Max–Plus Methods for Nonlinear H_∞ Control: Operating in the Transform Space

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Abstract

The solution of some forms of nonlinear H_∞/L_2-gain problems can be obtained via solution of the corresponding Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs). Alternatively, the solution of some classes HJB PDEs have representations as solutions of L_2-gain problems. Both can be obtained through solution of corresponding fixed-point problems – where the operators are the semigroups associated with the PDEs. In the linear/quadratic case, the solutions of these problems can be obtained simply by solution of associated Riccati equations. Here, an exploration of a way in which the operators for linear/quadratic problems can be combined (in the semiconvex dual space) to obtain operators, and hence solutions, for more general problems is begun.

Key words: dynamic programming, partial differential equations, max-plus algebra, Legendre transform, semiconvexity, Hamilton-Jacobi-Bellman equations.

1 Introduction

Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDE's) arise naturally when one attempts to solve many nonlinear, continuous time/space control and estimation problems via dynamic programming. They specifically occur for problems where one

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is maximizing or minimizing over some input process (typically a controller or distrbance process). We look at control/estimation problems where the inputs have finite energy, and so the PDE's are first-order. These types of problems occur in optimal control and Robust/H_∞ control and estimation. Some H_∞ control problems have state-space representations which take the form of a game (with both minimizing and maximizing input); some of these are equivalent to pure control problems (when one player sufficiently dominates) and so have corresponding PDE's which are HJB equations; in other cases, the PDE's are Hamilton-Jacobi-Isaacs equations. We will not be concerned with the latter here - only HJB equations.

These PDE's are nonlinear. For instance, in the H_∞ case, they typically contain a term which is quadratic in the gradient. Solution of such PDE's has presented a formidable challenge for many years. This challenge is most commonly summed up as the "curse-of-dimensionality".

We will be concerned here with steady-state, fully nonlinear, first–order HJB PDE's (although similar arguments could apply to time-dependent problems as well). The solutions are typically nonsmooth, and so one must look for solutions in the weaker class of viscosity solutions. There is generally a unique viscosity solution, however for some problems one must use and additional criterion to select the correct viscosity solution [29]. The computation of the solution of a nonlinear, steady-state, first-order PDE is typically quite difficult, and possibly even more so in the presence of the non-uniqueness mentioned above. Some previous works in the general area of numerical methods for these problems are [3], [6], [7], [11], [12], [18], and the references therein. All these methods suffer from the "curse-of-dimensionality", and are, at the core, in the class of finite element methods.

In recent years, we have begun consideration of an entirely new class of numerical methods for HJB PDE's based on the linearity of the semi-group over the max-plus algebra (cf. [9], [13], [22], [23], [21], [16], [24], [20], [25], [26]). (Alternatively, one uses the min-plus algebra for minimizing control/estimation problems, but we will work only with the max-plus algebra and maximizing control/estimation problems so as not to be repititous). This linearity had previously been noted in [19]. For purposes of completeness, we recall that the max-plus algebra is a commutative semi-field $(\mathbf{R} \cup \{-\infty\}, \oplus, \otimes)$ with the addition and multiplication operations given by

$$\begin{aligned} a \oplus b &= \max\{a, b\}, \\ a \otimes b &= a + b \end{aligned} \tag{1}$$

where the operations are defined for $-\infty$ in the obvious way. Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. Note that it is not a field since the additive inverses are missing. (See [2], [5] among a burgeoning mass of literature related to the max-plus algebra.)

Another key ingredient in the development of this new class of max-plus-based numerical methods was the development of an appropriate basis for the solution space over the max-plus algebra (i.e. with the max-plus algebra replacing the standard underlying field).

In fact this basis was first developed in [9] through the use of a semiconvex transform. The semiconvex transform is a slight modification of the Legendre transform and convex duality relationship [9] [30], [31]. The countable, infinite dimensional basis is formed by taking a countable dense subset of the transform space. In this paper, we will look more closely at this transform space, and the natural operations which may take place in it.

With max-plus-based methods for steady-state equations, one notes that the solutions of the HJB PDE are fixed points of the associated semi-group, that is

$$W = S_\tau[W] \tag{2}$$

where S_τ is the semi-group with time-step τ. Note that since 0 is the multiplicative identity, we can rewrite (2) as

$$0 \otimes W = S_\tau[W]. \tag{3}$$

In other words, W is an eigenvector for S_τ corresponding to eigenvalue 0. The solution is semiconvex ([24] among other references above). Letting e be the semiconvex transform of W (or, simply a truncation of the transform), one finds that e satisfies

$$e = B \otimes e, \quad \text{or,} \quad 0 \otimes e = B \otimes e \tag{4}$$

where B is the transform of the operator S_τ; when truncating the basis, B becomes a finite-dimensional matrix where $\otimes$ also represents max-plus matrix-vector multiplication.

In this paper, we will be concerned with natural operations on the transformed operator (specifically max-plus addition), and how these may be used as an aid in the solution of HJB PDE's. We note that in application of the max-plus-based numerical method given in the above references, there were two components to the computation: the computation of B, and the computation of e given B. The computational time for the former greatly dominated that of the latter by an order of magnitude. Here, we will develop some techniques for operating on the transform operators themselves which will allow us to construct a complex $\tilde{B}$ from other, more easily computed, B's. This will allow us to escape the "curse-of-dimensionality" *in the dominant portion of the computation*, for this class of HJB PDE's.

2 General Results

In this section, we will give some general results. Since the structure of the systems will be rather general, the assumptions are not readily verifiable. In a later section, we will apply the general results to more specific systems, in which case, one will have more explicit assumptions. In particular, we will consider some finite L_2-gain problems with more specific dynamics.

Throughout, we will work in the space of semiconvex functions, $\mathcal{S}$ (cf. [9], [22], [21], [16], [24], [26] among many more general references). As a reminder, recall that the space of semiconvex functions, $\mathcal{S}$, is defined as the set of $\psi : \mathbf{R}^n \to \mathbf{R}$ such that for any $R < \infty$ there exists $C_R < \infty$ such that $\psi(x) + \frac{C_R}{2}|x|^2$ is convex over $\overline{B_R(0)} = \{x \in \mathbf{R}^n : |x| \leq R\}$. We refer to such a C_R as a semiconvexity constant for ψ over $\overline{B_R(0)}$. Note that any $\psi \in \mathcal{S}$ is automatically locally Lipschitz [8]. For $R > 0$, we denote the space of semiconvex functions for which the semiconvexity constant over $\overline{B_R(0)}$ is C_R as $\mathcal{S}_{R,C_R}$.

Consider a finite set of possible system dynamics indexed by $m \in \mathcal{M} \doteq \{1, 2, \ldots, M\}$

$$\begin{aligned} \dot{X}^m &= F^m(X^m, w) \\ X_0^m &= x \in \mathbf{R}^n \end{aligned} \tag{5}$$

where we note that all the systems have the same initial condition, and $w \in \mathcal{W}$ (for all m) will be a (disturbance) input. $\mathcal{W}$ might be L_2 with range in in $\mathbf{R}^k$ for example. Specific assumptions on F^m will follow, but for now assume simply that the systems are sufficiently well-behaved such that there exist unique solutions for all $w \in \mathcal{W}$ and for all $x \in \mathbf{R}^n$.

Maintaining the high generality level in this section, we denote a corresponding cost functional for each system by $l_\tau^m(x, w.)$ where τ will denote the time horizon. For example, one may have

$$l_\tau^m(x, w) \doteq \int_0^\tau h^m(X_t^m) - \frac{\gamma^2}{2}|w_t|^2 \, dt \tag{6}$$

where X^m satisfies (5).

We will work with operators indexed by τ of the form

$$S_\tau^m[\phi](x) \doteq \sup_{w \in \mathcal{W}} \{l_\tau^m(x, w) + \phi(X_\tau^m)\}(x) \tag{7}$$

where X^m satisfies (5) where the domain of the S_τ^m operators will be defined explicitly for specific problems. Note that S_τ^m is a semigroup in the standard algebra sense (i.e. $S_{\tau_1+\tau_2}^m[\phi] = S_{\tau_1}^m\{S_{\tau_2}^m[\phi]\}$).

One typically finds that when the available storage (cf. [33], [14]), W, exists, it is a fixed point of the operator S_τ^m for any $\tau > 0$ (cf. [14], [21], [24] among many others). Specifically, one typically has that the storage

$$W^m(x) \doteq \sup_{0<\tau<\infty} \sup_{w \in L_2(0,\tau)} l_\tau^m(x, w) = \lim_{\tau \to \infty} \sup_{w \in L_2(0,\tau)} l_\tau^m(x, w) \tag{8}$$

is a fixed point of

$$\phi = S_\tau^m[\phi]. \tag{9}$$

In the H_∞/L_2-gain case under certain assumptions, it is known that the available storage is the unique fixed point in a class of continuous, nonnegative functions satisfying a certain quadratic growth condition, and we denote this class as $\mathcal{C}$ [29], [21], [24]. For the present, let us maintain a high level of generality, and not specify $\mathcal{C}$, but simply assume the existence of such a set (in which the available storage lies and is the unique fixed point); in Section 3, a specific example will be given.

The available storage is also the unique viscosity solution in $\mathcal{C}$ of the corresponding HJB equation

$$0 = H^m(x, \nabla\phi) \tag{10}$$

with boundary condition $\phi(0) = 0$ (i.e. ϕ being zero at the origin). More specifically, in the case of form (6), one would have HJB PDE

$$0 = \sup_{w \in \mathbf{R}^k} \left[F^m(x, w) \cdot \nabla\phi + h^m(x) - \frac{\gamma^2}{2}|w|^2 \right]. \tag{11}$$

In the case $F^m(x, w) = f^m(x) + \sigma^m(x)w$, this has the form

$$\begin{aligned} 0 &= \sup_{w \in \mathbf{R}^k} \left[f^m(x) \cdot \nabla\phi + (\sigma^m(x)w)^T \nabla\phi + h^m(x) - \frac{\gamma^2}{2}|w|^2 \right] \\ &= \sup_{w \in \mathbf{R}^k} \left[f^m(x) \cdot \nabla\phi + h^m(x) + \frac{1}{2\gamma^2} \nabla\phi^T \sigma^m(x)(\sigma^m)^T(x)\nabla\phi \right]. \end{aligned} \tag{12}$$

That W^m is the unique viscosity solution in a certain set $\mathcal{C}$ for this case (12) under certain conditions is given in [29], with a related result under weaker conditions appearing in [32].

Thus, we see that one "typically" expects (for well-defined) integral functional problems, that the fixed point of the operator S_τ^m is identical to the viscosity solution of the HJB PDE (both being the available storage).

We suppose for the remainder of this general section only that for integral functionals l_τ^m with dynamics (5), there exists a unique solution of the corresponding fixed point problem $\phi = S_\tau^m[\phi]$ in class $\mathcal{C}$, and that this is also the unique viscosity solution of $0 = H^m(x, \nabla\phi)$ in $\mathcal{C}$.

One may view the space of semiconvex functions as a "vector space" over the max-plus commutatuve semi-field, and we will henceforth refer to this space as a semi-vector space (where the "semi-" refers to the replacement of a field with a semi-field), and more particularly as a max-plus semi-vector space. In [2] a max-plus semi-vector space is denoted as a moduloid; we use the term max-plus semi-vector space as it seems more intuitive to this community. We now demonstrate a max–plus basis over $\mathcal{S}$. The following theorem is a minor variant of the semiconvex duality result given in [9]. It is derived from convex duality [30], [31] in a straight-forward manner. Refer to [9] for a proof; the only change is the replacement of a constant $c > C_R$ there by a symmetric matrix C such that $C - C_R I > 0$ where I is the (usual algebra) identity matrix. This replacement allows more freedom in the actual numerical implementation.

Theorem 2.1 *Let $\phi \in \mathcal{S}$. Let C be a symmetric matrix such that $C - C_R I > 0$ where $C_R > 0$ is a semiconvexity constant for ϕ, and let L_R be the Lipschitz constant for ϕ over*

$B_R(0)$. *Then for all* $x \in B_R(0)$,

$$\phi(x) = \max_{\widetilde{x} \in B_{L_R/C_R}} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) + a(\widetilde{x}) \Big] = \max_{\widetilde{x} \in \mathbf{R}^n} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) + a(\widetilde{x}) \Big]$$

where

$$a(\widetilde{x}) = -\max_{x \in B_R} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) - \phi(x) \Big].$$

Let $\phi \in \mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over $B_{L_R/C_R}(0)$, and let symmetric $C - C_R I > 0$ where (again) $C_R > 0$ is a semiconvexity constant for ϕ over $B_R(0)$. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x-x_i)^T C (x-x_i)$$

for each i. Then, using Theorem 2.1, one finds (see [9] for details)

$$\phi(x) = \bigoplus_{i=1}^{\infty} [a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{13}$$

where

$$a_i \doteq -\max_{x \in B_R} [\psi_i(x) - \phi(x)]. \tag{14}$$

This is a countable max–plus basis expansion for ϕ. More generally, the set $\{\psi_i\}$ forms a max–plus basis for the space of semiconvex functions over $B_R(0)$ with semiconvexity constant, C_R, and one might denote this space as $\mathcal{S}_{R,C_R}$.

It is often the case that in the case of the integral functional for instance, the available storage (a.k.a. fixed-point of the semi-group, a.k.a. solution of the HJB PDE) is semiconvex. Suppose there exists semiconvexity constant $C_R < \infty$ for the W^m over $B_R(0)$ (i.e. $W^m \in \mathcal{S}_{R,C_R}$ for all m), and a corresponding Lipschitz constant, L_R. Let $C - C_R I > 0$ and $\{x_i\}$ be dense over $B_{L_R/C_R}(0)$, and define the basis $\{\psi_i\}$ as above. Then

$$W^m(x) = \bigoplus_{i=1}^{\infty} [a_i^m \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{15}$$

where

$$a_i^m \doteq -\max_{x \in B_R} [\psi_i(x) - W^m(x)]. \tag{16}$$

Throughout this paper, we will assume that such available storage functions actually have *finite* max-plus expansions, i.e. that

$$W^m(x) = \bigoplus_{i=1}^{N} [a_i^m \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{17}$$

for some $N < \infty$. Although this assumption is completely unrealistic, it will greatly reduce the analysis – which seems acceptable in an introductory paper. Note that we used this same assumption in the early work on the use of max-plus linearity as a basis for a numerical method for solution of HJB equations (cf. [9], [21], [13], [24]), and later proved convergence as the number of terms in basis expansion went to infinity ([22], [20], [26]).

In this paper, we will not be concerned with a numerical method for solution of HJB PDE's based on the max-plus eigenvector problem solution, but rather on the construction of matrices (the B matrices of Section 1) for max-plus eigenvector problems from other matrices whose max-plus eigenvector problems are analytically tractable, and the relationship of the constructed matrices to corresponding HJB PDE's. In a **very** loose way, this is somewhat similar to working in the frequency domain for classical control problems; this analogy provides the inspriation for the direction of work being pursued here (although note that we are working over state space rather than the time dimension). The following theorem makes a critical connection between the problems over the corresponding domains.

Theorem 2.2 *Let S_τ^m be defined by (7) for each m in some finite set $\mathcal{M}$. Suppose that for each $i \in \{1, 2, \ldots, N\}$ and each $m \in \mathcal{M}$, there exists a finite basis expansion of $S_\tau^m[\psi_i]$, i.e. that*

$$S_\tau^m[\psi_i](x) = \bigoplus_{i=1}^{N} B_{j,i}^m \otimes \psi_j(x) \qquad \forall\, x \in \overline{B_R(0)}. \tag{18}$$

Define $\bar{S}_\tau[\phi]$ for any ϕ in the domain (to be specified for specific problems below) by

$$\bar{S}_\tau[\phi](x) = \sup_{w \in \mathcal{W}} \Big\{ \max_{m \in \mathcal{M}} [l_\tau^m(x, w) + \phi(X_\tau^m)] \Big\} \qquad \forall\, x \in \overline{B_R(0)} \tag{19}$$

where X^m satisfies (5). Then

$$\bar{S}_\tau[\psi_i](x) = \bigoplus_{j=1}^{N} \bar{B}_{j,i} \otimes \psi_j(x) \qquad \forall\, x \in \overline{B_R(0)} \tag{20}$$

where

$$\bar{B}_{j,i} = \max_{m \in \mathcal{M}} B_{j,i}^m = \bigoplus_{m \in \mathcal{M}} B_{j,i}^m \qquad \forall\, i, j \in \{1, 2, \ldots, N\}. \tag{21}$$

Remark 2.3 Again, note that the assumption of *finite* basis expansions is unrealistic. Proofs of convergence as the number of basis functions in the finitely-reuncated set goes to infinity (in a reasonable way) appear in [25], [27] and [26]. A full proof in this context is well beyond the scope of this paper, but the above references contain similar proofs.

Proof. The proof is a simple manipulation given by

$$\begin{aligned} \bar{S}_\tau[\psi_i](x) &= \sup_{w \in \mathcal{W}} \Big\{ \max_{m \in \mathcal{M}} [l_\tau^m(x, w) + \psi_i(X_\tau^m)] \Big\} \\ &= \max_{m \in \mathcal{M}} \Big\{ \sup_{w \in \mathcal{W}} [l_\tau^m(x, w) + \psi_i(X_\tau^m)] \Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max_{m\in\mathcal{M}} S_\tau^m[\psi_i](x) \\
&= \max_{m\in\mathcal{M}} \max_{j\in\{1,2,\ldots,N\}} B_{j,i}^m \otimes \psi_j(x) \\
&= \max_{j\in\{1,2,\ldots,N\}} [\max_{m\in\mathcal{M}} B_{j,i}^m] \otimes \psi_j(x) \\
&= \bigoplus_{j=1}^N \bar{B}_{j,i} \otimes \psi_j(x). \;\square
\end{aligned}
$$

Corollary 2.4 *Suppose that the solution of*

$$\overline{W} = \bar{S}_\tau[\overline{W}] \tag{22}$$

exists and also lies in $\mathcal{S}_{R,C_R}$. (Recall from the discussion above that we are already assuming that the W^m are in $\mathcal{S}_{R,C_R}$ and are fixed points of the S_τ^m.) Further, assume that the expansion for $\overline{W}$ is also finite with N coefficients which we denote as

$$\overline{W}(x) = \bigoplus_{j=1}^N \bar{e}_j \otimes \psi_j(x) \qquad \forall\, x \in \overline{B_R(0)}.$$

Also assume that each ψ_j is active in the sense that $\bigoplus_{i\neq j} \bar{e}_j \otimes \psi_j \neq \bigoplus_{j=1}^N \bar{e}_j \otimes \psi_j$ for any $i \leq N$. Then the vector of coefficients, $\bar{e}$ is the solution of the max-plus eigenvector equation

$$\bar{e} = \bar{B} \otimes \bar{e}$$

where $\bar{B}_{j,i} = \bigoplus_{m\in\mathcal{M}} B_{j,i}^m$ for all j, i.

Proof. By assumption, for all $x \in \overline{B_R(0)}$

$$
\begin{aligned}
\bigoplus_{j=1}^N \overline{e}_j \otimes \psi_j(x) = \overline{W}(x) &= \bar{S}_\tau[\overline{W}](x) \\
&= \bar{S}_\tau\left[\bigoplus_{i=1}^N \overline{e}_i \otimes \psi_i\right](x) \\
&= \bigoplus_{i=1}^N \overline{e}_i \otimes \bar{S}_\tau[\psi_i](x)
\end{aligned}
$$

which by Theroem 2.2

$$
\begin{aligned}
&= \bigoplus_{i=1}^N \overline{e}_i \otimes \left[\bigoplus_{j=1}^N \overline{B}_{j,i} \otimes \psi_j(x)\right] \\
&= \bigoplus_{j=1}^N \left[\overline{B}_{j,i} \otimes \overline{e}_i\right] \otimes \psi_j(x).
\end{aligned}
$$

Using the assumption that all the ψ_j are active, this implies that

$$\overline{e}_j = \bigoplus_{j=1}^{N} \overline{B}_{j,i} \otimes \overline{e}_i \qquad \forall\, j,$$

or equivalently,

$$\overline{e} = \overline{B} \otimes \overline{e}.$$

3 Solution of some HJB PDEs

Now suppose that instead of desiring to solve for fixed points of the semigroups, one desires to solve related HJB PDEs. The assumptions in the theorems and corollary of Section 2 are all assumed to hold throughout this section. Let us consider here the HJB PDE

$$\begin{aligned} 0 &= \widetilde{H}(x, \nabla\phi) && (23)\\ &\doteq \max_{m\in\mathcal{M}} \sup_{w\in\mathbf{R}^k} \left[F^m(x,w)\cdot\nabla\phi + h^m(x) - \frac{\gamma^2}{2}|w|^2 \right] && (24)\\ &= \max_{m\in\mathcal{M}} H^m(x,\nabla\phi). && (25) \end{aligned}$$

Consider the sets of measurable processes with values in $\mathcal{M}$ given by

$$\mathcal{M}^p = \{\mu :\ [0,\infty) \to \mathcal{M} \mid \text{measurable}\}$$

and

$$\mathcal{M}^p_\tau = \{\mu :\ [0,\tau) \to \mathcal{M} \mid \text{measurable}\}\,.$$

Also, for the sake of concreteness, let us take $\mathcal{W} \doteq L_2^{loc}([0,\infty),\mathbf{R}^k)$ and $\mathcal{W}_\tau \doteq L_2([0,\tau),\mathbf{R}^k)$. Then by standard dynamic programming results under typical assumptions (cf. [29], [21], [22], [24]), one obtains the following theorem. A specific example of a class of dynamics, cost and set $\mathcal{C}$ is given in the remark just below the theorem statement.

Theorem 3.1 *There exists a unique solution in some class $\mathcal{C}$ of PDE (23), and this viscosity solution is also the unique solution in $\mathcal{C}$ of*

$$\widetilde{W} = \widetilde{S}_\tau[\widetilde{W}] \tag{26}$$

where

$$\widetilde{S}_\tau[\phi](x) \doteq \sup_{\mu\in\mathcal{M}^p_\tau} \sup_{w\in\mathcal{W}_\tau} \left\{ \int_0^\tau h^{\mu_t}(\widetilde{X}_t) - \frac{\gamma^2}{2}|w_t|^2\,dt + \phi(\widetilde{X}_\tau) \right\}, \tag{27}$$

$$\dot{\widetilde{X}} = \tilde{f}(\widetilde{X}, w, \mu) \tag{28}$$

$$\tilde{f}(\widetilde{X}_t, w_t, \mu_t) \doteq f^{\mu_t}(\widetilde{X}_t, w_t). \tag{29}$$

This solution is also given by

$$\widetilde{W}(x) = \sup_{\mu\in\mathcal{M}^p} \sup_{w\in\mathcal{W}} \sup_{T<\infty} \int_0^T h^{\mu_t}(\widetilde{X}_t) - \frac{\gamma^2}{2}|w_t|^2\, dt \tag{30}$$

where $\widetilde{X}$ satisfies (28).

Remark 3.2 In order to ground the analysis, specific assumptions under which the above result holds are as follows. Suppose the dynamics are of the form $\dot{X} = f^m(X) + \sigma^m(X)w$. Assume that for each $m \in \mathcal{M}$, f^m is globally Lipschitz with some constant, K_f, that $(x-y)^T(f^m(x) - f^m(y)) \leq -c|x-y|^2$ for all x, y, and that $f^m(0) = 0$. Assume that for each $m \in \mathcal{M}$, σ^m is globally Lipschitz with some constant, K_σ, and that $|\sigma^m(x)| \leq M$ for all x. Suppose also that for each $m \in \mathcal{M}$, $0 \leq h^m(x) \leq \alpha|x|^2$. Finally, suppose that $(\alpha M^2)/(\gamma^2 c^2) < 1$. Then, the class $\mathcal{C}$ is the set of continuous, nonnegative functions satisfying growth condition $0 \leq \phi(x) \leq [c(\gamma-\delta)^2]/M^2$ for all x for some $\delta > 0$. See [21], [13], [24], [29] for details.

Note that operators $\widetilde{S}_\tau$ do not necessarily form a semigroup, although they do form a sub-semigroup (i.e. $\widetilde{S}_{\tau_1+\tau_2}[\phi](x) \leq \widetilde{S}_{\tau_1}\widetilde{S}_{\tau_2}[\phi](x)$ for all x and all ϕ in the domain). Further, it is easily seen that $S^m_\tau \leq \widetilde{S}_\tau \leq \bar{S}_\tau$ for all m.

Let τ act as a time-discretization step-size and define

$$\mathcal{M}^{p,\tau} = \Big\{\mu : [0,\infty) \to \mathcal{M} \,|\, \text{ for each } n \in \mathbf{N}\cup\{0\}, \text{ there exists } m_n \in \mathcal{M} \text{ such that } \mu(t) = m_n \text{ for } t \in [n\tau, (n+1)\tau)\Big\},$$

and for $T = \bar{n}\tau$ with $\bar{n} \in \mathbf{N}$ define

$$\mathcal{M}^{p,\tau}_T = \Big\{\mu : [0,T) \to \mathcal{M} \,|\, \text{ for each } n \in \{0,1,2,\ldots,\bar{n}-1\}, \text{ there exists } m_n \in \mathcal{M} \text{ such that } \mu(t) = m_n \text{ for } t \in [n\tau, (n+1)\tau)\Big\}.$$

Let $\mathcal{M}^N$ denote the outer product of $\mathcal{M}$ N times. Let $T = \bar{n}\tau$, and define

$$\bar{\bar{S}}^\tau_T[\phi](x) = max_{\{m_k\}_{k=0}^{\bar{n}-1}\in\mathcal{M}^N} \left\{\prod_{k=0}^{\bar{n}-1} S^{m_k}_\tau\right\}[\phi](x).$$

We make the claim in the following theorem without proof. Roughly speaking it simply states that any nealy optimal (worst cse) $w \in \mathcal{M}^p_T$ can be arbitrarily closely approximated (in terms of the cost) by a piecewise constant $w \in \mathcal{M}^{p,T}_\tau$ for some small τ.

Theorem 3.3 *Given $T < \infty$, $R < \infty$ and $\varepsilon > 0$, there exists $N \in \mathbf{N}$ sufficiently large such that letting $\tau = T/N$, one has*

$$\widetilde{S}_T[W^m](x) - \varepsilon \leq \bar{\bar{S}}^\tau_T[W^m](x) \qquad \forall\, x \in \overline{B_R(0)} \tag{31}$$

for all $m \in \mathcal{M}$.

In fact, we believe $\varepsilon \sim O(\tau^2)$, but that is not needed here.

Note that since $W^m, \overline{W} \in \mathcal{C}$, one has [29]

$$\lim_{T\to\infty} \widetilde{S}_T[W^m] = \widetilde{W} \quad \forall m \in \mathcal{M} \text{ and,} \quad \lim_{T\to\infty} \widetilde{S}_T[\overline{W}] = \widetilde{W}. \tag{32}$$

Also, for all $T < 0$,

$$\widetilde{W} = \widetilde{S}_T[\widetilde{W}] = \lim_{T\to\infty} \widetilde{S}_T[\widetilde{W}]. \tag{33}$$

By (32) and (33), given $R < \infty$ and $\varepsilon > 0$, there exists $\widehat{T} < \infty$ such that for all $T \ge \widehat{T}$ and all $m \in \mathcal{M}$,

$$\widetilde{S}_T[\widetilde{W}](x) - \varepsilon \le \widetilde{S}_T[W^m](x) \qquad \forall\, x \in \overline{B_R(0)}. \tag{34}$$

Also note that

$$\overline{W} = \bar{S}_\tau[\overline{W}] = \prod_{k=0}^{n-1} \bar{S}_\tau[\overline{W}] \ge \prod_{k=0}^{n-1} S_\tau^m[\overline{W}]. \tag{35}$$

Since this is true for all n,

$$\overline{W} \ge \lim_{T\to\infty} S_T^m[\overline{W}]$$

and since $\overline{W} \in \mathcal{C}$, one has [29]

$$= W^m \tag{36}$$

for any $m \in \mathcal{M}$. On the other hand,

$$\overline{W} = \prod_{k=0}^{n-1} \bar{S}_\tau[\overline{W}] \le \prod_{k=0}^{n-1} \widetilde{S}_\tau[\overline{W}] = \widetilde{S}_{n\tau}[\overline{W}]$$

which implies (using (32))

$$\overline{W} \le \lim_{T\to\infty} \widetilde{S}_T[\overline{W}] = \widetilde{W}. \tag{37}$$

Combining (36) and (37), one has

$$W^m \le \overline{W} \le \widetilde{W} \qquad \forall\, m \in \mathcal{M}. \tag{38}$$

Also, by definition it is obvious that

$$\bar{\bar{S}}_T^\tau[\phi] \le \widetilde{S}_T[\phi] \qquad \forall\, \phi \in \mathcal{C}. \tag{39}$$

Now, by Theorem 3.3 and (34), given $R < \infty$ and $\varepsilon > 0$, there exist $T < \infty$ and $\bar{n} < \infty$ such that with $\tau = T/\bar{n}$, one has

$$\widetilde{W}(x) - 2\varepsilon = \widetilde{S}_T[\widetilde{W}](x) - 2\varepsilon \le \bar{\bar{S}}_T^\tau[W^m](x)$$

which by (38) and (39)

$$\le \bar{\bar{S}}_T^\tau[\overline{W}](x) \le \widetilde{S}_T[\overline{W}](x) \le \widetilde{S}_T[\widetilde{W}](x) = \widetilde{W}(x) \qquad \forall\, x \in \overline{B_R(0)}. \tag{40}$$

Since $\overline{W}(x) = \bar{S}_\tau[\overline{W}](x) = (\bar{S}_\tau)^{\bar{n}}[\overline{W}](x) = \bar{\bar{S}}_T^\tau[\overline{W}](x)$ on $\overline{B_R(0)}$, (40) implies the following.

Theorem 3.4 *Given $R < \infty$ and $\varepsilon > 0$, there exists $\tau > 0$ such that*

$$\widetilde{W}(x) - 2\varepsilon \leq \overline{W}(x) \leq \widetilde{W}(x) \qquad \forall\, x \in \overline{B_R(0)} \tag{41}$$

where $\widetilde{W}$ and $\overline{W}$ satisfy

$$\widetilde{W} = \widetilde{S}_\tau[\widetilde{W}] \quad \textit{and} \quad \overline{W} = \bar{S}_\tau[\overline{W}].$$

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