\widetilde{W}|^2 \end{aligned} \tag{7.6}$$

in some neighborhood of 0, then the control $u^* = -(k_{12}^T k_{12})^{-1}\left(\frac{1}{2}g_2^T \nabla\widetilde{W} + k_{12}^T h_1\right)$ satisfies a local disturbance attenuation condition. (Note that u^* is the argmin in (7.6).) This means that under certain bounds of [23] on the deviation of the state from $x = 0$, (7.4) is satisfied. In other words, there is a controller which yields an H^∞ bound of γ. (7.6) is known as a dissipation inequality.

On the other hand, consider (6.9) with f and L given by (7.2). Also suppose that γ is large enough so that $\lambda \leq 0$. then (6.9) becomes

$$\begin{aligned} 0 = \min_{v\in U} & \left[(\bar{f} + g_2 v)^T \nabla W + (h_1 + k_{12}v)^T (h_1 + k_{12}v)\right] \\ & + \frac{1}{4\gamma^2}|\nabla W|^2 \end{aligned} \tag{7.7}$$

In particular, if $U \supset \left\{|(k_{12}^T k_{12})^{-1}\left(\frac{1}{2}g_2^T \nabla W + k_{12}^T h_1\right)| \quad \forall\,|x| \leq r\right\}$ for some $r \geq 0$, then (7.7) is the same as (7.6) on $\{|x| \leq r\}$ with an equality replacing the inequality. Thus, if W is in $C^1(\Re^n)$, and the conditions of Isidori are met, then we have an H^∞ control with attenuation γ. Now, by (6.6)–(6.9), this would require γ to be sufficiently large so that $\liminf_{\epsilon\downarrow 0}\lambda^\epsilon \leq 0$. In this context, that would imply that $z^T z$ satisfy a linear growth bound, which might require a cutoff of some type. It would also put restrictions on the nature of $\bar{f} + g_2 u$ which are stricter than those of [23].

The requirement that there be a C^1 solution to (7.6) is quite strict since, in general, value functions associated with variational problems are not globally smooth. James [28] has obtained a result similar to the result of [23] discussed above but with fixed (or no) control, where $\widetilde{W}$ is a viscosity solution to the dissipation inequality rather than a classical solution, and so is not required to be in $C^1(\Re^n)$.

4.0 APPENDIX

Here we bound

$$\epsilon \log \int_\Omega \exp\left[\frac{Q}{\epsilon}\right]\, d\hat{P}(\omega)$$

by $\alpha_1 + \alpha_2 R$ for $|x| \le R$, $\epsilon \le \epsilon_0(R)$ where α_1, α_2 are constants independent of R,

$$\begin{aligned}
Q &= 2C_4\sqrt{\frac{\epsilon}{2\gamma^2}}\left[(1+b_3)\int_s^T |B_t|\,dt + b_3\int_s^T\int_s^t |B_r|\,dr\,dt + |B_T|\right] \\
C_4 &= C(1+KTe^{KT}) \\
b_3 &= Ke^{KT}
\end{aligned}$$

and

$$\hat{P}(\mathcal{A}) = \frac{\int_{\mathcal{A}} \exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)\right]\, dP(\omega)}{\int_\Omega \exp\left[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)\right]\, dP(\omega)}.$$

First note that the expression is bounded below by zero, so we need only to bound it from above. Clearly

$$\frac{Q}{\epsilon} \le \frac{k_1}{\sqrt{\epsilon}} M_T$$

where

$$M_T = \max_{t\in[0,T]} |B_t|$$

and

$$k_1 = C_4\sqrt{\frac{2}{\gamma^2}}(b_3T+1)(T+1)$$

so we need only bound

$$\mathcal{Q} \equiv \epsilon \log \int_\Omega \exp\left[\frac{k_1}{\sqrt{\epsilon}}\right] M_T\, d\hat{P}(\omega). \tag{A.1}$$

We consider first the probability measure $\hat{P}$. We will find a simpler probability measure whose expectation with regard to the above integrand will bound that of $\hat{P}$. In particular, this new measure will have less weight than $\hat{P}$ for small values of M_T and more weight out near infinity. Thus we consider

$$\int_{M_T\in\mathcal{A}} \exp\left[\frac{1}{\epsilon}A^\epsilon(\omega)\right]\, dP(\omega).$$

We will bound

$$A = \int_{M_T \le R} \exp\left[\frac{1}{\epsilon} A^\epsilon\right] dP \quad \text{from below}$$

and

$$B = \int_{M_T \ge R} \exp\left[\frac{1}{\epsilon} A^\epsilon\right] dP \quad \text{from above.}$$

First, we consider B. By the usual arguments employing (AF1)–(AF3) and Gronwall (see, for instance, the proof of Lemma 2.3.5)

$$\exp\left[\frac{1}{\epsilon} A^\epsilon\right] \le e^{\frac{a_1}{\epsilon}} e^{a_2 M_T^2} \tag{A.2}$$

where

$$\begin{aligned} a_1 &= a_4 + a_6 |x|^2, \\ a_2 &= \frac{3C(1+T)\left(1+e^{KT}\right)^2}{2\gamma^2}, \end{aligned}$$

and a_4, a_6 depend on C, T, K, γ. Note that the expectation of the bound exists for $\gamma > \gamma_1$.

Note here that (since L, Ψ can grow quadratically in $|x|$) A^ϵ contains a quadratic term in $|x|$, while the bound on Q we desire has only a linear term in $|x|$. If it was not necessary to get this linear bound, then the following would be greatly simplified. To get the linear bound requires delicate estimates. This is not surprising, since the quadratic growth case is right on the limit of what can be proven for this problem. Let $F_e(z) = P(e^{a_2 M_T^2} \le z) - 1$. Then

$$B \le e^{a_1/\epsilon} \int_{e^{a_2 R^2}}^{\infty} z \, dF_e(z),$$

which upon integrating by parts

$$\begin{aligned} &= e^{a_1/\epsilon} \left[zF_e(z)\Big|_{e^{a_2 R^2}}^{\infty} - \int_{e^{a_2 R^2}}^{\infty} F_e(z)\, dz \right] \\ &= e^{a_1/\epsilon} \left[e^{a_2 R^2} P(M_T > R) + \int_{e^{a_2 R^2}}^{\infty} P\left(M_T > \sqrt{\frac{\log z}{a_2}} \right) dz \right]. \end{aligned} \tag{A.3}$$

But

$$P(M_t > a) = P\left(\bigcup_{i=1,n}\left[\{\max_{t\in[0,T]} B_t^i > a\} \cup \{\min_{t\in[0,T]} B_t^i < -a\}\right]\right)$$
$$\leq 2nP(\max_{t\in[0,T]} B_t^1 > a)$$

which as in [31]

$$= 4nP(B_T^1 > a) \qquad (A.4).$$

To simplify the notation, we take the scalar case, $n = 1$. The case with $n > 1$ simply involves carrying the constant n through the computations that follow. Since the expressions are already dense, we avoid this additional complexity.

Substituting (A.4) into (A.3) yields

$$B \leq 4e^{a_1/\epsilon}\left[e^{a_2R^2}P(B_T > R) + \int_{e^{a_2R^2}}^{\infty} P\left(B_T > \sqrt{\frac{\log z}{a_2}}\right) dz\right]. \qquad (A.5)$$

We consider the second term in (A.5) first. Let

$$\mathcal{P}(r) \equiv P\left(B_T > \sqrt{\frac{\log r}{a_2}}\right) = \int_{\sqrt{\frac{\log r}{a_2}}}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}}\, dz$$

and note that

$$\frac{d\mathcal{P}}{dr} = -\frac{1}{2}\frac{1}{\sqrt{2\pi T a_2}} r^{-1-\frac{1}{2Ta_2}} (\log r)^{-\frac{1}{2}}$$

and

$$\mathcal{P}(1) = \frac{1}{2}.$$

Consider

$$\tilde{\mathcal{P}}(r) \equiv \frac{1}{2} r^{-\frac{1}{2Ta_2}} \qquad r \in [1, \infty).$$

Note that $\tilde{\mathcal{P}}(1) = \frac{1}{2} = \mathcal{P}(1)$, $\lim_{r\to\infty}\tilde{\mathcal{P}} = 0 = \lim_{r\to\infty}\mathcal{P}$ and

$$\frac{d\mathcal{P}}{dr} - \frac{d\tilde{\mathcal{P}}}{dr} = r^{-1-\frac{1}{2Ta_2}}\left[k_1 - k_2(\log r)^{-\frac{1}{2}}\right]$$

for constants $k_1, k_2 > 0$, so that for some $b \in (1, \infty)$

$$\frac{d\mathcal{P}}{dr} - \frac{d\tilde{\mathcal{P}}}{dr} = \begin{cases} \leq 0 & \text{for all } r \leq b \\ > 0 & \text{for all } r > b. \end{cases}$$

Therefore $\tilde{\mathcal{P}} \geq \mathcal{P}$ for all $R \in [1, \infty)$ which implies

$$\int_{e^{a_2 R^2}}^{\infty} P\left(B_T > \sqrt{\frac{\log r}{a_2}}\right) dr \leq \frac{1}{2} \int_{e^{a_2 R^2}}^{\infty} r^{-\frac{1}{2T a_2}} \, dr. \tag{A.6}$$

Note that this will exist for $\gamma > \gamma_1$. Then, integrating the right hand side of (A.6), we have

$$\int_{e^{a_2 R^2}}^{\infty} P\left(B_T > \sqrt{\frac{\log r}{a_2}}\right) dr \leq \frac{T a_2}{a_3} e^{-\frac{a_3 R^2}{2T}} \tag{A.7}$$

where $a_3 = 1 - 2Ta_2$. This will be used to bound the second term in (A.5).

Now we turn to the first term in (A.5). Note that if $R > 1$

$$\begin{aligned} e^{a_2 R^2} \int_R^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}} \, dz &\leq e^{a_2 R^2} \left(\frac{1}{R}\right) \int_R^{\infty} \frac{1}{\sqrt{2\pi T}} z e^{-\frac{z^2}{2T}} \, dz \\ &\leq \sqrt{\frac{2T}{\pi}} e^{-\frac{a_3 R^2}{2T}}. \end{aligned}$$

We should note here that we are really only interested in the case where R is not bounded a priori. If we knew that R would be less than some number, then the required computations become relatively trivial.

Thus letting

$$\tilde{L}(R) = \begin{cases} e^{a_2} & \text{if } R < 1 \\ \sqrt{\frac{2T}{\pi}} e^{-\frac{a_3 R^2}{2T}} & \text{if } R \geq 1, \end{cases}$$

we have

$$e^{a_2 R^2} \int_R^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}} \, dz \leq \tilde{L}(R) \qquad \forall R \geq 0. \tag{A.8}$$

Employing (A.5), (A.7) and (A.8), we have

$$B \leq 4e^{a_1/\epsilon} \left[\tilde{L}(R) + \frac{a_2 T}{a_3} e^{-\frac{a_3 R^2}{2T}}\right] \equiv l_2(R). \tag{A.9}$$

Now we turn to bounding A from below. We have

$$
\begin{aligned}
A &= \int_{M_T \le R} \exp A^{\epsilon}\, dP \\
&\ge P(M_T \le R) \\
&= 1 - P(M_T > R)
\end{aligned}
$$

which from (A.4) (again taking $n = 1$ for simplicity)

$$
\begin{aligned}
&\ge 1 - 4P(B_T > R) \\
&= 1 - 4\frac{1}{\sqrt{2\pi T}} \int_R^{\infty} e^{-\frac{z^2}{2T}}\, dz \equiv l_1'(R).
\end{aligned}
$$

Note that there exists R_0 such that the last term is nonnegative for $R \ge R_0$. We define l_1 by $l_1 = 0$ if $R < R_0$ and $l_1 = l_1'$ if $R \ge R_0$. Then

$$
A \ge l_1(R) \qquad \forall R \ge 0 \tag{A.10}
$$

Now let

$$
l_3 = \begin{cases} 0 & \text{if } R < R_0 \\ 4\frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}} & \text{if } R \ge R_0 \end{cases}
$$

and

$$
l_4 = C_3 e^{a_1/\epsilon} z e^{-\frac{a_3 z^2}{2T}}
$$

where

$$
C_3 = 4\left(\sqrt{\frac{2T}{\pi}} + \frac{a_2 T}{a_3}\right)
$$

so that

$$
\begin{aligned}
l_3(z) &= \left.\frac{dl_1}{dR}\right|_{R=z} \\
l_4(z) &\ge \left.\frac{dl_2}{dR}\right|_{R=z}
\end{aligned} \tag{A.11}
$$

for $z \ge 1$ (where we again note that we are not interested here in small values of z).

Let

$$
f_{\epsilon,m}(z) = \begin{cases} l_3 & \text{if } 0 \le z \le m \\ l_4 & \text{if } z > m \end{cases} \tag{A.12}
$$

and choose $m(\epsilon, x)$ such that

$$\int_0^\infty f_{\epsilon,m}(z)\, dz = \int_\Omega e^{A^\epsilon}\, dP. \tag{A.13}$$

The distribution with density function $f_{\epsilon,m}/\int_0^\infty f_{\epsilon,m}(z)\, dz$ yields the promised new probability measure.

Lemma A.1:

$$\int_\Omega e^{kM_T} e^{A^\epsilon}\, dP \leq \int_0^\infty e^{kz} f_{\epsilon,m}(z)\, dz$$

for any $k \geq 0$.

proof: Define $F_M(z) = \int_{M_T \leq z} e^{A^\epsilon}\, dP$, and let

$$Q_1 = \int_{M_T \leq m} e^{kM_T} e^{A^\epsilon}\, dP - \int_0^m e^{kz} f_{\epsilon,m}(z)\, dz$$

which, abusing notation slightly, is then

$$= \int_0^m e^{kz}\, dF_M(z) - f_{\epsilon,m}(z)\, dz. \tag{A.14}$$

Also recall from (A.9),(A.10), and (A.13) that

$$\int_0^\infty dF_M(z) - f_{\epsilon,m}(z)\, dz = 0, \tag{A.15}$$

$$\int_0^x dF_M(z) - f_{\epsilon,m}(z)\, dz \geq 0 \qquad \forall x \leq m, \tag{A.16}$$

and

$$\int_x^\infty dF_M(z) - f_{\epsilon,m}(z)\, dz \leq 0 \qquad \forall x \geq m \geq 1, \tag{A.17}$$

Then

$$Q_1 = e^{km} \int_0^m dF_M(z) - f_{\epsilon,m}(z)\, dz + \int_0^m (e^{kz} - e^{km})(dF_M(z) - f_{\epsilon,m}(z)\, dz)$$

which, upon integrating the second term by parts, yields

$$= e^{km} \int_0^m dF_M(z) - f_{\epsilon,m}(z)\, dz - k \int_0^m e^{kz} \left[F_M(z) - \int_0^z f_{\epsilon,m}(x)\, dx \right] dz$$

and using (A.16) to bound the second term, we have

$$\leq e^{km} \int_0^m dF_m(z) - f_{\epsilon,m}(z)\, dz. \qquad (A.18)$$

Letting

$$\begin{aligned} Q_2 &= \int_{M_T \geq m} e^{kM_T} e^{A^\epsilon}\, dP - \int_m^\infty e^{kz} f_{\epsilon,m}(z)\, dz \\ &= \int_m^\infty e^{kz}\, dF_M(z) - f_{\epsilon,m}(z)\, dz, \qquad (A.19) \end{aligned}$$

and performing a similar analysis (but using (A.17) rather than (A.16)) yields

$$Q_2 \leq e^{km} \int_m^\infty dF_M(z) - f_{\epsilon,m}(z)\, dz$$

which by (A.15) is

$$= -e^{km} \int_0^m dF_M(z) - f_{\epsilon,m}(z)\, dz. \qquad (A.20)$$

Combining (A.19) and (A.20), we have $Q_1 + Q_2 \leq 0$. But this is the desired result.

■

By (A.1), (A.13) and the lemma, we now have

$$\begin{aligned} \mathcal{Q} &\leq \epsilon \log \frac{\int_0^\infty e^{\frac{k_1}{\sqrt{\epsilon}} z} f_{\epsilon,m}(z)\, dz}{\int_0^\infty f_{\epsilon,m}(z)\, dz} \\ &= \epsilon \log \frac{4 \int_{R_0}^m \frac{1}{\sqrt{2\pi T}} e^{\frac{k_1}{\sqrt{\epsilon}} z} e^{-\frac{z^2}{2T}}\, dz + k_1 e^{a_1/\epsilon} \int_m^\infty z e^{\frac{k_1}{\sqrt{\epsilon}} z} e^{-\frac{a_3 z^2}{2T}}\, dz}{4 \int_{R_0}^m \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}}\, dz + k_1 e^{a_1/\epsilon} \int_m^\infty z e^{-\frac{a_3 z^2}{2T}}\, dz} \qquad (A.21a) \\ &\equiv \epsilon \log \frac{D_1 + D_2}{E_1 + E_2}. \qquad (A.21b) \end{aligned}$$

All that remains now is to bound (A.21). As mentioned at the outset, we desire a bound of the form $\alpha_1 + \alpha_2 R$ for $|x| \leq R$. Now, although we have simplified the probability measure, we have introduced the new variable m, and we must be concerned with how this varies as $\epsilon \downarrow 0$. We consider some cases.

Consider a sequence $\epsilon_n \to 0$ where the corresponding m values satisfy $m_n \approx \frac{1}{\sqrt{\epsilon_n}}$, that is $m_n = O(\frac{1}{\sqrt{\epsilon_n}})$ and $\frac{1}{\sqrt{\epsilon_n}} = O(m_n)$. Given such a sequence, it is easily seen by compactness that given $\delta_0 > 0$, there exists some c and a subsequence (still denoted by ϵ_n) such that

$$m_n \in \left(\frac{c+\delta_0}{\sqrt{\epsilon_n}}, \frac{c-\delta_0}{\sqrt{\epsilon_n}}\right) \qquad \forall n.$$

With this in mind, we restrict our attention to sequences of the form

$$m_n = \sqrt{\frac{4Ta_1}{a_3 3\epsilon_n}} + (c+\delta)\frac{1}{\sqrt{\epsilon_n}} \tag{A.22}$$

where $c = c(x)$, $\delta = \delta(x,\epsilon)$ and $|\delta| \le \delta_0$.

Note that

$$\begin{aligned} D_2 &= k_2 \exp\left[\frac{1}{\epsilon}\left(a_1 + \frac{k^2T}{a_3}\right) - \frac{1}{2}\left(\sqrt{\frac{a_3}{2T}}m - k\sqrt{\frac{2T}{a_3\epsilon}}\right)^2\right] \\ &\quad + k_3 e^{\frac{1}{\epsilon}\left(a_1+\frac{k^2T}{a_3}\right)}\left[1 - \operatorname{erf}\left(\sqrt{\frac{a_3}{2T}}m - k\sqrt{\frac{2T}{a_3\epsilon}}\right)\right] \\ &\equiv \Lambda_1 + \Lambda_2, \end{aligned}$$

and

$$E_2 = k_4 e^{\frac{a_1}{\epsilon} - \frac{a_3 m^2}{4T}}.$$

It is easily shown that the exponential limiting behavior of Λ_2 is the same as that of Λ_1 (ignoring some irrelevant polynomial terms). Thus in the following, we will consider only Λ_1 in place of D_2 to make the calculations more reasonable.

Substituting (A.22) into these last two expressions yields

$$\Lambda_1 = k_2 \exp\left[-\frac{a_3}{4T}(c+\delta)^2 + (k_1 - \sqrt{\frac{a_1 a_3}{T}})\frac{\delta}{\sqrt{\epsilon}} + \sqrt{\frac{a_1 T}{a_3}}2k\frac{1}{\epsilon}\right] \tag{A.23}$$

$$E_2 = k_4 \exp\left[-\frac{a_3}{4T}(c+\delta)^2 - \sqrt{\frac{a_1 a_3}{T}}\frac{\delta}{\sqrt{\epsilon}}\right]. \tag{A.24}$$

Recall here that a_1 is quadratic in x.

Also recall

$$D_1 = \frac{4}{\sqrt{2\pi T}} e^{\frac{k^2 T}{2\epsilon}} \int_0^{m-\frac{k_1 T}{\sqrt{\epsilon}}} e^{-\frac{u^2}{2T}}\, du \tag{A.25a}$$

$$\leq 4e^{\frac{k^2 T}{2\epsilon}} \tag{A.25b}$$

and

$$E_1 = \frac{4}{\sqrt{2\pi T}} \int_0^{m} e^{-\frac{u^2}{2T}}\, du \tag{A.26a}$$

which for $m \geq 1$

$$\geq 4\mathrm{erf}(1) \tag{A.26b}$$

where $\mathrm{erf}(y)$ indicates $\frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{z^2}{2}}\, dz$. Note again that by continuity, there exists ϵ_0 such that $m \geq 1$ for all $\epsilon \leq \epsilon_0$, $|x| \leq R$.

Let ϵ_n, m_n be of the form (A.22). Then for the case where $(c+\delta) \geq 0$, we have

$$\Lambda_1 \leq k_2 \exp \frac{1}{\epsilon}\left[-\frac{a_3}{4T}(c+\delta)^2 + k_1(c+\delta) + \sqrt{\frac{a_1 T}{a_3}} 2k\right]$$

which upon maximizing the quadratic in $(c+\delta)$

$$\leq k_2 \exp \frac{1}{\epsilon}\left[\frac{2k_1^2 T}{a_3} + \sqrt{\frac{a_1 T}{a_3}} 2k\right]. \tag{A.27}$$

By (A.27), Λ_1 has the desired behavior, i.e. $\Lambda_1 \leq e^{[\alpha_1 + \alpha_2|x|]\frac{1}{\epsilon}}$. By (A.25), D_1 clearly has the desired behavior (without the $|x|$ dependence). For the denominator, note that by (A.26a) and the definition of E_2, $E_1 + E_2 \geq k_5 \int_0^\infty e^{-k_6 z^2}\, dz$ which is bounded away from zero independent of x, ϵ. Therefore, Q has the desired behavior.

On the other hand, for the case where $(c+\delta) \leq 0$, we have from (A.23), (A.24), (A.25b) and (A.26b)

$$\frac{D_1 + \Lambda_1}{E_1 + E_2} \leq \frac{4e^{\frac{k^2 T}{2\epsilon}} + k_2 \exp \frac{1}{\epsilon}\left[-\frac{a_3}{4T}(c+\delta)^2 - \sqrt{\frac{a_1 a_3}{T}}(c+\delta) + \sqrt{\frac{a_1 T}{a_3}} 2k\right]}{4\mathrm{erf}(1) + k_4 \exp \frac{1}{\epsilon}\left[-\frac{a_3}{4T}(c+\delta)^2 - \sqrt{\frac{a_1 a_3}{T}}(c+\delta)\right]}$$

which if $-\frac{a_3}{4T}(c+\delta)^2 - \sqrt{\frac{a_1 a_3}{T}}(c+\delta) < 0$

$$\leq \frac{4e^{\frac{k^2T}{2\epsilon}} + k_2 \exp\frac{1}{\epsilon}\left[\sqrt{\frac{a_1T}{a_3}}2k\right]}{4\mathrm{erf}(1)}$$

or if $-\frac{a_3}{4T}(c+\delta)^2 - \sqrt{\frac{a_1a_3}{T}}(c+\delta) \geq 0$

$$\leq \frac{4e^{\frac{k^2T}{2\epsilon}} + k_2 \exp\frac{1}{\epsilon}\left[\sqrt{\frac{a_1T}{a_3}}2k\right]}{k_4}.$$

Thus in either case

$$\frac{D_1 + \Lambda_1}{E_1 + E_2} \leq e^{[\alpha_1 + \alpha_2|x|]\frac{1}{\epsilon}}.$$

Now, suppose $m_n \prec \frac{1}{\sqrt{\epsilon_n}}$ (i.e. $m_n = o(\frac{1}{\sqrt{\epsilon_n}})$). Then (dropping the n subscript)

$$\frac{D_1 + D_2}{E_1 + E_2} \leq \frac{\int_0^m \frac{4}{\sqrt{2\pi T}} e^{\frac{k_1}{\sqrt{\epsilon}}z} e^{-\frac{z^2}{2T}}\, dz + k_1 \int_m^\infty z e^{\frac{k_1}{\sqrt{\epsilon}}z} e^{-\frac{a_3 z^2}{2T}}\, dz}{k_1 \int_m^\infty e^{-\frac{a_3 z^2}{2T}}\, dz}$$

which for any b and $\epsilon \leq \epsilon(b)$

$$\leq \frac{\frac{4}{\sqrt{2\pi T}} \int_0^\infty e^{\frac{k_1}{\sqrt{\epsilon}}z} e^{-\frac{z^2}{2T}}\, dz + k_1 \int_0^\infty z e^{\frac{k_1}{\sqrt{\epsilon}}z} e^{-\frac{a_3 z^2}{2T}}\, dz}{k_1 \int_{\frac{b}{\sqrt{\epsilon}}}^\infty e^{-\frac{a_3 z^2}{2T}}\, dz.}$$

(Note that it is easily shown that $m(x,\epsilon)$ is continuous, and so we can find such an ϵ_0 which is uniform for all $|x| \leq R$.) It is easily shown that this decays as $\frac{\alpha}{\epsilon}$ for some α which is independent of x.

Finally, suppose $m_n \succ \frac{1}{\sqrt{\epsilon_n}}$ (i.e. $\frac{1}{\sqrt{\epsilon_n}} = o(m_n)$). Then the same analysis as for $m_n \approx \frac{1}{\sqrt{\epsilon_n}}$ with $c + \delta \geq 0$ works.

Now consider the general case. Suppose there do not exist $\alpha_1, \alpha_2 < \infty$ such that $Q \leq \alpha_1 + \alpha_2 R$ for all $|x| \leq R$, $s \in [0,T]$, and $\epsilon \leq \epsilon_0$. Then there exists a sequence $\epsilon_n \downarrow 0$ which violates the bound, that is $\lim_{n\to\infty} Q(\epsilon_n) > \alpha_1 + \alpha_2 R$. Suppose $m_n \approx \frac{1}{\sqrt{\epsilon_n}}$. Then we showed that, in fact, $Q(\epsilon_n) \leq \alpha_1 + \alpha_2 R$. Thus we only need consider the case where $m_n \not\approx \frac{1}{\sqrt{\epsilon_n}}$. It is easily seen that if $m_n \not\approx \frac{1}{\sqrt{\epsilon_n}}$, then there exists a further subsequence (also denoted by ϵ_n, m_n) such that either $m_n \prec \frac{1}{\sqrt{\epsilon_n}}$ or $m_n \succ \frac{1}{\sqrt{\epsilon_n}}$. But we also showed that in either of these cases $Q(\epsilon_n) \leq \alpha_1 + \alpha_2 R$. Therefore, $Q \leq \alpha_1 + \alpha_2 R$ for all $|x| \leq R$, $s \in [0,T]$, and $\epsilon \leq \epsilon_0$.

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