Staticization and Associated Hamilton-Jacobi and Riccati Equations*

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Abstract

The use of stationary-action formulations for dynamical systems allows one to generate fundamental solutions for classes of two-point boundary-value problems (TPBVPs). One solves for stationary points of the payoff as a function of inputs, a task which is significantly different from that in optimal control problems. Both a dynamic programming principle (DPP) and a Hamilton-Jacobi partial differential equation (HJ PDE) are obtained for a class of problems subsuming the stationary-action formulation. Although convexity (or concavity) of the payoff may be lost as one propagates forward, stationary points continue to exist, and one must be able to use the DPP and/or HJ PDE to solve forward to such time horizons. In linear/quadratic models, this leads to a requirement for propagation of solutions of differential Riccati equations past finite escape times. Such propagation is also required in (nonlinear) n-body problem formulations where the potential is represented via semiconvex duality.

1 Introduction

The classical approach to solution of energy-conserving dynamical systems is integration of Newton's second law. An alternative viewpoint is that a system evolves along a path that makes the action functional stationary, i.e., such that the first-order differential around the path is the zero element. This latter viewpoint appears particularly useful in some applications in modern physics, including gravitational systems where relativistic effects are non-negligible, and systems in the quantum domain (cf. [6, 7, 8, 13]). Our interests are more pedestrian; the stationary-action formulation has recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems. For sufficiently short time horizons, stationarity of the action typically corresponds to minimization of the action. That is, the stationary point is a global minimum of that action (cf., [4, 10, 11]). For longer time horizons, the stationary point is more typically a saddle.

Our motivating interest is solution of TPBVPs for conservative dynamical systems, specifically including wave equations and n-body problems [4, 10, 11]. By appending a min-plus delta function terminal cost to the action functional, we obtain a fundamental solution object for such TPBVPs. Min-plus convolutions of this object with functionals associated to specific terminal conditions yield the solutions of the specific TPBVPs. As a change in the boundary data only requires convolution with a different functional, our object may best be termed a fundamental solution for TPBVPs, corresponding to the given time horizon.

As noted above, for sufficiently short time horizons, one may obtain the stationary action solution by minimization of the action functional, in which case it is obvious that the fundamental solution is derived from the value function for an optimal control formulation. However, for longer horizons, we must find the stationary point, and this requires a new set of tools. We define stationarity and value for such problems. Surprisingly, for a specific class of terminal costs, one may obtain a dynamic programming principle (DPP) for stationarity, where this is directly analogous to standard DPPs (for optimization). We also formally write the corresponding Hamilton-Jacobi partial differential equation (HJ PDE), and then obtain a verification result.

In the mass-spring case, which appears in Section 3 as a motivating example, the stationary-action problem is linear-quadratic, and the HJ PDE reduces to a differential Riccati equation (DRE). The wave equation [3, 4] also yields a DRE, albeit infinite dimensional. The n-body problem may be reduced to a parameterized set of time-dependent DREs [10, 11]. We see that in all cases, solutions of DREs form a critical building block. Of course, DREs can exhibit finite escape times, and do so in these cases. In classical optimal control, one is not interested in propagation of the solution past such escape times. However, in stationarity problems, these may correspond to points where one loses convexity [concavity] of the payoff. Although the minimum [maximum] may go to $-\infty$ [$+\infty$], the stationary value may be well-defined and finite past such asymptotes, and one must propagate the solution beyond them.

Although stationary action is the motivating prob-

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lem class, the theory developed below is applicable to wider classes of problems, where one is seeking a stationary point. An obvious example is that of certain differential games. Extensions to stochastic cases appear possible as well, but are not considered here. This last extension would be of particular note due to the applicability to solution of the Maslov dequantization of the Schrödinger equation [9], which is the second-order HJ PDE for $\bar{S} \doteq -i\hbar \log \bar{\psi}$ where $\bar{\psi}$ satisfies the Schrödinger equation.

Section 2 contains relevant definitions. Section 3 presents a simple mass-spring TPBVP motivating example. Section 4 contains the main results – the DPP and HJ PDE verification theorem. Section 5 reduces to the linear/quadratic case, and indicates a means for propagation of DREs past escape times.

2 Stationarity definitions

Recall that we are seeking stationary points of payoffs, which is unusual in comparison to the standard classes of problems in optimization. In analogy with the language for minimization and maximization, we will refer to the search for stationary points as staticization, with these points being statica (in analogy with minima/maxima) and a single such point being a staticum (in analogy with minimum/maximum). Prior to the development, we make the following definitions. Suppose $\mathcal{Y}$ is a generic normed vector space with $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$, and suppose $F : \mathcal{G}_\mathcal{Y} \to I\!R$. We say $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ if $\bar{y} \in \mathcal{G}_\mathcal{Y}$ and either

$$\limsup_{y \to \bar{y}, y \in \mathcal{G}_\mathcal{Y} \setminus \{\bar{y}\}} \frac{|F(y) - F(\bar{y})|}{|y - \bar{y}|} = 0, \tag{2.1}$$

or there exists $\delta > 0$ such that $\mathcal{G}_\mathcal{Y} \cap B_\delta(\bar{y}) = \{\bar{y}\}$ (where $B_\delta(\bar{y})$ denotes the ball of radius δ around $\bar{y}$). Further, we define

$$\begin{aligned} \operatorname*{stat}_{y \in \mathcal{G}_\mathcal{Y}} F(y) &\doteq \operatorname{stat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \\ &\doteq \left\{ F(\bar{y}) \,\middle|\, \bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \right\} \end{aligned} \tag{2.2}$$

if $\operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \neq \emptyset$. Throughout, we will abuse notation by writing $\bar{y} = \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ in the event that the argstat is the single point, $\{\bar{y}\}$, and similarly for stat.

In the case where $\mathcal{Y}$ is a Hilbert space, and $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$ is an open set, $F : \mathcal{G}_\mathcal{Y} \to I\!R$ is Fréchet differentiable at $\bar{y} \in \mathcal{G}_\mathcal{Y}$ with Riesz representation $F_y(\bar{y}) \in \mathcal{Y}$ if

$$\lim_{v \to 0,\, \bar{y}+v \in \mathcal{G}_\mathcal{Y} \setminus \{\bar{y}\}} \frac{|F(\bar{y}+v) - F(\bar{y}) - \langle F_y(\bar{y}), v \rangle|}{|v|} = 0. \tag{2.3}$$

The following is immediate from the above definitions.

Lemma 2.1. *Suppose $\mathcal{Y}$ is a Hilbert space, with open set $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$, and $\bar{y} \in \mathcal{G}_\mathcal{Y}$, $\mathcal{G}_\mathcal{Y} \neq \{\bar{y}\}$. Then, $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ if and only if $F_y(\bar{y}) = 0$.*

3 Motivational example

We examine the classic mass-spring example to provide motivation. Although the problem is essentially trivial, it provides a nice means for obtaining a sense of the stationary action principle as a tool for understanding system dynamics and TPBVPs. Further, as remarked above, the stationary action viewpoint is the accepted viewpoint in modern physics (cf., [6, 7, 8, 13]), and as such, it will be ultimately necessary for advanced applications. It also provides exceptional computational advantages for difficult classes of problems, such as TPBVPs in the gravitational n-body case [10, 11].

Remark 3.1. *Although the mass-spring model has an analytically solvable form due to the quadratic potential, this potential is not physically reasonable (the potential approaches $+\infty$ as $|x| \to \infty$), and induces degeneracies. Nonetheless, it is useful for building intuition.*

Consider the mass-spring problem with mass, m, and spring-constant, K (typically given as $\ddot{\xi} = -(K/m)\xi$). The associated stationary action TPBVP payoff, $J^\infty : \hat{T} \times I\!R \times \mathcal{U}_\infty \times I\!R \to I\!R \cup \{\infty\}$ with $\hat{T} \doteq \{(s,t) \in I\!R^2 \,|\, 0 \le s \le t < \infty\}$ and $\mathcal{U}_\infty \doteq \mathcal{L}_2^{loc}(0,\infty)$, is given by

$$\begin{aligned} J^\infty(s,t,x,u,z) &= \int_s^t \frac{m}{2} u^2(r) - \frac{K}{2} \xi^2(r)\, dr \\ &\quad + \psi^\infty(\xi(t), z), \\ \text{where} \quad \dot{\xi}(r) &= u(r), \quad r \in (s,t), \qquad \xi(s) = x, \end{aligned}$$

$$\psi^\infty(x,z) \doteq \begin{cases} 0 & \text{if } x = z, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Solution of this stationary-action problem will yield solution of the TPBVP given by dynamics $m\ddot{\xi}(r) = -\nabla V(\xi(r))$ with initial position $x \in I\!R^n$ and terminal position $z \in I\!R^n$ for the given duration t and given potential function [10, 11], which in this example is given by $V(x) = (K/2)x^2$.

The stationary action solution, u^*, is such that $J^\infty_u(s,t,x,u^*,z) = 0$, where J^∞_u denotes the Fréchet derivative of J^∞ with respect to u as per (2.3). Here, we take $K = m = 1$. In [10, 11], one notes that if $t - s < \pi/2$, then J^∞ is strictly convex in u, and defining $W^\infty(t-s,x,z) \doteq \inf_{u \in \mathcal{U}_\infty} J^\infty(s,t,x,u,z)$, one finds

$$W^\infty(t-s,x,z) = \tfrac{1}{2}\left[P(t)x^2 + 2Q(t)xz + R(t)z^2\right], \tag{3.5}$$

where, formally, $P(s) = R(s) = -Q(s) = +\infty$, $\dot{P} = -1 - P^2$, $\dot{Q} = -PQ$, and $\dot{R} = -Q^2$. Letting

$$\psi^c(x,z) \doteq \frac{c}{2}|x-z|^2 \tag{3.6}$$

for $c \in (0,\infty)$, and noting that $\psi^c \to \psi^\infty$ as $c \to \infty$, one may show that the solution does, in fact, have this form, and further that the solution is given by [10, 11]

$$P(t) = R(t) = \cot(t-s), \quad Q(t) = -1/\sin(t-s), \tag{3.7}$$

which is guaranteed by the aforementioned convexity to be valid on at least $t - s \in (0, \pi/2)$. From here, one may obtain the control and state solving the TPBVP, which are given, in the case $s = 0$ as $u^*(r) = P(t-r)\xi^*(r) + Q(t-r)z = \frac{\cos(t-r)}{\sin(t-r)}\xi^*(r) - \frac{1}{\sin(t-r)}z$, and $\xi^*(r) = x\cos(r) + \frac{z - x\cos(t)}{\sin(t)}\sin(r)$. However, at $t = \pi/2$, one loses convexity of J^∞ in u, and one must seek a staticum rather than a minimum.

One method for extending past $\pi/2$ to the stationary-over-u case is to break the interval into multiple segments of duration less than $\pi/2$, and then concatenate these. Suppose we wish to find the solution to TPBVPs for this mass-spring example with duration $3\pi/4$. As an illustration of this approach, let us break the interval up into two halves, where the payoff is then convex on each half-interval. Suppose we introduce an intermediate point, $\zeta \in I\!R$. Then, the stationary action problem with $s = 0$ is given by

$$\begin{aligned}
W^\infty(3\pi/4, x, z) &\doteq \operatorname*{stat}_{u\in\mathcal{L}_2(0,\frac{3\pi}{4})} J^\infty(0, 3\pi/4, x, u, z)\\
&= \operatorname*{stat}_{\zeta\in I\!R}\Big\{ \operatorname*{stat}_{u\in\mathcal{L}_2(0,\frac{3\pi}{8})} J^\infty(0, 3\pi/8, x, u, \zeta)\\
&\qquad + \operatorname*{stat}_{u\in\mathcal{L}_2(\frac{3\pi}{8},\frac{3\pi}{4})} J^\infty(3\pi/8, 3\pi/4, \zeta, u, z)\Big\}\\
&= \operatorname*{stat}_{\zeta\in I\!R}\Big\{ \operatorname*{stat}_{u\in\mathcal{L}_2(0,\frac{3\pi}{8})} J^\infty(0, 3\pi/8, x, u, \zeta)\\
&\qquad + \operatorname*{stat}_{u\in\mathcal{L}_2(0,\frac{3\pi}{8})} J^\infty(0, 3\pi/8, \zeta, u, z)\Big\},
\end{aligned}$$

where the second equality follows from time-invariance. Next, noting the convexity of each subsegment,

$$\begin{aligned}
W^\infty(3\pi/4, x, z) &= \operatorname*{stat}_{\zeta\in I\!R}\Big\{W^\infty(3\pi/8, x, \zeta)\\
&\qquad + W^\infty(3\pi/8, \zeta, z)\Big\}\\
&= \tfrac{1}{2}\operatorname*{stat}_{\zeta\in I\!R}\Big\{P(3\pi/8)x^2 + 2Q(3\pi/8)x\zeta + R(3\pi/8)\zeta^2\\
&\qquad + P(3\pi/8)\zeta^2 + 2Q(3\pi/8)\zeta z + R(3\pi/8)z^2\Big\}
\end{aligned}$$

and setting the derivative with respect to ζ equal to zero, we find this is

$$\begin{aligned}
&= \tfrac{1}{2}\big\{[P(3\pi/8) - Q^2(3\pi/8)/(P(3\pi/8) + R(3\pi/8))]\\
&\qquad \cdot(x^2 + z^2)\\
&\qquad - 2Q^2(3\pi/8)/(P(3\pi/8) + R(3\pi/8))xz\big\}
\end{aligned}$$

which by (3.7) and trigonometric identities,

$$\begin{aligned}
&= \tfrac{1}{2}\big\{\cot(3\pi/4)(x^2+z^2) - (2/\sin(3\pi/4))xz\big\}\\
&= \tfrac{1}{2}\big\{P(3\pi/4)x^2 + 2Q(3\pi/4)xz + R(3\pi/4)z^2\big\}.
\end{aligned}$$

That is, surprisingly, one finds that the solution is identical to that which one would obtain if one had naïvely propagated the analytical solution of the DRE forward. In fact, introducing additional intermediary points for increasingly long time durations, one continues to find that the solution is identical to that found by naïve application of the above analytical solution (3.5),(3.7) for arbitrarily long periods. We will see that this is not a coincidence, and solution of stationary action problems may be obtained in the linear-quadratic case by proper propagation of the DRE past escape times.

4 Dynamic programming for staticization

In this section, we will obtain a dynamic programming principle as well as a verification result for the appropriate Hamilton-Jacobi PDE.

REMARK 4.1. *For clarity, we provide several definitions. Let $\mathcal{Y}, \mathcal{Z}$ be Hilbert spaces, and let $F : \mathcal{Y} \times \mathcal{Z} \to I\!R$. We say F is strictly uniformly convex on $\mathcal{Y}$ with respect to $z \in \mathcal{Z}$, if there exists $C_F > 0$ such that for all $y, v \in \mathcal{Y}$, $z \in \mathcal{Z}$, $|F(y-v,z) - 2F(y,z) + F(y+v,z)| \geq C_F|v|^2$. We say F is coercive on $\mathcal{Y}$ if given $\bar{R}, M < \infty$, there exists $\hat{R} < \infty$ such that $F(y,z) \geq F(0,z) + M$ for all $|y| \geq \hat{R}$, $|z| \leq \bar{R}$. We say F has bounded second-order differences, if there exists $M_F < \infty$ such that*

$$\begin{aligned}
&|F(y + \delta_1^y + \delta_2^y, z + \delta_1^z + \delta_2^z) - F(y + \delta_1^y, z + \delta_1^z)\\
&-F(y + \delta_2^y, z + \delta_2^z) + F(y,z)| \leq M_F\big[|\delta_1|^2 + |\delta_2|^2\big]
\end{aligned}$$

for all $z, \delta_1^z, \delta_2^z \in \mathcal{Z}$, and all $y, \delta_1^y, \delta_2^y \in \mathcal{Y}$ where $\delta_k \doteq (\delta_k^y, \delta_k^z)$ for $k = 1, 2$.

We assume time-invariant dynamics given as

$$\dot{\xi}(r) = f(\xi(r), u(r)), \qquad \xi(s) = x \in I\!R^n,$$

where $f \in C^2(I\!R^n \times I\!R^m; I\!R^n)$ is globally Lipschitz. For $(s,t) \in \hat{T}$, let $\mathcal{U}_{s,t} \doteq \mathcal{L}_2([s,t); I\!R^m)$. The inclusion of the left endpoint in the domain of elements of $\mathcal{U}_{s,t}$, albeit a set of measure zero, will be helpful in definitions and discussions below. Then, for $x, z \in I\!R^n$ and $u \in \mathcal{U}_{s,t}$, we consider payoff $J : \hat{T} \times I\!R^n \times \mathcal{U}_{s,t} \times I\!R^n \to I\!R$ given by

$$J(s,t,x,u,z) \doteq \int_s^t L(\xi(r), u(r))\,dr + \psi(\xi(t), z), \tag{4.8}$$

where and $L \in C^2(I\!R^n \times I\!R^m)$, $\psi \in C^2(I\!R^n \times I\!R^n)$ with ψ convex and coercive in x. We also allow $\psi = \psi^\infty$ given by (3.4), generalized to domain $I\!R^n$. Define

$$W(t-s,x,z) = \operatorname*{stat}_{u\in\mathcal{U}_{s,t}} J(s,t,x,u,z). \tag{4.9}$$

Conditions guaranteeing the existence of W for specific classes of problems, particularly n-body problems, are given in [10, 11]. (These conditions were *not* satisfied by the mass-spring example.) The next step will be to obtain the DPP for W.

We make the following assumptions.

For any $(s,t) \in \hat{T}$ and $x,z \in I\!R^n$, there exists a unique staticum of $J(s,t,x,\cdot,z)$, $u^* = u^*(s,t,x,z)$, that is, $\{u^*\} = \operatorname{argstat}_{u\in\mathcal{U}_{s,t}} J(s,t,x,u,z)$. $(A.1)$

Given $x,z \in I\!R^n$, there exists $\delta^c > 0$ such that if $0 \le s \le t < s+\delta^c$, then for any $\zeta \in I\!R^n$, $J(s,t,\zeta,\cdot,z)$ is strictly uniformly convex on $\mathcal{U}_{s,t}$ with respect to $\zeta \in I\!R^n$ and coercive on $\mathcal{U}_{s,t}$. $(A.2)$

Remark 4.2. *We note that Assumption $(A.2)$ is motivated by the fact that for the standard stationary-action problems (cf., [10, 11] and the references therein), the action is strictly uniformly convex and coercive over sufficiently short time-horizons. Moreover, the stationary-action model is*

$$\dot{\xi}(r) = u(r), \qquad \xi(s) = x \in I\!R^n,$$

$$\psi(x,z) = \psi^\infty(x,z) \text{ and } L(x,v) = T(v) - V(x),$$

where $T(u(r)) = T(\dot{\xi}(r))$ denotes the kinetic energy at time $r \in [s,t]$, and $V(\xi(r))$ denotes the potential energy.

Let $(s,t) \in \hat{T}$ and $x,z \in I\!R^n$. For $u^0 \in \mathcal{U}_{0,s}$ and $u^1 \in \mathcal{U}_{s,t}$, let

$$\begin{aligned} J_d(s,t,x,u^0,u^1,z) &\doteq \int_0^s L(\xi^0(r),u^0(r))\,dr \\ &+ \int_s^t L(\xi^1(r),u^1(r))\,dr + \psi(\xi^1(t),z), \end{aligned} \tag{4.10}$$

where $\dot{\xi}^0(r) = u^0(r)$ on $(0,s)$ with $\xi^0(0) = x$ and $\dot{\xi}^1(r) = u^1(r)$ on (s,t) with $\xi^1(s) = \xi^0(s)$. Define the concatenation $\mathcal{C} : \mathcal{U}_{0,s} \times \mathcal{U}_{s,t} \to \mathcal{U}_{0,t}$ by

$$[\mathcal{C}(u^0,u^1)](r) \doteq \begin{cases} u^0(r) & \text{if } r \in [0,s), \\ u^1(r) & \text{if } r \in [s,t), \end{cases}$$

where we suppress the detail that these are equivalence classes of functions equal almost everywhere. One immediately sees that, letting $u^0 \in \mathcal{U}_{0,s}$, $u^1 \in \mathcal{U}_{s,t}$ and $\bar{u} \doteq \mathcal{C}(u^0,u^1)$, one has $\bar{u} \in \mathcal{U}_{0,t}$ and

$$J(0,t,x,\bar{u},z) = J_d(s,t,x,u^0,u^1,z). \tag{4.11}$$

Analogously, letting $\bar{u} \in \mathcal{U}_{0,t}$, $u^0(r) = \bar{u}(r)$ for $r \in [0,s)$ and $u^1(r) = \bar{u}(r)$ for $r \in [s,t)$, one again has (4.11). We take the norm on $\mathcal{U}_{0,s} \times \mathcal{U}_{s,t}$ to be $|(u^0,u^1)| = \{|u^0|^2 + |u^1|^2\}^{1/2}$, where the norms on the right-hand side are the respective $\mathcal{L}_2$ norms on $[0,s]$ and $[s,t]$. The following is easily obtained from the above.

Lemma 4.1. *For $(s,t) \in \hat{T}$, $\mathcal{C} : \mathcal{U}_{0,s} \times \mathcal{U}_{s,t} \to \mathcal{U}_{0,t}$ is a linear bijection. Further, the induced operator norm satisfies $|\mathcal{C}| = |\mathcal{C}^{-1}| = 1$.*

Lemma 4.2. *Let $(s,t) \in \hat{T}$ and $x,z \in I\!R^n$. Then,*

$$\begin{aligned} W(t,x,z) &= \operatorname*{stat}_{\bar{u}\in\mathcal{U}_{0,t}} J(0,t,x,\bar{u},z) \\ &= \operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} J_d(s,t,x,u^0,u^1,z). \end{aligned}$$

Proof. The left-hand equality is simply (4.9) with $s = 0$, where existence and uniqueness are guaranteed by Assumption $(A.1)$. In particular,

$$W(t,x,z) = J(0,t,x,\bar{u}^*,z), \tag{4.12}$$

where $\bar{u}^* = \operatorname{argstat}_{\bar{u}\in\mathcal{U}_{0,t}} J(0,t,x,\bar{u},z)$. Let $(s,t) \in \hat{T}$ and $x,z \in I\!R^n$. By (2.1),

$$0 = \lim_{\bar{u}\to\bar{u}^*} \frac{|J(0,t,x,\bar{u},z) - J(0,t,x,\bar{u}^*,z)|}{|\bar{u}-\bar{u}^*|}. \tag{4.13}$$

For any $\bar{u} \in \mathcal{U}_{0,t}$, let $(u^0,u^1) = \mathcal{C}^{-1}(\bar{u})$, and in particular, let $(u^{0,*},u^{1,*}) = \mathcal{C}^{-1}(\bar{u}^*)$. By (4.11), (4.13) and Lemma 4.1,

$$0 = \lim_{(u^0,u^1)\to(u^{0,*},u^{1,*})} \frac{|J_d(s,t,x,u^0,u^1,z) - J_d(s,t,x,u^{0,*},u^{1,*},z)|}{|(u^0,u^1)-(u^{0,*},u^{1,*})|},$$

which implies

$$(u^{0,*},u^{1,*}) \in \operatorname*{argstat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} J_d(s,t,x,u^0,u^1,z). \tag{4.14}$$

Similarly, $(\hat{u}^0,\hat{u}^1) \in \operatorname{argstat} J_d(s,t,x,u^0,u^1,z)$ implies that $\mathcal{C}(\hat{u}^0,\hat{u}^1) \in \operatorname{argstat}_{\bar{u}\in\mathcal{U}_{0,t}} J(0,t,x,\bar{u},z) = \bar{u}^*$. Consequently, $(u^{0,*},u^{1,*})$ is the unique staticizer, and

$$J_d(s,t,x,u^{0,*},u^{1,*},z) = \operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} J_d(s,t,x,u^0,u^1,z).$$

Also, by (4.11), $J_d(s,t,x,u^{0,*},u^{1,*},z) = J(0,t,x,\bar{u}^*,z)$. Combining these last two equalities with (4.12) completes the proof. $\square$

The DPP is obtained as follows:

THEOREM 4.1. *Let $(s,t) \in \hat{T}$, $t-s < \delta^c$ (where δ^c is given in Assumption (A.2)) and $x,z \in I\!R^n$. Suppose that for any $\zeta \in I\!R^n$, $J(0,t-s,\zeta,\cdot,z) \in C^2(\mathcal{U}_{0,t-s})$. Suppose $J(0,t,x,\mathcal{C}(\cdot,\cdot),z) \in C^2(\mathcal{U}_{0,s}\times\mathcal{U}_{s,t})$, and that it has bounded second-order differences. Then,*

$$W(t,x,z) = \operatorname*{stat}_{u^0\in\mathcal{U}_{0,s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + W(t-s,\xi^0(s),z)\right\},$$

where $\dot{\xi}^0(r) = f(\xi^0(r),u^0(r))$ for $r\in(0,s)$, with $\xi^0(0) = x$.

Proof. Let $(s,t)\in\hat{T}$ and $x,z\in I\!R^n$. By Lemma 4.2,

$$W(t,x,z) = \operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} J_d(s,t,x,u^0,u^1,z),$$

which by (4.10),

$$= \operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + \int_s^t L(\xi^1(r),u^1(r))\,dr + \psi(\xi^1(t),z)\right\}$$

$$= \operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{s,t}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + J(s,t,\xi^0(s),u^1,z)\right\}$$

$$= \operatorname*{stat}_{(u^0,\hat{u}^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{0,t-s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + J(0,t-s,\xi^0(s),\hat{u}^1,z)\right\}. \tag{4.15}$$

With some technical work regarding nested statica and minima (where the details are not included for reasons of space), one can show

$$\operatorname*{stat}_{(u^0,\hat{u}^1)\in\mathcal{U}_{0,s}\times\mathcal{U}_{0,t-s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + J(0,t-s,\xi^0(s),\hat{u}^1,z)\right\}$$

$$= \operatorname*{stat}_{u^0\in\mathcal{U}_{0,s}} \min_{\hat{u}^1\in\mathcal{U}_{0,t-s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + J(0,t-s,\xi^0(s),\hat{u}^1,z)\right\}$$

$$= \operatorname*{stat}_{u^0\in\mathcal{U}_{0,s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + \min_{\hat{u}^1\in\mathcal{U}_{0,t-s}} J(0,t-s,\xi^0(s),\hat{u}^1,z)\right\},$$

which by the assumed differentiability of $J(0,t-s,\xi^0(s),\cdot,z)$ and the definition of W,

$$= \operatorname*{stat}_{u^0\in\mathcal{U}_{0,s}} \left\{ \int_0^s L(\xi^0(r),u^0(r))\,dr + W(t-s,\xi^0(s),z)\right\}.$$

Combining this with (4.15) yields the desired result. □

In addition to the above DPP, we now obtain a verification theorem for the associated HJ PDE. The HJ PDE problem, for each $z\in I\!R^n$, is

$$0 = \operatorname*{stat}_{v\in I\!R^m} \left\{ L(x,v) - W_r(r,x,z) + W_x(r,x,z)\cdot f(x,v)\right\}, \quad (r,x)\in(0,t)\times I\!R^n, \tag{4.16}$$

$$W(0,x,z) = \psi(x,z), \qquad x\in I\!R^n. \tag{4.17}$$

THEOREM 4.2. *Let $t \in (0,\infty)$. Suppose $\overline{W} \in C^3((0,t)\times I\!R^{2n})\cap C([0,t]\times I\!R^{2n})$ satisfies (4.16), (4.17). Suppose there exist $K_L, K_f < \infty$ such that*

$$|L_{xx}(x,v)|, |L_{xv}(x,v)|, |L_{vv}(x,v)| \le K_L,$$
$$|f_x(x,v)|, |f_v(x,v)| \le K_f,$$
$$|f_{xx}(x,v)|, |f_{xv}(x,v)|, |f_{vv}(x,v)| \le K_f,$$

for all $(x,v)\in I\!R^n\times I\!R^m$. Suppose there exists $\tilde{u}:[0,t]\times I\!R^n \to I\!R^n$ satisfying $\tilde{u}(r,x) \in \operatorname{argstat}_{v\in I\!R^m}[\overline{W}_x(t-r,x,z)\cdot f(x,v) + L(x,v)]$, such that $\tilde{u}$ is bounded on bounded sets, and such that there exists $\tilde{K}_f < \infty$ such that $|f(x,\tilde{u}(r,x)) - f(y,\tilde{u}(r,y))| \le \tilde{K}_f|x-y|$ for all $x,y\in I\!R^n$ and $r\in[0,t]$. Then, $\overline{W}(t,x,z) = W(t,x,z) = \operatorname{stat}_{u\in\mathcal{U}_{0,t}} J(0,t,x,u,z)$.

Proof. Due to space limitations, only a partial proof is included. Let $t,x,z,\overline{W},\tilde{u}$ be as assumed. By standard results, there exists unique absolutely continuous $\tilde{\xi}:[0,t]\to I\!R^n$ satisfying $\dot{\tilde{\xi}}(r) = f(\tilde{\xi}(r),\tilde{u}(r,\tilde{\xi}(r)))$, $\tilde{\xi}(0) = x$ such that $\tilde{u}(\cdot,\tilde{\xi}(\cdot))\in\mathcal{U}_{0,t}$. By the Fundamental Theorem of Calculus,

$$\overline{W}(t,x,z) = \overline{W}(0,\tilde{\xi}(t),z) + \int_0^t \overline{W}_r(r,\tilde{\xi}(t-r),z) - \overline{W}_x(r,\tilde{\xi}(t-r),z)\cdot f(\tilde{u}(t-r,\tilde{\xi}(t-r)))\,dr,$$

which by (4.16),(4.17) and the choice of $\tilde{u}$,

$$= \psi(\tilde{\xi}(t),z) + \int_0^t L(\tilde{\xi}(t-r),\tilde{u}(t-r,\tilde{\xi}(t-r)))\,dr$$

$$= \int_0^t L(\tilde{\xi}(r),\tilde{u}(r,\tilde{\xi}(r)))\,dr + \psi(\tilde{\xi}(t),z). \tag{4.18}$$

Let $u^*(r) \doteq \tilde{u}(r,\tilde{\xi}(r))$, $\xi^*(r) = \tilde{\xi}(r)$ for all $r\in[0,t]$. Then, (4.18) may be rewritten as

$$\overline{W}(t,x,z) = \int_0^t L(\xi^*(r),u^*(r))\,dr + \psi(\xi^*(t),z)$$

$$= J(0,t,x,u^*,z). \tag{4.19}$$

Now we must demonstrate that u^* is the *argstat* of $J(0,t,x,\cdot,z)$. Let $\hat{u} \in \mathcal{U}_{0,t}$, $\delta \doteq \hat{u} - u^* \in \mathcal{L}_2((0,t);I\!R^n)$ with $|\delta| \le 1$, and let $\hat{\xi}$ be the corresponding trajectory (i.e., $\hat{\xi}(r) = x + \int_0^r f(\hat{\xi}(r),\hat{u}(r))\,dr$). By (4.19) and the Fundamental Theorem of Calculus again,

$$
\begin{aligned}
&J(0,t,x,u^*,z) = \overline{W}(t,x,z)\\
&= \overline{W}(0,\hat{\xi}(t),z) + [\overline{W}(t,x,z) - \overline{W}(0,\hat{\xi}(t),z)]\\
&= \overline{W}(0,\hat{\xi}(t),z) + \int_0^t \overline{W}_r(r,\hat{\xi}(t-r),z)\\
&\qquad -\overline{W}_x(r,\hat{\xi}(t-r),z)\cdot f(\hat{\xi}(t-r),\hat{u}(t-r))\,dr,
\end{aligned}
$$

which by (4.16),(4.17) (implying that the last term in brackets below is zero),

$$
\begin{aligned}
&= \psi(\hat{\xi}(t),z) + \int_0^t L(\hat{\xi}(t-r),\hat{u}(t-r))\,dr\\
&+ \int_0^t \big[-L(\hat{\xi}(t-r),\hat{u}(t-r)) + \overline{W}_r(r,\hat{\xi}(t-r),z)\\
&\qquad -\overline{W}_x(r,\hat{\xi}(t-r),z)\cdot f(\hat{\xi}(t-r),\hat{u}(t-r))\big]\\
&-\big[-L(\xi^*(t-r),u^*(t-r)) + \overline{W}_r(r,\xi^*(t-r),z)\\
&\qquad -\overline{W}_x(r,\xi^*(t-r),z)\cdot f(\xi^*(t-r),u^*(t-r))\big]\,dr,
\end{aligned}
$$

and noting that the first and second terms comprise the payoff for $\hat{u}$,

$$
\begin{aligned}
&= J(0,t,x,\hat{u},z)\\
&+ \int_0^t \big[-L(\hat{\xi}(t-r),\hat{u}(t-r)) + \overline{W}_r(r,\hat{\xi}(t-r),z)\\
&\qquad -\overline{W}_x(r,\hat{\xi}(t-r),z)\cdot f(\hat{\xi}(t-r),\hat{u}(t-r))\big]\\
&-\big[-L(\xi^*(t-r),u^*(t-r)) + \overline{W}_r(r,\xi^*(t-r),z)\\
&\qquad -\overline{W}_x(r,\xi^*(t-r),z)\cdot f(\xi^*(t-r),u^*(t-r))\big]\,dr.
\end{aligned}
$$

That is,

$$
\begin{aligned}
(4.20)\quad &|J(0,t,x,\hat{u},z) - J(0,t,x,u^*,z)|\\
&\le \int_0^t \Big| \big[-L(\hat{\xi}(t-r),\hat{u}(t-r)) + \overline{W}_r(r,\hat{\xi}(t-r),z)\\
&\quad -\overline{W}_x(r,\hat{\xi}(t-r),z)\cdot f(\hat{\xi}(t-r),\hat{u}(t-r))\big]\\
&\quad -\big[-L(\xi^*(t-r),u^*(t-r)) + \overline{W}_r(r,\xi^*(t-r),z)\\
&\quad -\overline{W}_x(r,\xi^*(t-r,z))\cdot f(\xi^*(t-r),u^*(t-r))\big]\Big|\,dr.
\end{aligned}
$$

We must show that this is bounded by $\bar{C}|\delta|^2$ for some $\bar{C} = \bar{C}(t,x,z) < \infty$. This requires a significant technical argument; the details are not included. Recalling (2.1), we see that this implies that u^* is the argstat of $J(0,t,x,\cdot,z)$. □

5 Linear-quadratic example

We consider the linear-quadratic problem given by

$$
\begin{aligned}
(5.21)\quad & L(x,v) = \tfrac{1}{2}v'Dv - \tfrac{1}{2}x'Bx, \quad f(x,v) = v,\\
(5.22)\quad & \psi(x,z) = \psi^c(x,z),
\end{aligned}
$$

for all $x,v,z \in I\!R^n$, where $D \succ dI$ (where we write $A \succ B$ if $A-B$ is positive definite), $d > 0$, B symmetric, and $c \in (0,\infty)$. We look for $\overline{W}$ of the form

$$
(5.23)\quad \overline{W}(t,x,z) = \tfrac{1}{2}\big[x'P(t)x + 2x'Q(t)z + z'R(t)z\big].
$$

With the above quadratic cost and given dynamics, the HJ PDE (4.16) takes the form

$$
\begin{aligned}
0 &= \operatorname*{stat}_{v\in I\!R^n} \big[\tfrac{1}{2}v'Dv - \tfrac{1}{2}x'Bx - \overline{W}_r(r,x,z)\\
&\qquad + \overline{W}_x(r,x,z)\cdot v\big]\\
&= \min_{v\in I\!R^n} \big[\tfrac{1}{2}v'Dv - \tfrac{1}{2}x'Bx - \overline{W}_r(r,x,z)\\
(5.24)\quad &\qquad + \overline{W}_x(r,x,z)\cdot v\big]\\
&= -\tfrac{1}{2}x'Bx - \overline{W}_r(r,x,z)\\
(5.25)\quad &\qquad -\tfrac{1}{2}\overline{W}_x'(r,x,z)D^{-1}\overline{W}_x(r,x,z)\big].
\end{aligned}
$$

REMARK 5.1. *That (5.24) is a minimum, in spite of the fact that, for sufficiently long duration problems, the value is obtained as a staticum, may appear at first glance to be contradictory. The consistent minimum in the HJ PDE is due to the infinitesimal limit implicit there. If one examines the DPP (Theorem 4.1), we see that there are two terms inside the outer staticization, where for sufficiently short durations, $t-s$, the payoff underlying the second term is convex with respect to the input in $\mathcal{U}_{s,t}$. It is also worth noting that at the outset of the proof of the DPP, one is already working with a value, W, which is defined as a staticum rather than a minimum.*

Substituting form (5.23) in (5.25), one obtains

$$
\begin{aligned}
0 = &-\tfrac{1}{2}x'Bx - \tfrac{1}{2}\big[x'\dot{P}(t)x + 2x'\dot{Q}(t)z + z'\dot{R}(t)z\big]\\
&-\tfrac{1}{2}(P(t)x + Q(t)z)'D^{-1}(P(t)x + Q(t)z).
\end{aligned}
$$

Equating like terms yields

$$
\begin{aligned}
(5.26)\quad & \dot{P}(t) = -B - P(t)D^{-1}P(t),\\
(5.27)\quad & \dot{Q}(t) = -P(t)D^{-1}Q(t),\\
(5.28)\quad & \dot{R}(t) = -Q'(t)D^{-1}Q(t),
\end{aligned}
$$

and the initial condition (i.e., (5.22), (4.17) and (3.6)) imply

$$
(5.29)\quad P(0) = R(0) = cI = -Q(0).
$$

Note that if P,Q,R are well-defined on $(0,t)$, then the optimal control is given by $u^*(r) = P(t-r)x + Q(t-r)z$ for $r \in (0,t)$, and one has $u^* \in \mathcal{U}_{0,t}$. Further, one can verify that the assumptions of verification Theorem 4.2 are valid on this interval. Consequently, $\overline{W}$ is the value function on this interval.

We are now faced with the prospect that P, Q and R may exhibit finite escape times, while there may exist an argstat remaining finite (a.e.) indefinitely. In fact, one finds that $\overline{W}$ continues to have form (5.23) past such escape times, and one must obtain the means to correctly propagate the solution of (5.26)–(5.28) past such times. The means for this was indicated in the specific mass-spring example above, and we now proceed to obtain this in a more general context. There are two ways to proceed past such escape times: via the staticization DPP and through what will be termed static-duality. These two approaches are discussed below.

5.1 DPP-based propagation Suppose we have successfully propagated forward to time $s \in (0,\infty)$, and wish to propagate to $t > s$, where $t - s < \delta^c$ (defined in $(A.2)$), and one might have an escape in (s,t). We will use the staticization DPP (i.e., Theorem 4.1) to propagate from s up to t. For $y \in I\!\!R^n$, define

$$\mathcal{U}^y_{0,s} \doteq \Big\{ u \in \mathcal{U}_{0,s} \,\Big|\, \int_0^s u(r)\,dr = y \Big\}. \tag{5.30}$$

We need to verify that the conditions of Theorem 4.1 hold for sufficiently small $\delta^c > 0$ (see [10, 11] for similar computations). We have:

Lemma 5.1. *Suppose $\delta^c < \sqrt{2d/\max\{1,\hat\lambda\}}$ where $\hat\lambda$ is the maximal eigenvalue of B in (5.21). Let $\tau \in (0,\delta^c)$ and $\zeta, z \in I\!\!R^n$. Let J be given by (4.8), where L, f are given by (5.21). $J(0,\tau,\zeta,\cdot,z)$ is a convex quadratic function on $\mathcal{U}_{0,\tau}$.*

Proof. We prove only the convexity. Let $u, \delta \in \mathcal{U}_{0,\tau}$, $u^+ \doteq u + \delta$, $u^- \doteq u - \delta$; $\xi(0) = \xi^+(0) = \xi^-(0) = \zeta$; $\dot\xi(r) = u(r)$, $\dot\xi^+(r) = u^+(r)$, $\dot\xi^-(r) = u^-(r)$ on $(0,\tau)$. One easily sees that

$$\begin{aligned}
&J(0,\tau,\zeta,u^+,z) - 2J(0,\tau,\zeta,u,z) + J(0,\tau,\zeta,u^-,z)\\
&= \tfrac12 \int_0^\tau \delta(r)'D\delta(r) - \big(\textstyle\int_0^r \delta(\rho)\,d\rho\big)'B\big(\int_0^r \delta(\rho)\,d\rho\big)\,dr\\
&\quad + \frac{c}{2}\Big|\int_0^\tau \delta(r)\,dr\Big|^2\\
&\ge \frac{d}{2}|\delta|^2_{\mathcal{L}_2(0,\tau)} - \frac{\hat\lambda}{2}\int_0^\tau \big|\textstyle\int_0^r \delta(\rho)\,d\rho\big|^2\,dr\\
&\ge \frac{d}{2}|\delta|^2_{\mathcal{L}_2(0,\tau)} - \frac{\hat\lambda}{2}\int_0^\tau r\int_0^r |\delta(\rho)|^2\,d\rho\,dr\\
&\ge \Big[\frac{d}{2} - \frac{\hat\lambda\tau^2}{4}\Big]|\delta|^2_{\mathcal{L}_2(0,\tau)}. \quad \square
\end{aligned}$$

The assumptions of bounded second-order differences and C^2 behavior are not difficult to verify, and we do not include verifications of these. Uniqueness assumption $(A.1)$ does not always hold for purely quadratic problems, and we simply assume it here. (One may note the nonuniqueness of trajectories for the scalar mass-spring system when $x = z = 0$ and the duration is a half-period, as an example.) Applying Theorem 4.1, and recalling notation (5.30),

$$\begin{aligned}
W(t,x,z) &= \operatorname*{stat}_{u^0\in\mathcal{U}_{0,s}} \Big\{ \int_0^s L(\xi^0(r),u^0(r))\,dr\\
&\qquad\qquad + W(t-s,\xi^0(s),z)\Big\}\\
&= \operatorname*{stat}_{\zeta\in I\!\!R^n}\ \operatorname*{stat}_{u^0\in\mathcal{U}^{\zeta-x}_{0,s}} \Big\{ \int_0^s L(\xi^0(r),u^0(r))\,dr\\
&\qquad\qquad + W(t-s,\zeta,z)\Big\}\\
&= \operatorname*{stat}_{\zeta\in I\!\!R^n}\Big\{ \operatorname*{stat}_{u^0\in\mathcal{U}^{\zeta-x}_{0,s}} \Big[\int_0^s L(\xi^0(r),u^0(r))\,dr\\
&\qquad\qquad + \psi^\infty(\xi^0(s),\zeta)\Big] + W(t-s,\zeta,z)\Big\}\\
&= \operatorname*{stat}_{\zeta\in I\!\!R^n}\Big\{ W(s,x,\zeta) + W(t-s,\zeta,z)\Big\}.
\end{aligned} \tag{5.31}$$

As we have already propagated forward to s, we have

$$W(s,x,\zeta) = \tfrac12\big[x'P(s)x + 2x'Q(s)\zeta + \zeta'R(s)\zeta\big]. \tag{5.32}$$

Further, as $t - s < \delta^c$, we have strict convexity on the $t-s$ duration segment, and consequently we have

$$\begin{aligned}
|W(t-s,\zeta,z) = \tfrac12\big[&\zeta'P(t-s)\zeta + 2\zeta'Q(t-s)z\\
&+ z'R(t-s)z\big].
\end{aligned} \tag{5.33}$$

Combining (5.31)–(5.33), one finds staticizing point $\zeta^* = -[R(s) + P(t-s)]^{-1}[Q'(s)x + Q(t-s)z]$, and consequently,

$$W(t,x,z) = \tfrac12\big[x'P(t)x + x'Q(t)z + z'Q(t)x + z'R(t)z\big], \tag{5.34}$$

where $P(t), Q(t), R(t)$ are given by

$$P(t) = P(s) - Q(s)[R(s) + P(t-s)]^{-1}Q'(s), \tag{5.35}$$

$$Q(t) = Q(s)[R(s) + P(t-s)]^{-1}Q(t-s), \tag{5.36}$$

$$\begin{aligned}
R(t) = R(t-s)&\\
&- Q'(t-s)[R(s) + P(t-s)]^{-1}Q(t-s).
\end{aligned} \tag{5.37}$$

That is, (5.35)–(5.37) allows us to propagate past the finite escape time occurring in interval (s,t). We remark that forward propagation by repeated application of updates similar to (5.35)–(5.37) for general DREs is discussed in [12] for the finite-dimensional case, and in [4, 5] for the infinite-dimensional case.

5.2 Static duality Another means for propagation past escape times is through what will be termed static-duality and stat-quad duality. At points in time where the DRE (5.26)–(5.28) solutions escape, the stat-quad dual DRE solutions are well-behaved and vice-versa. Associated to solutions of the DREs at each moment in time is the above quadratic functional, $W(t,x,z) = \frac{1}{2}[x'P(t)x + 2x'Q(t)z + z'R(t)z]$, and the coefficients of the stat-quad dual of W over x form the solution to the stat-quad dual DREs. Here, we briefly describe static duality, and indicate the stat-quad dual DREs.

THEOREM 5.1. *Let $\mathcal{A}, \mathcal{B} \subseteq I\!R^n$ be open. Suppose $\phi \in C^1(\mathcal{A}; I\!R)$ and $[\nabla\phi]^{-1} \in C^1(\mathcal{B}; \mathcal{A})$. Then,*

$$\phi(u) = \operatorname*{stat}_{v\in\mathcal{B}}[a(v) + v'u] \quad \forall\, u \in \mathcal{A},$$

where

$$a(v) = \operatorname*{stat}_{u\in\mathcal{A}}[\phi(u) - v'u] \quad \forall\, v \in \mathcal{B}.$$

We will refer to a as the static dual of ϕ. As an example, let $\mathcal{A} = \mathcal{B} = I\!R^n \setminus \{0\}$ and $\phi(u) = 1/|u|$, where $|u|$ denotes the Euclidean norm of u. Then, $\nabla\phi = -u/|u|^3$, $[\nabla\phi]^{-1}(v) = -v/(|v|^{3/2})$ and $a(v) = 2|v|^{1/2}$.

Let $S_n^{\neq 0}$ be the space of real, symmetric, nonsingular $n \times n$ matrices. In analogy to the semiconvex [semiconcave] generalization of convex [concave] duality, static duality may be generalized to stat-quad duality. In particular, we have the following.

THEOREM 5.2. *Let $\mathcal{A}, \hat{\mathcal{B}} \subseteq I\!R^n$ be open. Let $C \in S_n^{\neq 0}$. Suppose $\phi \in C^1(\mathcal{A}; I\!R)$. Letting $\eta(u) \doteq \nabla\phi(u) - Cu$ for all $u \in \mathcal{A}$, suppose $\eta^{-1} \in C^1(\hat{\mathcal{B}}; \mathcal{A})$. Then,*

$$\phi(u) = \operatorname*{stat}_{v\in\mathcal{B}} \left[a(v) + \tfrac{1}{2}(v-u)'C(v-u)\right] \quad \forall\, u \in \mathcal{A},$$

where

$$a(v) = \operatorname*{stat}_{u\in\mathcal{A}} \left[\phi(u) - \tfrac{1}{2}(v-u)'C(v-u)\right] \quad \forall\, v \in \mathcal{B},$$

where $\mathcal{B} \doteq \{v \in I\!R^n \mid -Cv \in \hat{\mathcal{B}}\}$.

We will refer to a as the stat-quad dual of ϕ (with respect to duality matrix, C). As an example, let $P, C \in S_n^{\neq 0}$, and suppose $P - C$ is nonsingular. Let $\mathcal{A} = \mathcal{B} = I\!R^n$, and let $\phi \in C^1(I\!R^n)$ be given by $\phi(u) = \frac{1}{2}u'Pu$. Then, ϕ has the stat-dual (with respect to C) given by

$$a(v) = \tfrac{1}{2}v'C(C-P)^{-1}Pv = \tfrac{1}{2}v'P(C-P)^{-1}Cv. \tag{5.38}$$

With some work, one can show that the DREs satisfied by the stat-quad dual coefficients corresponding to P, Q, R (satisfying (5.26)–(5.28)) with respect to duality matrix C, say α, β, γ, are given by

$$\begin{aligned}\dot{\alpha} &= -\alpha(D^{-1} + C^{-1}BC^{-1})\alpha - \alpha C^{-1}B - BC^{-1}\alpha \\ &\quad - B,\\ \dot{\beta} &= -\alpha(D^{-1} + C^{-1}BC^{-1})\beta - BC^{-1}\beta,\\ \dot{\gamma} &= -\beta'(D^{-1} + C^{-1}BC^{-1})\beta,\end{aligned}$$

where, at any time t such that both are finite, one can transform between the original and stat-quad dual coefficients through

$$\begin{aligned}\alpha(t) &= -C - C(P(t)-C)^{-1}C = -C(P(t)-C)^{-1}P(t),\\ \beta(t) &= -C(P(t)-C)^{-1}Q(t),\\ \gamma(t) &= R(t) - Q'(t)(P(t)-C)^{-1}Q(t).\end{aligned}$$

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