Static Duality and a Stationary-Action Application *

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Abstract

Conservative dynamical systems propagate as stationary points of the action functional. Using this representation, it has previously been demonstrated that one may obtain fundamental solutions for two-point boundary value problems for some classes of conservative systems via solution of an associated dynamic program. Further, such a fundamental solution may be represented as a set of solutions of differential Riccati equations (DREs), where the solutions may need to be propagated past escape times. Notions of "static duality" and "stat-quad duality" are developed, where the relationship between the two is loosely analogous to that between convex and semiconvex duality. Static duality is useful for smooth functionals where one may not be guaranteed of convexity or concavity. Some simple properties of this duality are examined, particularly commutativity. Application to stationary action is considered, which leads to propagation of DREs past escape times via propagation of stat-quad dual DREs.

Key words. dynamic programming, stationary action, convex duality, semiconvexity, staticization, two-point boundary value problem, optimal control.

1 Introduction

The classical approach to solution of energy-conserving dynamical systems is integration of Newton's second law. An alternative postulate is that a system evolves along a path that makes the action functional stationary, i.e., such that the first-order differential around the path is the zero element. This latter viewpoint appears particularly useful in some applications in modern physics, including systems where relativistic effects are non-negligible and systems in the quantum domain (cf. [7, 8, 10, 23, 26]). The stationary-action formulation has also recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems (cf. [3, 4, 17, 18, 20]).

To give a sense of this latter application domain, consider a finite-dimensional action-functional formulation of such a TPBVP. Let the path of the conservative system be denoted by ξ_r for $r \in [0, t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(\bar{x}, t, u) \doteq \int_0^t T(u_r) - V(\xi_r)\, dr + \psi(\xi_t), \tag{1}$$

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where $\dot{\xi} = u$, $u \in \mathcal{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2} v^T \mathcal{M} v$ throughout, with $\mathcal{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\psi(x) \doteq -\bar{v}^T \mathcal{M} x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes ψ to be a min-plus delta function centered at z (see Section 5.1), then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$, cf. [4]. In the early work of Hamilton, it was formulated as the least-action principle [11], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases, cf. [10] and the references therein. In such short-duration cases, optimization methods and semiconvex duality are quite useful [3, 4, 20, 21]. However, in order to extend to longer-duration problems, it becomes necessary to apply concepts of stationarity [17, 19].

It is worth noting that if one defines $\operatorname{stat}_{x \in \mathcal{X}} \phi(x)$ to be the critical value of ϕ (defined rigorously in Section 2), then a gravitational potential given as $V(x) = -\mu/|x|$ for $x \neq 0$ and constant $\mu > 0$, has the representation $V(x) = -(\frac{3}{2})^{3/2} \mu \operatorname{stat}_{\alpha > 0}\{\alpha[1 - \frac{\alpha^2 |x|^2}{2}]\}$, where we note that the argument of the stat operator is polynomial. Although stationarity-based representations for spherical-body gravitational potentials are inside the integral in (1), they may be moved outside through the introduction of α-valued processes, cf. [12, 20]. In particular, not only does one seek the stationary path for action J, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in gravitational systems, cf. [12, 17, 20].

It has also been demonstrated that this stationary-action approach may be applied to TPBVPs for infinite-dimensional conservative systems described by classes of lossless wave equations, see for example [3, 4]. There, stat is used in the construction of fundamental solution groups for these wave equations by appealing to stationarity of action on longer horizons.

Lastly, it has recently been demonstrated that stationarity may be employed to obtain a Feynman-Kac type of representation for solutions of the Schrödinger initial value problem (IVP) for certain classes of initial conditions and potentials [16, 18]. As with the conservative-system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation, such representations are valid only on time intervals such that the action remains convex, which is always a bounded duration and potentially zero.

Important to this stationary-action approach for solution of conservative-system TPBVPs and Schrödinger IVPs is the ability to propagate the stationary-action value function forward in time for indefinitely long durations. Further, for the harmonic oscillator, the quantum harmonic oscillator, the wave equation and the above stat-based representations for the gravitational and Coulomb potentials, propagation forward in the case of linear-quadratic action functionals is a key tool. This is the underlying motivation for the effort at hand.

Convex duality and semiconvex duality have proven to be quite useful in solution of optimization problems. In addressing stationary-action problems, we have found a need for what will be termed static duality in this context, and which is obtained via the Legendre transform. The transform will be specifically applied in the case where the functionals are C^1 but not necessarily convex, and this case has been well-discussed in [6], which also considered similar issues of stationarity. In particular, we will find a minor generalization of this that is analogous to the generalization of convex duality to semiconvex duality (cf. [9, 22]), where this generalization will be referred to as static-quadratic, or more compactly, "stat-quad" duality. In a certain class of smooth cases, static duality generalizes both convex and concave duality. However, as currently conceived, static duality is not applicable to general nonsmooth examples. Nonetheless, it appears to be a useful aid in solution of some stationary-action problems, specifically the TPBVPs indicated above.

2 Stationarity definitions

As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note for example, [6, 24]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization", with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. One might note that here that the term staticization is being derived from a Latin root, staticus (presumably originating from the Greek, statikós), in analogy with the Latin root, maximus, of "maximization". We note that Ekeland [6] employed the term "extremization" for what is essentially the same notion that is being referred to here as staticization. We make the following definitions. Suppose $\mathcal{U}$ is a normed vector space with $\mathcal{G} \subseteq \mathcal{U}$, and suppose $F : \mathcal{G} \to I\!R$. We say $\bar{v} \in \operatorname{argstat}\{F(v) \,|\, v \in \mathcal{G}\}$ if $\bar{v} \in \mathcal{G}$ and either

$$\limsup_{v \to \bar{v}, v \in \mathcal{G} \setminus \{\bar{v}\}} \frac{|F(v) - F(\bar{v})|}{|v - \bar{v}|} = 0, \tag{2}$$

or there exists $\delta > 0$ such that $\mathcal{G} \cap B_\delta(\bar{v}) = \{\bar{v}\}$ (where $B_\delta(\bar{v})$ denotes the ball of radius δ around $\bar{v}$). If $\operatorname{argstat}\{F(v) \,|\, v \in \mathcal{G}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\operatorname*{stat}^s_{v \in \mathcal{G}} F(v) \doteq \operatorname{stat}^s\{F(v) \,|\, v \in \mathcal{G}\} \doteq \left\{ F(\bar{v}) \,\middle|\, \bar{v} \in \operatorname{argstat}\{F(v) \,|\, v \in \mathcal{G}\} \right\}. \tag{3}$$

If $\operatorname{argstat}\{F(v) \,|\, v \in \mathcal{G}\} = \emptyset$, $\operatorname{stat}^s_{v \in \mathcal{G}} F(v)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s). In particular, if there exists $a \in I\!R$ such that $\operatorname{stat}^s_{v \in \mathcal{G}} F(v) = \{a\}$, then $\operatorname{stat}_{v \in \mathcal{G}} F(v) \doteq a$; otherwise, $\operatorname{stat}_{v \in \mathcal{G}} F(v)$ is undefined. At times, we may abuse notation by writing $\bar{v} = \operatorname{argstat}\{F(v) \,|\, v \in \mathcal{G}\}$ in the event that the argstat is the single point $\{\bar{v}\}$.

In the case where $\mathcal{U}$ is a Hilbert space, and $\mathcal{G} \subseteq \mathcal{U}$ is an open set, $F : \mathcal{G} \to I\!R$ is Fréchet differentiable at $\bar{v} \in \mathcal{G}$ with Riesz representation $DF(\bar{v}) \in \mathcal{U}$ if

$$\lim_{w \to 0,\, \bar{v}+w \in \mathcal{G} \setminus \{\bar{v}\}} \frac{|F(\bar{v}+w) - F(\bar{v}) - \langle DF(\bar{v}), w \rangle|}{|w|} = 0. \tag{4}$$

The following is immediate from the above definitions.

Lemma 1 *Suppose $\mathcal{U}$ is a Hilbert space, with open set $\mathcal{G} \subseteq \mathcal{U}$, and that F is Fréchet differentiable at $\bar{v} \in \mathcal{G}$. Then, $\bar{v} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}\}$ if and only if $DF(\bar{v}) = 0$.*

3 The Legendre transform and stat-quad duality

Throughout this section, we let $\mathcal{U}$ denote a real Hilbert space. Also, henceforth, for functions on real Hilbert spaces, we do not distinguish between Fréchet derivatives and their Riesz representations, denoted by D, which may be subscripted when necessary to avoid confusion. The following Legendre-transform result in the case of C^1, but possibly nonconvex, functionals on $\mathcal{U} = I\!R^n$ appeared in [6]. (See also [2, 24], and in a somewhat different direction, [25].) Although the extension is essentially trivial, a proof in the Hilbert-space case is included in the appendix.

Theorem 2 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{U}$ be open. Suppose $\phi \in C^1(\mathcal{A}; I\!R)$ and $[D\phi]^{-1} \in C^1(\mathcal{B}; \mathcal{A})$. Then,*

$$\phi(u) = \operatorname*{stat}_{v\in\mathcal{B}} [a(v) + \langle v, u\rangle] \quad \forall\, u \in \mathcal{A},$$

where

$$a(v) = \operatorname*{stat}_{u\in\mathcal{A}} [\phi(u) - \langle v, u\rangle] \quad \forall\, v \in \mathcal{B}.$$

We will refer to a as the static dual of ϕ.

Remark 3 *As an example, let $\mathcal{A} = \mathcal{B} = I\!R^n \setminus \{0\}$ and $\phi(u) = 1/|u|$, where $|u|$ denotes the Euclidean norm of u. Then, $D\phi = -u/|u|^3$, $[D\phi]^{-1}(v) = -v/(|v|^{3/2})$ and $a(v) = 2|v|^{1/2}$.*

Let $S^{\neq 0}$ denote the space of the space of symmetric, invertible elements of $\mathcal{L}(\mathcal{U};\mathcal{U})$. In analogy to the semiconvex [semiconcave] generalization of convex [concave] duality, static duality may be generalized to stat-quad duality. In particular, we have the following.

Theorem 4 *Let $\mathcal{A}, \hat{\mathcal{B}} \subseteq \mathcal{U}$ be open. Let $C \in S^{\neq 0}$. Suppose $\phi \in C^1(\mathcal{A}; I\!R)$. Letting $\eta(u) \doteq D\phi(u) - Cu$ for all $u \in \mathcal{A}$, suppose $\eta^{-1} \in C^1(\hat{\mathcal{B}}; \mathcal{A})$. Then,*

$$\phi(u) = \operatorname*{stat}_{v\in\mathcal{B}} \left[a(v) + \tfrac{1}{2}\langle v-u, C(v-u)\rangle\right] \quad \forall\, u \in \mathcal{A},$$

where

$$a(v) = \operatorname*{stat}_{u\in\mathcal{A}} \left[\phi(u) - \tfrac{1}{2}\langle v-u, C(v-u)\rangle\right] \quad \forall\, v \in \mathcal{B},$$

and where $\mathcal{B} \doteq \{v \in \mathcal{U} \mid -Cv \in \hat{\mathcal{B}}\}$.

We will refer to a as the stat-quad dual of ϕ (with respect to duality operator, C).

Proof: Let $\bar{\phi}(u) \doteq \phi(u) - \frac{1}{2}\langle u, Cu\rangle$ for all $u \in \mathcal{A}$. By assumption, $[D\bar{\phi}]^{-1} = \eta^{-1} \in C^1(\hat{\mathcal{B}}; \mathcal{A})$. Therefore, by Theorem 2, we have the static duality relationship

$$\bar{\phi}(u) = \operatorname*{stat}_{z\in\hat{\mathcal{B}}} [\bar{a}(z) + \langle z, u\rangle] \quad \forall\, u \in \mathcal{A}, \qquad \bar{a}(z) = \operatorname*{stat}_{u\in\mathcal{A}} [\bar{\phi}(u) - \langle z, u\rangle] \quad \forall\, z \in \hat{\mathcal{B}}.$$

That is,

$$\bar{\phi}(u) = \operatorname*{stat}_{v\in\mathcal{B}} [\bar{a}(-Cv) - \langle Cv, u\rangle] \quad \forall\, u \in \mathcal{A}, \qquad \bar{a}(-Cv) = \operatorname*{stat}_{u\in\mathcal{A}} [\bar{\phi}(u) + \langle Cv, u\rangle] \quad \forall\, v \in \mathcal{B},$$

or equivalently,

$$\phi(u) = \operatorname*{stat}_{v\in\mathcal{B}} [\bar{a}(-Cv) + \tfrac{1}{2}\langle u, Cu\rangle - \langle v, Cu\rangle] \quad \forall\, u \in \mathcal{A},$$

$$\bar{a}(-Cv) = \operatorname*{stat}_{u\in\mathcal{A}} [\phi(u) - \tfrac{1}{2}\langle u, Cu\rangle + \langle v, Cu\rangle] \quad \forall\, v \in \mathcal{B}.$$

Letting $a(v) \doteq \bar{a}(-Cv) - \frac{1}{2}\langle v, Cv\rangle$ for all $v \in \mathcal{B}$, one obtains the asserted result. □

Remark 5 *As an example, let $\mathcal{A} = \mathcal{B} = \mathcal{U} = I\!R^n$, $P, C \in S^{\neq 0}$, and suppose $P - C$ is nonsingular. Let $\phi \in C^1(I\!R^n; I\!R)$ be given by $\phi(u) = \frac{1}{2}u'Pu$. Then, ϕ has the stat-dual (with respect to C) given by*

$$a(v) = \tfrac{1}{2}v'C(C-P)^{-1}Pv = \tfrac{1}{2}v'P(C-P)^{-1}Cv. \tag{5}$$

4 Nested statica

For the application to follow, we will need to demonstrate an invariance to order in nested staticization of quadratic functionals. We note that nested staticization appears in the use of the dynamic programming principle for the control problems that yield solution of the TPBVPs for conservative systems, where this may be seen in the proof of Theorem 17 below. It may also be seen in the development at the start of Section 5.2 and in the approach to gravitational problems indicated in the introduction (although not for quadratic functionals in this last case).

4.1 Definitions and examples

Let $\mathcal{U}, \mathcal{V}$ be normed vector spaces, $\mathcal{G} \subseteq \mathcal{U}$, $\mathcal{H} \subseteq \mathcal{V}$, and $F : \mathcal{G} \times \mathcal{H} \to I\!R$. Given $v \in \mathcal{H}$, let $f^v : \mathcal{G} \to I\!R$ be given by

$$f^v(u) = F(u,v) \quad \forall\, u \in \mathcal{G}. \tag{6}$$

Let

$$\mathcal{H}_F^s \doteq \bigl\{ v \in \mathcal{H} \,\big|\, \operatorname*{stat}^s_{u\in\mathcal{G}} f^v(u) \neq \emptyset \bigr\} \quad \text{and} \quad \mathcal{H}_F \doteq \bigl\{ v \in \mathcal{H} \,\big|\, \operatorname*{stat}_{u\in\mathcal{G}} f^v(u) \text{ exists} \bigr\}.$$

We also let $\operatorname{stat}^s_{u\in\mathcal{G}} F(u,\cdot) : \mathcal{H}_F^s \to \mathcal{P}(I\!R)$ and $\operatorname{stat}_{u\in\mathcal{G}} F(u,\cdot) : \mathcal{H}_F \to I\!R$ be given by

$$[\operatorname*{stat}^s_{u\in\mathcal{G}} F(u,\cdot)](v) \doteq \operatorname*{stat}^s_{u\in\mathcal{G}} f^v(u) \quad \text{and} \quad [\operatorname*{stat}_{u\in\mathcal{G}} F(u,\cdot)](v) \doteq \operatorname*{stat}_{u\in\mathcal{G}} f^v(u),$$

for all $v \in \mathcal{H}_F^s$, where $\mathcal{P}(I\!R)$ denotes the power set of $I\!R$. We define the analogs $f^u(v) \doteq F(u,v)$ for all $v \in \mathcal{H}$, $\mathcal{G}_F^s$, $\mathcal{G}_F$, $\operatorname{stat}^s_{v\in\mathcal{H}}$ and $\operatorname{stat}_{v\in\mathcal{H}}$ similarly, and do not include the details. Also, we will abuse notation slightly by writing $\operatorname{argstat}_{u\in\mathcal{U}} F(u,v)$ for $\operatorname{argstat}_{u\in\mathcal{U}} f^v(u)$ et cetera, where convenient.

Remark 6 *It is worth noting that* $\operatorname{dom}\bigl(\operatorname{stat}^s{}_{u\in\mathcal{G}} F(u,\cdot)\bigr) = \{v \in \mathcal{H} \,|\, \operatorname{argstat}_{u\in\mathcal{G}} f^v(u) \neq \emptyset\}$, *while*

$$\operatorname{dom}\bigl(\operatorname*{stat}_{u\in\mathcal{G}} F(u,\cdot)\bigr) = \Bigl\{ v \in \operatorname{dom}\bigl(\operatorname*{stat}^s_{u\in\mathcal{G}} F(u,\cdot)\bigr) \,\Big|\, \operatorname*{stat}^s_{u\in\mathcal{G}} F(u,\cdot) \ \textit{is single-valued} \Bigr\} = \mathcal{H}_F,$$

with analogous domain definitions for stat *over $v \in \mathcal{H}$.*

The term, nested statica, refers to expressions such as $\operatorname{stat}_{v\in\mathcal{H}_F}\{[\operatorname{stat}_{u\in\mathcal{G}} F(u,\cdot)](v)\}$ and $\operatorname{stat}_{u\in\mathcal{G}_F}\{[\operatorname{stat}_{v\in\mathcal{H}} F(\cdot,v)](u)\}$, whereas the corresponding stat operation over the product space is $\operatorname{stat}_{(u,v)\in\mathcal{G}\times\mathcal{H}} F(u,v)$, and the natural questions regard conditions guaranteeing existence of, and/or equivalences between, these objects.

We will find that the ordering of nesting is irrelevant when the function under consideration is quadratic, whereas this invariance to ordering does not hold more generally. We begin with a simple quadratic example that illustrates the invariance regardless of a certain degeneracy. This is followed by a similarly simple cubic example where the invariance does not hold.

Example 1: Let $\mathcal{U} = \mathcal{V} = I\!R$. Let $F : \mathcal{U} \times \mathcal{V} \to I\!R$ be given by $F(u, v) = uv + v^2/2$. Using Lemma 1, we see that $\operatorname{argstat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u, v) = \{(0, 0)\}$ and

$$\operatorname*{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u, v) = 0. \tag{7}$$

Considering nested statica for this same function, note that when $v = 0$, $f^v(u) = f^0(u) = F(u, 0) = 0$ for all $u \in I\!R$, which implies $\operatorname{argstat}_{u\in I\!R} f^0(u) = I\!R$ and $\operatorname{stat}_{u\in I\!R} f^0(u) = 0$. Alternatively, if $v \neq 0$, then $\operatorname{argstat}_{u\in I\!R} f^v(u) = \emptyset$. Consequently, $\mathcal{H}_F = \{0\}$ and $\operatorname{stat}_{u\in I\!R} F(u, \cdot) : \{0\} \to I\!R$ is given by $\operatorname{stat}_{u\in I\!R} F(u, 0) = 0$. Then, by definition,

$$\operatorname*{stat}_{v\in\mathcal{H}_F} \{[\operatorname*{stat}_{u\in I\!R} F(u, \cdot)](v)\} = 0. \tag{8}$$

Next, we reverse the order of the statica in the example. Given $u \in I\!R$, let $f^u(v) \doteq F(u, v)$ for all $v \in I\!R$. We find $\operatorname{argstat}_{v\in I\!R} f^u(v) = \{-u\}$ and $\operatorname{stat}_{v\in I\!R} f^u(v) = -u^2/2$. This implies $\operatorname{dom}(\operatorname{stat}_{v\in I\!R} F(\cdot, v)) = \mathcal{G}_F \doteq I\!R$, and on this domain, $[\operatorname{stat}_{v\in I\!R} F(\cdot, v)](u) = -u^2/2$. Consequently, using Lemma 1, $\operatorname{stat}_{u\in\mathcal{G}_F}\{[\operatorname{stat}_{v\in I\!R} F(\cdot, v)](u)\} = 0$. Comparing this with (7) and (8), we see that both orderings of the nesting yield the full stat over the product space in this example.

Example 2: In order to see that the ordering of nesting can be relevant in non-quadratic cases, again let $\mathcal{U} = \mathcal{V} = I\!R$, but now take $F(u, v) = u(v^2 - 1)$. In this case, we find that $\operatorname{stat}_{u\in\mathcal{U}} F(u, v) = 0$ if $v = \pm 1$, and does not exist otherwise, while $\operatorname{stat}_{v\in\mathcal{V}} F(u, v) = -u$ for all $u \in \mathcal{U}$. Hence, $\mathcal{H}_F = \{-1, 1\}$ and $\mathcal{G}_F = I\!R = \mathcal{U}$. Using this, we easily find that $\operatorname{stat}_{v\in\mathcal{H}_F} \operatorname{stat}_{u\in\mathcal{U}} F(u, v) = 0 = \operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u, v)$, while $\operatorname{stat}_{u\in\mathcal{G}_F} \operatorname{stat}_{v\in\mathcal{V}} F(u, v)$ does not exist.

4.2 Nesting with Quadratics

In order to obtain a general result regarding independence of nesting order for quadratic functionals, we first indicate the form of quadratics that will be considered. Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces. Let $F : \mathcal{U} \times \mathcal{V} \to I\!R$ have the quadratic form

$$\begin{aligned} F(u, v) &= \tfrac{c}{2} + \langle w, u\rangle_{\mathcal{U}} + \langle y, v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_1 u, u\rangle_{\mathcal{U}} + \langle \bar{B}_2 v, u\rangle_{\mathcal{U}} + \tfrac{1}{2}\langle \bar{B}_3 v, v\rangle_{\mathcal{V}} \\ &= \tfrac{c}{2} + \langle w, u\rangle_{\mathcal{U}} + \langle y, v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_1 u, u\rangle_{\mathcal{U}} + \langle \bar{B}_2' u, v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3 v, v\rangle_{\mathcal{V}}, \end{aligned} \tag{9}$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $\bar{B}_1 \in \mathcal{L}(\mathcal{U})$, $\bar{B}_2 \in \mathcal{L}(\mathcal{V}, \mathcal{U})$, $\bar{B}_3 \in \mathcal{L}(\mathcal{V})$, $w \in \mathcal{U}$, $y \in \mathcal{V}$, $c \in I\!R$, and $\langle \cdot, \cdot\rangle_{\mathcal{X}}$ denotes the inner product in space $\big(\mathcal{X}, \langle \cdot, \cdot\rangle_{\mathcal{X}}\big)$ throughout.

Throughout the remainder of this section, F will refer to a quadratic functional of the form in (9), and we assume there exists $(\bar{u}, \bar{v}) \in \mathcal{U}\times\mathcal{V}$ such that $\{(\bar{u}, \bar{v})\} = \operatorname{argstat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u, v)$, which implies

$$\operatorname*{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u, v) = F(\bar{u}, \bar{v}). \tag{10}$$

Lemma 7 *Let $\hat{v} \in \mathcal{V}$. Then $\hat{u} \in \operatorname{argstat}_{u\in\mathcal{U}} F(u,\hat{v})$ if and only if $\bar{B}_1\hat{u} + \bar{B}_2\hat{v} + w = 0$.*

Lemma 8 *Given $\bar{u}$, $\bar{v}$ as per (10), $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{U}} F(u,\bar{v})$ and $\bar{v} \in \operatorname{argstat}_{v\in\mathcal{V}} F(\bar{u},v)$.*

Proof: We only prove the first assertion. By (10) and the definition of argstat for the case $\operatorname{dom}\big(F(\cdot,\bar{v})\big) \equiv \mathcal{U}$, given $\epsilon > 0$, there exists $\delta > 0$ such that $|F(u,\bar{v}) - F(\bar{u},\bar{v})|/|u-\bar{u}| < \epsilon$ for all $u \in B_\delta(\bar{u})$, and this yields the assertion. □

Lemma 9 *Let $\bar{u}$, $\bar{v}$ as per (10). Both $\operatorname{stat}^s_{u\in\mathcal{U}} F(u,\bar{v})$ and $\operatorname{stat}^s_{v\in\mathcal{V}} F(\bar{u},v)$ are single-valued. In addition, $\bar{v} \in \operatorname{dom}\big(\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot)\big)$ and $\bar{u} \in \operatorname{dom}\big(\operatorname{stat}_{v\in\mathcal{V}} F(\cdot,v)\big)$.*

Proof: We prove only the first and third assertions, as the other two follow by symmetry. By Lemma 8, using the notation of (6), $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{U}} F(u,\bar{v}) = \operatorname{argstat}_{u\in\mathcal{U}} f^{\bar{v}}(u)$. Therefore, by Remark 6, we need only demonstrate that

$$\operatorname*{stat}^s_{u\in\mathcal{U}} F(u,\bar{v}) \text{ is single-valued} \tag{11}$$

in order to show both the first and third claims. Let $\hat{u} \in \operatorname{argstat}_{u\in\mathcal{U}} F(u,\bar{v})$. By (11), it is sufficient to show that $F(\hat{u},\bar{v}) = F(\bar{u},\bar{v})$. Let $\hat{\nu} \in \mathcal{U}$. Then by Lemma 7, $\langle \bar{B}_1\hat{u} + \bar{B}_2\bar{v} + w, \hat{\nu}\rangle_{\mathcal{U}} = 0$. Combining this with (9), we find

$$F(u,\bar{v}) - F(\hat{u},\bar{v}) = \tfrac{1}{2}\langle \bar{B}_1(u-\hat{u}), u-\hat{u}\rangle_{\mathcal{U}} \quad \forall\, u \in \mathcal{U}. \tag{12}$$

Similarly, as $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{U}} F(u,\bar{v})$,

$$F(u,\bar{v}) - F(\bar{u},\bar{v}) = \tfrac{1}{2}\langle \bar{B}_1(u-\bar{u}), u-\bar{u}\rangle_{\mathcal{U}} \quad \forall\, u \in \mathcal{U}. \tag{13}$$

Taking $u = \bar{u}$ in (12) and $u = \hat{u}$ in (13), we see that $F(\bar{u},\bar{v}) - F(\hat{u},\bar{v}) = F(\hat{u},\bar{v}) - F(\bar{u},\bar{v})$, and hence, $F(\hat{u},\bar{v}) - F(\bar{u},\bar{v}) = 0$. □

Lemma 10 *Let $\hat{v} \in \mathcal{V}$, $\hat{u} \in \mathcal{U}$, $\tilde{\mathcal{G}}(\hat{v}) \doteq \operatorname{argstat}_{u\in\mathcal{U}} F(u,\hat{v})$ and $\tilde{\mathcal{H}}(\hat{u}) \doteq \operatorname{argstat}_{v\in\mathcal{V}} F(\hat{u},v)$. Then $\tilde{\mathcal{G}}(\hat{v})$ is an affine subspace of $\mathcal{U}$, and $F(\cdot,\hat{v})$ is constant on $\tilde{\mathcal{G}}(\hat{v})$. Similarly, $\tilde{\mathcal{H}}(\hat{u})$ is an affine subspace of $\mathcal{V}$, and $F(\hat{u},\cdot)$ is constant on $\tilde{\mathcal{H}}(\hat{u})$. Further, if $\tilde{\mathcal{G}}(\hat{v}) \neq \emptyset$, then $\hat{v} \in \mathcal{H}_F$, and if $\tilde{\mathcal{H}}(\hat{u}) \neq \emptyset$, then $\hat{u} \in \mathcal{G}_F$.*

Proof: That $\tilde{\mathcal{G}}(\hat{v})$ and $\tilde{\mathcal{H}}(\hat{u})$ are affine subspaces is standard and easily demonstrated, while the second and fourth assertions follow from Lemma 9. □

Theorem 11 *Let $\mathcal{U},\mathcal{V}$ be Hilbert spaces, and let $F : \mathcal{U}\times\mathcal{V} \to I\!R$ have quadratic form (9). Then, $\operatorname{stat}_{u\in\mathcal{G}_F}\left\{\left[\operatorname{stat}_{v\in\mathcal{V}} F(\cdot,v)\right](u)\right\} = F(\bar{u},\bar{v}) = \operatorname{stat}_{v\in\mathcal{H}_F}\left\{\left[\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot)\right](v)\right\}$.*

Proof: We prove only the second equality. Suppose $\hat{v} \in \mathcal{H}_F$. Then, by definition, there exists $\tilde{u} \in \tilde{\mathcal{G}}(\hat{v}) \doteq \operatorname{argstat}_{u\in\mathcal{U}} F(u,\hat{v})$. By Lemma 7, this is equivalent to $u = \tilde{u}$ being a solution of

$$\bar{B}_1 u + \bar{B}_2\hat{v} + w = 0. \tag{14}$$

By Lemma 9, $\tilde{\mathcal{G}}(\bar{v}) \neq \emptyset$, and in particular,

$$\bar{u} \in \tilde{\mathcal{G}}(\bar{v}). \tag{15}$$

Also by Lemma 9, we have $\bar{v} \in \mathcal{H}_F$.

Two cases must be considered, namely, $\mathcal{H}_F = \{\bar{v}\}$ and $\mathcal{H}_F \neq \{\bar{v}\}$. First, suppose $\mathcal{H}_F = \{\bar{v}\}$. Then, by the definition of stat, $\{\bar{v}\} = \operatorname{argstat}_{v\in\mathcal{V}} \left\{ \left[\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](v) \right\}$, and

$$\operatorname*{stat}_{v\in\mathcal{V}} \left\{ \left[\operatorname*{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](v) \right\} = \left[\operatorname*{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](\bar{v}),$$

which by (15) and Lemma 10,

$$= F(\bar{u},\bar{v}),$$

thereby yielding the right-hand equality in the theorem statement for the first case.

Now we proceed to the second case. Suppose $\mathcal{H}_F \neq \{\bar{v}\}$, that is, there exists $\hat{v} \in \mathcal{H}_F \setminus \{\bar{v}\}$. By Lemma 10, $\tilde{\mathcal{G}}(\hat{v})$ is an affine subspace, and $\operatorname{stat}_{u\in\mathcal{U}} F(u,\hat{v}) = F(\tilde{u},\hat{v})$ where $\tilde{u}$ is any element of the subspace, which is nonempty by the fact that $\hat{v} \in \mathcal{H}_F$. Noting (14), we take the element of the space given by $\tilde{u} = -\bar{B}_1^{\#}(\bar{B}_2\hat{v} + w)$, where the $\#$ superscript indicates the Moore-Penrose pseudo-inverse, cf. [1]. For $v \in \mathcal{H}_F$, define the affine transformation $Av + \hat{w} \doteq -\bar{B}_1^{\#}(\bar{B}_2 v + w)$. (The authors remark that the Moore-Penrose pseudo-inverse was employed only for the convenience of selecting a specific affine transformation from the space; the minimum-norm property is not used.) Subsequently, define

$$\bar{F}(v) \doteq \operatorname*{stat}_{u\in\mathcal{U}}[F(u,\cdot)](v) = F(Av + \hat{w}, v),$$

which using (9),

$$\begin{aligned} &= \tfrac{c}{2} + \langle w, Av + \hat{w}\rangle_{\mathcal{U}} + \langle y, v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_1(Av + \hat{w}), Av + \hat{w}\rangle_{\mathcal{U}} + \langle \bar{B}_2 v, Av + \hat{w}\rangle_{\mathcal{U}} + \tfrac{1}{2}\langle \bar{B}_3 v, v\rangle_{\mathcal{V}}, \\ &= \tfrac{\bar{c}}{2} + \langle \bar{y}, v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_4 v, v\rangle_{\mathcal{V}} \quad \forall\, v \in \mathcal{H}_F, \end{aligned} \tag{16}$$

where $\bar{c} = c + 2\langle w, \hat{w}\rangle + \langle \bar{B}_1\hat{w}, \hat{w}\rangle$, $\bar{y} = y + A'w + (A'\bar{B}_1 + \bar{B}_2')\hat{w}$, and $\bar{B}_4 = A'\bar{B}_1 A + 2A'\bar{B}_2 + \bar{B}_3$. Note that $\bar{F}$ is a quadratic form, $\operatorname{dom}(\bar{F}) = \mathcal{H}_F$ and $\bar{v}, \hat{v} \in \mathcal{H}_F$. Let $\hat{\mathcal{H}}_F \doteq \operatorname{argstat}_{v\in\mathcal{H}_F} \left\{ \left[\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](v) \right\}$.

Now let $\delta \in \mathcal{V}$ be such that $\bar{v} + \delta \in \mathcal{H}_F$. By Lemma 7, $\tilde{u} \in \tilde{\mathcal{G}}(\bar{v} + \delta)$ if and only if $\bar{B}_1\tilde{u} + \bar{B}_2(\bar{v} + \delta) + w = 0$. Recalling from above that $\bar{v} \in \mathcal{H}_F$ with $\bar{u} \in \tilde{\mathcal{G}}(\bar{v})$, by Lemma 7 again, $\bar{B}_1\bar{u} + \bar{B}_2\bar{v} + w = 0$. Combining these last two equalities yields $\bar{B}_1(\tilde{u} - \bar{u}) + \bar{B}_2\delta = 0$ and we may take $\tilde{u} = \bar{u} - \bar{B}_1^{\#}\bar{B}_2\delta$. Then, recalling Lemma 10 and noting that $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} F(u,v)$,

$$\left| [\operatorname*{stat}_{u\in\mathcal{U}} F(u,\cdot)](\bar{v} + \delta) - [\operatorname*{stat}_{u\in\mathcal{U}} F(u,\cdot)](\bar{v}) \right| = \left| F(\bar{u} - \bar{B}_1^{\#}\bar{B}_2\delta, \bar{v} + \delta) - F(\bar{u},\bar{v}) \right| \le \mathcal{O}\big(|\delta|_{\mathcal{V}}^2\big),$$

which implies $\bar{v} \in \hat{\mathcal{H}}_F$.

Suppose $\breve{v} \in \hat{\mathcal{H}}_F \setminus \{\bar{v}\}$. By Lemma 10 applied to $\bar{F}$, and then using (16),

$$\bar{F}(\breve{v}) = \bar{F}(\bar{v}) = \operatorname*{stat}_{u\in\mathcal{U}}[F(u,\cdot)](\bar{v}),$$

which by (15) and Lemma 10,

$$= F(\bar{u},\bar{v}).$$

As this is true for all $\breve{v} \in \hat{\mathcal{H}}_F$, $\operatorname{stat}_{u\in\mathcal{U}}[F(u,\cdot)](v) = F(\bar{u},\bar{v}) \quad \forall\, v \in \hat{\mathcal{H}}_F$. Consequently, $\operatorname{stat}_{v\in\mathcal{H}_F} \left\{ \left[\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](v) \right\}$ exists, and in particular, $\operatorname{stat}_{v\in\mathcal{H}_F} \left\{ \left[\operatorname{stat}_{u\in\mathcal{U}} F(u,\cdot) \right](v) \right\} = F(\bar{u},\bar{v})$. $\square$

5 Application to Stationary Action

As indicated in the introduction, a primary motivation for this effort has been the solution of TPBVPs in stationary action, and we now indicate the approach to this application. Applying staticization and stat-quad duality in both the harmonic oscillator and quantum harmonic oscillator problems [20, 21, 16, 18] as well as wave-equation problem classes [3, 4], a linear-quadratic problem formulation is possible. Further, as noted in the introduction, in the n-body problem class, linear-quadratic problems appear through a numerically useful duality-based representation of the gravitational potential [12, 20, 21]. In all these cases, one is interested in a quadratic functional of the control variable. Further, staticization over the control yields a functional that is quadratic over space, where the time-dependent coefficients satisfy differential Riccati equation (DREs) [4, 20, 21]. At times where the action functional has changes in convexity/concavity along subspaces, the norm of the DRE solution typically escapes to infinity. Now, for DREs associated to optimal control and estimation, it is meaningless to attempt to propagate past such asymptotes. However, in the case of stationary action, propagation past such points is required for solution of the dynamical system problems. We will find that by propagating both the solutions of the DREs for the static value and the solutions of the DREs for the stat-quad duals of the static value, we may propagate past these escape times.

5.1 Action Problem Formulation

We consider dynamics

$$\dot{\xi}_r = u_r, \tag{17}$$

where $\xi_r \in I\!R^n$ denotes the state at time $r \in (-\infty, 0]$, evolved forward from an initial state $\xi_s = x$, $s \in (-\infty, 0]$, via input $u \in L_2\big((s,0); I\!R^n\big)$. Let $\mathcal{T} \doteq \{(s,t) \in I\!R^2 \mid -\infty < s \le t \le 0\}$. For $(s,t) \in \mathcal{T}$, we also define

$$\mathcal{U}_{s,t} \doteq L_2\big((s,t); I\!R^n\big), \tag{18}$$

and we let $\langle \cdot, \cdot \rangle_{\mathcal{U}_{s,t}}$ denote the inner product on $\mathcal{U}_{s,t}$. For $(s,t) \in \mathcal{T}$, we define the basic action functional, $J^0(s,t,\cdot,\cdot) : I\!R^n \times \mathcal{U}_{s,t} \to I\!R$, by

$$J^0(s,t,x,u) \doteq \int_s^t T(u_r) - V(r, \xi_r)\, dr, \tag{19}$$

where ξ satisfies (17) with initial condition $\xi_s = x \in I\!R^n$, T denotes the kinetic energy and V denotes the potential. Further, we will assume that the kinetic energy takes the form

$$T(v) \doteq \tfrac{1}{2} v' \mathcal{M} v \quad \forall\, v \in I\!R^n, \tag{20}$$

where $\mathcal{M} \in S_n^{>0}$ and $S_n^{>0}$ $[S_n^{\geq 0}]$ denotes the space of positive-definite [positive-semidefinite], symmetric $n \times n$ matrices. Recalling, for example, the duality-based representation for the gravitational potential [20, 21], we let

$$-V(r,x) \doteq \tfrac{1}{2}\omega_r - \tfrac{1}{2} x' \Omega_r x \quad \forall\, r \in (-\infty, 0],\ x \in I\!R^n, \tag{21}$$

where $\omega \in C\big(I\!R; I\!R\big)$ and $\Omega \in C\big(I\!R; S_n^{\geq 0}\big)$.

We will append a terminal cost to the functional J^0 for the purposes of using the action to solve TPBVPs; see [4, 5, 12, 20, 21] for more information on how this allows solution of TPBVPs.

For $C \in S_n^{>0}$ and $c \in (0,\infty)$, let $\hat{\psi}^C, \psi^c : I\!\!R^n \times I\!\!R^n \to I\!\!R$ be given by

$$\hat{\psi}^C(x,z) \doteq \frac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}' \begin{pmatrix} C & -C \\ -C & C \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \qquad \psi^c(x,z) \doteq \hat{\psi}^{cI}(x,z), \tag{22}$$

while for the $c = \infty$ case, we take

$$\psi^\infty(x,z) \doteq \begin{cases} 0 & \text{if } x = z, \\ \infty & \text{otherwise,} \end{cases} \tag{23}$$

which is the min-plus delta function (cf. [13]). Throughout, for $a \in I\!\!R$, we let $[a,\infty]$ denote $[a,\infty) \cup \{\infty\}$. For $C \in S_n^{>0}$ and $c \in (0,\infty]$, the (appended) action functionals are

$$\hat{J}^C(s,t,x,u,z) \doteq J^0(s,t,x,u) + \hat{\psi}^C(\xi_t,z), \quad J^c(s,t,x,u,z) \doteq J^0(s,t,x,u) + \psi^c(\xi_t,z). \tag{24}$$

For $C \in S_n^{>0}$ and $c \in (0,\infty]$, the static value functionals are $\widehat{W}^C, W^c : [0,\infty) \times I\!\!R^n \times I\!\!R^n \to I\!\!R$ given by

$$\widehat{W}^C(t,x,z) \doteq \operatorname*{stat}_{u \in \mathcal{U}_{-t,0}} \hat{J}^C(-t,0,x,u,z), \qquad W^c(t,x,z) \doteq \operatorname*{stat}_{u \in \mathcal{U}_{-t,0}} J^c(-t,0,x,u,z), \tag{25}$$

where existence is addressed in the assumptions to follow.

For fixed $x, z \in I\!\!R^n$, $(s,t) \in \mathcal{T}$, $c \in (0,\infty]$, we say $J^c(s,t,x,\cdot,z)$ is uniformly strictly convex on $\mathcal{U}_{s,t}$ if there exists $k > 0$ such that $J^c(s,t,x,u+\hat{u},z) + J^c(s,t,x,u-\hat{u},z) - 2J^c(s,t,x,u,z) \geq k|\hat{u}|^2$ for all $u, \hat{u} \in \mathcal{U}_{s,t}$.

As our concern regards propagation of the static value past points where the DRE solutions escape, we assume:

For any $\bar{t} > 0$, there exists $\hat{\delta} = \hat{\delta}(\bar{t}) > 0$ such that W^c exists on $(\max\{\bar{t} - \hat{\delta}, 0\}, \bar{t}) \cup (\bar{t}, \bar{t} + \hat{\delta}) \times I\!\!R^n \times I\!\!R^n$. $\quad (A.1)$

We remark that Assumption $(A.1)$ requires only that the stationary value exist outside of a nowhere dense set of times. One can look to the example in Section 5.3 for a sense of the motivation for the particular assumption.

Using a straightforward adaptation of [20] Lemma 4.17, we have the following.

Lemma 12 *There exists $\bar{\delta} > 0$ and $\bar{c} \in (0,\infty)$ such that for all $x, z \in I\!\!R^n$, $s \in (-\infty, 0)$, $\delta \in (0,\bar{\delta})$ such that $s + \delta \leq 0$, $C \succeq \bar{c}I$ and $c \in [\bar{c},\infty]$, $\hat{J}^C(s,s+\delta,x,\cdot,z)$ and $J^c(s,s+\delta,x,\cdot,z)$ are uniformly strictly convex and coercive on $\mathcal{U}_{s,s+\delta}$.*

Theorem 13 *Let $c \in [0,\infty)$; $x, z \in I\!\!R^n$; $(s,t) \in \mathcal{T}$. There exist $\bar{\nu} = \bar{\nu}(x,z) \in I\!\!R$, $\bar{\bar{B}} \in \mathcal{L}(\mathcal{U}_{s,t})$ and $\tilde{B}^1, \tilde{B}^2 \in \mathcal{L}(I\!\!R^n; \mathcal{U}_{s,t})$ such that*

$$J^c(s,t,x,u,z) = \tfrac{\bar{\nu}}{2} + \langle \bar{\bar{B}} u, u \rangle_{\mathcal{U}_{s,t}} + \langle \tilde{B}^1 x + \tilde{B}^2 z, u \rangle_{\mathcal{U}_{s,t}}$$

for all $u \in \mathcal{U}_{s,t}$, where in particular, $\bar{\nu} = \nu_1 + c|x-z|^2_{I\!\!R^n}$ with $\nu_1 \doteq \int_s^t \omega_\rho \, d\rho$.

Remark 14 *For brevity of expressions, we do not explicitly indicate, in the notation, the dependence on c of objects such as $\bar{\nu}, \bar{\bar{B}}, \tilde{B}^1, \tilde{B}^2$. This policy is continued where appropriate throughout.*

Proof: We examine each of the terms on the right-hand side of the definition of J^c in (24) separately. Letting $[\hat{B}_1 u]_r \doteq \mathcal{M} u_r$ for all $r \in (s,t)$, we see $\hat{B}_1 \in \mathcal{L}(\mathcal{U}_{s,t})$, and

$$\int_s^t T(u_r)\, dr = \langle \hat{B}_1 u, u \rangle_{\mathcal{U}_{s,t}} \quad \forall\, u \in \mathcal{U}_{s,t}. \tag{26}$$

Next, using (21), we have

$$\int_s^t -V(r,\xi_r)\, dr = \tfrac{\nu_1}{2} - \tfrac{1}{2} \int_s^t \xi_r' \Omega_r \xi_r \, dr, \tag{27}$$

where $\nu_1 \doteq \int_s^t \omega_\rho \, d\rho$. For $u \in \mathcal{U}_{s,t}$ and $r \in (s,t)$, let $[A_1 u]_r \doteq \int_s^r u_\rho \, d\rho$ for all $r \in (s,t)$, and note that $A_1 \in \mathcal{L}(\mathcal{U}_{s,t})$. Further, $\xi_r = x + [A_1 u]_r$ for all $r \in (s,t)$. Also define $A_0 \in \mathcal{L}(I\!R^n; \mathcal{U}_{s,t})$ by $[A_0 x]_r \doteq x$ for all $r \in (0,t)$ and $x \in I\!R^n$. With these definitions, $\xi = A_0 x + A_1 u$ for all $x \in I\!R^n$ and $u \in \mathcal{U}_{s,t}$. Recalling $\Omega \in C(I\!R; S_n^{\geq 0})$, we let $[\hat{B}_2 \xi]_r \doteq \Omega_r \xi_r$ for all $r \in (s,t)$, and note that $\hat{B}_2 \in \mathcal{L}(\mathcal{U}_{s,t})$. With these definitions, (27) becomes

$$\int_s^t -V(r,\xi_r)\, dr = \tfrac{\nu_1}{2} + \tfrac{1}{2} \langle \hat{B}_2 A_0 x + \hat{B}_2 A_1 u, A_0 x + A_1 u \rangle_{\mathcal{U}_{s,t}}. \tag{28}$$

Lastly, we turn to $\psi^c(\xi_t, z)$. Note that

$$\psi^c(\xi_t, z) = \tfrac{c}{2} \Big| x - z + \int_s^t u_\rho \, d\rho \Big|^2_{I\!R^n} = \tfrac{c}{2} |x-z|^2_{I\!R^n} + \tfrac{c}{2} \langle \overline{A}_1 u, u \rangle_{\mathcal{U}_{s,t}} + c \langle (x-z) \tilde{i}, u \rangle_{\mathcal{U}_{s,t}}, \tag{29}$$

where $[\overline{A}_1 u]_r \doteq \int_s^t u_\rho \, d\rho$ and $\tilde{i}_r \doteq 1$ for all $r \in (s,t)$. Combining (24), (26), (28) and (29), and performing straight-forward manipulations, one obtains the result. □

Let $x, z \in I\!R^n$; $\bar{t} > 0$; $\delta \in (0, \bar{t})$; $\mathcal{U}^0 \doteq \mathcal{U}_{-(\bar{t}+\delta), -(\bar{t}-\delta)}$ and $\mathcal{U}^1 \doteq \mathcal{U}_{-(\bar{t}-\delta), 0}$. For any $u^0 \in \mathcal{U}^0$ and $u^1 \in \mathcal{U}^1$, let

$$\begin{aligned} F(u^0,u^1) &= F(u^0,u^1; \bar{t}, \delta, x, z, c) & (30) \\ &\doteq J^0(-(\bar{t}+\delta), -(\bar{t}-\delta), x, u^0) + J^c(-(\bar{t}-\delta), 0, \xi^0_{-(\bar{t}-\delta)}, u^1, z) \\ &= \int_{-(\bar{t}+\delta)}^{-(\bar{t}-\delta)} T(u^0_r) - V(r, \xi^0_r)\, dr + \int_{-(\bar{t}-\delta)}^{0} T(u^1_r) - V(r,\xi^1_r)\, dr + \psi^c(\xi^1_0, z), & (31) \end{aligned}$$

where

$$\dot{\xi}^0_r = u^0_r, \;\; \xi^0_{-(\bar{t}+\delta)} = x, \;\; \dot{\xi}^1_r = u^1_r \;\text{ and }\; \xi^1_{-(\bar{t}-\delta)} = \xi^0_{-(\bar{t}-\delta)}. \tag{32}$$

Theorem 15 *Let $x, z \in I\!R^n$; $\bar{t} > 0$ and $\delta \in (0, \min\{\bar{\delta}/2, \hat{\delta}, \bar{t}\})$. There exist $\eta \in I\!R$, $y^0 \in \mathcal{U}^0$, $y^1 \in \mathcal{U}^1$, $\bar{B}_1 \in \mathcal{L}(\mathcal{U}^0)$, $\bar{B}_2 \in \mathcal{L}(\mathcal{U}^1, \mathcal{U}^0)$ and $\bar{B}_3 \in \mathcal{L}(\mathcal{U}^1)$ (where these depend on x, z, δ, t, c) such that*

$$F(u^0,u^1) = \tfrac{\eta}{2} + \langle y^0, u^0 \rangle_{\mathcal{U}^0} + \langle y^1, u^1 \rangle_{\mathcal{U}^1} + \tfrac{1}{2} \langle \bar{B}_1 u^0, u^0 \rangle_{\mathcal{U}^0} + \langle \bar{B}_2 u^1, u^0 \rangle_{\mathcal{U}^0} + \tfrac{1}{2} \langle \bar{B}_3 u^1, u^1 \rangle_{\mathcal{U}^1} \tag{33}$$

for all $u^0 \in \mathcal{U}^0$ and $u^1 \in \mathcal{U}^1$.

Proof: The right-hand side of (31) is a sum of five terms, and we may address each term in a similar way as was done in the proof of Theorem 13. There is some additional complication due to the dependence of ξ^1 on u^0 (as well as u^1). Because of the similarities, we examine only the $\int_{-(\bar{t}-\delta)}^{0} -V(r,\xi_r^1)\,dr$ term; the other terms are similar and simpler. For $r \in (-(\bar{t}-\delta), 0]$, define $A_1^1 : \mathcal{U}^1 \to \mathcal{U}^1$ by $[A_1^1 u^1]_r \doteq \int_{-(\bar{t}-\delta)}^{r} u_\rho^1\,d\rho$. By the same method as used in the proof of Theorem 13 (for A_1), we find $A_1^1 \in \mathcal{L}(\mathcal{U}^1)$. Next, define $A_0^1 : \mathcal{U}^0 \to \mathcal{U}^1$ by $[A_0^1 u^0]_r \doteq \int_{-(\bar{t}+\delta)}^{-(\bar{t}-\delta)} u_\rho^0\,d\rho$ for all $r \in (-(\bar{t}-\delta), 0]$. A similar argument to that for A_1^1 implies that $A_0^1 \in \mathcal{L}(\mathcal{U}^0, \mathcal{U}^1)$. Using the same A_0 definition as in the proof of Theorem 13, we have $\xi^1 = A_0 x + A_0^1 u^0 + A_1^1 u^1$. Also as in the proof of Theorem 13, let $[\hat{B}_2^1 \xi]_r \doteq \Omega_r \xi_r$ for all $r \in (-(\bar{t}-\delta), 0)$. Recalling (27) and the above definitions, we have

$$\int_{-(\bar{t}-\delta)}^{0} -V(r,\xi_r^1)\,dr = \tfrac{\nu^1}{2} - \tfrac{1}{2}\Big\langle \hat{B}_2^1(A_0 x + A_0^1 u^0 + A_1^1 u^1), A_0 x + A_0^1 u^0 + A_1^1 u^1 \Big\rangle_{\mathcal{U}^1}, \tag{34}$$

where $\nu^1 \doteq \int_{-(\bar{t}-\delta)}^{0} \omega_r\,dr$. Proceeding similarly with the other terms, and performing standard manipulations, one obtains the result. □

Theorem 16 *Let $x, z \in I\!R^n$; $\bar{t} > 0$; $\delta \in (0, \min\{\bar{\delta}/2, \hat{\delta}, \bar{t}\})$; $t \in (0, \bar{\delta}) \cup (\bar{t}-\delta, \bar{t}) \cup (\bar{t}, \bar{t}+\delta)$ and $c \in [\bar{c}, \infty)$. Then, there exist $\bar{\nu} \in I\!R$ and $P_t, Q_t, R_t \in I\!R^{n\times n}$ such that*

$$W^c(t,x,z) = \tfrac{\bar{\nu}}{2} + \tfrac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}' \begin{pmatrix} P_t & Q_t \\ Q_t' & R_t \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad \forall\, x, z \in I\!R^n,$$

where $\bar{\nu} = \nu_1 + c|x-z|_{I\!R^n}^2$ with $\nu_1 \doteq \int_s^t \omega_\rho\,d\rho$.

Proof: Assumption $(A.1)$ and Lemma 12 yield existence and uniqueness for $W^c(t,\cdot,\cdot)$ (defined in (25)). Then, by Theorem 13,

$$W^c(t,x,z) = \operatorname*{stat}_{u \in \mathcal{U}_{-t,0}} \Big\{ \tfrac{\bar{\nu}}{2} + \tfrac{1}{2}\langle \bar{\bar{B}} u, u\rangle_{\mathcal{U}_{-t,0}} + \langle \tilde{B}^1 x + \tilde{B}^2 z, u\rangle_{\mathcal{U}_{-t,0}} \Big\},$$

where $\bar{\nu} = \nu_1 + c|x-z|_{I\!R^n}^2$. By Lemmas 1 and 10, and letting $u^* \doteq -\big(\bar{\bar{B}}\big)^{\#}\big[\tilde{B}^1 x + \tilde{B}^2 z\big]$, this implies

$$\begin{aligned} W^c(t,x,z) &= \tfrac{\bar{\nu}}{2} + \tfrac{1}{2}\langle \bar{\bar{B}} u^*, u^*\rangle_{\mathcal{U}_{-t,0}} + \langle \tilde{B}^1 x + \tilde{B}^2 z, u^*\rangle_{\mathcal{U}_{-t,0}} \\ &= \tfrac{\bar{\nu}}{2} - \tfrac{1}{2}\Big\langle \big(\bar{\bar{B}}\big)^{\#}\big[\tilde{B}^1 x + \tilde{B}^2 z\big], \big[\tilde{B}^1 x + \tilde{B}^2 z\big] \Big\rangle_{\mathcal{U}_{-t,0}} \quad \forall\, x, z \in I\!R^n. \end{aligned}$$

□

The next result is an adaption of the Dynamic Programming Principle to the staticization class of problems. It is also specifically designed to to allow us to propagate the stat value function across at time, $\bar{t}$, where that value may not exist.

Theorem 17 *Let $x, z \in I\!R^n$; $\bar{t} > 0$; $\delta \in (0, \min\{\bar{\delta}/2, \hat{\delta}, \bar{t}\})$ and $c \in [\bar{c}, \infty)$. Then,*

$$W^c(\bar{t}+\delta, x, z) = \operatorname*{stat}_{u \in \mathcal{U}^0} \big[J^0(-(\bar{t}+\delta), -(\bar{t}-\delta), x, u) + W^c(\bar{t}-\delta, \xi^0_{-(\bar{t}-\delta)}, z) \big] \quad \forall\, x, z \in I\!R^n,$$

where $\xi^0_{-(\bar{t}+\delta)} = x$ and $\dot{\xi}^0(r) = u(r)$ for all $r \in (-(\bar{t}+\delta), -(\bar{t}-\delta))$.

Proof: By Lemma 6 of [17],

$$W^c(\bar{t}+\delta,x,z)=\operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}^0\times\mathcal{U}^1}\big[J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)\\ +J^c(-(\bar{t}-\delta),0,\xi^0_{-(\bar{t}-\delta)},u^1,z)\big],\tag{35}$$

where Assumptions $(A.1)$ and Lemma 12 replace Assumptions $(A.1)$ and $(A.2)$ of [17]. By (30) and (35),

$$W^c(\bar{t}+\delta,x,z)=\operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}^0\times\mathcal{U}^1}F(u^0,u^1;\bar{t},\delta,x,z,c)=\operatorname*{stat}_{(u^0,u^1)\in\mathcal{U}^0\times\mathcal{U}^1}F(u^0,u^1)$$

which by Theorem 11,

$$=\operatorname*{stat}_{u^0\in\mathcal{G}_F}\big\{\big[\operatorname*{stat}_{u^1\in\mathcal{U}^1}F(\cdot,u^1)\big](u^0)\big\},$$

where $\mathcal{G}_F=\{u^0\in\mathcal{U}^0\mid \operatorname{stat}_{u^1\in\mathcal{U}^1}F(u^0,u^1)\text{ exists}\}$. Noting that $J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)$ is independent of u^1, this is

$$=\operatorname*{stat}_{u^0\in\mathcal{G}_F}\big\{J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)+\operatorname*{stat}_{u^1\in\mathcal{U}^1}J^c(-(\bar{t}-\delta),0,\xi^0_{-(\bar{t}-\delta)},u^1,z)\big\},\tag{36}$$

and we note that $\mathcal{G}_F=\{u^0\in\mathcal{U}^0\mid \operatorname{stat}_{u^1\in\mathcal{U}^1}J^c(-(\bar{t}-\delta),0,\xi^0_{-(\bar{t}-\delta)},u^1,z)\text{ exists}\}$.

Now, by our assumption concerning δ (in the theorem statement) and Lemma 12, $\mathcal{G}_F=\mathcal{U}^0$. Combining this with (36), we have

$$\begin{aligned}W^c(\bar{t}+\delta,x,z)&=\operatorname*{stat}_{u^0\in\mathcal{U}^0}\big\{J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)+\operatorname*{stat}_{u^1\in\mathcal{U}^1}J^c(-(\bar{t}-\delta),0,\xi^0_{-(\bar{t}-\delta)},u^1,z)\big\}\\&=\operatorname*{stat}_{u^0\in\mathcal{U}^0}\big\{J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)+W^c(\bar{t}-\delta,\xi^0_{-(\bar{t}-\delta)},z)\big\}.\end{aligned}$$

□

5.2 Stat-quad duality-based propagation

We now begin the process of obtaining Riccati differential equations for the coefficient matrices in the stat-quad dual of W^c.

We assume there exists $C\in S_n^{\geq 0}$ and $\tilde{\delta}\in(0,\hat{\delta})$ such that P_r-C is nonsingular $\forall r\in(\bar{t}-\tilde{\delta},\bar{t})\cup(\bar{t}\cup\bar{t}+\tilde{\delta})\cup(0,\bar{\delta})$. $(A.2)$

Let $\tilde{\mathcal{T}}\doteq(\bar{t}-\tilde{\delta},\bar{t})\cup(\bar{t},\bar{t}+\tilde{\delta})\cup(0,\hat{\delta}\wedge\bar{\delta}/2)$, where $a\wedge b\doteq\min\{a,b\}$ for all $a,b\in I\!R$. Note that by Assumption $(A.1)$ and Lemma 12, $W^c(r,x,z)$ exists for all $r\in(\bar{t}-\hat{\delta},\bar{t})\cup(\bar{t},\bar{t}+\hat{\delta})\cup(0,\bar{\delta})\supseteq\tilde{\mathcal{T}}$. For $r\in\tilde{\mathcal{T}}$ and $z\in I\!R^n$, let $\bar{\eta}_{r,z}(x)\doteq D_xW^c(r,x,z)-Cx$ for all $x\in I\!R^n$. Then, by Theorem 16 and Assumption $(A.2)$, $\bar{\eta}_{r,z}^{-1}\in C^1(I\!R^n;I\!R^n)$. Consequently, by Theorem 4

$$W^c(t,x,z)=\operatorname*{stat}_{y\in I\!R^n}\big\{B(t,y,z)+\hat{\psi}^C(x,y)\big\}\quad\forall\,x,z\in I\!R^n,\tag{37}$$

where

$$B(t,y,z)=\operatorname*{stat}_{x\in I\!R^n}\big\{W^c(t,x,z)-\hat{\psi}^C(x,y)\big\}\quad\forall\,y,z\in I\!R^n,\tag{38}$$

where we note that $B(t,\cdot,\cdot)$ is quadratic (see Remark 5).

Let $\delta\in(0,\tilde{\delta}\wedge\bar{\delta}/2)$. By Theorem 17 and (38)

$$B(\bar{t}+\delta,y,z)=\operatorname*{stat}_{x\in I\!R^n}\operatorname*{stat}_{u^0\in\mathcal{U}^0}\big[W^c(\bar{t}-\delta,\xi^0_{-(\bar{t}-\delta)},z)+J^0(-(\bar{t}+\delta),-(\bar{t}-\delta),x,u^0)-\hat{\psi}^C(x,y)\big],$$

which by (37),

$$= \operatorname*{stat}_{x\in\mathbb{R}^n} \operatorname*{stat}_{u^0\in\mathcal{U}^0} \operatorname*{stat}_{\eta\in\mathbb{R}^n} \big[B(\bar t-\delta,\eta,z) + \hat\psi^C(\xi^0_{-(\bar t-\delta)},\eta) + J^0(-(\bar t+\delta),-(\bar t-\delta),x,u) - \hat\psi^C(x,y)\big].$$

Note that the term in brackets is a quadratic function of x, u^0, η, y, z, and that in the case of a quadratic functional, the existence of the nested stat over $u^0 \in \mathcal{U}^0$ and $\eta \in \mathbb{R}^n$ implies the existence of the stat over the product space. Consequently, by Theorem 11, we can reorder the stat operations, and this becomes (using the symmetry of $\hat\psi^C$ in its arguments as well)

$$\begin{aligned} B(\bar t+\delta,y,z) = \operatorname*{stat}_{x\in\mathbb{R}^n} \operatorname*{stat}_{\eta\in\tilde{\mathcal{G}}_\eta} \Big\{ \operatorname*{stat}_{u^0\in\mathcal{U}^0} \big[J^0(-(\bar t+\delta),-(\bar t-\delta),x,u) \\ + \hat\psi^C(\xi^0_{-(\bar t-\delta)},\eta)\big] + B(\bar t-\delta,\eta,z) - \hat\psi^C(x,y)\Big\}, \end{aligned}$$

for some affine subspace, $\tilde{\mathcal{G}}_\eta \subseteq \mathbb{R}^n$, and this is,

$$= \operatorname*{stat}_{x\in\mathbb{R}^n} \operatorname*{stat}_{\eta\in\tilde{\mathcal{G}}_\eta} \Big\{\widehat{W}^C(2\delta,x,\eta) + B(\bar t-\delta,\eta,z) - \hat\psi^C(x,y)\Big\}, \tag{39}$$

where existence and uniqueness of $\widehat{W}^C(2\delta,x,\eta)$ is guaranteed by the indicated bounds on δ and Lemma 12.

Lemma 18 *Let $r \in (0, 2\tilde\delta \wedge \bar\delta)$ and $x, \eta \in \mathbb{R}^n$. Then,*

$$\widehat{W}^C(r,x,\eta) = \tfrac{\hat\nu_r}{2} + \tfrac12 \begin{pmatrix} x \\ \eta \end{pmatrix}' \begin{pmatrix} \hat P_r & \hat Q_r \\ \hat Q_r' & \hat R_r \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \tag{40}$$

where

$$\hat P_0 = -\hat Q_0 = \hat R_0 = C, \tag{41}$$

$$\dot{\hat P}_r = -\hat P_r \mathcal{M}^{-1} \hat P_r - \Omega_{-r-(\bar t-\delta)}, \qquad \dot{\hat Q}_r = -\hat P_r \mathcal{M}^{-1} \hat Q_r, \tag{42}$$

$$\dot{\hat R}_r = -\hat Q_r' \mathcal{M}^{-1} \hat Q_r, \qquad \hat\nu_r = \int_0^r \omega_{-\rho-(\bar t-\delta)}\, d\rho. \tag{43}$$

Proof: This is a minor application of existing results [17]. For $r \in (0, 2\tilde\delta \wedge \bar\delta)$, let $\hat P, \hat Q, \hat R, \hat\nu$ be given by (41)–(43), where existence is guaranteed by Assumption $(A.2)$. Let $\widetilde{W}(r,x,\eta)$ be given by the right-hand side of (40) for all $r \in (0, 2\tilde\delta \wedge \bar\delta)$ and $x, \eta \in \mathbb{R}^n$. One easily sees that $\widetilde{W}$ satisfies

$$0 = \operatorname*{stat}_{v\in\mathbb{R}^n} \big[T(v) - V(r,x) - W_r(r,x,\eta) + (W_x(r,x,\eta))'v\big]; \quad r \in (0, 2\tilde\delta \wedge \bar\delta); \;\; x,\eta \in \mathbb{R}^n;$$

$$W(0,x,\eta) = \hat\psi^C(x,\eta); \quad x,\eta \in \mathbb{R}^n.$$

Then, by Theorem 8 of [17], $\widetilde{W}(r,x,\eta) = \widehat{W}^C(r,x,\eta)$ for all $r \in (0, 2\tilde\delta \wedge \bar\delta)$; $x, \eta \in \mathbb{R}^n$. □

Similarly to Lemma 18, for all $r \in (0, \bar\delta)$, $P, Q, R, \bar\nu$ of Theorem 16 satisfy

$$P_0 = -Q_0 = R_0 = cI, \tag{44}$$

$$\dot P_r = -P_r \mathcal{M}^{-1} P_r - \Omega_{-r}, \qquad \dot Q_r = -P_r \mathcal{M}^{-1} Q_r, \tag{45}$$

$$\dot R_r = -Q_r' \mathcal{M}^{-1} Q_r, \qquad \bar\nu_r = \int_0^r \omega_{-\rho}\, d\rho. \tag{46}$$

Note that without loss of generality, we may take $\bar{\delta} = 2\tilde{\delta} = 2\hat{\delta}$. Combining (39) and (40), we have

$$B(\bar{t}+\delta, y, z) = \operatorname*{stat}_{x\in I\!R^n} \operatorname*{stat}_{\eta\in\tilde{\mathcal{G}}_\eta} \left\{ B(\bar{t}-\delta, \eta, z) + \tfrac{1}{2} \begin{pmatrix} x \\ \eta \end{pmatrix}' \begin{pmatrix} \hat{P}_{2\delta} & \hat{Q}_{2\delta} \\ \hat{Q}'_{2\delta} & \hat{R}_{2\delta} \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} \right.$$
$$\left. - \tfrac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} C & -C \\ -C & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\} + \tfrac{\hat{\nu}_{2\delta}}{2}, \tag{47}$$

We need an expression for the coefficients of quadratic $B(r,\cdot,\cdot)$ obtained from those of quadratics $\widehat{W}^C(r,\cdot,\cdot)$ and/or $W^c(r,\cdot,\cdot)$ for r such that the objects exist. The following is easily verified.

Lemma 19 *Suppose*

$$\mathcal{Q}^1(x,y) \doteq \tfrac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} Q^1_{1,1} & Q^1_{1,2} \\ (Q^1_{1,2})' & Q^1_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \forall\, x,y \in I\!R^n,$$

and

$$\mathcal{Q}^2(y,z) \doteq \tfrac{1}{2} \begin{pmatrix} y \\ z \end{pmatrix}' \begin{pmatrix} Q^2_{1,1} & Q^2_{1,2} \\ (Q^2_{1,2})' & Q^2_{2,2} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \qquad \forall\, y,z \in I\!R^n,$$

where $\bar{S} \doteq Q^1_{2,2} + Q^2_{1,1}$ *is nonsingular. Let* $\mathcal{Q}^3(x,z) = \operatorname{stat}_{y\in I\!R^n} \left[\mathcal{Q}^1(x,y) + \mathcal{Q}^2(y,z)\right]$. *Then,*

$$\mathcal{Q}^3(x,z) = \tfrac{1}{2} \begin{pmatrix} x \\ z \end{pmatrix}' \begin{pmatrix} Q^3_{1,1} & Q^3_{1,2} \\ (Q^3_{1,2})' & Q^3_{2,2} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \qquad \forall\, x,z \in I\!R^n,$$

where

$$Q^3_{1,1} = Q^1_{1,1} - Q^1_{1,2}\bar{S}^{-1}(Q^1_{1,2})', \qquad Q^3_{1,2} = -Q^1_{1,2}\bar{S}^{-1}Q^2_{1,2},$$
$$Q^3_{2,2} = Q^2_{2,2} - (Q^2_{1,2})'\bar{S}^{-1}Q^2_{1,2}.$$

The following is immediate from Theorem 4 and Lemma 19.

Lemma 20 *Suppose* $W(r,\cdot,\cdot)$ *has form*

$$W(r,x,z) = \tfrac{1}{2} \begin{pmatrix} x \\ z \end{pmatrix}' \begin{pmatrix} P_r & Q_r \\ Q'_r & R_r \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \tfrac{\nu_r}{2},$$

and let $B(r,\cdot,\cdot)$ *be the stat-quad dual of* W *given by (38). Then,*

$$B(r,y,z) = \tfrac{1}{2} \begin{pmatrix} y \\ z \end{pmatrix}' \begin{pmatrix} \alpha_r & \beta_r \\ \beta'_r & \gamma_r \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \tfrac{\hat{\nu}^B_r}{2},$$

where the dual coefficients are given by

$$\alpha_r \doteq -C - C(P_r - C)^{-1}C = -C(P_r - C)^{-1}P_r = -P_r(P_r - C)^{-1}C, \tag{48}$$
$$\beta_r \doteq -C(P_r - C)^{-1}Q_r, \qquad \gamma_r \doteq R_r - Q'_r(P_r - C)^{-1}Q_r, \tag{49}$$
$$\hat{\nu}^B_r \doteq \nu_r.$$

Further,

$$P_r = C - C(\alpha_r + C)^{-1}C = C(\alpha_r + C)^{-1}\alpha_r = \alpha_r(\alpha_r + C)^{-1}C,$$
$$Q_r = C(\alpha_r + C)^{-1}\beta_r, \qquad R_r = \gamma_r - \beta'_r(\alpha_r + C)^{-1}\beta_r.$$

We will develop dual DREs for the α, β, γ coefficient functions of time. We begin by applying Lemma 20 to (47), we have

$$B(\bar{t}+\delta, y, z) \doteq \frac{1}{2}\begin{pmatrix} y \\ z \end{pmatrix}' \begin{pmatrix} \alpha_{\bar{t}+\delta} & \beta_{\bar{t}+\delta} \\ \beta'_{\bar{t}+\delta} & \gamma_{\bar{t}+\delta} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \frac{\bar{\nu}^B_{\bar{t}+\delta}}{2}$$
$$= \operatorname*{stat}_{x \in I\!R^n} \operatorname*{stat}_{\eta \in \tilde{\mathcal{G}}_\eta} \left\{ \frac{1}{2}\begin{pmatrix} \eta \\ z \end{pmatrix}' \begin{pmatrix} \alpha_{\bar{t}-\delta} & \beta_{\bar{t}-\delta} \\ \beta'_{\bar{t}-\delta} & \gamma_{\bar{t}-\delta} \end{pmatrix} \begin{pmatrix} \eta \\ z \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x \\ \eta \end{pmatrix}' \begin{pmatrix} \hat{P}_{2\delta} & \hat{Q}_{2\delta} \\ \hat{Q}'_{2\delta} & \hat{R}_{2\delta} \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} \right.$$
$$\left. - \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} C & -C \\ -C & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\} + \frac{\hat{\nu}^B_{\bar{t}-\delta}}{2} + \frac{\hat{\nu}_{2\delta}}{2},$$

which by Theorem 11,

$$= \operatorname*{stat}_{\eta \in \hat{\mathcal{G}}_\eta} \left\{ \frac{1}{2}\begin{pmatrix} \eta \\ z \end{pmatrix}' \begin{pmatrix} \alpha_{\bar{t}-\delta} & \beta_{\bar{t}-\delta} \\ \beta'_{\bar{t}-\delta} & \gamma_{\bar{t}-\delta} \end{pmatrix} \begin{pmatrix} \eta \\ z \end{pmatrix} + \operatorname*{stat}_{x \in I\!R^n} \left[\frac{1}{2}\begin{pmatrix} x \\ \eta \end{pmatrix}' \begin{pmatrix} \hat{P}_{2\delta} & \hat{Q}_{2\delta} \\ \hat{Q}'_{2\delta} & \hat{R}_{2\delta} \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} \right. \right.$$
$$\left. \left. - \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} C & -C \\ -C & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] \right\} + \frac{\hat{\nu}^B_{\bar{t}-\delta}}{2} + \frac{\hat{\nu}_{2\delta}}{2}, \tag{50}$$

where $\hat{\mathcal{G}}_\eta$ is an affine subspace of $\tilde{\mathcal{G}}_\eta$. Also, using this, (43) and (46), we see that

$$\bar{\nu}^B_{\bar{t}+\delta} = \hat{\nu}^B_{\bar{t}-\delta} + \hat{\nu}_{2\delta} = \int_0^{\bar{t}-\delta} \omega_{-\rho}\, d\rho + \int_0^{2\delta} \omega_{-(\rho+\bar{t}-\delta)}\, d\rho = \int_0^{\bar{t}+\delta} \omega_{-\rho}\, d\rho. \tag{51}$$

Now, applying the the expression given in Lemma 19 for the stat-quad dual (38) on horizon 2δ, to the last two terms in the brackets in (50), we have

$$\frac{1}{2}\begin{pmatrix} y \\ z \end{pmatrix}' \begin{pmatrix} \alpha_{\bar{t}+\delta} & \beta_{\bar{t}+\delta} \\ \beta'_{\bar{t}+\delta} & \gamma_{\bar{t}+\delta} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$
$$= \operatorname*{stat}_{\eta \in \hat{\mathcal{G}}_\eta} \left\{ \frac{1}{2}\begin{pmatrix} \eta \\ z \end{pmatrix}' \begin{pmatrix} \alpha_{\bar{t}-\delta} & \beta_{\bar{t}-\delta} \\ \beta'_{\bar{t}-\delta} & \gamma_{\bar{t}-\delta} \end{pmatrix} \begin{pmatrix} \eta \\ z \end{pmatrix} + \frac{1}{2}\begin{pmatrix} y \\ \eta \end{pmatrix}' \begin{pmatrix} \hat{\alpha}_{2\delta} & \hat{\beta}_{2\delta} \\ \hat{\beta}'_{2\delta} & \hat{\gamma}_{2\delta} \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix} \right\}, \tag{52}$$

where we note that the zeroth-order terms have cancelled and

$$\hat{\alpha}_{2\delta} = -C - C(\hat{P}_{2\delta} - C)^{-1}C = -C(\hat{P}_{2\delta} - C)^{-1}\hat{P}_{2\delta} = -\hat{P}_{2\delta}(\hat{P}_{2\delta} - C)^{-1}C, \tag{53}$$
$$\hat{\beta}_{2\delta} = -C(\hat{P}_{2\delta} - C)^{-1}\hat{Q}_{2\delta} \tag{54}$$
$$\hat{\gamma}_{2\delta} = \hat{R}_{2\delta} - \hat{Q}'_{2\delta}(\hat{P}_{2\delta} - C)^{-1}\hat{Q}_{2\delta}. \tag{55}$$

Noting that the term in brackets in (52) is quadratic and defined for all $\eta, y, z \in I\!R^n$, we see that $\hat{\mathcal{G}}_\eta = I\!R^n$, and consequently, we may replace $\operatorname{stat}_{\eta \in \hat{\mathcal{G}}_\eta}$ in (52) with $\operatorname{stat}_{\eta \in I\!R^n}$. Then applying Lemma 19 to the right-hand side of (52) with this change in the set over which stat is taken, and equating like terms, we have

$$\begin{pmatrix} \alpha_{\bar{t}+\delta} & \beta_{\bar{t}+\delta} \\ \beta'_{\bar{t}+\delta} & \gamma_{\bar{t}+\delta} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{2\delta} - \hat{\beta}_{2\delta}(\alpha_{\bar{t}-\delta} + \hat{\gamma}_{2\delta})^{-1}\hat{\beta}'_{2\delta} & -\hat{\beta}_{2\delta}(\alpha_{\bar{t}-\delta} + \hat{\gamma}_{2\delta})^{-1}\beta_{\bar{t}-\delta} \\ -\beta'_{\bar{t}-\delta}(\alpha_{\bar{t}-\delta} + \hat{\gamma}_{2\delta})^{-1}\hat{\beta}'_{2\delta} & \gamma_{\bar{t}-\delta} - \beta'_{\bar{t}-\delta}(\alpha_{\bar{t}-\delta} + \hat{\gamma}_{2\delta})^{-1}\beta_{\bar{t}-\delta} \end{pmatrix}, \tag{56}$$

which implies

$$\alpha_{\bar{t}+\delta} - \alpha_{\bar{t}-\delta} = -\alpha_{\bar{t}-\delta} + \hat{\alpha}_{2\delta} - \hat{\beta}_{2\delta}(\alpha_{\bar{t}-\delta} + \hat{\gamma}_{2\delta})^{-1}\hat{\beta}'_{2\delta}, \tag{57}$$

$$\beta_{\bar t+\delta}-\beta_{\bar t-\delta}=-\big[I+\hat\beta_{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\big]\beta_{\bar t-\delta}, \tag{58}$$
$$\gamma_{\bar t+\delta}-\gamma_{\bar t-\delta}=-\beta'_{\bar t-\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\beta_{\bar t-\delta}. \tag{59}$$

Therefore,

$$\dot\alpha_{\bar t}=\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[-\alpha_{\bar t-\delta}+\hat\alpha_{2\delta}-\hat\beta_{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\hat\beta'_{2\delta}\big], \tag{60}$$
$$\dot\beta_{\bar t}=\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[-\big(I+\hat\beta_{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\big)\beta_{\bar t-\delta}\big], \tag{61}$$
$$\dot\gamma_{\bar t}=\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[-\beta'_{\bar t-\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\beta_{\bar t-\delta}\big], \tag{62}$$

if the limits involved exist.

Theorem 21 *Let α,β,γ be the time-dependent coefficients in the stat-quad dual of W (i.e., given by (48)–(49)). If α,β,γ are continuous at $\bar t$, then they are differentiable there, with derivatives given by*

$$\dot\alpha_{\bar t}=-\alpha_{\bar t}\big[\mathcal{M}^{-1}+C^{-1}\Omega_{-\bar t}C^{-1}\big]\alpha_{\bar t}-\alpha_{\bar t}C^{-1}\Omega_{-\bar t}-\Omega_{-\bar t}C^{-1}\alpha_{\bar t}-\Omega_{-\bar t}, \tag{63}$$
$$\dot\beta_{\bar t}=-\alpha_{\bar t}\big[\mathcal{M}^{-1}+C^{-1}\Omega_{-\bar t}C^{-1}\big]\beta_{\bar t}-\Omega_{-\bar t}C^{-1}\beta_{\bar t}, \tag{64}$$
$$\dot\gamma_{\bar t}=-\beta'_{\bar t}\big[\mathcal{M}^{-1}+C^{-1}\Omega_{-\bar t}C^{-1}\big]\beta_{\bar t}. \tag{65}$$

Proof: We consider each of the three assertions separately, and begin with (65). By (41) and (42),

$$\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat P_{2\delta}-C)=\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}[\hat P_{2\delta}-\hat P_0]=\dot{\hat P}_0=-C\mathcal{M}^{-1}C-\Omega_{-\bar t}. \tag{66}$$

Using (53)–(55),

$$\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}=\lim_{\delta\downarrow 0}(2\delta\alpha_{\bar t-\delta}+2\delta\hat\gamma_{2\delta})^{-1}$$
$$=\lim_{\delta\downarrow 0}\big[2\delta\alpha_{\bar t-\delta}+2\delta\hat R_{2\delta}-\hat Q'_{2\delta}\big[2\delta(\hat P_{2\delta}-C)^{-1}\big]\hat Q_{2\delta}\big]^{-1},$$

which by the assumption of continuity of α at $\bar t$ and (41), and then using (66),

$$=-C^{-1}\big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat P_{2\delta}-C)\big]C^{-1}=\mathcal{M}^{-1}+C^{-1}\Omega_{-\bar t}C^{-1}. \tag{67}$$

By (62) and the assumption of continuity of β at $\bar t$, and then using (67),

$$\dot\gamma_{\bar t}=-\beta'_{\bar t}\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\beta_{\bar t}=-\beta'_{\bar t}\big[\mathcal{M}^{-1}+C^{-1}\Omega_{-\bar t}C^{-1}\big]\beta_{\bar t}, \tag{68}$$

which is (65).

Next, by the assumption of continuity of the dual coefficients applied to (61), and operating under the assumption that the limits exist (until reaching expressions guaranteeing their existence),

$$\dot\beta_{\bar t}=-\big\{\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[I+\hat\beta_{2\delta}(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\big]\big\}\beta_{\bar t}$$
$$=-\big\{\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta}+\hat\beta_{2\delta})(\alpha_{\bar t-\delta}+\hat\gamma_{2\delta})^{-1}\big]\big\}\beta_{\bar t},$$

which by the continuity assumption and (67),

$$= -\big\{\big[\alpha_{\bar t} + \lim_{\delta\downarrow 0}(\hat\gamma_{2\delta} + \hat\beta_{2\delta})\big]\big[\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar t}C^{-1}\big]\big\}\beta_{\bar t}, \tag{69}$$

Now, by (53)–(55),

$$\lim_{\delta\downarrow 0}(\hat\gamma_{2\delta} + \hat\beta_{2\delta}) = \lim_{\delta\downarrow 0}\big[\hat R_{2\delta} - (\hat Q'_{2\delta} + C)(\hat P_{2\delta} - C)^{-1}\hat Q_{2\delta}\big],$$

which by (41) and (66),

$$\begin{aligned} &= C + \big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat Q'_{2\delta} + C)\big]\big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat P_{2\delta} - C)\big]^{-1}C \\ &= C - \big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat Q'_{2\delta} + C)\big]\big[C\mathcal{M}^{-1}C + \Omega_{-\bar t}\big]^{-1}C. \end{aligned} \tag{70}$$

Using (41) again,

$$\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat Q'_{2\delta} + C) = \dot{\hat Q}_0 = C\mathcal{M}^{-1}C. \tag{71}$$

Substituting (71) into (70), we have

$$\lim_{\delta\downarrow 0}(\hat\gamma_{2\delta} + \hat\beta_{2\delta}) = C - C\mathcal{M}^{-1}C\big(C\mathcal{M}^{-1}C + \Omega_{-\bar t}\big)^{-1}C. \tag{72}$$

Substituting (72) into (69) yields

$$\begin{aligned} \dot\beta_{\bar t} &= -\big\{\big[\alpha_{\bar t} + C - C\mathcal{M}^{-1}C\big(C\mathcal{M}^{-1}C + \Omega_{-\bar t}\big)^{-1}C\big]\big[\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar t}C^{-1}\big]\big\}\beta_{\bar t} \\ &= -\alpha_{\bar t}\big(\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar t}C^{-1}\big)\beta_{\bar t} - \Omega_{-\bar t}C^{-1}\beta_{\bar t}, \end{aligned} \tag{73}$$

which is (64).

Lastly, we turn to the first assertion. From (60), and again operating under the assumption that the limits exist (until reaching expressions guaranteeing their existence),

$$\dot\alpha_{\bar t} = \big\{\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[(\hat\alpha_{2\delta} - \alpha_{\bar t-\delta})(\hat\beta'_{2\delta})^{-1}(\alpha_{\bar t-\delta} + \hat\gamma_{2\delta}) - \hat\beta_{2\delta}\big]\big\}\big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\alpha_{\bar t-\delta} + \hat\gamma_{2\delta})^{-1}\big]\big[\lim_{\delta\downarrow 0}2\delta\hat\beta'_{2\delta}\big]. \tag{74}$$

By (54),

$$\lim_{\delta\downarrow 0}2\delta\hat\beta_{2\delta} = -C\big[\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}(\hat P_{2\delta} - C)\big]^{-1}\lim_{\delta\downarrow 0}\hat Q_{2\delta},$$

which by (41) and (66),

$$= -C(C\mathcal{M}^{-1}C + \Omega_{-\bar t})^{-1}C = -(\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar t}C^{-1})^{-1}. \tag{75}$$

Substituting (67) and (75) into (74), we have

$$\dot\alpha_{\bar t} = -\big\{\lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[(\hat\alpha_{2\delta} - \alpha_{\bar t-\delta})(\hat\beta'_{2\delta})^{-1}(\alpha_{\bar t-\delta} + \hat\gamma_{2\delta}) - \hat\beta_{2\delta}\big]\big\},$$

which upon expanding and applying (75) again,

$$\begin{aligned} &= -\alpha_{\bar t}(\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar t}C^{-1})\alpha_{\bar t} + \lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[\alpha_{\bar t-\delta}(\hat\beta'_{2\delta})^{-1}\hat\gamma_{2\delta} - \hat\alpha_{2\delta}(\hat\beta'_{2\delta})^{-1}\alpha_{\bar t-\delta}\big] \\ &\quad + \lim_{\delta\downarrow 0}\tfrac{1}{2\delta}\big[\hat\beta_{2\delta} - \hat\alpha_{2\delta}(\hat\beta'_{2\delta})^{-1}\hat\gamma_{2\delta}\big]. \end{aligned} \tag{76}$$

Applying (53)–(55) to terms in the first limit in (76), we have

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta}(\hat{\beta}'_{2\delta})^{-1}\hat{\gamma}_{2\delta} - \hat{\alpha}_{2\delta}(\hat{\beta}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right] \\
&= \alpha_{\bar{t}} C^{-1} \lim_{\delta\downarrow 0}\left[\tfrac{1}{2\delta}(\hat{P}_{2\delta} - C)\right] + \lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta} C^{-1}\hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right],
\end{aligned}
$$

which by (66),

$$
= -\alpha_{\bar{t}} C^{-1}(C\mathcal{M}^{-1}C + \Omega_{-\bar{t}}) + \lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta} C^{-1}\hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right]. \tag{77}
$$

However, note that $\lim_{\delta\downarrow 0}\left[\alpha_{\bar{t}-\delta} C^{-1}\hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right] = 0$, and consequently,

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta} C^{-1}\hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right] \\
&= \lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta} C^{-1}\hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta} - \left(\alpha_{\bar{t}} C^{-1}\hat{Q}_0 - \hat{P}_0(\hat{Q}'_0)^{-1}\alpha_{\bar{t}}\right)\right] \\
&= \frac{d}{dr}\left[\alpha_{\bar{t}-r/2} C^{-1}\hat{Q}_r - \hat{P}_r(\hat{Q}'_r)^{-1}\alpha_{\bar{t}-r/2}\right]\Big|_{r=0} \\
&= \alpha_{\bar{t}} C^{-1}\dot{\hat{Q}}_0 + \dot{\hat{P}}_0 C^{-1}\alpha_{\bar{t}} + \dot{\hat{Q}}'_0 C^{-1}\alpha_{\bar{t}},
\end{aligned}
$$

which by (41)–(42),

$$
= \alpha_{\bar{t}}\mathcal{M}^{-1}C - (C\mathcal{M}^{-1}C + \Omega_{-\bar{t}})C^{-1}\alpha_{\bar{t}} + C\mathcal{M}^{-1}\alpha_{\bar{t}}. \tag{78}
$$

Combining (77) and (78) yields

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\alpha_{\bar{t}-\delta}(\hat{\beta}'_{2\delta})^{-1}\hat{\gamma}_{2\delta} - \hat{\alpha}_{2\delta}(\hat{\beta}'_{2\delta})^{-1}\alpha_{\bar{t}-\delta}\right] \\
&= -\alpha_{\bar{t}} C^{-1}(C\mathcal{M}^{-1}C + \Omega_{-\bar{t}}) + \alpha_{\bar{t}}\mathcal{M}^{-1}C - (C\mathcal{M}^{-1}C + \Omega_{-\bar{t}})C^{-1}\alpha_{\bar{t}} + C\mathcal{M}^{-1}\alpha_{\bar{t}}.
\end{aligned} \tag{79}
$$

Now we turn to the second limit in (76). Note that by (53)–(55),

$$
\hat{\beta}_{2\delta} - \hat{\alpha}_{2\delta}(\hat{\beta}'_{2\delta})^{-1}\hat{\gamma}_{2\delta} = \hat{Q}_{2\delta} - \hat{P}_{2\delta}(\hat{Q}'_{2\delta})^{-1}\hat{R}_{2\delta} \to 0, \quad \text{as} \quad \delta \downarrow 0.
$$

Consequently,

$$
\begin{aligned}
&\lim_{\delta\downarrow 0} \tfrac{1}{2\delta}\left[\hat{\beta}_{2\delta} - \hat{\alpha}_{2\delta}(\hat{\beta}'_{2\delta})^{-1}\hat{\gamma}_{2\delta}\right] = \tfrac{d}{dr}\left[\hat{Q}_r - \hat{P}_r(\hat{Q}'_r)^{-1}\hat{R}_r\right]\Big|_{r=0} \\
&= \dot{\hat{Q}}_0 - \dot{\hat{P}}_0(\hat{Q}'_0)^{-1}\hat{R}_0 + \hat{P}_0(\hat{Q}'_0)^{-1}\dot{\hat{Q}}'_0(\hat{Q}'_0)^{-1}\hat{R}_0 - \hat{P}_0(\hat{Q}'_0)^{-1}\dot{\hat{R}}_0,
\end{aligned}
$$

which by (41)–(43),

$$
= -(C\mathcal{M}^{-1}C + \Omega_{-\bar{t}}) + C\mathcal{M}^{-1}C = -\Omega_{-\bar{t}}. \tag{80}
$$

Substituting (79) and (80) into (76), we obtain

$$
\dot{\alpha}_{\bar{t}} = -\alpha_{\bar{t}}(\mathcal{M}^{-1} + C^{-1}\Omega_{-\bar{t}}C^{-1})\alpha_{\bar{t}} - \alpha_{\bar{t}} C^{-1}\Omega_{-\bar{t}} - \Omega_{-\bar{t}} C^{-1}\alpha_{\bar{t}} - \Omega_{-\bar{t}}.
$$

□

5.3 Propagation Example

Even simple stationary-action problems tend to exhibit finite escape times. The classical mass-spring problem with mass, m, and spring-constant, K, is an example. As noted in [17, 19, 20], the stationary-value, with terminal payoff ψ^∞, is easily calculated to be $W^\infty(t,x,z) = c_1[\cot(\bar\omega t)(x^2+z^2) - \mathrm{cosec}(\bar\omega t)xz]$ where $c_1 = \sqrt{Km}$ and $\bar\omega = \sqrt{K/m}$, which has finite escape times, t, satisfying $\bar\omega t = \bar k\pi$, $\bar k \in I\!N$. Examples with time-dependent $\Omega.$ and $\omega.$ in the potential-energy term can be found in [20, 21].

To indicate the usage of stat-quad duality as a means for propagation of such solutions, we consider the simple example given by $T(v) = \frac{1}{2}|v|^2$ for $v \in I\!R^2$ and

$$-V(r,x) = -\tfrac{1}{2}x'\Omega x = -\tfrac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

In this case, $W^c(t,x,z) = \frac{1}{2}x'P_tx - x'Q_tz + \frac{1}{2}z'R_tz$, where, taking $c = 100$ in this example, $P_0 = -Q_0 = R_0 = cI = 100I$, and

$$\dot P = -P_t^2 - \Omega, \quad \dot Q_t = -P_tQ_t, \quad \dot R_t = -Q_t'Q_t. \tag{81}$$

Fixing dualizing matrix, $C \in S_n^{>0}$, which in this example is taken to be $C = 2I$, the stat-quad dual coefficients satisfy

$$\dot\alpha_t = -\alpha_t\tilde K\alpha_t - \alpha_tC^{-1}\Omega - \Omega C^{-1}\alpha_t - \Omega, \tag{82}$$

$$\dot\beta_t = -\alpha_t\tilde K\beta_t - \Omega C^{-1}\beta_t, \qquad \dot\gamma_t = -\beta_t'\tilde K\beta_t, \tag{83}$$

where $\tilde K \doteq I + C^{-1}\Omega C^{-1}$. Noting the Bernoulli substitution form (cf. [14]) for the DREs generating P, Q, R, which is a double-dimension linear system, the solutions for P, Q, R will be periodic. (It is worthwhile mentioning that the Benoulli substitution and symplectic semigroup approach can be used to obtain other means for longterm propagation of solutions to DREs, cf. [15].) Given the algebraic expressions for the coefficients of the stat-quad dual from Lemma 20, the escape times for the DREs generating α, β, γ will be periodic as well. Let the escape times for P, Q, R be $\{\tilde t_k\}_{k=1}^\infty$ and those for α, β, γ be $\{\hat t_k\}_{k=1}^\infty$. Suppose we wish to compute the solution on $[0, \hat T]$. We suppose that there do not exist finite escape times that are common to both the primal and dual space quantities (i.e., there do not exist k_P, k_α such that $\tilde t_{k_P} = \hat t_{k_\alpha} \in (0, \hat T)$). In this case, we may propagate by (42)–(43) ((81) in the example) away from the $\tilde t_k$, and by (63)–(65) ((82)–(83) in the example) in neighborhoods around the $\tilde t_k$. One may transform back and forth between the original and dual coefficients via the equations of Lemma 20. Representative components of the solutions for this example, computed using this method, appear in Figures 1 and 2. The figures are intended to give a sense of the ability to propagate across escape times.

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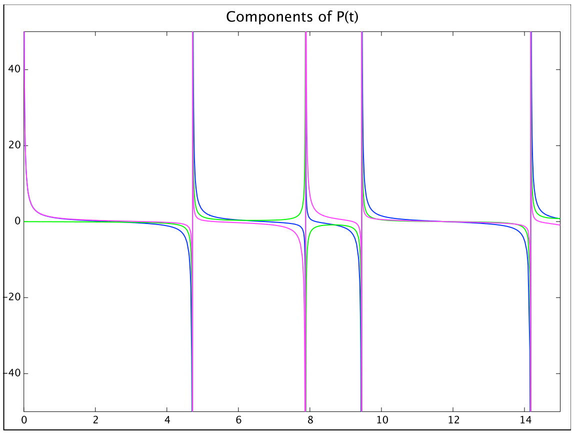

Figure 1: Three distinct components of P_t.

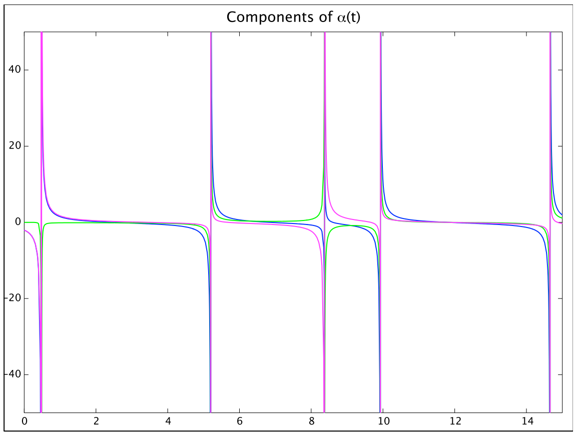

Figure 2: Three distinct components of α_t.

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6 Appendix: Proof of Theorem 2

Proof: Let $u_v^* \doteq [D\phi]^{-1}(v)$ for all $v \in \mathcal{B}$. Noting that this implies $u_v^* = \operatorname{argstat}_{u\in\mathcal{A}}[\phi(u) - \langle v,u\rangle]$, we have

$$a(v) \doteq \operatorname*{stat}_{u\in\mathcal{A}}[\phi(u) - \langle v,u\rangle] = \phi(u_v^*) - \langle v, u_v^*\rangle \quad \forall\, v \in \mathcal{B}. \tag{84}$$

By assumption and the definition of u_v^*,

$$D_v u_v^* = D_v\big\{[D\phi]^{-1}(v)\big\} \text{ exists } \forall v \in \mathcal{B}. \tag{85}$$

Fix $u \in \mathcal{A}$. Suppose $\bar{v} \doteq D\phi(u) \in \mathcal{B}$, and note that this implies

$$u_{\bar{v}}^* = u. \tag{86}$$

By (84), (85) and the assumptions, for any $v \in \mathcal{B}$,

$$D_v[a(v) + \langle v,u\rangle] = D_v[\phi(u_v^*) - \langle v, u_v^*\rangle + \langle v,u\rangle] = [D_v u_v^*](D\phi(u_v^*) - v) + u - u_v^*,$$

which by the definition of u_v^*,

$$= u - u_v^*. \tag{87}$$

By (86) and (87), $\bar{v} \in \operatorname{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle]$, which implies

$$\operatorname*{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle] \neq \emptyset, \quad \text{and} \quad u - u_{\tilde{v}}^* = 0 \;\forall \tilde{v} \in \operatorname*{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle]. \tag{88}$$

Noting that (87) implies existence, and then applying (84) and then (88), yields

$$\begin{aligned}
\operatorname*{stat}^s_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle] &= \Big\{ a(\tilde{v}) + \langle \tilde{v}, u\rangle \,\Big|\, \tilde{v} \in \operatorname*{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle] \Big\} \\
&= \Big\{ \phi(u_{\tilde{v}}^*) - \langle \tilde{v}, u_{\tilde{v}}^*\rangle + \langle \tilde{v}, u\rangle \,\Big|\, \tilde{v} \in \operatorname*{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle] \Big\} \\
&= \Big\{ \phi(u) \,\Big|\, \tilde{v} \in \operatorname*{argstat}_{v\in\mathcal{B}}[a(v) + \langle v,u\rangle] \Big\} = \phi(u).
\end{aligned}$$

□