# The principle of least action and two-point boundary value problems in orbital mechanics

Seung Hak Han and William M. McEneaney

***Abstract*— We consider a two-point boundary value problem (TPBVP) in orbital mechanics involving a small body (e.g., a spacecraft or asteroid) and $N$ larger bodies. The least action principle TPBVP formulation is converted into an initial value problem via the addition of an appropriate terminal cost to the action functional. The latter formulation is used to obtain a fundamental solution, which may be used to solve the TPBVP for a variety of boundary conditions within a certain class. In particular, the method of convex duality allows one to interpret the least action principle as a differential game, where an opposing player maximizes over an indexed set of quadratics to yield the gravitational potential. The fundamental solution is obtained as a set of solutions of associated Riccati equations.**

## I. INTRODUCTION

We examine the motion of a single body under the influence of the gravitational potential generated by $N$ other celestial bodies, where the mass of the first body is negligible relative to the masses of the other bodies, and we suppose that the $N$ large bodies are on known trajectories. The single, small body follows a trajectory satisfying the principle of stationary action (c.f., [5], [6]), where under certain conditions, the stationary-action trajectory coincides with the least-action trajectory. This allows problems in dynamics to be posed, instead, in terms of optimal control problems with vastly simplified dynamics. In particular, one may convert two-point boundary-value problems (TPBVPs) for a dynamical system into initial value problems. From the solution of a reachability problem, we develop the fundamental solution for a class of TPBVPs.

Although the gravitational potential does not take the form of a quadratic function, one may take a dynamic game approach, which allows the inner optimization problem to be posed in a linear-quadratic form. In particular, the problem is converted into a differential game where one player controls the velocity, and the opposing player controls the potential energy term (c.f., [1], [3]). It will be demonstrated that for the case where the time duration is less than a specified bound, the action functional is strictly convex in the velocity control. For any potential energy control, the minimizing trajectory is the unique stationary point, and the least action is obtained by solution of associated Riccati equations. The fundamental solution takes the form of a finite-dimensional set of Riccati equation solutions. For *any* TPBVP in a certain class, the solution is obtained by taking the maximum of a corresponding linear functional over that *same* finite-dimensional set.

Research partially supported by AFOSR.

S. H. Han and W.M. McEneaney are with the Dept. of Mechanical and Aerospace Engineering, University California San Diego, La Jolla, CA 92093-0411,USA. Emails: shh013@ucsd.edu and wmcneaney@ucsd.edu

## II. PROBLEM STATEMENT AND FUNDAMENTAL SOLUTION

We consider a single body, moving among a set of $N$ other bodies in space, $I\!R^n = I\!R^3$. The only forces to be considered are gravitational. The single body has negligible mass in relation to the masses of the other bodies, and consequently has no effect on their motion. In particular, we suppose that the $N$ bodies are moving along already-known trajectories. We will obtain fundamental solutions of TPBVPs for the motion of the small body. Although the small body might be any object, for the sake of concreteness, henceforth we will refer to it as the "spacecraft".

The set of $N$ bodies may be indexed as $\mathcal{N} \doteq ]1, N[ \doteq \{1, 2, \cdots, N\}$. Throughout, for integers $a \le b$, we will use $]a, b[$ to denote $\{a, a+1, \cdots, b-1, b\}$. Let $\rho_i$ and $R_i$ denote the (uniform) density and radius of each body for $i \in \mathcal{N}$. Obviously, the mass of each body is given by $m_i = \frac{4}{3}\pi\rho_i R_i^3$. Let $\zeta_r^i \doteq \zeta^i(r)$ denote the position of the center of body $i$ at time $r \in [0, \infty)$. We suppose that $\zeta \doteq \{\zeta^i\}_{i\in\mathcal{N}} \in \widehat{\mathcal{Z}} \doteq \{\{\zeta^i\}_{i\in\mathcal{N}} \,|\, \zeta^i \in C([0,\infty); I\!R^n) \quad \forall i \in \mathcal{N}\}$. We assume that collision between bodies does not occur. So, letting $\mathcal{Y} \doteq \{\{y^i\}_{i\in\mathcal{N}} \in I\!R^{nN} \,|\, |y^i - y^j| > 2\overline{R} \;\; \forall i \neq j\}$, we define the subset of $\widehat{\mathcal{Z}}$ as $\mathcal{Z} \doteq \{\zeta \in \widehat{Z} \,|\, \zeta \in C([0,\infty); \mathcal{Y})\}$ where $\overline{R} \doteq \max_{i\in\mathcal{N}} R_i$.

For simplicity, the spacecraft is considered as a point particle with mass $m_s$. Suppose that the position of the spacecraft at time $r$ is denoted by $\xi_r$, where also, we will use $x \in I\!R^n$ to denote generic position values. Let $t > 0$. We model the dynamics of the spacecraft position as

$$\dot{\xi}_r = u_r, \quad \xi_0 = x, \quad \forall r \in (0,t), \tag{1}$$

where $u = u_\cdot \in \mathcal{U}^{0,t}$ with $\mathcal{U}^{s,t} \doteq L_2([s,t);\mathbb{R}^n)$.

The kinetic energy, $\widehat{T}$, is given by

$$\widehat{T}(v) \doteq \tfrac{1}{2} v^T [m_s I_n] v = \tfrac{1}{2} m_s |v|^2 \quad \forall v \in \mathbb{R}^n, \tag{2}$$

where $I_n$ denotes the identity matrix of size $n$.

Given $i \in \mathcal{N}$ and $Y \doteq \{y^i\}_{i\in\mathcal{N}} \in \mathcal{Y}$, the potential energy between the spacecraft at $x$ and body $i$ at $y^i$, $\widehat{\widehat{V}}_i(x,y^i)$, is given by

$$-\widehat{\widehat{V}}_i(x,y^i) \doteq \begin{cases} G m_i m_s \frac{3R_i^2 - |x-y^i|^2}{2R_i^3} & \text{if } x \in B_{R_i}(y^i), \\ \frac{G m_i m_s}{|x-y^i|} & \text{if } x \notin B_{R_i}(y^i), \end{cases}$$

where $G$ is the universal gravitational constant. We define the total potential energy $\widehat{\widehat{V}} : \mathbb{R}^n \times \mathcal{Y} \to \mathbb{R}^n$ as $\widehat{\widehat{V}}(x,Y) \doteq \sum_{i\in\mathcal{N}} \widehat{\widehat{V}}_i(x,y^i)$. We remark that we include the gravitational potential here within the extended bodies as the finiteness and smoothness of the potential are relevant at technical points in the theory, in spite of the infeasibility of spacecraft trajectories that pass through the bodies.

We remind the reader that we will obtain fundamental solutions for the TPBVPs through a game-theoretic formulation. The game will appear through application of a generalization of convex duality to a control-problem formulation. With that in mind, we define the action functional, $J^0 : [0,\infty) \times \mathbb{R}^n \times \mathcal{U}^{0,t} \times \mathcal{Z} \to \mathbb{R}$, as

$$J^0(t,x,u,\zeta) \doteq \int_0^t T(u_r) - \overline{V}(\xi_r,\zeta_r)\, dr, \tag{3}$$

where $\overline{V} \doteq \widehat{\widehat{V}}/m_s$ and $T \doteq \widehat{T}/m_s$.

Attaching a terminal cost to $J^0$ will yield a TPBVP, where we will control the terminal condition in the TPBVP with this terminal cost, and we will have initial condition $\xi_0 = x$. For background on this approach to TPBVPs for conservative systems, see [11]. Given generic terminal cost, $\bar{\psi} : \mathbb{R}^n \to \mathbb{R}$, let

$$\begin{aligned} \bar{J}(t,x,u,\zeta) &\doteq J^0(t,x,u,\zeta) + \bar{\psi}(\xi_t), \\ \overline{W}(t,x,\zeta) &\doteq \inf_{u\in\mathcal{U}^{0,t}} \left\{ \bar{J}(t,x,u,\zeta) \right\}. \end{aligned} \tag{4}$$

Suppose that the desired destination of the spacecraft is denoted by $z \in \mathbb{R}^n$, i.e., $\xi_t = z$. Then, we define a reachability problem of interest via the value function $\widetilde{W} : [0,\infty) \times \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} &\widetilde{W}(t,x,\zeta,z) \\ &\doteq \inf_{u\in\mathcal{U}^{0,t}} \left\{ J^0(t,x,u,\zeta) \;\middle|\; \begin{array}{c} \text{(1) holds with} \\ \xi_0 = x, \xi_t = z \end{array} \right\}, \end{aligned} \tag{5}$$

and the function $\widehat{W} : [0,\infty) \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}$ by

$$\widehat{W}(t,x,\zeta) \doteq \inf_{z\in\mathbb{R}^n} \left\{ \widetilde{W}(t,x,\zeta,z) + \bar{\psi}(z) \right\}. \tag{6}$$

*Theorem 2.1:* The value function $\overline{W}$ of (4) and the function $\widehat{W}$ of (6) are equivalent. That is,

$$\overline{W}(t,x,\zeta) = \widehat{W}(t,x,\zeta)$$

for all $t \geq 0$, $x \in \mathbb{R}^n$, and $\zeta \in \mathcal{Z}$.

*We note that due to paper-length issues, we include only a subset of the proofs here. In particular, we include critical, non-trivial proofs only.*

It is seen that given $\widetilde{W}$, the value function $\overline{W}$ of (4) for any terminal cost $\bar{\psi}$ can be evaluated via (6). Therefore, $\widetilde{W}$ may be regarded as fundamental to the solution of TPBVPs arising from the given orbital dynamics. For the development of the fundamental solution, it is useful to firstly introduce a terminal cost which takes the form of min-plus delta-function.

Let $\psi^\infty : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ (where we take $[0,\infty] \doteq [0,\infty) \cup \{+\infty\}$) be given by

$$\psi^\infty(y,z) = \delta^-(y-z),$$

where $\delta^-$ denotes the min-plus "delta-function" (c.f., [4], [10]) given by

$$\delta^-(y) \doteq \begin{cases} 0 & \text{if } y = 0, \\ \infty & \text{otherwise.} \end{cases}$$

We define the finite time-horizon payoff, $\bar{J}^\infty : [0,\infty) \times \mathbb{R}^n \times \mathcal{U}^{0,t} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, by

$$\bar{J}^\infty(t,x,u,\zeta,z) \doteq J^0(t,x,u,\zeta) + \psi^\infty(\xi_t,z),$$

and the corresponding value function as

$$\overline{W}^\infty(t,x,\zeta,z) = \inf_{u\in\mathcal{U}^{0,t}} \bar{J}^\infty(t,x,u,\zeta,z), \tag{7}$$

where $\xi$ satisfies (1). Then, we have:

*Theorem 2.2:* The value function $\overline{W}^\infty$ of (7) and $\widetilde{W}$ of (5) are equivalent. That is,

$$\overline{W}^\infty(t,x,\zeta,z) = \widetilde{W}(t,x,\zeta,z) < \infty \tag{8}$$

for all $t > 0$, $x, z \in \mathbb{R}^n$ and $\zeta \in \mathcal{Z}$.

## III. Optimal control problem

For the development of fundamental solutions to optimal control problem (4), we will define the value function $\overline{W}^c$ with the quadratic form of terminal cost $\psi^c$ and demonstrate that the limit property such that $\lim_{c\to\infty} \overline{W}^c = \overline{W}^\infty$ holds.

For $c \in [0,\infty)$, let $\psi^c : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be given by

$$\psi^c(x,z) \doteq \frac{c}{2}|x-z|^2.$$

We define the finite time-horizon payoff, $\bar{J}^c : [0,\infty) \times I\!R^n \times \mathcal{U}^{0,t} \times \mathcal{Z} \times I\!R^n \to I\!R$, by

$$\bar{J}^c(t,x,u,\zeta,z) \doteq J^0(t,x,u,\zeta) + \psi^c(\xi_t,z), \quad (9)$$

where $J^0$ is given by (3) and its corresponding value function by

$$\overline{W}^c(t,x,\zeta,z) \doteq \inf_{u\in\mathcal{U}^{0,t}} \bar{J}^c(t,x,u,\zeta,z), \quad (10)$$

where (1) holds.

From the definition of $\overline{V}$, one can easily see that there exist $K_L, K_L^1 < \infty$ such that

$$|-\overline{V}(x,Y) + \overline{V}(\hat{x},Y)| \le K_L|x-\hat{x}|, \quad (L1)$$

$$|\overline{V}(x,Y)| \le K_L^1(1+|x|) \quad (L2)$$

for all $x, \hat{x} \in I\!R^n$ and $Y \in \mathcal{Y}$.

### A. A limit property

Let $t > 0$. Suppose that given $x, z \in I\!R^n$, the straight-line control from $x$ to $z$ is given by $u_r^s \doteq (1/t)[z-x]$ for all $r \in [0,t]$, and we let the corresponding trajectory be denoted by $\xi^s$. Then,

$$\begin{aligned}|\xi_r^s| &= \left|x + \int_0^r u_\rho^s\,d\rho\right| \quad (11)\\ &\le |x| + \frac{r}{t^2}|z-x| \le \left(1+\frac{r}{t^2}\right)|x| + \frac{r}{t^2}|z| \\ &\le (1+t^{-1})(|x|+|z|) \doteq D_1(t)(|x|+|z|)\end{aligned}$$

for all $r \in [0,t]$. Hence, by $(L2)$,

$$\begin{aligned}-\int_0^t \overline{V}(\xi_r^s,\zeta_r)\,dr &\le K_L^1 \int_0^t (1+|\xi_r^s|)\,dr \\ &= K_L^1\left[t + \int_0^t |\xi_r^s|\,dr\right],\end{aligned}$$

which by (11),

$$\le K_L^1\left[t + tD_1(t)(|x|+|z|)\right],$$

which since $D_1(t) > 1$ for all $t > 0$,

$$\begin{aligned}&\le K_L^1 tD_1(t)(1+|x|+|z|)\\ &= K_L^1(t+1)(1+|x|+|z|)\\ &\doteq D_2(t)(1+|x|+|z|). \quad (12)\end{aligned}$$

Noting that $\xi_t^s = z$, given $c \in [0,\infty)$,

$$\widetilde{W}^s(t,x,\zeta,z) \doteq \bar{J}^c(t,x,u^s,\zeta,z) = J^0(t,x,u^s,\zeta),$$

which by the definition of $u^s$ and (12),

$$\begin{aligned}&\le \tfrac{1}{2t}|z-x|^2 + D_2(t)(1+|x|+|z|)\\ &\le \tfrac{1}{2t}||x|+|z||^2 + D_2(t)(1+|x|+|z|),\end{aligned}$$

which since $2ab \le a^2+b^2$ for all $a,b \ge 0$,

$$\begin{aligned}&\le \tfrac{1}{t}[|x|^2+|z|^2]\\ &\quad + D_2(t)(2+\tfrac{1}{2}|x|^2+\tfrac{1}{2}|z|^2),\end{aligned}$$

which implies that there exists $D_3 = D_3(t) < \infty$ such that

$$\widetilde{W}^s(t,x,\zeta,z) \doteq \bar{J}^c(t,x,u^s,\zeta,z) \le D_3(1+|x|^2+|z|^2). \quad (13)$$

Suppose that given $c \in [0,\infty)$ and $\varepsilon \in (0,1]$, there exists $u^{c,\varepsilon} \in \mathcal{U}^{0,t}$ such that

$$\bar{J}^c(t,x,u^{c,\varepsilon},\zeta,z) \le \overline{W}^c(t,x,\zeta,z) + \varepsilon. \quad (14)$$

Let $\xi^{c,\varepsilon}$ be the trajectory corresponding to $u^{c,\varepsilon}$. By the non-negativity of $T$ and $-\overline{V}$,

$$\begin{aligned}\tfrac{c}{2}|\xi_t^{c,\varepsilon} - z|^2 &\le \bar{J}^c(t,x,u^{c,\varepsilon},\zeta,z)\\ &\le \overline{W}^c(t,x,\zeta,z) + \varepsilon,\end{aligned}$$

which by the sub-optimality of $u^s$ with respect to $\overline{W}^c$,

$$\le \widetilde{W}^s(t,x,\zeta,z) + \varepsilon,$$

which by (13),

$$\le D_3(1+|x|^2+|z|^2) + 1,$$

which since $a^2+b^2 \le (a+b)^2$ for $a,b \ge 0$,

$$\begin{aligned}&\le D_3(1+|x|+|z|)^2 + 1\\ &\le (\sqrt{D_3}(1+|x|+|z|)+1)^2, \quad (15)\end{aligned}$$

which implies that there exists $\tilde{D} = \tilde{D}(t) < \infty$ such that

$$|\xi_t^{c,\varepsilon} - z| \le \frac{\tilde{D}(1+|x|+|z|)}{\sqrt{c}}. \quad (16)$$

Let

$$\hat{u}_r^{c,\varepsilon} \doteq u_r^{c,\varepsilon} + \tfrac{1}{t}[z - \xi_t^{c,\varepsilon}], \quad \forall r \in [0,t], \quad (17)$$

which yields $\hat{\xi}_t^{c,\varepsilon} = z$ where $\hat{\xi}^{c,\varepsilon}$ denotes the trajectory corresponding to $\hat{u}^{c,\varepsilon}$. Then, by (16) and (17),

$$|\xi_r^{c,\varepsilon} - \hat{\xi}_r^{c,\varepsilon}| \le \frac{1}{t}\int_0^r |z - \xi_t^{c,\varepsilon}|\,d\rho = \frac{r\tilde{D}(1+|x|+|z|)}{t\sqrt{c}} \quad (18)$$

for all $r \in [0,t]$. Also, by $(L1)$ and (18),

$$\begin{aligned}&\left|\int_0^t -\overline{V}(\xi_r^{c,\varepsilon},\zeta_r) + \overline{V}(\hat{\xi}_r^{c,\varepsilon},\zeta_r)\,dr\right| \quad (19)\\ &\le K_L\int_0^t |\xi_r^{c,\varepsilon} - \hat{\xi}_r^{c,\varepsilon}|\,dr \le \frac{K_L\tilde{D}(1+|x|+|z|)t}{2\sqrt{c}}.\end{aligned}$$

By non-negativity of $-\overline{V}$ and $\psi^c$,

$$\begin{aligned}\tfrac{1}{2}\|u^{c,\varepsilon}\|^2_{L_2(0,t)} = \int_0^t T(u_r^{c,\varepsilon})\,dr &\le \bar{J}^c(t,x,u^{c,\varepsilon},\zeta,c)\\ &\le \overline{W}^c(t,x,\zeta,z) + \varepsilon\end{aligned}$$

which implies that by (15) and (16),

$$\|u^{c,\varepsilon}\|_{L_2(0,t)} \le \tilde{D}(1+|x|+|z|). \quad (20)$$

Noting that $\left||a|^2-|b|^2\right| < |a-b|\,[|a|+|b|]$ for $a,b \in I\!R^n$,

$$\begin{aligned}
&\left|\int_0^t T(u_r^{c,\varepsilon}) - T(\hat{u}_r^{c,\varepsilon})\,dr\right| \\
&\quad \le \frac{1}{2}\int_0^t |u_r^{c,\varepsilon}-\hat{u}_r^{c,\varepsilon}|[|u_r^{c,\varepsilon}|+|\hat{u}_r^{c,\varepsilon}|]\,dr,
\end{aligned}$$

which by (17) and the triangle inequality,

$$\le \frac{1}{2t}|z-\xi_t^{c,\varepsilon}|\int_0^t 2|u_r^{c,\varepsilon}| + \frac{1}{t}|z-\xi_t^{c,\varepsilon}|\,dr,$$

which by applying Hölder's inequality,

$$\le \frac{1}{2t}|z-\xi_t^{c,\varepsilon}|\left[2\sqrt{t}\|u^{c,\varepsilon}\|_{L_2(0,t)} + |z-\xi_t^{c,\varepsilon}|\right],$$

which by (16) and (20),

$$\begin{aligned}
&\le \left[\frac{\tilde{D}^2}{\sqrt{c\,t}} + \frac{\tilde{D}^2}{2c\,t}\right](1+|x|+|z|)^2 \\
&\le \frac{\hat{D}(t)(1+|x|+|z|)^2}{\sqrt{c}}, \qquad (21)
\end{aligned}$$

for all $x,z\in I\!R^n$ and all $c\in[\frac{1}{4},\infty)$ with an appropriate choice of $\hat{D}=\hat{D}(t)<\infty$. Therefore, by (19), (21), and non-negativity of $\psi^c$, we have

$$\begin{aligned}
&\bar{J}^c(t,x,u^{c,\varepsilon},\zeta,z) - \bar{J}^c(t,x,\hat{u}^{c,\varepsilon},\zeta,z) \\
&\quad \ge -\frac{\hat{D}(1+|x|+|z|)^2}{\sqrt{c}} - \frac{K_L\tilde{D}(1+|x|+|z|)t}{2\sqrt{c}} \\
&\quad \ge -\frac{D_4(t)(1+|x|+|z|)^2}{\sqrt{c}}
\end{aligned}$$

for proper choice of $D_4(t)<\infty$. This implies

$$\begin{aligned}
&\bar{J}^c(t,x,u^{c,\varepsilon},\zeta,z) \\
&\quad \ge \overline{W}^\infty(t,x,\zeta,z) - \frac{D_4(t)(1+|x|+|z|)^2}{\sqrt{c}},
\end{aligned}$$

and further, since this is true for all $\varepsilon\in(0,1]$, by (14),

$$\begin{aligned}
&\overline{W}^c(t,x,\zeta,z) \\
&\quad \ge \overline{W}^\infty(t,x,\zeta,z) - \frac{D_4(t)(1+|x|+|z|)^2}{\sqrt{c}}. \qquad (22)
\end{aligned}$$

Further, by the definitions of $\overline{W}^c$ and $\overline{W}^\infty$ and (22),

$$\begin{aligned}
&\overline{W}^\infty(t,x,\zeta,z) - \frac{D_4(t)(1+|x|+|z|)^2}{\sqrt{c}} \\
&\qquad \le \overline{W}^c(t,x,\zeta,z) \le \overline{W}^\infty(t,x,\zeta,z)
\end{aligned}$$

for all $x,z\in I\!R^n$, $\zeta\in\mathcal{Z}$, $t>0$, and $c\ge\frac{1}{4}$. This immediately implies:

*Lemma 3.1:* For $t\in(0,\bar{t})$,

$$\begin{aligned}
\overline{W}^\infty(t,x,\zeta,z) &= \lim_{c\to\infty}\overline{W}^c(t,x,\zeta,z) \\
&= \sup_{c\in[0,\infty)}\overline{W}^c(t,x,\zeta,z).
\end{aligned}$$

where the convergence is uniform on compact subsets of $[0,\bar{t})\times I\!R^n\times\mathcal{Z}\times I\!R^n$.

### *B. Dynamic game approach to gravitation*

Recall that $\overline{V}$ does not take a linear-quadratic form in the position variable. In the case where the potential energy takes a linear-quadratic form, the fundamental solution will be obtained through the solution of associated Riccati equations. Therefore, one may take a dynamic game approach to gravitation, whereby opponent controls maximize over an indexed set of quadratics to yield the potential. This requires an additional max-plus integral, over the opponent controls, beyond that which is required in the purely linear-quadratic potential case. Consider $f:(0,\infty)\to I\!R$ is given by $f(\hat{d})=\hat{d}^{-1/2}$. Then, by the convex duality (c.f., [7], [8], [9]), we may represent

$$f(\hat{d}) = \sup_{\hat{\beta}<0}\left[\hat{\beta}\hat{d}+a(\hat{\beta})\right] \quad \forall\hat{d}\in(0,\infty), \qquad (23)$$

where

$$a(\hat{\beta}) = -\sup_{\hat{d}>0}\left[\hat{\beta}\hat{d}-f(\hat{d})\right] \quad \forall\hat{\beta}\in(-\infty,0).$$

Further, since $\hat{\beta}\hat{d}-f(\hat{d})$ is strictly convex, the maximum is achieved at $\hat{d}=(2\hat{\beta})^{-3/2}$ so that

$$a(\hat{\beta}) = -\tfrac{3}{2}(2\hat{\beta})^{1/3} \quad \forall\hat{\beta}\in(-\infty,0).$$

Letting $\beta\doteq-\hat{\beta}$ and $d^2=\hat{d}$, (23) becomes

$$\tfrac{1}{d} = \sup_{\beta>0}\left[\tfrac{3}{2}(2\beta)^{1/3}-\beta d^2\right], \quad \forall d>0. \qquad (24)$$

We note that $\frac{3}{2}(2\beta)^{1/3}-\beta d^2$ is strictly concave in $\beta>0$ and the maximum is attained at $\beta=\frac{1}{2}d^{-3}$. Therefore, we may rewrite (24) as

$$\tfrac{1}{d} = \sup_{\beta\in(0,1/2d^{-3}]}\left[\tfrac{3}{2}(2\beta)^{1/3}-\beta d^2\right], \quad \forall d>0. \qquad (25)$$

For the case that $|x-y^i|\ge R_i$, applying (25) to (3),

$$\begin{aligned}
-\overline{V}_i(x,y^i) &= \frac{Gm_i}{|x-y^i|} \qquad (26)\\
&= Gm_i\sup_{\beta\in\left(0,1/2R_i^{-3}\right]}\left[\tfrac{3}{2}(2\beta)^{1/3}-\beta|x-y^i|^2\right].
\end{aligned}$$

We note that for $x\in B_{R_i}(y^i)$, letting $\bar{\beta}\doteq\frac{1}{2}R_i^{-3}$, we also obtain the same form as the argument of supremum in (26), i.e.,

$$\begin{aligned}
-\overline{V}_i(x,y^i) &= Gm_i\frac{3R_i^2-|x-y^i|^2}{2R_i^3} \\
&= Gm_i\left[\tfrac{3}{2}(2\bar{\beta})^{1/3}-\bar{\beta}|x-y^i|^2\right]. \qquad (27)
\end{aligned}$$

Therefore, combining (26) and (27) yields

$$\begin{aligned}&-\overline{V}_i(x,y^i)\\&\quad = Gm_i \sup_{\beta\in\left(0,1/2R_i^{-3}\right]}\left[\tfrac{3}{2}(2\beta)^{1/3}-\beta|x-y^i|^2\right]\end{aligned}$$

for all $x, y^i \in I\!R^n$. Letting $\hat{\alpha} \doteq \sqrt{\frac{2}{3}}(2\beta)^{1/3}$, we find

$$-\overline{V}_i(x,y^i) = \sup_{\hat{\alpha}\in(0,\sqrt{2/3}R_i^{-1}]} \mu_i\left[\hat{\alpha}-\frac{\hat{\alpha}^3}{2}|x-y^i|^2\right] \tag{28}$$

for all $x, y^i \in I\!R^n$ where $\mu_i \doteq Gm_i\left(\frac{3}{2}\right)^{3/2}$.

### *C. Revisit payoff*

Let

$$\mathcal{A} \doteq \{\alpha = \{\alpha^i\}_{i\in\mathcal{N}} \,|\, \alpha^i \in (0,\sqrt{2/3}R_i^{-1}] \quad \forall i\in\mathcal{N}\}.$$

Then, using (28), the potential energy, $-\overline{V}$, may be represented by

$$-\overline{V}(x,Y) = -\sum_{i\in\mathcal{N}}\overline{V}_i(x,Y) = \max_{\alpha\in\mathcal{A}}\{-\widehat{V}(x,Y,\alpha)\} \tag{29}$$

where

$$-\widehat{V}(x,Y,\alpha) \doteq \sum_{i\in\mathcal{N}}\mu_i\left[\alpha^i-\frac{(\alpha^i)^3}{2}|x-y^i|^2\right].$$

Further, (9) may be written as

$$\begin{aligned}&\bar{J}^c(t,x,u,\zeta,z)\\&= \int_0^t T(u_r)-\overline{V}(\xi_r,\zeta_r)\,dr+\psi^c(\xi_t,z) \qquad (30)\\&= \int_0^t T(u_r)+\max_{\alpha\in\mathcal{A}}\{-\widehat{V}(\xi_r,\zeta_r,\alpha)\}\,dr+\psi^c(\xi_t,z). \qquad (31)\end{aligned}$$

Given $t>0$, let $\mathcal{A}^t \doteq C([0,t];\mathcal{A})$. Also, we replace the time-indenpendent potential energy function, $\widehat{V}$, with

$$\begin{aligned}-V^\alpha(r,x,Y) &\doteq -\widehat{V}(x,Y,\alpha_r)\\&= \sum_{i=1}^N \mu_i\left[\alpha_r^i-\frac{(\alpha_r^i)^3}{2}|x-y^i|^2\right]. \qquad (32)\end{aligned}$$

Given $c\in[0,\infty)$, let $J^c : [0,\infty)\times I\!R^n\times\mathcal{U}^{0,t}\times\mathcal{A}^t\times\mathcal{Z}\times I\!R^n \to I\!R$ be given by

$$\begin{aligned}&J^c(t,x,u,\alpha,\zeta,z)\\&\quad\doteq \int_0^t T(u_r)-V^\alpha(r,\xi_r,\zeta_r)\,dr+\psi^c(\xi_t,z). \qquad (33)\end{aligned}$$

*Theorem 3.2:* Let $t\geq 0$, $x,z\in I\!R^n$, and $\zeta\in\mathcal{Z}$. For any $u\in\mathcal{U}^{0,t}$,

$$\bar{J}^c(t,x,u,\zeta,z) = \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) \tag{34}$$

Further,

$$\overline{W}^c(t,x,\zeta,z) = \inf_{u\in\mathcal{U}^{0,t}}\max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z). \tag{35}$$

*Proof:* Given $u\in\mathcal{U}^{0,t}$, $\xi$ denotes the state trajectory corresponding to $u$ with $\xi_0 = x\in I\!R^n$. Given $t\geq 0$, $x,z\in I\!R^n$ and $\zeta\in\mathcal{Z}$, any $\alpha_r$ is suboptimal in the maximization in (31) for any $r\in[0,t]$ and any $\alpha\in\mathcal{A}^t$, and in particular,

$$\begin{aligned}&\bar{J}^c(t,x,u,\zeta,z)\\&\geq \int_0^t T(u_r)+\max_{\alpha\in\mathcal{A}^t}\{-\widehat{V}(\xi_r,\zeta_r,\alpha_r)\}\,dr+\psi^c(\xi_t,z)\\&= \max_{\alpha\in\mathcal{A}^t}\int_0^t T(u_r)-V^\alpha(r,\xi_r,\zeta_r)\,dr+\psi^c(\xi_t,z)\\&= \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z). \qquad (36)\end{aligned}$$

Let $Y\in\mathcal{Y}$ and $\bar{\alpha}^* : I\!R^n\times I\!R^{nN}\to\mathcal{A}$ be given by $\bar{\alpha}^*(x,Y)\doteq\{[\bar{\alpha}^*]^i(x,y^i)\}_{i\in\mathcal{N}}$ where

$$[\bar{\alpha}^*]^i(x,y^i) \doteq \operatorname*{argmax}_{\alpha\in(0,\sqrt{2/3}R_i^{-1}]}\mu_i\left[\alpha-\frac{\alpha^3}{2}|x-y^i|^2\right] \tag{37}$$

for all $x,y^i\in I\!R^n$ and all $i\in\mathcal{N}$. Let $\alpha_r^* = \alpha^*(r;u_\cdot,\zeta_\cdot) = \{[\alpha_r^*]^i|\ i\in\mathcal{N}\}$ where $i^{th}$ element of $\alpha^*$ is given by

$$[\alpha_r^*]^i = [\bar{\alpha}^*]^i(\xi_r,\zeta_r^i) \quad \forall r\in[0,t]. \tag{38}$$

Given $s,r\in[0,t]$ and $i\in\mathcal{N}$, let $d_r^i\doteq|\xi_r-\zeta_r^i|$ and $d_s^i\doteq|\xi_s-\zeta_s^i|$. The continuity of trajectories implies that given $\varepsilon_d>0$, there exists $\delta_d^i=\delta_d^i(\varepsilon_d)<\infty$ such that $|r-s|<\delta_d^i$ implies $|d_r^i-d_s^i|<\varepsilon_d$. Assuming $R_i>1$, one can easily see that given $\varepsilon_d>0$, $|[\alpha_r^*]^i-[\alpha_s^*]^i|<\varepsilon_d$ if $|r-s|<\delta_d^i$. Since this is true for all $r,s\in[0,t]$ and $i\in\mathcal{N}$, $\alpha^*$ is continuous on $[0,t]$ so that $\alpha^*\in\mathcal{A}_t^c$. Hence, by (29), (32), (37) and (38),

$$-\overline{V}(\xi_r,\zeta_r) = -V^{\alpha^*}(r,\xi_r,\zeta_r) \quad \forall r\in[0,t] \tag{39}$$

and by (30)–(31), (33), and (39),

$$\begin{aligned}\bar{J}^c(t,x,u,\zeta,z) &= J^c(t,x,u,\alpha^*,\zeta,z)\\&\leq \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) \qquad (40)\end{aligned}$$

Consequently, combining (36) and (40) yields (34), which immediately implies (35) ∎

### *D. Existence of optimal u*

*Lemma 3.3:* Let $t\in(0,\infty)$, $x,z\in I\!R^n$, $c\in[0,\infty)$, and $\zeta\in\mathcal{Z}$. Let $u^\dagger\in\mathcal{U}^{0,t}$ and the corresponding trajectory be denoted by $\xi^\dagger$. Let $\alpha_r^* = \alpha^*(r;u_\cdot^\dagger,\zeta_\cdot)\doteq\alpha^*(\xi_r^\dagger,\zeta_r)$ for all $r\in[0,t]$ where $\bar{\alpha}^*$ is given in (37). Then, $u^\dagger$ is a critical point of $\bar{J}^c(t,x,\cdot,\zeta,z)$ if and only if $u^\dagger$ is a critical point of $J^c(t,x,\cdot,\alpha^*,\zeta,z)$.

*Lemma 3.4:* Let

$$\bar{t} \doteq \left[ \sum_{i\in\mathcal{N}} \frac{Gm_i}{R_i^3} \right]^{-1/2}.$$

Let $x, z \in I\!\!R^n$, $c \in [0,\infty)$, $\alpha \in \mathcal{A}^t$, and $\zeta \in \mathcal{Z}$. Then, if $t \in (0,\bar{t})$, $J^c(t,x,u,\alpha,\zeta,z)$ is strictly convex in $u$.

*Proof:* Let $u, \nu \in \mathcal{U}^{0,t}$ and $\delta > 0$. In order to prove the convexity of $J^c$, we need to examine second-order differences in the direction of $\nu$ from $u$. Note that $\xi$ denotes the state trajectory corresponding to $u$ and $\xi^+$ denotes the one corresponding to $u+\delta\nu$ given by

$$\xi_r^+ = \xi_r + \delta \int_0^r \nu_\rho \, d\rho \quad \forall r \in (0,t). \tag{41}$$

We examine each of differences separately. The first-order difference of kinetic energy term is given by

$$\begin{aligned} T(u_r+\delta\nu_r) - T(u_r) &= \tfrac{1}{2}|u_r+\delta\nu_r|^2 - \tfrac{1}{2}|u_r|^2 \\ &= \delta v_r^T u_r + \frac{\delta^2}{2}|\nu_r|^2 \end{aligned} \tag{42}$$

for $r \in [0,t]$. Given $\alpha \in \mathcal{A}^t$,

$$\begin{aligned} &-V^\alpha(r,\xi_r^+,\zeta_r) + V^\alpha(r,\xi_r,\zeta_r) \\ &= \sum_{i\in\mathcal{N}} \frac{\mu_i(\alpha_r^i)^3}{2} \left[ |\xi_r-\zeta_r^i|^2 - |\xi_r^+-\zeta_r^i|^2 \right] \end{aligned}$$

which by (41),

$$\begin{aligned} &= -\delta \sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 (\xi_r-\zeta_r^i)^T \left( \int_0^r \nu_\rho \, d\rho \right) \\ &\quad -\frac{\delta^2}{2} \sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 \left| \int_0^r \nu_\rho \, d\rho \right|^2 \end{aligned} \tag{43}$$

Lastly, the first-order difference of terminal cost term is given by

$$\begin{aligned} &\psi^c(\xi_t^+,z) - \psi^c(\xi_t,z) \\ &= \delta c(\xi_t - z)^T \int_0^t \nu_r \, dr + \frac{\delta^2 c}{2} \left| \int_0^t \nu_r \, dr \right|^2. \end{aligned} \tag{44}$$

Using (42) – (44) and its analogous version with $-\delta\nu$ replacing $\delta\nu$, we obtains

$$\begin{aligned} &[J^c(t,x,u+\delta\nu,\alpha,\zeta,z) + J^c(t,x,u-\delta\nu,\alpha,\zeta,z) \\ &\quad -2J^c(t,x,u,\alpha,\zeta,z)]/2 \\ &= \frac{\delta^2}{2} \int_0^t |v_r|^2 - \sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 \left| \int_0^r \nu_\rho \, d\rho \right|^2 dr \\ &\quad + \frac{\delta^2 c}{2} \left| \int_0^t \nu_r \, dr \right|^2 \\ &\geq \frac{\delta^2}{2} \int_0^t |v_r|^2 - 2 \sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 \left[ \int_0^r |\nu_\rho| \, d\rho \right]^2 dr \end{aligned}$$

which by Hölder inequality for the inner integral,

$$\geq \frac{\delta^2}{2} \int_0^t |\nu_r|^2 - 2r \sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 \int_0^r |\nu_\rho|^2 \, d\rho \, dr. \tag{45}$$

By integration by parts,

$$\begin{aligned} &\int_0^t 2r \int_0^r |\nu_\rho|^2 \, d\rho \, dr \\ &= t^2 \int_0^t |\nu_r|^2 \, dr - \int_0^t r^2 \frac{d}{dr} \int_0^r |\nu_\rho|^2 \, d\rho \, dr \\ &= t^2 \int_0^t |\nu_r|^2 \, dr - \int_0^t r^2 |\nu_r|^2 \, dr \leq t^2 \int_0^t |\nu_r|^2 \, dr. \end{aligned} \tag{46}$$

We note that since $\alpha_r^i \in (0, \sqrt{2/3}R_i^{-1}]$ for all $r \in [0,t]$ and $i \in \mathcal{N}$,

$$\sum_{i\in\mathcal{N}} \mu_i(\alpha_r^i)^3 \leq \sum_{i\in\mathcal{N}} \frac{Gm_i}{R_i^3} \doteq K_R. \tag{47}$$

Employing (46) and (47) into (45) yields

$$\begin{aligned} &[J^c(t,x,u+\delta\nu,\alpha,\zeta,z) + J^c(t,x,u-\delta\nu,\alpha,\zeta,z) \\ &\quad -2J^c(t,x,u,\alpha,\zeta,z)]/2 \\ &\geq \frac{\delta^2}{2} \left\{ \int_0^t |\nu_r|^2 \, dr - K_R t^2 \int_0^t |\nu_r|^2 \, dr \right\} \\ &\geq \frac{\delta^2}{2} \left[ 1 - K_R t^2 \right] \int_0^t |\nu_r|^2 \, dr > 0 \end{aligned}$$

if $t < \bar{t} = \sqrt{1/K_R}$ and $\nu \neq 0$. Since this is true for all direction $\nu$ from any $u \in \mathcal{U}^{0,t}$, we see that $J^c(t,x,u,\alpha,\zeta,z)$ is strictly convex in $u$. ■

*Corollary 3.5:* Let $x, z \in I\!\!R^n$, $c \in [0,\infty)$, and $\zeta \in \mathcal{Z}$. Then, if $t \in (0,\bar{t})$, $\bar{J}^c(t,x,u,\zeta,z)$ is strictly convex in $u$.

Combining Lemma 3.3, 3.4 and Corollay 3.5 immediately yields the following.

*Lemma 3.6:* For $t \in (0,\bar{t})$ and $c \in [0,\infty)$, $u^*$ is the optimal control for $\bar{J}^c(t,x,\cdot,\zeta,z)$ if and only if $u^*$ is the optimal control for $J^c(t,x,\cdot,\alpha^*,\zeta,z)$ where $\alpha^*$ is given by (38).

We now proceed to consider the game where the order of infimum and supremum are reversed. For $c \in [0,\infty]$, let

$$\begin{aligned} \underline{W}^c(t,x,\zeta,z) &\doteq \sup_{\alpha\in\mathcal{A}^t} \inf_{u\in\mathcal{U}^{0,t}} J^c(t,x,u,\alpha,\zeta,z) \\ &\doteq \sup_{\alpha\in\mathcal{A}^t} \mathcal{W}^{\alpha,c}(t,x,\zeta,z). \end{aligned} \tag{48}$$

*Lemma 3.7:* For all $t > 0$, $c > 0$, $x, z \in I\!\!R^n$, and $\zeta \in \mathcal{Z}$, both $\mathcal{W}^{\alpha,c}(t,x,\zeta,z)$ (as well as $J^c(t,x,u,\alpha,\zeta,z)$) and $\mathcal{W}^{\alpha,\infty}(t,x,\zeta,z)$ are strictly concave in $\alpha$.

Combining Lemma 3.3, 3.4, 3.7 and Corollay 3.5 immediately yields the following.

*Theorem 3.8:* If $t \in (0, \bar{t})$, then

$$\begin{aligned}\underline{W}^c(t,x,\zeta,z) &= \sup_{\alpha\in\mathcal{A}^t} \inf_{u\in\mathcal{U}^{0,t}} J^c(t,x,u,\alpha,\zeta,z) \\ &= \inf_{u\in\mathcal{U}^{0,t}} \sup_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) \\ &= \overline{W}^c(t,x,\zeta,z)\end{aligned}$$

for all $x, z \in I\!R^n$ and $\zeta \in \mathcal{Z}$.

By the monotonicity of $\mathcal{W}^{\alpha,c}$ with respect to $c$ and Lemma 3.1, the following is easily obtained.

*Lemma 3.9:* For $t \in (0, \bar{t})$,

$$\overline{W}^{\infty}(t,x,\zeta,z) = \sup_{\alpha\in\mathcal{A}^t} \mathcal{W}^{\alpha,\infty}(t,x,\zeta,z)$$

for all $x, z \in I\!R^n$ and $\zeta \in \mathcal{Z}$.

### E. Hamilton-Jacobi-Bellman PDE

The Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) associated with our problem is

$$\begin{aligned}0 = &-\tfrac{\partial}{\partial r} W^{\alpha}(r,x,\zeta,z) \\ &-H^{\alpha}(t-r,x,\zeta,\nabla_x W^{\alpha}(r,x,\zeta,z)) \qquad (49)\\ W^{\alpha}&(0,x,\zeta,z) = \psi^c(x,z) \quad \forall x \in I\!R^n. \qquad (50)\end{aligned}$$

where the corresponding Hamiltonian is given by

$$H^{\alpha}(r,x,\zeta,p) = V^{\alpha}(r,x,\zeta_r) + \tfrac{1}{2}|p|^2.$$

For $t > 0$, let

$$\mathcal{D}_t \doteq C([0,t] \times I\!R^n) \cap C^1((0,t) \times I\!R^n).$$

*Lemma 3.10:* Let $c \in [0,\infty), z \in I\!R^n, \zeta \in \mathcal{Z}$, and $\alpha \in \mathcal{A}^t$. Let $s \in [0,t]$ and $r \doteq t - s$. Suppose that $W^{\alpha}(\cdot,\cdot,\zeta,z) \in \mathcal{D}_t$ satisfies (49) and (50). Then, $W^{\alpha}(t,x,\zeta,z) \le J^c(t,x,u,\alpha,\zeta,z)$ for all $x \in I\!R^n$ and $u \in \mathcal{U}^{0,t}$. Further, $W^{\alpha}(t,x,\zeta,z) = J^c(t,x,u^*,\alpha,\zeta,z)$ for the input $u^*_s = \tilde{u}(s,\tilde{\xi}_s)$ with $\tilde{\xi}_s$ given by (1) with $\tilde{u}(s,x) = -\nabla_x W^{\alpha}(t-s,x,\zeta,z)$ and $\tilde{\xi}_0 = x$. Consquently, $W^{\alpha}(t,x,\zeta,z) = \mathcal{W}^{\alpha,c}(t,x,\zeta,z)$.

## IV. Fundamental solution as set of Riccati solution

We will find that the fundamental solution may be given in terms of a set of solutions of associated Riccati equations.

### A. Differential Riccati Equation

Given $c \in [0,\infty)$, $r \le t$ and $\alpha \in \mathcal{A}^t$, $\zeta \in \mathcal{Z}$, we look for a solution, $\breve{\mathcal{W}}^{\alpha,c}$, of the form

$$\begin{aligned}\breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) \doteq \tfrac{1}{2}\big[&x^T P^c_r x + 2x^T Q^c_r z + z^T R^c_r z \\ &+2p^c_r \cdot x + 2q^c_r \cdot z + \gamma^c_r\big] \qquad (51)\end{aligned}$$

where $P^c_\cdot$, $Q^c_\cdot$, $R^c_\cdot$, $p^c_\cdot$, $q^c_\cdot$, and $\gamma^c_\cdot$ depend implicitly on given $\zeta$ and the choice of $\alpha \in \mathcal{A}^t$ and satisfy the respective initial value problems $(R1)$:

$$\begin{aligned}&\dot{P}^c_r = -[P^c_r]^2 - \textstyle\sum_{i\in\mathcal{N}} A^i_{t-r}, && P^c_0 = cI_n,\\ &\dot{Q}^c_r = -P^c_r Q^c_r, && Q^c_0 = -cI_n,\\ &\dot{R}^c_r = -[Q^c_r]^T Q^c_r, && R^c_0 = cI_n,\\ &\dot{p}^c_r = -P^c_r p^c_r + \textstyle\sum_{i\in\mathcal{N}} A^i_{t-s}\zeta^i_{t-s}, && p^c_0 = 0_{n\times 1},\\ &\dot{q}^c_r = -[Q^c_r]^T p^c_r, && q^c_0 = 0_{n\times 1},\end{aligned}$$

and

$$\dot{r}^c_r = -|p^c_r|^2 + \sum_{i\in\mathcal{N}} \mu_i \left\{2\alpha^i_{t-s} - (\alpha^i_{t-s})^3 |\zeta^i_{t-s}|^2\right\}$$

with $r^c_0 = 0$ where $A^i_s \doteq \mu_i[\alpha^i_s]^3 I_n$ for all $i \in \mathcal{M}$, $r \in [0,t]$ and $0_{m\times n}$ denotes the zero matrix of size $m \times n$. Upon examining $(R1)$, we see that the solutions, $P^c_r, Q^c_r, R^c_r$, are symmetric matrices, more precisely, diagonal matrices of size $n \times n$.

*Lemma 4.1:* For $t \in [0, \bar{t})$, the value function $\mathcal{W}^{\alpha,c}$ of (48) and the explicit function $\breve{\mathcal{W}}^{\alpha,c}$ of (51) satisfying $(R1)$ are equivalent. That is, $\mathcal{W}^{\alpha,c}(r,x,\zeta,z) = \breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z)$ for all $x,z \in I\!R^n$, $\zeta \in \mathcal{Z}$, and $r \in [0,t]$.

*Proof:* It will be sufficient to show that $\breve{\mathcal{W}}^{\alpha,c}$ satisfies the condition of Lemma 3.10. It is useful to rewrite (51) as

$$\begin{aligned}\breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) &= \frac{1}{2}\begin{bmatrix}x\\ z\\ 1\end{bmatrix}^T \begin{bmatrix}P^c_r & Q^c_r & p^c_r\\ Q^T_r & R^c_r & q^c_r\\ [p^c_r]^T & [q^c_r]^T & \gamma^c_r\end{bmatrix}\begin{bmatrix}x\\ z\\ 1\end{bmatrix}\\ &\doteq \tfrac{1}{2} X_z^T \mathcal{P}^c_r X_z. \qquad (52)\end{aligned}$$

From (51) and (52), we have

$$\tfrac{\partial}{\partial r}\breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) = \tfrac{1}{2} X_z^T \dot{\mathcal{P}}^c_r X_z, \qquad (53)$$

$$\nabla_x \breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) = [P^c_r \; Q^c_r \; p^c_r] X_z \doteq \widetilde{\mathcal{P}}^c_r X_z, \quad (54)$$

and

$$\begin{aligned}-|\nabla_x \breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z)|^2 &= -X_z^T [\widetilde{\mathcal{P}}^c_r]^T \widetilde{\mathcal{P}}^c_r X_z\\ &\doteq -X_z^T \mathcal{P}^c_r \Lambda \mathcal{P}^c_r X_z \qquad (55)\end{aligned}$$

where

$$\Lambda \doteq \begin{bmatrix} I_n & 0_{n\times(n+1)}\\ 0_{(n+1)\times n} & 0_{(n+1)\times(n+1)}\end{bmatrix}.$$

Also, collecting like terms, we have

$$\begin{aligned}&-V^{\alpha}(t-r,x,\zeta_{t-r})\\ &\quad= \frac{1}{2}x^T\Big[-\sum_{i\in\mathcal{N}} A^i_{t-r}\Big]x + \sum_{i\in\mathcal{N}} [\zeta^i_{t-r}]^T A^i_{t-r} x\\ &\qquad + \sum_{i\in\mathcal{N}}\Big\{\mu_i\alpha^i_{t-r} - \frac{1}{2}[\zeta^i_{t-r}]^T A^i_{t-r}\zeta^i_{t-r}\Big\}\\ &\quad\doteq \tfrac{1}{2} X_z^T \Gamma(\alpha_{t-r},\zeta_{t-r}) X_z, \qquad (56)\end{aligned}$$

where $\Gamma(\alpha_{t-r}, \zeta_{t-r})$ has the block structure with 3 row partitions and 3 column partitions as

$$\begin{aligned}
\Gamma_{11} &= -\sum_{i\in\mathcal{N}} A^i_{t-r}, \quad \Gamma_{13} = \Gamma_{31}^T = \sum_{i\in\mathcal{N}} A^i_{t-r}\zeta^i_{t-r}, \\
\Gamma_{33} &= \sum_{i\in\mathcal{N}} \mu_i \left\{2\alpha^i_{t-r} - (\alpha^i_{t-r})^3 |\zeta^i_{t-r}|^2\right\}, \\
\Gamma_{12} &= \Gamma_{21} = \Gamma_{22} = 0_{n\times n}, \quad \Gamma_{23} = \Gamma_{32}^T = 0_{n\times 1}.
\end{aligned}$$

Here, we note that $(R1)$ is equivalent to

$$\dot{\mathcal{P}}^c_r = -\mathcal{P}^c_r \Lambda \mathcal{P}^c_r + \Gamma(\alpha_{t-r}, \zeta_{t-r}) \quad \text{with} \quad \mathcal{P}^c_0 = \mathcal{C}.$$

Consequently, substituting (53) – (56) in the right-hand side of the PDE (49) yields

$$\begin{aligned}
&-\tfrac{\partial}{\partial r}\breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) \\
&-H^\alpha(t-r, x, \zeta, \nabla_x \breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z)) \\
&= \tfrac{1}{2} X_z^T \left[-\dot{\mathcal{P}}^c_r - \mathcal{P}^c_r \Lambda \mathcal{P}^c_r + \Gamma(\alpha_{t-r}, \zeta_{t-r})\right] X_z = 0,
\end{aligned}$$

which implies (51) is a solution of HJB PDE (49), and by Lemma 3.4 and 3.10, $\mathcal{W}^{\alpha,c}(r,x,\zeta,z) = \breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z)$ for all $r \in [0,t]$, with $t \in [0,\bar{t})$. ■

Recall from Lemma 3.1 and 3.9 that the fundamental solution of interest is obtained as the $c \to \infty$ limit of $\mathcal{W}^{\alpha,c}$. Consequently, we have that for $t < \bar{t}$,

$$\begin{aligned}
\overline{W}^\infty(t,x,\zeta,z) &= \sup_{\alpha\in\mathcal{A}^t} \mathcal{W}^{\alpha,\infty}(t,x,\zeta,z) \\
&= \sup_{\alpha\in\mathcal{A}^t} \lim_{c\to\infty} \mathcal{W}^{\alpha,c}(t,x,\zeta,z),
\end{aligned}$$

which by Lemma 4.1,

$$\begin{aligned}
&= \sup_{\alpha\in\mathcal{A}^t} \lim_{c\to\infty} \tfrac{1}{2} X_z^T \mathcal{P}^c_t X z \\
&= \sup_{\alpha\in\mathcal{A}^t} \tfrac{1}{2} X_z^T \mathcal{P}^\infty_t X z. \qquad (57)
\end{aligned}$$

Let $\mathcal{G}_t = \mathcal{G}(t, \{m_i\}_{i\in\mathcal{N}}, \zeta)$ be given by

$$\mathcal{G}_t \doteq \{\mathcal{P}^\infty_t(\alpha) \mid \alpha \in \mathcal{A}^t\}.$$

Then, the fundamental solution (57) can be represented by

$$\overline{W}^\infty(t,x,\zeta,z) = \sup_{\mathcal{P}\in\mathcal{G}_t} \tfrac{1}{2} X_z^T \mathcal{P} X_z.$$

### *B. An approximated solution*

In order to compute the suprema in (48) and (57), piecewise linear approximations to continuous function $\alpha \in \mathcal{A}^t$ are used to generate a tractable algorithm. Consequently, we need to demonstrate that such approximations converge to the value function of interest.

Given $t < \bar{t}$ and $K \in I\!N$, let $\mathcal{A}^t_K$ be the set of continuous piecewise linear functions contained in $\mathcal{A}^t$, in particuar, $\alpha \in \mathcal{A}^t_K$ is a piecewise linear function on a uniform grid with $K+1$ elements on the interval $[0,t]$. Given $K_0 \in I\!N$, let $K_\eta \doteq K_0^\eta$ with $\eta \in I\!N$ and

$$\overline{W}^c_\eta(t,x,\zeta,z) \doteq \sup_{\alpha\in\mathcal{A}^t_{K_\eta}} \mathcal{W}^{\alpha,c}(t,x,\zeta,z).$$

Since $\{\mathcal{A}^t_{K_\eta}\}_{\eta\in I\!N}$ is an increasing sequence of subsets contained in $\mathcal{A}^t$, for all $\eta \in I\!N$,

$$\overline{W}^c_\eta(t,x,\zeta,z) \le \overline{W}^c_{\eta+1}(t,x,\zeta,z) \le \overline{W}^c(t,x,\zeta,z).$$

Therefore,

$$\begin{aligned}
\overline{W}^c(t,x,\zeta,c) &= \lim_{\eta\to\infty} \overline{W}^c_\eta(t,x,\zeta,z) \\
&= \lim_{\eta\to\infty} \sup_{\alpha\in\mathcal{A}^\eta_{K_\eta}} \mathcal{W}^{\alpha,c}(t,x,\zeta,z). \quad (58)
\end{aligned}$$

Since (58) holds for all subsequences $\{K_\eta\}_{\eta\in I\!N} \subset I\!N$, we have the following.

*Lemma 4.2:* Given $K \in I\!N$, let $\bar{\alpha} \doteq \{\hat{\alpha}^i\}_{i\in\mathcal{N}} \in \mathcal{A}^{N\times(K+1)}$ where $k^{th}$ element of $\hat{\alpha}^i$ is given by $\hat{\alpha}^i_k \in \mathcal{A}$ for all $i \in \mathcal{N}$ and all $k \in ]0, K[$. Assume that there exists a one to one and onto mapping $\mathcal{L}^K$ such that $\mathcal{L}^K : \mathcal{A}^{N\times(K+1)} \to \mathcal{A}^t_K$. Then,

$$\overline{W}^c(t,x,\zeta,z) = \lim_{K\to\infty} \sup_{\bar{\alpha}\in\mathcal{A}^{N\times(K+1)}} \mathcal{W}^{\mathcal{L}^K(\bar{\alpha}),c}(t,x,\zeta,z).$$

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