A MAX-PLUS DUAL SPACE FUNDAMENTAL SOLUTION FOR A CLASS OF OPERATOR DIFFERENTIAL RICCATI EQUATIONS*

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Abstract. A new fundamental solution semigroup for operator differential Riccati equations is developed. This fundamental solution semigroup is constructed via an auxiliary finite horizon optimal control problem whose value functional growth with time horizon is determined by a particular solution of the operator differential Riccati equation of interest. By exploiting semiconvexity of this value functional, and the attendant max-plus linearity and semigroup properties of the associated dynamic programming evolution operator, a semigroup of max-plus integral operators is constructed in a dual space defined via the Legendre-Fenchel transform. It is demonstrated that this semigroup of max-plus integral operators can be used to propagate all solutions of the operator differential Riccati equation that are initialized from a specified class of initial conditions. As this semigroup of max-plus integral operators can be identified with a semigroup of quadratic kernels, an explicit recipe for the aforementioned solution propagation is also rendered possible.

Key words. Infinite dimensional systems, operator differential Riccati equations, fundamental solution, semigroups, max-plus methods, Lengendre-Fenchel transform, optimal control.

AMS subject classifications. 49L20, 49M29, 15A80, 93C20, 47F05, 47D06.

1. Introduction.

The objective of this paper is to develop a new fundamental solution semigroup for operator differential Riccati equations of the form

$$\dot{\mathcal{P}}(t) = \mathcal{P}(t)\,\mathcal{A} + \mathcal{A}'\,\mathcal{P}(t) + \mathcal{P}(t)\,\sigma\,\sigma'\,\mathcal{P}(t) + \mathcal{C}\,, \tag{1.1}$$

where $\mathcal{P}(t)$ is a self-adjoint bounded linear operator evolved to time t from some initial operator, $\mathcal{A}$ is an unbounded densely defined and boundedly invertible linear operator that generates a C_0-semigroup of bounded linear operators, and σ and $\mathcal{C}$ are bounded linear operators, all defined with respect to an underlying Hilbert space $\mathscr{X}$. The fundamental solution semigroup obtained directly generalizes the finite dimensional case presented in [12], and unifies the specific infinite dimensional cases documented in [6, 7]. Preliminary results in this direction also appear in [8].

Development of the new fundamental solution semigroup for (1.1) proceeds by considering an infinite dimensional optimal control problem on a finite time horizon t. This control problem is constructed such that the value functional obtained exhibits quadratic growth with respect to the state variable, where the growth is determined by the solution $\mathcal{P}(t)$ of the operator differential Riccati equation (1.1) at time t. Consequently, evolution of the solution $\mathcal{P}(t)$ of (1.1) with time t can be identified with evolution of the value functional with respect to time horizon t, with dynamic programming [2, 3] providing a mechanism for the latter. As the value functional obtained is demonstrably semiconvex, its evolution via dynamic programming can be identified with a corresponding evolution in a dual space defined via the Legendre-Fenchel transform. Critically, by exploiting max-plus linearity of the dynamic programming evolution operator, this dual space evolution can be decoupled from the terminal payoff employed in the optimal control problem, and hence from the

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initial data that defines a solution $\mathcal{P}$ of (1.1). That is, a set of time horizon indexed dual space evolution operators is well-defined via this decoupling. Furthermore, as dynamic programming naturally endows the value functional with a semigroup property, this set of time horizon indexed dual space evolution operators naturally defines a semigroup, thereby yielding the claimed *max-plus dual space fundamental solution semigroup* for (1.1).

In terms of organization, the operator differential Riccati equation of interest is posed in Section 2, along with results concerning existence and uniqueness of its solution. Construction of the max-plus fundamental solution semigroup is presented in detail in Section 3, with the steps involved in applying this fundamental solution to evaluate solutions of (1.1) elucidated in Section 4. This is followed by some brief conclusions in Section 5, and appendices containing deferred technical details.

2. Operator differential Riccati equation. Attention is initially restricted to the operator differential Riccati equation of interest. Two auxiliary operator differential equations of utility later are considered subsequently.

2.1. Riccati equation. Consider the operator differential Riccati equation posed with respect to Hilbert spaces $\mathscr{X}$ and $\mathscr{W}$ by

$$\dot{\mathcal{P}}(t) = \mathcal{A}'\,\mathcal{P}(t) + \mathcal{P}(t)\,\mathcal{A} + \mathcal{P}(t)\,\sigma\,\sigma'\,\mathcal{P}(t) + \mathcal{C}\,, \tag{1.1}$$

in which $\mathcal{A} : \mathsf{dom}(\mathcal{A}) \subset \mathscr{X} \to \mathscr{X}$ is unbounded and densely defined on $\mathscr{X}$, $\sigma \in \mathcal{L}(\mathscr{W};\mathscr{X})$, $\mathcal{C} \in \mathcal{L}(\mathscr{X})$ is self-adjoint and non-negative, and $\mathcal{A}'$ and σ' denote the respective adjoints of $\mathcal{A}$ and σ.

Assumption 2.1. *$\mathcal{A}$ is boundedly invertible and generates a C_0-semigroup of bounded linear operators.*

With a view to describing a notion of solution for the operator differential Riccati equation (1.1), it is convenient to define two spaces of self-adjoint bounded linear operators by

$$\Sigma(\mathscr{X}) \doteq \left\{ \mathcal{P} \in \mathcal{L}(\mathscr{X}) \,\middle|\, \mathcal{P} \text{ is self-adjoint} \right\}, \tag{2.1}$$

$$\Sigma_{\mathcal{M}}(\mathscr{X}) \doteq \left\{ \mathcal{P} \in \Sigma(\mathscr{X}) \,\middle|\, \begin{array}{c} \mathcal{P} - \mathcal{M} \text{ is coercive} \\ \text{on } \mathsf{dom}(\mathcal{A}) \end{array} \right\}, \quad \mathcal{M} \in \Sigma(\mathscr{X})\,. \tag{2.2}$$

In (2.2), an operator $\mathcal{F} : \mathsf{dom}(\mathcal{F}) \subset \mathscr{X} \to \mathscr{X}$ is defined to be coercive if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $\langle x, \mathcal{F}\,x\rangle \ge \epsilon\,\|x\|^2 \ \forall\ x \in \mathsf{dom}(\mathcal{F})$. Spaces of continuous and strongly continuous operator-valued functions defined on $I \doteq [0,T] \subset \mathbb{R}_{\ge 0}$, $T \in \mathbb{R}_{>0}$, and taking values in $\Sigma(\mathscr{X})$ and $\Sigma_{\mathcal{M}}(\mathscr{X})$ respectively, are defined by

$$C(I;\mathcal{L}(\mathscr{X})) \doteq \left\{ \mathcal{F} : I \to \mathcal{L}(\mathscr{X}) \,\middle|\, \begin{array}{c} \mathcal{F} \text{ is} \\ \text{continuous} \end{array} \right\}, \tag{2.3}$$

$$C_0(I;\mathcal{L}(\mathscr{X})) \doteq \left\{ \mathcal{F} : I \to \mathcal{L}(\mathscr{X}) \,\middle|\, \begin{array}{c} \mathcal{F} \text{ is strongly} \\ \text{continuous} \end{array} \right\}. \tag{2.4}$$

Remark 2.2. By the definition, the space $C(I;\mathcal{L}(\mathscr{X}))$ includes all uniformly continuous semigroups of bounded linear operators, see p.1 of [13]. Similarly, the space $C_0(I;\mathcal{L}(\mathscr{X}))$ includes all strongly continuous (C_0-) semigroups of bounded linear operators. A linear operator $\mathcal{A} : \mathsf{dom}(\mathcal{A}) \subset \mathscr{X} \to \mathscr{X}$ is the generator of a uniformly continuous semigroup if and only if $\mathsf{dom}(\mathcal{A}) \equiv \mathscr{X}$, in which case $\mathcal{A}$ is bounded (see

Theorem 1.2 of [13]). However, if $\mathcal{A}$ is the generator of a C_0-semigroup that is not uniformly continuous, then $\mathsf{dom}(\mathcal{A})$ is merely dense in $\mathscr{X}$, and must be unbounded (but closed, see Corollary 2.5 of [13]). In either case, the semigroup of bounded linear operators generated is denoted for each $t \in I \subset \mathbb{R}_{\ge 0}$ by $e^{\mathcal{A}\,t} \in \mathcal{L}(\mathscr{X})$, and satisfies the usual semigroup properties $e^{\mathcal{A}\,0} = \mathcal{I}$, where $\mathcal{I} \in \mathcal{L}(\mathscr{X})$ denotes the identity, and $e^{\mathcal{A}\,t}\, e^{\mathcal{A}\,s} = e^{\mathcal{A}\,(t+s)}$ for all $s, t \in \mathbb{R}_{\ge 0}$ such that $s, t, s+t \in I$.

A *mild* solution of the operator differential Riccati equation (1.1) on a time interval $[0,T]$, $T \in \mathbb{R}_{>0}$, is any operator-valued function $\mathcal{P} \in C_0([0,T];\Sigma(\mathscr{X}))$ that satisfies

$$\mathcal{P}(t)\, x = \gamma(\mathcal{P})(t)\, x\,, \tag{2.5}$$

for all $x \in \mathscr{X}$, $t \in [0,T]$, with operator γ defined for every $Q \in C_0([0,T];\Sigma(\mathscr{X}))$ by $Q \mapsto \gamma(Q)$, where

$$\gamma(\mathcal{Q})(t)\, x \doteq e^{\mathcal{A}'\,t}\, \mathcal{Q}(0)\, e^{\mathcal{A}\,t}\, x + \int_0^t e^{\mathcal{A}'\,(t-s)}\, [\mathcal{Q}(s)\, \sigma\, \sigma'\, \mathcal{Q}(s) + \mathcal{C}]\, e^{\mathcal{A}\,(t-s)}\, x\, ds \tag{2.6}$$

for all $t \in [0,T]$, $x \in \mathscr{X}$, where $e^{\mathcal{A}'\,\cdot}$ denotes the C_0-semigroup generated by the operator adjoint $\mathcal{A}'$ (see, for example, Theorem 2.2.6 of [5]). As per [4], it is convenient to introduce an analogous operator differential Riccati equation to (1.1), defined with respect to the Yosida approximation $\mathcal{A}_n \in \mathcal{L}(\mathscr{X})$ of $\mathcal{A}$ for all $n \in \mathbb{N}$. In particular,

$$\dot{\mathcal{P}}_n(t) = \mathcal{A}_n'\, \mathcal{P}_n(t) + \mathcal{P}_n(t)\, \mathcal{A}_n + \mathcal{P}_n(t)\, \sigma\, \sigma'\, \mathcal{P}_n(t) + \mathcal{C}\,. \tag{2.7}$$

A mild solution of (2.7) on a time interval $[0,T]$, $T \in \mathbb{R}_{>0}$, is any operator-valued function $\mathcal{P}_n \in C([0,T];\Sigma(\mathscr{X}))$ that satisfies the corresponding equation to (2.5), i.e.,

$$\mathcal{P}_n(t)\, x = \gamma_n(\mathcal{P}_n)(t)\, x\,, \tag{2.8}$$

for all $x \in \mathscr{X}$, $t \in [0,T]$, where operator γ_n applied to $Q \in C([0,T];\Sigma(\mathscr{X}))$ is defined by

$$\gamma_n(Q)(t)\, x \doteq e^{\mathcal{A}_n'\,t}\, \mathcal{Q}(0)\, e^{\mathcal{A}_n\,t}\, x + \int_0^t e^{\mathcal{A}_n'\,(t-s)}\, [\mathcal{Q}(s)\, \sigma\, \sigma'\, \mathcal{Q}(s) + \mathcal{C}]\, e^{\mathcal{A}_n\,(t-s)}\, x\, ds \tag{2.9}$$

for all $t \in [0,T]$, $x \in \mathscr{X}$, and $e^{\mathcal{A}_n'\,\cdot}$ denotes the uniformly continuous semigroup generated by the adjoint of the Yoshida approximation $\mathcal{A}_n \in \mathcal{L}(\mathscr{X})$.

THEOREM 2.3. *Given any $\mathcal{P}_0 \in \Sigma(\mathscr{X})$, there exists a $\tau \in \mathbb{R}_{>0}$ such that the operator differential Riccati equations (1.1), (2.7) exhibit respective unique mild solutions $\mathcal{P} \in C_0([0,\tau];\Sigma(\mathscr{X}))$, $\mathcal{P}_n \in C([0,\tau];\Sigma(\mathscr{X}))$ satisfying $\mathcal{P}(0) = \mathcal{P}_0 = \mathcal{P}_n(0)$. Furthermore, for all $x \in \mathscr{X}$,*

$$\lim_{n\to\infty} \mathcal{P}_n(\cdot)\, x = \mathcal{P}(\cdot)\, x\,, \tag{2.10}$$

where the limit is defined with respect to the Banach space $(C([0,\tau];\mathscr{X}),\ \|\cdot\|_{C([0,\tau];\mathscr{X})})$.

REMARK 2.4. Note that the limit (2.10) is a statement of *strong* (operator) convergence defined on $C([0,\tau];\mathscr{X})$. This is strictly weaker than *uniform* (operator) convergence defined on $C([0,\tau];\mathcal{L}(\mathscr{X}))$ via the norm $\|\cdot\|_{C[0,\tau]}$ (see, for example, p.263 of [10]). As $(C([0,\tau];\mathcal{L}(\mathscr{X})),\ \|\cdot\|_{C[0,\tau]})$ defines a Banach space, this weaker form of convergence allows the limit to reside in $C_0([0,\tau];\mathcal{L}(\mathscr{X})) \setminus C([0,\tau];\mathcal{L}(\mathscr{X}))$ should $\mathcal{A}$ be unbounded.

Remark 2.5. It may also be noted that, by an analogous argument to Proposition 2.1 on p.391 of [4], $\mathcal{P} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ is a mild solution of (2.5) if and only if it is a *weak* solution (see Definition 2.1 on p.390 of [4]).

Proof. [Theorem 2.3] The proof follows that of Lemma 2.2 on p. 391 of [4], while including global uniqueness on a finite horizon. It is not extended to the infinite horizon due to the possibility of finite escape. Fix $T \in \mathbb{R}_{>0}$ and define as per [4]

$$M_T \doteq \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \left\| e^{\mathcal{A}_n\, t} \right\|_{\mathcal{L}(\mathscr{X})} \tag{2.11}$$

where $\|\cdot\|$ denotes the induced operator norm on $\mathcal{L}(\mathscr{X})$. Fix any $\mathcal{P}_0 \in \Sigma(\mathscr{X})$, $r \in \mathbb{R}_{>0}$, and $\tau \in (0,T]$ such that

$$r > 2\, M_T^2\, a\,,\ \tau < \min\left(\frac{a}{r^2 b + \|\mathcal{C}\|_{\mathcal{L}(\mathscr{X})}},\ \frac{1}{4\, r\, M_T^2\, b} \right). \tag{2.12}$$

where $a \doteq \|\mathcal{P}_0\|_{\mathcal{L}(\mathscr{X})}$ and $b \doteq \|\sigma\,\sigma'\|_{\mathcal{L}(\mathscr{X})}$. Let $B_{C[0,\tau]}(r)$ and $B_{C_0[0,\tau]}(r)$ denote respective balls of radius r in $C([0,\tau];\mathcal{L}(\mathscr{X}))$ and $C_0([0,\tau];\mathcal{L}(\mathscr{X}))$, defined with respect to the norms $\|\cdot\|_{C[0,\tau]}$ and $\|\cdot\|_{C_0[0,\tau]}$ of (A.3). That is,

$$\begin{aligned} B_{C[0,\tau]}(r) &\doteq \left\{ \mathcal{F} \in C([0,\tau];\Sigma(\mathscr{X})) \,\middle|\, \|\mathcal{F}\|_{C[0,\tau]} \le r \right\}, \\ B_{C_0[0,\tau]}(r) &\doteq \left\{ \mathcal{F} \in C_0([0,\tau];\Sigma(\mathscr{X})) \,\middle|\, \|\mathcal{F}\|_{C_0[0,\tau]} \le r \right\}. \end{aligned}$$

(Note by Lemma A.1 that $B_{C[0,\tau]}(r) \subset B_{C_0[0,\tau]}(r)$.) Fix any $\mathcal{P} \in B_{C[0,\tau]}(r)$ satisfying $\mathcal{P}(0) = \mathcal{P}_0$. Applying the operator γ_n of (2.9) to $\mathcal{P}$, evaluating at time $t \in [0,\tau]$, and applying the norm $\|\cdot\|_{\mathcal{L}(\mathscr{X})}$ (while dropping the $\mathcal{L}(\mathscr{X})$ subscript),

$$\begin{aligned} \|\gamma_n(\mathcal{P})(t)\| &\le \|e^{\mathcal{A}_n' t}\|\, \|\mathcal{P}_0\|\, \|e^{\mathcal{A}_n\, t}\| + \int_0^t \|e^{\mathcal{A}_n'(t-s)}\| \left[\|\mathcal{P}\|\, \|\sigma\,\sigma'\|\, \|\mathcal{P}\| + \|\mathcal{C}\| \right] \|e^{\mathcal{A}_n(t-s)}\|\, ds \\ &\le M_T^2 \left(a + \tau \left[r^2\, \|\sigma\,\sigma'\| + \|\mathcal{C}\| \right] \right) \le 2\, M_T^2\, a \le r\,, \end{aligned}$$

where (2.11) and (2.12) have been applied. So, taking the supremum over $t \in [0,\tau]$ (and restoring the norm subscripts),

$$\|\gamma_n(\mathcal{P})\|_{C([0,\tau])} = \sup_{t\in[0,\tau]} \|\gamma_n(\mathcal{P})(t)\|_{\mathcal{L}(\mathscr{X})} \le r\,.$$

Hence, as $\mathcal{P} \in B_{C[0,\tau]}(r)$ is arbitrary, it follows immediately that $\gamma_n : B_{C[0,\tau]}(r) \to B_{C[0,\tau]}(r)$. In order to show that γ_n is a contraction, fix any $\widehat{\mathcal{P}} \in B_{C[0,\tau]}(r)$ satisfying $\widehat{\mathcal{P}}(0) = \mathcal{P}_0$. Applying (2.9) for $t \in [0,\tau]$,

$$\|\gamma_n(\mathcal{P})(t) - \gamma_n(\widehat{\mathcal{P}})(t)\|_{\mathcal{L}(\mathscr{X})} \le M_T^2 \int_0^t \left\| \mathcal{P}(s)\,\sigma\,\sigma'\,\mathcal{P}(s) - \widehat{\mathcal{P}}(s)\,\sigma\,\sigma'\,\widehat{\mathcal{P}}(s) \right\|_{\mathcal{L}(\mathscr{X})} ds\,, \tag{2.13}$$

where for all $s \in [0,t]$,

$$\begin{aligned} &\left\| \mathcal{P}(s)\,\sigma\,\sigma'\,\mathcal{P}(s) - \widehat{\mathcal{P}}(s)\,\sigma\,\sigma'\,\widehat{\mathcal{P}}(s) \right\|_{\mathcal{L}(\mathscr{X})} \\ &= \left\| \mathcal{P}(s)\,\sigma\,\sigma' \left[\mathcal{P}(s) - \widehat{\mathcal{P}}(s) \right] + \left[\mathcal{P}(s) - \widehat{\mathcal{P}}(s) \right] \sigma\,\sigma'\,\widehat{\mathcal{P}}(s) \right\|_{\mathcal{L}(\mathscr{X})} \\ &\le \left(\|\mathcal{P}\|_{C[0,\tau]} + \|\widehat{\mathcal{P}}\|_{C[0,\tau]} \right) b\, \|\mathcal{P} - \widehat{\mathcal{P}}\|_{C[0,\tau]} < 2\, r\, b\, \|\mathcal{P} - \widehat{\mathcal{P}}\|_{C[0,\tau]}\,. \end{aligned}$$

Combining the above two inequalities and taking the supremum over $t \in [0, \tau]$,

$$\|\gamma_n(\mathcal{P}) - \gamma_n(\widehat{\mathcal{P}})\|_{C[0,\tau]} \leq 2\, r\, \tau\, M_T^2\, b\, \|\mathcal{P} - \widehat{\mathcal{P}}\|_{C[0,\tau]} < \tfrac{1}{2}\, \|\mathcal{P} - \widehat{\mathcal{P}}\|_{C[0,\tau]}\,.$$

Hence, $\gamma_n : B_{C[0,\tau]}(r) \to B_{C[0,\tau]}(r)$ defines a contraction on $B_{C[0,\tau]}(r)$. Consequently, the Banach Fixed Point Theorem (for example, Theorem 5.1-4 on p.303 of [10]) implies that there exists a unique solution $\mathcal{P}_n \in B_{C[0,\tau]}(r)$ of (2.8) for all $t \in [0, \tau]$ and $x \in \mathscr{X}$. In order to conclude global uniqueness in $C([0, \tau]; \Sigma(\mathscr{X}))$, suppose there exists a second mild solution $\widehat{\mathcal{P}}_n \in C([0, \tau]; \Sigma(\mathscr{X}))$ of (2.7) satisfying $\widehat{\mathcal{P}}_n(0) = \mathcal{P}_0$. That is, $\mathcal{P}_n = \gamma_n(\mathcal{P}_n)$ and $\widehat{\mathcal{P}}_n = \gamma_n(\widehat{\mathcal{P}}_n)$, where γ_n is as per (2.9). Given any $x \in \mathscr{X}$, using an inequality analogous to (2.13),

$$\|\mathcal{P}_n(t)\, x - \widehat{\mathcal{P}}_n(t)\, x\| \leq M_T^2, b \int_0^t \left(r + \|\widehat{\mathcal{P}}_n\|_{C[0,\tau]}\right) \|\mathcal{P}_n(s)\, x - \widehat{\mathcal{P}}_n(s)\, x\|\, ds\,.$$

where $\|\mathcal{P}_n\|_{C[0,\tau]} \leq r$ has been used. As $\widehat{\mathcal{P}}_n \in \mathcal{L}(\mathscr{X}; \mathcal{L}([0, \tau]; \mathscr{X}))$ by Lemma A.1, there exists a $K \in \mathbb{R}_{\geq 0}$ such that $\|\widehat{\mathcal{P}}_n(\cdot)\, x\|_{C([0,\tau];\mathscr{X})} \leq K\, \|x\|$ for all $x \in \mathscr{X}$. Consequently, by Lemma A.2,

$$\|\widehat{\mathcal{P}}_n\|_{C[0,\tau]} = \|\widehat{\mathcal{P}}_n\|_{C_0[0,\tau]} = \sup_{\|x\|=1} \|\widehat{\mathcal{P}}_n(\cdot)\, x\|_{C([0,\tau];\mathscr{X})} \leq K < \infty\,.$$

Combining these facts yields the inequality

$$\|\mathcal{P}_n(t)\, x - \widehat{\mathcal{P}}_n(t)\, x\| \leq M_T^2\, b\, (r + K) \int_0^t \|\mathcal{P}_n(s)\, x - \widehat{\mathcal{P}}_n(s)\, x\|\, ds\,.$$

As $\mathcal{P}_n - \widehat{\mathcal{P}}_n \in C_0([0, \tau]; \mathcal{L}(\mathscr{X}))$ is strongly continuous, the attendant function $\|\mathcal{P}_n(\cdot)\, x - \widehat{\mathcal{P}}_n(\cdot)\, x\| : [0, \tau] \to \mathbb{R}_{\geq 0}$ is continuous by definition. This admits a straightforward application of Gronwall's inequality, yielding

$$\|\mathcal{P}_n(t)\, x - \widehat{\mathcal{P}}_n(t)\, x\|_{\mathcal{L}(\mathscr{X})} \leq 0 \quad \forall\, t \in [0, \tau],\, x \in \mathscr{X}\,.$$

That is, $\mathcal{P}_n = \widehat{\mathcal{P}}_n$, so the asserted uniqueness is indeed global on $C([0, \tau]; \mathcal{L}(\mathscr{X}))$. An almost identical argument, using the same $\tau \in [0, T]$ and $r \in \mathbb{R}_{>0}$, implies the existence of a unique solution $\mathcal{P} \in C_0([0, \tau]; \mathcal{L}(\mathscr{X}))$ of (2.5). The fact that (2.10) holds follows as per Lemma 2.1 on p. 389 of [4]. □

Assumption 2.6. *An operator $\mathcal{M} \in \Sigma(\mathscr{X})$ exists such that the following holds:*

1. *The operator $\Gamma(\mathcal{M})$ defined by*

$$\Gamma(\mathcal{M}) \doteq \mathcal{A}'\, \mathcal{M} + \mathcal{M}\, \mathcal{A} + \mathcal{M}\, \sigma\, \sigma' \mathcal{M} + \mathcal{C}\,, \tag{2.14}$$

   is coercive, where $\mathcal{A}$, σ, and $\mathcal{C}$ are as per (1.1); and

2. *The unique mild solution $\mathcal{P} \in C_0([0, \tau_0]; \Sigma(\mathscr{X}))$ of (1.1) satisfying $\mathcal{P}(0) = \mathcal{M}$ that exists for some $\tau_0 \in \mathbb{R}_{>0}$ by Theorem 2.3 is such that $\mathcal{P}(t) - \mathcal{M}$ is coercive for all $t \in (0, \tau_0]$. That is,*

$$\mathcal{P} \in C_0([0, \tau_0]; \Sigma(\mathscr{X})) \cap C_0((0, \tau_0]; \Sigma_{\mathcal{M}}(\mathscr{X})). \tag{2.15}$$

Theorem 2.7. *Given any self-adjoint bounded linear operator $\mathcal{M} \in \Sigma(\mathscr{X})$ and $\tau_0 \in \mathbb{R}_{>0}$ satisfying Assumption 2.6, and any self-adjoint bounded linear operator $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, there exists a $\tau_1 \in (0, \tau_0] \subset \mathbb{R}_{>0}$ such that a unique mild solution*

$$\widetilde{\mathcal{P}} \in C_0([0, \tau_1]; \Sigma(\mathscr{X})) \cap C_0((0, \tau_1]; \Sigma_{\mathcal{M}}(\mathscr{X})) \tag{2.16}$$

of (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ *exists.*

Proof. Fix any $\mathcal{M} \in \Sigma(\mathscr{X})$ such that Assumption 2.6 holds, and any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$. Note that a unique mild solution $\mathcal{P} \in C_0([0,\tau_0];\Sigma(\mathscr{X}))$ of (1.1) satisfying $\mathcal{P}(0) = \mathcal{M}$ exists, as per Theorem 2.3, where τ_0 is as specified by Assumption 2.6. Note also that the coercivity property (2.15) holds. Again applying Theorem 2.3, note that a unique mild solution $\widetilde{\mathcal{P}} \in C_0([0,\tilde{\tau}_0];\Sigma(\mathscr{X}))$ of (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ also exists for some $\tilde{\tau}_0 \in \mathbb{R}_{>0}$. Define $\tau_1 \doteq \tau_0 \wedge \tilde{\tau}_0 \in \mathbb{R}_{>0}$, and note that $\mathcal{P}, \widetilde{\mathcal{P}} \in C_0([0,\tau_1];\Sigma(\mathscr{X}))$. Hence, operators $\mathcal{E}, \widetilde{\mathcal{E}} \in C_0([0,\tau_1];\Sigma(\mathscr{X}))$ are well-defined for all $t \in [0,\tau_1]$ by

$$\mathcal{E}(t) \doteq \mathcal{P}(t) - \mathcal{M}\,, \quad \widetilde{\mathcal{E}}(t) \doteq \widetilde{\mathcal{P}}(t) - \mathcal{M}\,, \tag{2.17}$$

respectively. Note that $\mathcal{E}(0) = 0$ (the zero operator), while $\widetilde{\mathcal{E}}(0) = \widetilde{\mathcal{M}} - \mathcal{M}$ (which is coercive by definition). Substitution for $\mathcal{P}$ (or $\widetilde{\mathcal{P}}$) in (1.1), using $\mathcal{E}$ (respectively $\widetilde{\mathcal{E}}$), yields the evolution equation

$$\dot{\mathcal{E}}(t) = \mathcal{A}(t)'\,\mathcal{E}(t) + \mathcal{E}(t)\,\mathcal{A}(t) + \Gamma(\mathcal{M})\,, \tag{2.18}$$

which holds for all $t \in [0,\tau_1]$, where $\Gamma(\mathcal{M})$ is as per (2.14). Operator $\mathcal{A}(t)$ (resp. $\widetilde{\mathcal{A}}(t)$) is defined for all $t \in [0,\tau_1]$ by

$$\mathcal{A}(t) \doteq \mathcal{A} + \sigma\,\sigma'\left(\mathcal{M} + \tfrac{1}{2}\,\mathcal{P}(t)\right)\,. \tag{2.19}$$

Note that the $\Gamma(\mathcal{M})$ term in (2.18) is unchanged in moving from $\mathcal{E}$ to $\widetilde{\mathcal{E}}$. That is, it is *not* replaced with $\Gamma(\widetilde{\mathcal{M}})$. As the evolution equation (2.18) is simply a rearranged version of (1.1), it exhibits a unique mild solution corresponding to each of the two initial conditions $\mathcal{E}(0)$ and $\widetilde{\mathcal{E}}(0)$ described above. (These unique solutions of course correspond to $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ respectively.) Note that (2.18) is of the form of (3.17) on p.138 of [4], see (in particular) the proof of Theorem 2.1 on p.393 of [4]. Consequently, the mild solutions $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ may be represented in terms of an *evolution operator* $\mathcal{U} : \Delta_{\tau_1} \to \mathcal{L}(\Sigma(\mathscr{X}))$, where $\Delta_{\tau_1} \doteq \left\{(t,s) \in \mathbb{R}^2 \,\middle|\, t \in [0,\tau_1]\,,\ s \in [0,t]\right\}$. In particular,

$$\begin{aligned}\mathcal{E}(t)\,x &= \mathcal{U}(t,0)\,\mathcal{E}(0)\,\mathcal{U}(t,0)'\,x + \int_0^t \mathcal{U}(t,s)\,\Gamma(\mathcal{M})\,\mathcal{U}(t,s)'\,x\,ds \\ &= \int_0^t \mathcal{U}(t,s)\,\Gamma(\mathcal{M})\,\mathcal{U}(t,s)'\,x\,ds \end{aligned}\tag{2.20}$$

for all $x \in \mathsf{dom}(\mathcal{A})$, as $\mathcal{E}(0) = 0$. (The evolution operator $\mathcal{U}$ has standard properties that are set out in Proposition 3.6 on p.138 of [4]. The key property employed here is that $\frac{\partial}{\partial s}[\mathcal{U}(t,s)] = -\mathcal{U}(t,s)\,\mathcal{A}(s)'$ for all $(t,s) \in \Delta_{\tau_1}$.) Similarly,

$$\begin{aligned}\widetilde{\mathcal{E}}(t)\,x &= \mathcal{U}(t,0)\,\widetilde{\mathcal{E}}(0)\,\mathcal{U}(t,0)'\,x + \int_0^t \mathcal{U}(t,s)\,\Gamma(\mathcal{M})\,\mathcal{U}(t,s)'\,x\,ds \\ &= \mathcal{U}(t,0)\,(\widetilde{\mathcal{M}} - \mathcal{M})\,\mathcal{U}(t,0)'\,x + \mathcal{E}(t)\,x\,, \end{aligned}\tag{2.21}$$

where the second equation follows from (2.20), and by noting that the evolution operator $\mathcal{U}$ is independent of initial conditions in (2.18). Hence, taking the inner product of both sides of (2.21) with any $x \in \mathsf{dom}(\mathcal{A}) \subset \mathscr{X}$ yields

$$\langle x,\, \widetilde{\mathcal{E}}(t)\,x\rangle = \langle x,\, \mathcal{E}(t)\,x\rangle + \langle \mathcal{U}(t,0)'x,\, (\widetilde{\mathcal{M}} - \mathcal{M})\,\mathcal{U}(t,0)'x\rangle$$

As $\mathcal{E} \in C_0([0,\tau_1];\Sigma(\mathscr{X})) \cap C_0((0,\tau_1];\Sigma_{\mathcal{M}}(\mathscr{X}))$ by definition (2.17) and Assumption 2.6 (in particular, (2.15)), and $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, it follows immediately there exist $\epsilon_1, \epsilon_2 \in \mathbb{R}_{>0}$ such that

$$\langle x,\, (\widetilde{\mathcal{P}}(t) - \mathcal{M})\, x\rangle \geq \epsilon_1 \|x\|^2 + \epsilon_2\, \|\mathcal{U}(t,0)' x\|^2$$

for all $t \in (0,\tau_1]$, $x \in \mathsf{dom}(A) \subset \mathscr{X}$. That is, $\widetilde{\mathcal{P}}(t) - \mathcal{M}$ is coercive for all $t \in (0,\tau_1]$, so that (2.16) follows. ☐

Remark 2.8. In obtaining forms (2.20) and (2.21) for the respective solutions $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ of the evolution equation (2.18), it is important to note that a generalization of Proposition 3.4 on p.136 of [4] is applied. This generalization allows the trajectory $\mathcal{E}$ of (2.18) to be an operator-valued function in $C_0([0,\tau_1];\Sigma(\mathscr{X}))$. While this is straightforward, the forcing term $f(t) \doteq \Gamma(\mathcal{M})$, $t \in [0,\tau_1]$, in (2.18) must satisfy the corresponding generalized condition

$$f(\cdot) \doteq \Gamma(\mathcal{M}) \in \mathcal{L}_2([0,\tau_1];\Sigma(\mathscr{X}))\,. \tag{2.22}$$

While f is a constant valued function on $[0,\tau_1]$ by inspection of (2.14), its value $\Gamma(\mathcal{M})$ is not in $\Sigma(\mathscr{X})$ for arbitrary $\mathcal{M} \in \Sigma(\mathscr{X})$ as $\mathcal{A}$ is unbounded. However, by Assumption 2.1, it is possible to choose $\mathcal{M} = (\mathcal{A}^{-1})'\, \mathcal{M}_0\, \mathcal{A}^{-1}$ for any $\mathcal{M}_0 \in \Sigma(\mathscr{X})$, thereby yielding $\Gamma(\mathcal{M}) \in \mathcal{L}(\mathscr{X})$ (possibly after extension from $\mathsf{dom}(\mathcal{A})$ to $\mathscr{X}$), in which case (2.22) holds.

2.2. Auxiliary operator differential equations. In proposing a max-plus dual space fundamental solution to the differential operator Riccati equation (1.1), two (additional) auxiliary operator differential equations are of interest. These equations, also defined with respect to Hilbert spaces $\mathscr{X}$ and $\mathscr{W}$, are given by

$$\dot{\mathcal{Q}}(t) = \mathcal{A}'\, \mathcal{Q}(t) + \mathcal{P}(t)\, \sigma\, \sigma'\, \mathcal{Q}(t)\,, \tag{2.23}$$

$$\dot{\mathcal{R}}(t) = \mathcal{Q}'(t)\, \sigma\, \sigma'\, \mathcal{Q}(t)\,, \tag{2.24}$$

in which $\mathcal{A} : \mathsf{dom}(\mathcal{A}) \subset \mathscr{X} \to \mathscr{X}$ and $\sigma \in \mathcal{L}(\mathscr{W};\mathscr{X})$ are defined as per (1.1). Also as per (1.1), any operator-valued functions $\mathcal{Q} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ and $\mathcal{R} \in C_0([0,\tau];\Sigma(\mathscr{X}))$ satisfying

$$\begin{aligned}
\mathcal{Q}(t)\, x &= e^{\mathcal{A}'\, t}\, \mathcal{Q}(0)\, x + \int_0^t e^{\mathcal{A}'\,(t-s)}\, [\mathcal{P}(s)\, \sigma\, \sigma'\, \mathcal{Q}(s)]\, x\, ds\,, \\
\mathcal{R}(t)\, x &= \mathcal{R}(0)\, x + \int_0^t \mathcal{Q}(s)'\, \sigma\, \sigma'\, \mathcal{Q}(s)\, x\, ds\,,
\end{aligned} \tag{2.25}$$

for all $x \in \mathscr{X}$, $t \in [0,\tau]$, $\tau \in \mathbb{R}_{>0}$, are defined to be *mild solutions* of (2.23) and (2.24) (respectively) on $[0,\tau]$. With regard to the range of $\mathcal{Q}$, note $\mathcal{Q}(t) \in \mathcal{L}(\mathscr{X})$ is not self-adjoint by inspection of (2.23) or (2.25). Hence, $\mathcal{Q} \in C_0([0,\tau];\mathcal{L}(\mathscr{X}))$ (rather than $C_0([0,\tau];\Sigma(\mathscr{X}))$). On the other hand, $\mathcal{R}(t) \in \Sigma(\mathscr{X})$ is self-adjoint by inspection of (2.24) or (2.25). Following the arguments used in the proofs of Theorems 2.3 and 2.7, existence of unique mild solutions of (2.23) and (2.24) can also be established for specific initial conditions. Note that coercivity of these operators is not sought or required.

Theorem 2.9. *Given any $\mathcal{M} \in \Sigma(\mathscr{X})$ and any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, there exists a $\tau_2 \in \mathbb{R}_{>0}$ such that unique mild solutions $\mathcal{Q}, \widetilde{\mathcal{Q}} \in C_0([0,\tau_2];\mathcal{L}(\mathscr{X}))$ and $\mathcal{R}, \widetilde{\mathcal{R}} \in C_0([0,\tau_2];\Sigma(\mathscr{X}))$ of (2.23) and (2.24) (respectively) satisfying $\mathcal{Q}(0) = -\mathcal{M}$, $\widetilde{\mathcal{Q}}(0) = -\widetilde{\mathcal{M}}$ and $\mathcal{R}(0) = \mathcal{M}$, $\widetilde{\mathcal{R}}(0) = \widetilde{\mathcal{M}}$ (resp.) exist.*

Proof. The proof is similar to that of Theorem 2.3 and is omitted. ☐

2.3. Common horizon of existence of mild solutions. For the remainder, it is convenient to define a common horizon $\tau^* \in \mathbb{R}_{>0}$ of existence for the unique mild solutions $\mathcal{P}$, $\widetilde{\mathcal{P}}$, $\mathcal{Q}$, $\widetilde{\mathcal{Q}}$, $\mathcal{R}$, $\widetilde{\mathcal{R}}$ of the respective operator differential equations (1.1), (2.23), (2.24) corresponding to the respective initializations $\mathcal{P}(0) = \mathcal{M} = \mathcal{R}(0)$, $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}} = \widetilde{\mathcal{R}}(0)$, $\mathcal{Q}(0) = -\mathcal{M}$, $\widetilde{\mathcal{Q}}(0) = -\widetilde{\mathcal{M}}$. In particular, define

$$\tau^* \doteq \tau_1 \wedge \tau_2 \,, \tag{2.26}$$

where $\tau_1, \tau_2 \in \mathbb{R}_{>0}$ are the horizons of existence for $\mathcal{P}$, $\widetilde{\mathcal{P}}$ and $\mathcal{Q}$, $\widetilde{\mathcal{Q}}$, $\mathcal{R}$, $\widetilde{\mathcal{R}}$ as guaranteed by Theorems 2.7 and 2.9 in the company of Assumption 2.6, and $\wedge$ denotes the min operation.

3. Max-plus dual space fundamental solution semigroup. A max-plus dual space fundamental solution semigroup for the operator differential Riccati equation (1.1) is constructed by exploiting the semigroup property that attends the dynamic programming evolution operator of a related optimal control problem. In particular, by employing the Legendre-Fenchel transform of a particular solution of the operator differential Riccati equation (1.1), a max-plus integral operator is defined in a corresponding max-plus dual space. It is demonstrated that this max-plus integral operator defines the aforementioned fundamental solution semigroup for the operator differential Riccati equation (1.1), which allows the realization of any solution of (1.1). This construction generalizes the finite dimensional case documented in [12], and the infinite dimensional cases of [6, 7] in that it does not assume an explicit representation for the operator-valued solution of the operator differential Riccati equation (1.1).

3.1. Optimal control problem. With $\tau^* \in \mathbb{R}_{>0}$ as per (2.26), an optimal control problem is defined with respect to the abstract Cauchy problem [4, 5, 13]

$$\dot{\xi}(t) = \mathcal{A}\,\xi(t) + \sigma\, w(t)\,, \tag{3.1}$$

where $\xi(t) \in \mathsf{dom}(\mathcal{A}) \subset \mathscr{X}$ denotes the (possibly infinite dimensional) state at time $t \in [0, \tau^*]$, evolved from an initial state $\xi(0) = x \in \mathsf{dom}(\mathcal{A})$ in the presence of an input signal $w \in \mathcal{L}_2([0,t];\mathscr{W})$. Given such an input, a function $\xi \in C([0,t];\mathscr{X})$ is a *mild* solution of the abstract Cauchy problem (3.1) on $[0,t]$ if it satisfies

$$\xi(t) = e^{\mathcal{A}\,t}\,\xi(0) + \int_0^t e^{\mathcal{A}\,(t-s)}\, w(s)\, ds\,, \tag{3.2}$$

(see for example Definition 3.1 on p.129 of [4]), where $e^{\mathcal{A}\,t} \in \mathcal{L}(\mathscr{X})$ denotes the corresponding element of the C_0-semigroup of bounded linear operators generated by $\mathcal{A}$. Note that $\xi \in C([0,t);\mathscr{X})$ is in fact implied by (3.2), see for example Lemma 3.1.5 of [5]. Indeed, given any $\xi(0) = x \in \mathscr{X}$ and $w \in \mathcal{L}_2([0,t];\mathscr{W})$, the abstract Cauchy problem (3.1) has a unique *strong* solution which is also the mild solution (see for example Definition 3.1 and Proposition 3.1 on pp.129 – 130 of [4]). The optimal control problem of interest is well-defined via the value functional $W^z : [0,\tau^*] \times \mathscr{X} \to \mathbb{R}$ for each $z \in \mathscr{X}$ by

$$W^z(t,x) \doteq \sup_{w \in \mathcal{L}_2([0,t];\mathscr{W})} J_{\psi(\cdot,z)}(t,x;w)\,, \tag{3.3}$$

where the payoff $J_\Psi : [0,t] \times \mathscr{X} \times \mathcal{L}_2([0,t];\mathscr{W}) :\to \mathbb{R}$ is defined with respect to the unique mild solution (3.2) corresponding to $\xi(0) = x \in \mathscr{X}$, and a generic terminal payoff $\Psi : \mathscr{X} \to \mathbb{R}$, by

$$J_\Psi(t,x;w) \doteq \int_0^t \tfrac{1}{2}\langle \xi(s),\, \mathcal{C}\,\xi(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\, ds + \Psi(\xi(t))\,. \tag{3.4}$$

A specific terminal payoff $\Psi(\cdot) = \psi(\cdot, z) : \mathscr{X} \to \mathbb{R}$ of interest is defined for each $z \in \mathscr{X}$ via the self-adjoint bounded linear operator $\mathcal{M} \in \Sigma(\mathscr{X})$ of Assumption 2.6, with

$$\psi(x,z) \doteq \tfrac{1}{2}\langle x - z,\, \mathcal{M}\,(x-z)\rangle\,. \tag{3.5}$$

Solutions of the operator differential Riccati equation (1.1), and the auxiliary operator differential equations (2.23) and (2.24), are fundamentally related to the optimal control problem of (3.3). To explore and exploit this relationship, define the operator-valued map $\mathcal{F}_t : [0,t] \to \mathcal{L}(\mathscr{X};\mathscr{W})$ by

$$\mathcal{F}_t(s)\,x \doteq \sigma'\left(\mathcal{P}(t-s)\,x + \mathcal{Q}(t-s)\,z\right), \tag{3.6}$$

where $\mathcal{P} \in C_0([0,\tau^*];\Sigma(\mathscr{X}))$ and $\mathcal{Q} \in C_0([0,\tau^*];\mathcal{L}(\mathscr{X}))$ denote the unique mild solutions of (1.1) and (2.23) satisfying $\mathcal{P}(0) = \mathcal{M}$ and $\mathcal{Q}(0) = -\mathcal{M}$ respectively (as per Theorems 2.3 and 2.9). The map (3.6) can be regarded as a feedback for the abstract Cauchy problem (3.1), yielding the closed-loop abstract Cauchy problem

$$\dot{\xi}(s) = (\mathcal{A} + \sigma\,\mathcal{F}_t(s))\,\xi(s)\,, \quad s \in [0,t]\,, \tag{3.7}$$

where $\xi(0) = x \in \mathscr{X}$ and $t \in [0,\tau^*]$.

Theorem 3.1. *Given any $t \in [0,\tau^*]$, the closed-loop abstract Cauchy problem (3.7) has a unique mild solution $\xi^* \in C([0,t];\mathscr{X})$. Furthermore, the input $w^* \in C([0,t];\mathscr{W})$ defined by*

$$w^*(s) \doteq \mathcal{F}_t(s)\,\xi^*(s) = \sigma'(\mathcal{P}(t-s)\,\xi^*(s) + \mathcal{Q}(t-s)\,z) \tag{3.8}$$

is optimal with respect to (3.3), (3.4), with

$$\begin{aligned} J_{\psi(\cdot,z)}(t,x;w) &\le J_{\psi(\cdot,z)}(t,x;w^*) = W^z(t,x) \\ &= \tfrac{1}{2}\langle x,\, \mathcal{P}(t)\,x + \langle x,\, \mathcal{Q}(t)\,z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}(t)\,z\rangle \end{aligned} \tag{3.9}$$

for all $w \in \mathcal{L}_2([0,t];\mathscr{W})$, $x \in \mathscr{X}$.

Proof. Fix any $t \in [0,\tau^*]$. The abstract Cauchy problem (3.7) exhibits a unique mild solution $\xi^* \in C([0,t];\mathscr{X})$ via a straightforward modification of Proposition 6.1 on p.409 of [4]. The fact that input w^* defined by (3.8) is optimal follows by a modification of the proof of Proposition 6.2 on p. 409 of [4]. In particular, let $\mathcal{P}_n \in C([0,\tau_n];\Sigma(\mathscr{X}))$ denote the unique mild solution of (2.7) corresponding to the Yoshida approximation $\mathcal{A}_n$ of $\mathcal{A}$ that exists on some interval $[0,\tau_n]$, where $n \in \mathbb{N}$ is selected large enough so that $t \in [0,\tau_n]$. (Do the same with $\mathcal{Q}$, etc.) Let $\xi_n \in C([0,t];\mathscr{X})$ denote the unique mild solution of the corresponding abstract Cauchy problem (3.1) for arbitrary $w \in \mathcal{L}_2([0,t];\mathscr{W})$. Define $\pi_n : [0,t] \to \mathbb{R}$ by

$$\pi_n(s) \doteq p_n(s) + q_n(s) + r_n(s) \tag{3.10}$$

where $p_n, q_n, r_n : [0,t] \to \mathbb{R}$ are given by

$$p_n(s) \doteq \tfrac{1}{2}\langle \xi_n(s),\, \mathcal{P}_n(t-s)\,\xi_n(s)\rangle\,, \tag{3.11}$$

$$q_n(s) \doteq \langle \xi_n(s),\, \mathcal{Q}_n(t-s)\,z\rangle\,, \tag{3.12}$$

$$r_n(s) \doteq \tfrac{1}{2}\langle z,\, \mathcal{R}_n(t-s)\,z\rangle\,. \tag{3.13}$$

Differentiating (formally) and applying (1.1), (2.23), and (2.24), it is straightforward to show that

$$\begin{aligned} \dot p_n(s) &= -\tfrac{1}{2}\langle \xi_n(s),\, [\mathcal{C} + \mathcal{P}_n(t-s)\,\sigma\,\sigma'\,\mathcal{P}_n(t-s)]\ \xi_n(s)\rangle \\ &\qquad + \langle w(s),\, \sigma'\mathcal{P}_n(t-s)\,\xi_n(s)\rangle\,, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \dot q_n(s) &= -\langle \xi_n(s),\, \mathcal{P}_n(t-s)\,\sigma\,\sigma'\,\mathcal{Q}_n(t-s)\,z\rangle \\ &\qquad + \langle w(s),\, \sigma'\mathcal{Q}_n(t-s)\,z\rangle\,, \end{aligned} \tag{3.15}$$

$$\dot r_n(s) = -\tfrac{1}{2}\langle z,\, \mathcal{Q}_n(t-s)'\,\sigma\,\sigma'\,\mathcal{Q}_n(t-s)\,z\rangle \tag{3.16}$$

Define $\overline{w}(s) \doteq \sigma'\left(\mathcal{P}_n(t-s)\,\xi_n(s) + \mathcal{Q}_n(t-s)\,z\right)$ for all $s \in [0,t]$. Differentiation of (3.10), substitution of (3.14), (3.15), (3.16), followed by completion of squares, yields

$$\dot\pi(s) = -\left[\tfrac{1}{2}\langle \xi_n(s)\,\mathcal{C}\,\xi_n(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\right] - \tfrac{1}{2}\|w(s) - \overline{w}(s)\|^2 \tag{3.17}$$

for all $s \in [0,t]$. As $\mathcal{P}_n \in C([0,t];\Sigma(\mathscr{X}))$ and $\mathcal{Q}_n \in C([0,t];\mathcal{L}(\mathscr{X}))$, note that $\overline{w} \in C([0,t];\mathscr{W}) \subset \mathcal{L}_2([0,t];\mathscr{W})$ by definition. Note also by (3.10), and that $\mathcal{P}_n(0) = \mathcal{M} = \mathcal{R}_n(0)$ and $\mathcal{Q}_n(0) = -\mathcal{M}$, so that

$$\pi_n(t) = \tfrac{1}{2}\langle \xi_n(t),\, \mathcal{M}\,\xi_n(t)\rangle - \langle \xi_n(t),\, \mathcal{M}\,z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{M}\,z\rangle = \psi(\xi_n(t), z)\,, \tag{3.18}$$

where ψ is the terminal payoff (3.5). Similarly, as $\xi_n(0) = x$,

$$\pi_n(0) = \tfrac{1}{2}\langle x,\, \mathcal{P}_n(t)\,x\rangle + \langle x,\, \mathcal{Q}_n(t)\,z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}_n(t)\,z\rangle\,.$$

So, integrating (3.17) with respect to $s \in [0,t]$ and applying (3.18) yields that

$$\pi_n(0) = \int_0^t \tfrac{1}{2}\langle \xi_n(s),\, \mathcal{C}\,\xi_n(s)\rangle - \tfrac{1}{2}\|w(s)\|^2\,ds + \psi(\xi_n(t),\, z) + \tfrac{1}{2}\int_0^t \|w(s) - \overline{w}(s)\|^2\,ds\,,$$

Taking the limit as $n \to \infty$, setting $p_\infty \doteq \lim_{n\to\infty} p_n$, and applying (3.4),

$$\pi_\infty(0) - \tfrac{1}{2}\int_0^t \|w(s) - \overline{w}(s)\|^2\,ds = J_{\psi(\cdot,z)}(t,x;w)\,.$$

Finally, taking the supremum over $w \in \mathcal{L}_2([0,t];\mathscr{W})$ yields

$$\begin{aligned} W^z(t,x) &= \sup_{w\in\mathcal{L}_2([0,t];\mathscr{W})} J_{\psi(\cdot,z)}(t,x;w) \\ &= \pi_\infty(0) = \tfrac{1}{2}\langle x,\, \mathcal{P}(t)\,x\rangle + \langle x,\, \mathcal{Q}(t)\,z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}(t)\,z\rangle\,, \end{aligned}$$

in which the optimal input is $w^* = \overline{w}$, as per (3.8). ☐

Theorem 3.1 is crucial to the development of a max-plus fundamental solution to the operator differential Riccati equation (1.1). In particular, it demonstrates that the unique mild solution $\mathcal{P} \in C_0([0,\tau^*];\Sigma(\mathscr{X}))$ of (1.1) may be propagated forward in time via propagation of the value function $W^z(t,\cdot)$ of (3.3) with respect to its time horizon $t \in [0,\tau^*]$. This is significant as propagation of $W^z(t,\cdot)$ is possible via the dynamic programming [2, 3] evolution operator. In particular, W^z may be written as

$$W^z(t,x) = (\mathcal{S}_t\,\psi(\cdot,z))(x) \tag{3.19}$$

for all $t \in [0,\tau^*]$, $x \in \mathscr{X}$, where $\mathcal{S}_t$ denotes the aforementioned dynamic programming evolution operator. This operator is defined by

$$(\mathcal{S}_t\,\Psi)(x) \doteq \sup_{w\in\mathcal{L}_2([0,t];\mathscr{W})} J_\Psi(t,x;w)\,. \tag{3.20}$$

and satisfies the semigroup property

$$\mathcal{S}_{t+s} = \mathcal{S}_s \, \mathcal{S}_t = \mathcal{S}_t \, \mathcal{S}_s \tag{3.21}$$

for all $s, t \in [0, \tau^*]$, $s + t \in [0, \tau^*]$. It is this semigroup property, in combination with (3.19), that allows $W^z(t, \cdot)$ to be propagated to longer time horizons via the dynamic programming evolution operator of (3.20).

3.2. Max-plus dual space representation of W^z. The semigroup property (3.21) describes how the value functional $W^z(t, \cdot)$ of (3.3) can be propagated from any initial horizon $t \in [0, \tau^*]$ to any final longer time horizon $t + s \in [0, \tau^*]$, $s \in [0, \tau^* - t]$, via dynamic programming. As this value functional is identified with the operator differential Riccati equation solution $\mathcal{P}$ of (1.1) via Theorem 3.1, this value functional propagation corresponds to evolution of $\mathcal{P}$ from its initial condition $\mathcal{M} \in \Sigma(\mathscr{X})$ satisfying Assumption 2.6. By appealing to semiconvex duality [15] of the value functional, and max-plus linearity of the dynamic programming evolution operator, this evolution can be represented via a dual space evolution operator that is defined independently of the terminal payoff ψ of (3.5), and hence the initial data $\mathcal{M} \in \Sigma(\mathscr{X})$. The dual space evolution operator obtained is subsequently shown to propagate the solution of the operator differential Riccati equation (1.1) from any arbitrary initial condition $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ satisfying the conditions of Theorem 2.7.

This development relies on concepts and results from convex analysis and idempotent analysis. In particular, semiconvex duality [15] is introduced using operators defined with respect to the max-plus algebra, c.f. [11]. The max-plus algebra is a commutative semifield over $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$ equipped with the addition and multiplication operations $\oplus$ and $\otimes$ that are defined by $a \oplus b \doteq \max(a, b)$ and $a \otimes b \doteq a + b$. It is also an idempotent semifield as $\oplus$ is an idempotent operation (i.e. $a \oplus a = a$) with no inverse. The respective spaces $\mathscr{S}^{\mathcal{K}}(\mathscr{X})$ and $\mathscr{S}_-^{\mathcal{K}}(\mathscr{X})$ of semiconvex and semiconcave functionals are defined with respect to a self-adjoint bounded linear operator $\mathcal{K} \in \Sigma(\mathscr{X})$ by

$$\mathscr{S}^{\mathcal{K}}(\mathscr{X}) \doteq \left\{ f : \mathscr{X} \to \mathbb{R} \;\middle|\; f(\cdot) + \tfrac{1}{2} \langle \cdot, \mathcal{K} \cdot \rangle \text{ is convex on } \mathscr{X} \right\}, \tag{3.22}$$

$$\mathscr{S}_-^{\mathcal{K}}(\mathscr{X}) \doteq \left\{ f : \mathscr{X} \to \mathbb{R} \;\middle|\; f(\cdot) - \tfrac{1}{2} \langle \cdot, \mathcal{K} \cdot \rangle_{\mathscr{X}} \text{ is concave on } \mathscr{X} \right\}. \tag{3.23}$$

It may be shown that $\mathscr{S}^{\mathcal{K}}(\mathscr{X})$ is a max-plus vector space of functionals defined on $\mathscr{X}$, see [11] for the analogous details in the finite dimensional case. Semiconvex duality [15] is formalized as follows, in which max-plus integration of a functional f over $\mathscr{X}$ is defined by $\int_{\mathscr{X}}^{\oplus} f(z)\, dz \doteq \sup_{z \in \mathscr{X}} f(z)$.

THEOREM 3.2. *Let $\phi \in \mathscr{S}^{\mathcal{K}}(\mathscr{X})$ be a closed semiconvex functional on $\mathscr{X}$, where $\mathcal{K} \in \Sigma(\mathscr{X})$ is a self-adjoint bounded linear operator satisfying $\mathcal{K} < -\mathcal{M}$ with $\mathcal{M} \in \Sigma(\mathscr{X})$ specified as per Assumption 2.6. Then,*

$$\phi = \mathcal{D}_\psi^{-1} \, a \in \mathscr{S}^{\mathcal{K}}(\mathscr{X}), \quad a = \mathcal{D}_\psi \, \phi \in \mathscr{S}_-^{\mathcal{K}}(\mathscr{X}), \tag{3.24}$$

where ψ is the quadratic terminal payoff (3.5), and $\mathcal{D}_\psi$, $\mathcal{D}_\psi^{-1}$ denote respectively the semiconvex dual and inverse dual operators [15] defined by

$$\mathcal{D}_\psi \, \phi = (\mathcal{D}_\psi \, \phi)(\cdot) \doteq - \int_{\mathscr{X}}^{\oplus} \psi(x, \cdot) \otimes (-\phi(x)) \, dx, \tag{3.25}$$

$$\mathcal{D}_\psi^{-1} \, a = (\mathcal{D}_\psi^{-1} \, a)(\cdot) \doteq \int_{\mathscr{X}}^{\oplus} \psi(\cdot, z) \otimes a(z) \, dz. \tag{3.26}$$

Proof. (3.24) follows by Lemma B.1 and Theorem 5 of [15]. ◻

In order to demonstrate that the semiconvex dual of $\mathcal{S}_t\,\psi(\cdot,z)$ is well-defined for each $t\in(0,\tau^*]$ and $z\in\mathscr{X}$, define the self-adjoint bounded linear operator $\mathcal{K}_t\in\Sigma(\mathscr{X})$ by $\mathcal{K}_t\doteq-\alpha\,\mathcal{P}(t)-(1-\alpha)\,\mathcal{M}$, with $\alpha\in(0,1)$ fixed, where $\mathcal{P}(t)$ and $\mathcal{M}$ are as per Assumption 2.6. Theorem 3.1 and assertion *(ii)* of Lemma B.1 imply that

$$\begin{aligned}(\mathcal{S}_t\,\psi(\cdot,z))\,(x)+\tfrac12\,\langle x,\,\mathcal{K}_t\,x\rangle&=W^z(t,x)+\tfrac12\,\langle x,\,\mathcal{K}_t\,x\rangle\\&=\tfrac12\,\langle x,\,(\mathcal{P}(t)+\mathcal{K}_t)\,x\rangle+\langle x,\,\mathcal{Q}(t)\,z\rangle+\tfrac12\,\langle z,\,\mathcal{R}(t)\,z\rangle\,.\end{aligned}\tag{3.27}$$

With $t\in(0,\tau^*]$, note that $\mathcal{P}(t)>\mathcal{M}$, so that $\mathcal{P}(t)+\mathcal{K}_t=(1-\alpha)\,(\mathcal{P}(t)-\mathcal{M})>0$, and $-\mathcal{K}_t-\mathcal{M}=\alpha\,(\mathcal{P}(t)-\mathcal{M})>0$. That is, $\mathcal{K}_t$ is self-adjoint and satisfies $-\mathcal{P}(t)<\mathcal{K}_t<-\mathcal{M}$. Hence, the right-hand side of (3.27) is the sum of a non-negative quadratic functional and an affine functional. As any non-negative quadratic functional is convex by assertion (ii) of Lemma B.1, and any affine functional is convex by definition, the right-hand side of (3.27) is also convex. Hence,

$$\mathcal{S}_t\,\psi(\cdot,z)\in\mathscr{S}^{\mathcal{K}_t}(\mathscr{X})\,.\tag{3.28}$$

for all $t\in(0,\tau^*]$. Also note that as $\mathcal{S}_t\,\psi(\cdot,z)$ is closed by Theorem 3.1 and assertion *(i)* of Lemma B.1. Consequently, Theorem 3.2 implies that the semiconvex dual of $\mathcal{S}_t\,\psi(\cdot,z)$ is well-defined for any $z\in\mathscr{X}$. Denote this dual by the functional $B_t(\cdot,z):\mathscr{X}\to\mathbb{R}$ for each $z\in\mathscr{X}$ fixed, so that (3.24) yields

$$(\mathcal{S}_t\,\psi(\cdot,z))\,(x)=(\mathcal{D}_\psi^{-1}\,B_t(\cdot,z))\,(x)\,,\tag{3.29}$$

$$B_t(y,z)=(\mathcal{D}_\psi\,\mathcal{S}_t\,\psi(\cdot,z))\,(y)\,.\tag{3.30}$$

for all $t\in(0,\tau^*]$, $x,y,z\in\mathscr{X}$. Theorem 3.1 also ensures that an explicit quadratic form for the functional $B_t(\cdot,z)$ is inherited from $\mathcal{S}_t\,\psi(\cdot,z)$, as formalized below.

LEMMA 3.3. *$B_t:\mathscr{X}\times\mathscr{X}\to\mathbb{R}$ is a quadratic functional given explicitly by*

$$B_t(y,z)=\tfrac12\langle y,\,\mathcal{B}_t^{1,1}\,y\rangle+\langle z,\,\mathcal{B}_t^{1,2}\,y\rangle+\tfrac12\langle z,\,\mathcal{B}_t^{2,2}\,z\rangle\tag{3.31}$$

for all $t\in(0,\tau^*]$, $y,z\in\mathscr{X}$, where $\mathcal{B}_t^{1,1}$, $\mathcal{B}_t^{1,2}$ and $\mathcal{B}_t^{2,2}$ denote the well-defined operators

$$\mathcal{B}_t^{1,1}\doteq-\mathcal{M}-\mathcal{M}\,(\mathcal{P}(t)-\mathcal{M})^{-1}\,\mathcal{M}\,,\tag{3.32}$$

$$\mathcal{B}_t^{1,2}\doteq-\mathcal{Q}(t)'\,(\mathcal{P}(t)-\mathcal{M})^{-1}\,\mathcal{M}\,,\tag{3.33}$$

$$\mathcal{B}_t^{2,2}\doteq-\mathcal{Q}(t)'\,(\mathcal{P}(t)-\mathcal{M})^{-1}\,\mathcal{Q}(t)+\mathcal{R}(t)\,.\tag{3.34}$$

Proof. With $\tau^*\in\mathbb{R}_{>0}$ fixed as per (2.26), recall by (3.28) , (3.29) and (3.30) that $B_t(\cdot,z)$ is the well-defined dual of $\mathcal{S}_t\,\psi(\cdot,z)$. In particular, $B_t(y,z)$ is finite for all $t\in(0,\tau^*]$, $y,z\in\mathscr{X}$. Applying (3.25), (3.30), and Theorem 3.1, $B_t(y,z)=-\int_{\mathscr{X}}^{\oplus}\pi_t^{y,z}(x)\,dx$ for all $t\in(0,\tau^*]$, $y,z\in\mathscr{X}$, where

$$\begin{aligned}\pi_t^{y,z}(x)&\doteq\psi(x,y)\otimes(-(\mathcal{S}_t\,\psi(\cdot,z))(x))\\&=\tfrac12\langle x-y,\,\mathcal{M}\,(x-y)\rangle-\tfrac12\langle x,\,\mathcal{P}(t)\,x\rangle-\langle x,\,\mathcal{Q}(t)\,z\rangle-\tfrac12\langle z,\,\mathcal{R}(t)\,z\rangle\\&=\tfrac12\langle x,\,(\mathcal{M}-\mathcal{P}(t))\,x\rangle+\langle x,\,-(\mathcal{M}\,y+\mathcal{Q}(t)\,z)\rangle+\tfrac12\langle y,\,\mathcal{M}\,y\rangle-\tfrac12\langle z,\,\mathcal{R}(t)\,z\rangle\\&=b(x)+\tfrac12\langle y,\,\mathcal{M}\,y\rangle-\tfrac12\langle z,\,\mathcal{R}(t)\,z\rangle\,,\end{aligned}$$

and $b(x) \doteq \frac{1}{2}\langle x,\, (\mathcal{M} - \mathcal{P}(t))\, x\rangle + \langle x,\, -(\mathcal{M}\, y + \mathcal{Q}(t)\, z)\rangle$. That is,

$$B_t(y,z) = -\tfrac{1}{2}\langle y,\, \mathcal{M}\, y\rangle + \tfrac{1}{2}\langle z,\, \mathcal{R}(t)\, z\rangle - \int_{\mathscr{X}}^{\oplus} b(x)\, dx\,, \tag{3.35}$$

where $b : \mathscr{X} \to \mathbb{R}$ is a quadratic functional. Note that $\mathcal{M} - \mathcal{P}(t)$ is a self-adjoint, bounded linear operator, while $-(\mathcal{M}\, y + \mathcal{Q}(t)\, z) \in \mathscr{X}$. As B_t is finite as previously indicated, (B.2) of Lemma B.2 implies that the supremum in (3.35) is attained at $x^* = -(\mathcal{P}(t) - \mathcal{M})^{-1}(\mathcal{M}\, y + \mathcal{Q}(t)\, z)$, with

$$\begin{aligned}
\int_{\mathscr{X}}^{\oplus} b(x)\, dx = b(x^*) &= \tfrac{1}{2}\langle \mathcal{M}\, y + \mathcal{Q}(t)\, z,\, (\mathcal{P}(t) - \mathcal{M})^{-1}(\mathcal{M}\, y + \mathcal{Q}(t)\, z)\rangle \\
&= \tfrac{1}{2}\langle y,\, \mathcal{M}\,(\mathcal{P}(t) - \mathcal{M})^{-1}\mathcal{M}\, y\rangle + \langle z,\, \mathcal{Q}(t)'\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{M}\, y\rangle \\
&\qquad + \tfrac{1}{2}\langle z,\, \mathcal{Q}(t)'\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{Q}(t)\, z\rangle\,,
\end{aligned}$$

where it may be noted that the inverse (rather than the pseudo-inverse) $(\mathcal{P}(t) - \mathcal{M})^{-1}$ exists as $\mathcal{P}(t) - \mathcal{M}$ is self-adjoint and coercive for all $t \in (0, \tau^*]$ by Assumption 2.6. (See also [5], Examples A.4.2 and A.4.3, p.609.) So, recalling (3.35),

$$\begin{aligned}
B_t(y,z) = &\tfrac{1}{2}\langle y,\, -(\mathcal{M} + \mathcal{M}\,(\mathcal{P}(t) - \mathcal{M})^{-1}\mathcal{M})\, y\rangle + \langle z,\, -\mathcal{Q}(t)'\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{M}\, y\rangle \\
&+ \langle z,\, (-\mathcal{Q}(t)'\,(\mathcal{P}(t) - \mathcal{M})^{-1}\,\mathcal{Q}(t) + \mathcal{R}(t))\, z\rangle\,,
\end{aligned}$$

which is as per (3.31) via definitions (3.32), (3.33) and (3.34). ☐

3.3. Fundamental solution semigroup. The functional B_t of (3.31) may be used as the kernel in defining a max-plus integral operator $\mathcal{B}_t^{\oplus}$ on the dual-space of functionals generated by the semiconvex dual operator $\mathcal{D}_\psi$ of (3.24). Specifically,

$$\mathcal{B}_t^{\oplus}\, a = \left(\mathcal{B}_t^{\oplus}\, a\right)(\cdot) \doteq \int_{\mathscr{X}}^{\oplus} B_t(\cdot, z) \otimes a(z)\, dz \tag{3.36}$$

for all $t \in (0, \tau^*]$. This operator will be identified as the new max-plus dual space fundamental solution semigroup for the operator differential Riccati equation (1.1). To this end, fix any operator $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ as per Theorem 2.7, and define the functional $\widetilde{\psi} : \mathscr{X} \to \mathbb{R}$ by

$$\widetilde{\psi}(x) \doteq \tfrac{1}{2}\,\langle x,\, \widetilde{\mathcal{M}}\, x\rangle\,. \tag{3.37}$$

By replacing the terminal payoff ψ of (3.5) with this functional in the value functional W^z of (3.3), note that the unique solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) initialized with $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ and defined on $[0, \tau^*]$ viz (2.26) may be characterized in an analogous way to Theorem 3.1. That is, $\widetilde{\mathcal{P}}(t)$ may be identified with the propagated value functional $\mathcal{S}_t\, \widetilde{\psi}$ for all $t \in (0, \tau^*]$. Furthermore, this value functional can be represented in terms of the max-plus integral operator $\mathcal{B}_t^{\oplus}$ of (3.36).

Theorem 3.4. *The value functional $\mathcal{S}_t\, \widetilde{\psi}$ defined via evolution operator $\mathcal{S}_t$ of (3.20) and terminal payoff functional $\widetilde{\psi}$ of (3.37) may be represented equivalently by*

$$(\mathcal{S}_t\, \widetilde{\psi})(x) = \tfrac{1}{2}\langle x,\, \widetilde{\mathcal{P}}(t)\, x\rangle = (\mathcal{D}_\psi^{-1}\, \mathcal{B}_t^{\oplus}\, \mathcal{D}_\psi\, \widetilde{\psi})(x)\,, \quad x \in \mathscr{X}\,, \tag{3.38}$$

for all $t \in (0, \tau^*]$, where $\widetilde{\mathcal{P}}$ is the solution of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ as per Theorem 2.7, and $\mathcal{D}_\psi$, $\mathcal{D}_\psi^{-1}$, $\mathcal{B}_t^{\oplus}$ denote the semiconvex dual operators (3.25), (3.26), and the max-plus integral operator (3.36).

Proof. Applying the (omitted) analog of Theorem 3.1 for the terminal cost functional $\widetilde{\psi}$ of (3.37), the value functional $\mathcal{S}_t\,\widetilde{\psi}$ enjoys the explicit quadratic representation (3.9) with $\mathcal{P}$ replaced with $\widetilde{\mathcal{P}}$ and $z\equiv 0$. That is, the left-hand equality in (3.38) holds. By an analogous argument to (3.28), the self-adjoint bounded linear operator $\widetilde{\mathcal{K}}_t \doteq -\alpha\,\widetilde{\mathcal{P}}(t)-(1-\alpha)\,\mathcal{M}$ defined for any $\alpha\in(0,1)$ satisfies $\widetilde{\mathcal{P}}(t)+\widetilde{\mathcal{K}}_t = (1-\alpha)\,(\widetilde{\mathcal{P}}(t)-\mathcal{M})>0$ and $-\widetilde{\mathcal{K}}_t-\mathcal{M}=\alpha\,(\widetilde{\mathcal{P}}(t)-\mathcal{M})>0$ for all $t\in(0,\tau^*]$, where the inequalities follow by Theorem 2.7 and (2.26). That is, $\mathcal{S}_t\,\widetilde{\psi}\in\mathscr{S}^{\widetilde{\mathcal{K}}_t}(\mathscr{X})$ for all $t\in(0,\tau^*]$. As $\mathcal{S}_t\,\widetilde{\psi}$ is a quadratic functional (as demonstrated above), it is closed via Lemma B.1. Consequently, the semiconvex dual of both $\widetilde{\psi}$ and $\mathcal{S}_t\,\widetilde{\psi}$ are well-defined by Theorem 3.2. Set $\tilde{a}\doteq\mathcal{D}_\psi\,\widetilde{\psi}$. Recalling (3.3), (3.4), (3.19), and (3.20), define $I_t:\mathscr{X}\times\mathcal{L}_2([0,\tau^*];\mathscr{W})\to\mathbb{R}$ by

$$I_t(x;w)\doteq\int_0^t \tfrac{1}{2}\langle\xi(s),\,\mathcal{C}\,\xi(s)\rangle-\tfrac{1}{2}\|w(s)\|^2\,ds\;\left|\;\begin{array}{c}\text{(3.1) holds with}\\ \xi(0)=x\,.\end{array}\right.$$

Using max-plus integral notation, (3.20), (3.26) and the definition of $\tilde{a}$ imply that

$$\begin{aligned}
(\mathcal{S}_t\,\widetilde{\psi})(x)&=\int_{\mathcal{L}_2([0,t];\mathscr{W})}^{\oplus} I_t(x;w)\otimes\widetilde{\psi}(\xi(t))\,dw\\
&=\int_{\mathcal{L}_2([0,t];\mathscr{W})}^{\oplus} I_t(x;w)\otimes(\mathcal{D}_\psi^{-1}\,\tilde{a})(\xi(t))\,dw\\
&=\int_{\mathcal{L}_2([0,t];\mathscr{W})}^{\oplus} I_t(x;w)\otimes\left[\int_{\mathscr{X}}^{\oplus}\psi(\xi(t),z)\otimes\tilde{a}(z)\,dz\right]dw\\
&=\int_{\mathscr{X}}^{\oplus}\left[\int_{\mathcal{L}_2([0,t];\mathscr{W})}^{\oplus} I_t(x;w)\otimes\psi(\xi(t),z)\,dw\right]\otimes\tilde{a}(z)\,dz\\
&=\int_{\mathscr{X}}^{\oplus}(\mathcal{S}_t\,\psi(\cdot,z))(x)\otimes\tilde{a}(z)\,dz\,,
\end{aligned}$$

where the second last equation follows by swapping the order of the max-plus integrals (i.e. suprema), while the last equation follows by applying definition (3.20) of $\mathcal{S}_t$ with terminal cost $\psi(\cdot,z)$ of (3.5). Subsequently applying (3.26) and (3.29), swapping the order of the max-plus integrals, and applying (3.36) yields

$$\begin{aligned}
\int_{\mathscr{X}}^{\oplus}(\mathcal{S}_t\,\psi(\cdot,z))(x)\otimes\tilde{a}(z)\,dz&=\int_{\mathscr{X}}^{\oplus}(\mathcal{D}_\psi^{-1}\,B_t(\cdot,z))(x)\otimes\tilde{a}(z)\,dz\\
&=\int_{\mathscr{X}}^{\oplus}\left[\int_{\mathscr{X}}^{\oplus}\psi(x,y)\otimes B_t(y,z)\,dy\right]\otimes\tilde{a}(z)\,dz\\
&=\int_{\mathscr{X}}^{\oplus}\psi(x,y)\otimes\left[\int_{\mathscr{X}}^{\oplus}B_t(y,z)\otimes\tilde{a}(z)\,dz\right]dy\\
&=\int_{\mathscr{X}}^{\oplus}\psi(x,y)\otimes[\mathcal{B}_t^{\oplus}\,\tilde{a}](y)\,dy\\
&=(D_\psi^{-1}\,\mathcal{B}_t^{\oplus}\,\tilde{a})(x)\,.
\end{aligned}$$

Hence, combining the above two equations and recalling the definition of $\tilde{a}$ yields

$$(\mathcal{S}_t\,\widetilde{\psi})(x)=(D_\psi^{-1}\,\mathcal{B}_t^{\oplus}\,\mathcal{D}_\psi\,\widetilde{\psi})(x)$$

for all $x \in \mathscr{X}$, as required by the right-hand equality in (3.38). □

The dual-space representation provided by Theorem 3.4 allows the general solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1), defined with respect to any initialization $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ as per Theorem 2.7, to be represented in terms of the max-plus integral operator $\mathcal{B}_t^\oplus$ of (3.36). However, $\mathcal{B}_t^\oplus$ is defined via the max-plus kernel B_t of (3.30), (3.31), which is itself derived from the particular solution $\mathcal{P}$ of the operator differential Riccati equation (1.1) that is defined with respect to the initialization $\mathcal{M} \in \Sigma(\mathscr{X})$ as per Assumption 2.6. That is, Theorem 3.4 provides a representation for the general solution of the operator differential Riccati equation (1.1) in terms of a particular solution of the same equation, and implies the following commutation diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{M}} \arrow[r, "(3.37)"] & \widetilde{\psi} \arrow[r, "\mathcal{S}_t"] \arrow[d, "\mathcal{D}_\psi"] & \mathcal{S}_t\, \widetilde{\psi} \arrow[r, "(3.38)"] & \widetilde{\mathcal{P}}(t) \\
 & \mathcal{D}_\psi\, \widetilde{\psi} \arrow[r, "\mathcal{B}_t^\oplus"] & \mathcal{B}_t^\oplus\, \mathcal{D}_\psi\, \widetilde{\psi} \arrow[u, "\mathcal{D}_\psi^{-1}"] &
\end{tikzcd} \tag{3.39}$$

In subsequent applications of (3.39), an explicit evaluation of $\mathcal{D}_\psi\, \widetilde{\psi}$ and $\mathcal{D}_\psi\, \mathcal{S}_t\, \widetilde{\psi}$ is useful. These evaluations are provided in the following two lemmas.

LEMMA 3.5. *Given the semiconvex dual operator $\mathcal{D}_\psi$ of (3.25), the semiconvex dual $\mathcal{D}_\psi\, \widetilde{\psi}$ of the terminal cost $\widetilde{\psi}$ of (3.37) is given for all $z \in \mathscr{X}$ by*

$$(\mathcal{D}_\psi\, \widetilde{\psi})(z) = -\tfrac{1}{2}\, \langle z,\, \widetilde{\mathcal{N}}\, z\rangle \tag{3.40}$$

where $\widetilde{\mathcal{N}} \in \Sigma(\mathscr{X})$ is the nonnegative self-adjoint bounded linear operator

$$\widetilde{\mathcal{N}} \doteq \mathcal{M} + \mathcal{M}\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \mathcal{M} = \mathcal{M}\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \widetilde{\mathcal{M}}\,, \tag{3.41}$$

and $\mathcal{M}$, $\widetilde{\mathcal{M}}$ are as per Assumption 2.6 and Theorem 2.7.

Proof. Applying (3.25) to (3.37), $(\mathcal{D}_\psi\, \widetilde{\psi})(z) = -\int_{\mathscr{X}}^\oplus \pi^z(x)\, dx$, where

$$\begin{aligned}
\pi^z(x) &\doteq \tfrac{1}{2}\langle x - z,\, \mathcal{M}\, (x - z)\rangle - \tfrac{1}{2}\langle x,\, \widetilde{\mathcal{M}}\, x\rangle \\
&= \tfrac{1}{2}\langle x,\, (\mathcal{M} - \widetilde{\mathcal{M}})\, x\rangle + \langle x,\, -\mathcal{M}\, z\rangle + \tfrac{1}{2}\langle z,\, \mathcal{M}\, z\rangle \\
&= b(x) + \tfrac{1}{2}\langle z,\, \mathcal{M}\, z\rangle\,,
\end{aligned}$$

where $b(x) \doteq \frac{1}{2}\langle x,\, (\mathcal{M} - \widetilde{\mathcal{M}})\, x\rangle + \langle x,\, -\mathcal{M}\, z\rangle$. That is,

$$(\mathcal{D}_\psi\, \widetilde{\psi})(z) = -\tfrac{1}{2}\langle z,\, \mathcal{M}\, z\rangle - \int_{\mathscr{X}}^\oplus b(x)\, dx\,. \tag{3.42}$$

Here, $\mathcal{M} - \widetilde{\mathcal{M}} \in \Sigma(\mathscr{X})$ is a self-adjoint bounded linear operator, while $-\mathcal{M}\, z \in \mathscr{X}$. Furthermore, $\widetilde{\mathcal{M}} > \mathcal{M}$, as $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ as per Theorem 2.7. Hence, $\sup_{x\in\mathscr{X}} b(x) < \infty$, with Lemma B.2 requiring that the supremum in (3.42) be attained at $x^* = -(\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \mathcal{M}\, z$, where invertibility of $\widetilde{\mathcal{M}} - \mathcal{M}$ follows by the coercivity condition $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ specified in Theorem 2.7. Consequently, $\int_{\mathscr{X}}^\oplus b(x)\, dx = b(x^*) = \frac{1}{2}\langle \mathcal{M}\, z,\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \mathcal{M}\, z\rangle_{\mathscr{X}}$, so that (3.42) implies that (3.40) holds with $\widetilde{\mathcal{N}}$ given by the left-hand equality in (3.41). Furthermore,

$$\begin{aligned}
\widetilde{\mathcal{N}} = \mathcal{M} + \mathcal{M}\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \mathcal{M} &= \mathcal{M} - \mathcal{M}\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \left((\widetilde{\mathcal{M}} - \mathcal{M}) - \widetilde{\mathcal{M}}\right) \\
&= \mathcal{M} - \mathcal{M} + \mathcal{M}\, (\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \widetilde{\mathcal{M}}\,,
\end{aligned}$$

which yields the right-hand equality in (3.41). Selecting a self-adjoint bounded linear operator $\mathcal{K} \in \Sigma(\mathscr{X})$ such that $\mathcal{K} + \mathcal{M} > 0$, and applying (3.40),

$$-\left[(\mathcal{D}_\psi\,\widetilde{\psi})(z) - \tfrac{1}{2}\langle z,\, \mathcal{K}\, z\rangle\right] = \tfrac{1}{2}\langle z,\, \mathcal{M}\,(\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\,\mathcal{M}\, z\rangle + \tfrac{1}{2}\langle z,\, (\mathcal{K} + \mathcal{M})\, z\rangle$$

for all $z \in \mathscr{X}$. By inspection, this functional is positive, and hence convex by Lemma B.1. That is, $(\mathcal{D}_\psi\,\widetilde{\psi})(z) - \frac{1}{2}\langle z,\, \mathcal{K}\, z\rangle_{\mathscr{X}}$ defines a concave functional, so that $\mathcal{D}_\psi\,\widetilde{\psi} \in \mathscr{S}_-^{\mathcal{K}}(\mathscr{X})$ is semiconcave by (3.23). ◻

LEMMA 3.6. *Given the semiconvex dual operator $\mathcal{D}_\psi$ of (3.25), the semiconvex dual $\mathcal{D}_\psi\,\mathcal{S}_t\,\widetilde{\psi}$ of the value functional $\mathcal{S}_t\,\widetilde{\psi}$ of (3.19) and (3.38) corresponding to the terminal cost $\widetilde{\psi}$ of (3.37) is given for all $z \in \mathscr{X}$ and $t \in (0, \tau^*]$ by*

$$(\mathcal{D}_\psi\,\mathcal{S}_t\,\widetilde{\psi})(z) = -\tfrac{1}{2}\,\langle z,\, \widetilde{\mathcal{N}}_t\, z\rangle \tag{3.43}$$

where $\widetilde{\mathcal{N}}_t \in \Sigma(\mathscr{X})$ is the self-adjoint bounded linear operator defined by

$$\widetilde{\mathcal{N}}_t \doteq \mathcal{M} + \mathcal{M}\,(\widetilde{\mathcal{P}}(t) - \mathcal{M})^{-1}\,\mathcal{M} = \mathcal{M}\,(\widetilde{\mathcal{P}}(t) - \mathcal{M})^{-1}\,\widetilde{\mathcal{P}}(t)\,, \tag{3.44}$$

and $\mathcal{M}$ is as per Assumption 2.6.

Proof. As $\widetilde{\mathcal{P}}(t) > \mathcal{M}$ by Theorem 2.7, an analogous argument to that yielding (3.28) follows, with $\mathcal{P}(t)$ replaced with $\widetilde{\mathcal{P}}(t)$. Consequently, there exists a self-adjoint bounded linear operator $\widetilde{\mathcal{K}}_t \in \Sigma(\mathscr{X})$ such that $\mathcal{S}_t\,\widetilde{\psi} \in \mathscr{S}^{\widetilde{\mathcal{K}}_t}(\mathscr{X})$. Similarly, assertion (i) of Lemma B.1 and (3.38) imply that $\mathcal{S}_t\,\widetilde{\psi}$ is closed. Hence, the semiconvex dual $\mathcal{D}_\psi\,\mathcal{S}_t\,\widetilde{\psi}$ is well-defined by Theorem 3.2. So, applying (3.25) to (3.38) in an analogous fashion to the proof of Lemma 3.5 (i.e. replacing $\widetilde{\mathcal{M}}$ with $\widetilde{\mathcal{P}}(t)$ and noting that $\widetilde{\mathcal{P}}(t) - \mathcal{M}$ is coercive and hence invertible) yields (3.43). ◻

Theorem 3.4 states that the value functional $\mathcal{S}_t\,\widetilde{\psi}$ of (3.38) may be identified with the solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ for any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$. Hence, $\widetilde{\mathcal{P}}$ may be propagated to longer time horizons via the dynamic programming evolution operator $\mathcal{S}_t$ of (3.20). Furthermore, Theorem 3.4 also states that this propagation can be represented in a max-plus dual space via the max-plus integral operator $\mathcal{B}_t^\oplus$ of (3.36). As $\mathcal{S}_t$ of (3.20) satisfies the semigroup property (3.21), it follows that $\mathcal{B}_t^\oplus$ of (3.36) inherits a similar semigroup property. Consequently, as $\mathcal{B}_t^\oplus$ of (3.36) can be used to propagate any solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ corresponding to any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, it follows that the set of time-indexed max-plus integral operators

$$\bigcup \left\{ \mathcal{B}_t^\oplus \;\middle|\; t \in \mathbb{R}_{>0},\ \sum t \le \tau^* \right\} \tag{3.45}$$

defines the claimed *max-plus dual space fundamental solution semigroup* for the operator differential Riccati equation (1.1) on the interval $(0, \tau^*] \subset \mathbb{R}_{>0}$. This is formalized by combining the following theorem with Theorem 3.4.

THEOREM 3.7. *The max-plus integral operator $\mathcal{B}_t^\oplus$ of (3.36) satisfies the semigroup property*

$$\mathcal{B}_{\tau+t}^\oplus\,\tilde{a} = \mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,\tilde{a} \tag{3.46}$$

for all $\tau, t \in (0, \tau^*]$ such that $t + \tau \in (0, \tau^*]$, and for any $\tilde{a} = \mathcal{D}_\psi\,\widetilde{\psi}$, where $\widetilde{\psi}$ is the terminal payoff (3.37) corresponding to any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ as per Theorem 2.7, and $\tau^* \in \mathbb{R}_{>0}$ is as per (2.26).

Proof. Applying the semigroup property (3.21) and Theorem 3.4,

$$\begin{aligned}\mathcal{D}_\psi^{-1}\,\mathcal{B}_{\tau+t}^\oplus\,\mathcal{D}_\psi\,\widetilde{\psi} &= \mathcal{S}_{\tau+t}\,\widetilde{\psi} \\ &= \mathcal{S}_\tau\,\mathcal{S}_t\,\widetilde{\psi} = \mathcal{D}_\psi^{-1}\,\mathcal{B}_\tau^\oplus\,\mathcal{D}_\psi\,\mathcal{D}_\psi^{-1}\,\mathcal{B}_t^\oplus\,\mathcal{D}_\psi\,\widetilde{\psi} = \mathcal{D}_\psi^{-1}\,\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,\mathcal{D}_\psi\,\widetilde{\psi}\,,\end{aligned}$$

That is, $\mathcal{D}_\psi^{-1}\,\mathcal{B}_{\tau+t}^\oplus\,\tilde{a} = \mathcal{D}_\psi^{-1}\,\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,\tilde{a}$, where $\tilde{a} \doteq \mathcal{D}_\psi\,\widetilde{\psi}$. Applying the semiconvex dual operator $\mathcal{D}_\psi$ of (3.25) to both sides thus yields the semigroup property (3.46). ∎

COROLLARY 3.8. *The composition $\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus$ of max-plus integral operators of the form (3.36) is a well-defined operator of the same form, with*

$$\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,a = (\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,a)(\cdot) = \int_{\mathscr{X}}^\oplus B_{\tau,t}(\cdot,z)\otimes a(z)\,dz \tag{3.47}$$

for all $\tau, t \in (0,\tau^*]$, $\tau + t \in (0,\tau^*]$, in which the kernel $B_{\tau,t} : \mathscr{X}\times\mathscr{X} \to \mathbb{R}$ is a functional defined by

$$B_{\tau,t}(y,z) \doteq \tfrac{1}{2}\,\langle y,\,\mathcal{B}_{\tau,t}^{1,1}\,y\rangle + \langle z,\,\mathcal{B}_{\tau,t}^{1,2}\,y\rangle + \tfrac{1}{2}\,\langle z,\,\mathcal{B}_{\tau,t}^{2,2}\,z\rangle\,, \tag{3.48}$$

with $\mathcal{B}_{\tau,t}^{1,1}$, $\mathcal{B}_{\tau,t}^{1,2}$ and $\mathcal{B}_{\tau,t}^{2,2}$ denoting the self-adjoint bounded linear operators

$$\mathcal{B}_{\tau,t}^{1,1} \doteq \mathcal{B}_\tau^{1,1} - (\mathcal{B}_\tau^{1,2})'\,\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\,, \tag{3.49}$$

$$\mathcal{B}_{\tau,t}^{1,2} \doteq -\mathcal{B}_t^{1,2}\,\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\,, \tag{3.50}$$

$$\mathcal{B}_{\tau,t}^{2,2} \doteq \mathcal{B}_t^{2,2} - \mathcal{B}_t^{1,2}\,\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,(\mathcal{B}_t^{1,2})'\,, \tag{3.51}$$

each of which are well-defined via (3.32), (3.33) and (3.34).

Proof. Given any functional $a : \mathscr{X} \to \mathbb{R}$, definition (3.36) of $\mathcal{B}_t^\oplus$ requires that

$$\begin{aligned}\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus\,a &= \int_{\mathscr{X}}^\oplus B_\tau(\cdot,\xi)\otimes\left[\int_{\mathscr{X}}^\oplus B_t(\xi,z)\otimes a(z)\,dz\right]d\xi \\ &= \int_{\mathscr{X}}^\oplus\left[\int_{\mathscr{X}}^\oplus B_\tau(\cdot,\xi)\otimes B_t(\xi,z)\,d\xi\right]\otimes a(z)\,dz\,.\end{aligned} \tag{3.52}$$

That is, operator $\mathcal{B}_\tau^\oplus\,\mathcal{B}_t^\oplus$ is of the form (3.47), with $B_{\tau,t}(y,z) \doteq \int_{\mathscr{X}}^\oplus B_\tau(y,\xi)\otimes B_t(\xi,z)\,d\xi$. However, Lemma 3.3 states that B_τ and B_t are quadratic functionals with the explicit form (3.31), implying that the functional $\pi_{\tau,t}^{y,z}(\xi) \doteq B_\tau(y,\xi)\otimes B_t(\xi,z)$ is also quadratic. In particular,

$$\begin{aligned}\pi_{\tau,t}^{y,z}(\xi) &= \tfrac{1}{2}\,\langle y,\,\mathcal{B}_\tau^{1,1}\,y\rangle_{\mathscr{X}} + \langle \xi,\,\mathcal{B}_\tau^{1,2}\,y\rangle_{\mathscr{X}} + \tfrac{1}{2}\,\langle \xi,\,\mathcal{B}_\tau^{2,2}\,\xi\rangle_{\mathscr{X}} \\ &\quad + \tfrac{1}{2}\,\langle \xi,\,\mathcal{B}_t^{1,1}\,\xi\rangle_{\mathscr{X}} + \langle z,\,\mathcal{B}_t^{1,2}\,\xi\rangle_{\mathscr{X}} + \tfrac{1}{2}\,\langle z,\,\mathcal{B}_t^{2,2}\,z\rangle_{\mathscr{X}} \\ &= b(\xi) + \tfrac{1}{2}\,\langle y,\,\mathcal{B}_\tau^{1,1}\,y\rangle_{\mathscr{X}} + \tfrac{1}{2}\,\langle z,\,\mathcal{B}_t^{2,2}\,z\rangle_{\mathscr{X}}\,,\end{aligned}$$

where

$$b(\xi) \doteq \tfrac{1}{2}\,\langle \xi,\,(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1})\,\xi\rangle_{\mathscr{X}} + \langle \xi,\,\mathcal{B}_\tau^{1,2}\,y + (\mathcal{B}_t^{1,2})'\,z\rangle_{\mathscr{X}}\,. \tag{3.53}$$

That is,

$$B_{\tau,t}(y,z) = \tfrac{1}{2}\,\langle y,\,\mathcal{B}_\tau^{1,1}\,y\rangle_{\mathscr{X}} + \tfrac{1}{2}\,\langle z,\,\mathcal{B}_t^{2,2}\,z\rangle_{\mathscr{X}} + \int_{\mathscr{X}}^\oplus b(\xi)\,d\xi\,. \tag{3.54}$$

In order to derive the form (3.48) for $B_{\tau,t}$, the supremum on the right-hand side of (3.54) must be shown to be finite, and subsequently evaluated. To do this, first note that $B_{\tau,t}$ as written in (3.54) is defined entirely in terms of the particular solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) defined by Assumption 2.6, via (3.32), (3.33), and (3.34). However, by Theorem 2.7 and the definition (2.26) of $\tau^* \in \mathbb{R}_{>0}$, any solution $\widetilde{\mathcal{P}}$ of (1.1) defined by the initialization $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ corresponding to any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ also exists on the interval $[0, \tau^*]$. Hence, Theorem 3.4 and Lemma 3.6 imply respectively that $\mathcal{S}_{\tau+t}\,\widetilde{\psi}$ and (hence) $\mathcal{D}_\psi\,\mathcal{S}_{\tau+t}\,\widetilde{\psi} = \mathcal{B}_{\tau+t}^{\oplus}\,\mathcal{D}_\psi\,\widetilde{\psi}$ are well-defined quadratic functionals (i.e. finite-valued everywhere on $\mathscr{X}$) for all $\tau, t \in (0, \tau^*]$, $\tau + t \in (0, \tau^*]$. Indeed, these functionals are given explicitly by (3.38) and (3.43), with

$$(\mathcal{S}_{\tau+t}\,\widetilde{\psi})(x) = \tfrac{1}{2}\,\langle x,\, \widetilde{\mathcal{P}}_{\tau+t}\,x\rangle\,, \quad (\mathcal{D}_\psi\,\mathcal{S}_{\tau+t}\,\widetilde{\psi})(z) = -\tfrac{1}{2}\,\langle x,\, \widetilde{\mathcal{N}}_{\tau+t}\,z\rangle\,,$$

where $\widetilde{\mathcal{P}}_{\tau+t}$ is as per Theorem 2.7 and $\widetilde{\mathcal{N}}_{\tau+t}$ is as per (3.44). So, Theorem 3.7 states that $\mathcal{B}_\tau^{\oplus}\,\mathcal{B}_t^{\oplus}\,\tilde{a} = \mathcal{B}_{\tau+t}^{\oplus}\,\tilde{a} = \mathcal{D}_\psi\,\mathcal{S}_{\tau+t}\,\widetilde{\psi}$ is a well-defined quadratic functional, where $\tilde{a} \doteq \mathcal{D}_\psi\,\widetilde{\psi}$. That is, there exists a functional $\tilde{a} : \mathscr{X} \to \mathbb{R}$ such that the functionals $\tilde{a}, (\mathcal{B}_\tau^{\oplus}\,\mathcal{B}_t^{\oplus})\,\tilde{a} : \mathscr{X} \to \mathbb{R}$ are finite-valued everywhere on $\mathscr{X}$. By inspection of (3.52), also recall that $\mathcal{B}_\tau^{\oplus}\,\mathcal{B}_t^{\oplus}$ of (3.47) is a max-plus integral operator of the form (B.5). Hence, Lemma B.3 implies that the kernel $B_{\tau,t}$ of $\mathcal{B}_\tau^{\oplus}\,\mathcal{B}_t^{\oplus}$ defined via (3.47) and (3.54) must be finite-valued. Hence, the integral on the right-hand side of (3.54) must be finite. Recalling the definition (3.53) of the quadratic functional $b : \mathscr{X} \mapsto \mathbb{R}$, observe that $\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}$ is self-adjoint by (3.32) and (3.34), while $\mathcal{B}_\tau^{1,2}\,y + (\mathcal{B}_t^{1,2})'\,z \in \mathscr{X}$. Hence, applying Lemma B.2, the pseudo-inverse of $\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}$ must exist, with the supremum in (3.54) attained at $\xi^* \doteq -(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1})^+\,(\mathcal{B}_\tau^{1,2}\,y + (\mathcal{B}_t^{1,2})'\,z)$. That is,

$$\int_{\mathscr{X}}^{\oplus} b(\xi)\,d\xi = b(\xi^*) = -\tfrac{1}{2}\left\langle \mathcal{B}_\tau^{1,2}\,y + (\mathcal{B}_t^{1,2})'\,z,\, \left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,(\mathcal{B}_\tau^{1,2}\,y + (\mathcal{B}_t^{1,2})'\,z)\right\rangle .$$

So, substituting in (3.54) yields that

$$\begin{aligned} B_{\tau,t}(y,z) = \tfrac{1}{2}&\left\langle y,\, \left[\mathcal{B}_\tau^{1,1} - (\mathcal{B}_\tau^{1,2})'\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\right] y\right\rangle \\ &+ \left\langle z,\, \left[-\mathcal{B}_t^{1,2}\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\right] y\right\rangle \\ &+ \tfrac{1}{2}\left\langle z,\, \left[\mathcal{B}_t^{2,2} - \mathcal{B}_t^{1,2}\left(\mathcal{B}_\tau^{2,2} + \mathcal{B}_t^{1,1}\right)^+\,(\mathcal{B}_t^{1,2})'\right] z\right\rangle , \end{aligned}$$

which is as per (3.48) via the operator definitions (3.49), (3.50), and (3.51). ∎

The specific details of how the max-plus dual space fundamental solution semigroup (3.45) may be applied to solve (1.1) follow in the next section.

4. Solving the operator differential Riccati equation. The remaining objective is to illustrate how solutions of the operator differential Riccati equation (1.1) can be evaluated at some time $\hat{t} \in (0, \tau^*]$ using the max-plus dual space fundamental solution semigroup (3.45). Three main steps are involved. First, the max-plus integral operator $\mathcal{B}_\tau^{\oplus}$ is obtained for some intermediate time $\tau \doteq \hat{t}/\kappa \in (0, \tau^*]$, $\kappa \in \mathbb{Z}_{>0}$, from the particular solution $\mathcal{P}$ initialized with $\mathcal{P}(0) = \mathcal{M}$, where $\mathcal{M} \in \Sigma(\mathscr{X})$ is as per Assumption 2.6. Second, $\mathcal{B}_{\hat{t}}^{\oplus}$ is derived from $\mathcal{B}_\tau^{\oplus}$ via the fundamental solution semigroup property of Theorem 3.7 and Corollary 3.8. Third, $\mathcal{B}_{\hat{t}}^{\oplus}$ is applied to evaluate the solution $\widetilde{\mathcal{P}}$ initialized with $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ at time $\hat{t}$.

It is important to note that where solutions $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ are to be evaluated at the same time $\hat{t} \in (0, \tau^*]$ for a collection of initializations $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$, the first two steps need only be performed once, whereupon step three can be repeated for each initialization $\widetilde{\mathcal{M}}$ of interest.

In the following three subsections, these three main steps are addressed separately. A summarized recipe for computing $\widetilde{\mathcal{P}}(\hat{t})$ from $\mathcal{P}_\tau$ is also provided. For reasons of brevity, an error analysis for this recipe is not included.

4.1. Step 1 – Obtaining $\mathcal{B}_\tau^\oplus$ from $\mathcal{P}_\tau$. With $\hat{t} \in (0, \tau^*]$ fixed, select $\kappa \in \mathbb{Z}_{>1}$ and define $\tau \doteq \hat{t}/\kappa$. Recall that the max-plus integral operator $\mathcal{B}_\tau^\oplus$ of (3.36) is defined via kernel B_τ. Lemma 3.3 provides an explicit quadratic functional representation (3.31) for B_τ in terms of the operators $\mathcal{B}_\tau^{1,1}$, $\mathcal{B}_\tau^{1,2}$ and $\mathcal{B}_\tau^{2,2}$ of (3.32), (3.33) and (3.34). These are bounded linear operators that are defined directly in terms of $\mathcal{P}_\tau$, with

$$\mathcal{B}_\tau^{1,1} \doteq -\mathcal{M} - \mathcal{M}\,(\mathcal{P}_\tau - \mathcal{M})^{-1}\,\mathcal{M}\,, \tag{3.32}$$

$$\mathcal{B}_\tau^{1,2} \doteq -\mathcal{Q}_\tau'\,(\mathcal{P}_\tau - \mathcal{M})^{-1}\,\mathcal{M}\,, \tag{3.33}$$

$$\mathcal{B}_\tau^{2,2} \doteq -\mathcal{Q}_\tau'\,(\mathcal{P}_\tau - \mathcal{M})^{-1}\,\mathcal{Q}_\tau + \mathcal{R}_\tau\,. \tag{3.34}$$

These operators completely describe the max-plus integral operator $\mathcal{B}_\tau^\oplus$ in terms of the particular solution $\mathcal{P}_\tau$ of the operator differential Riccati equation (1.1) provided by Assumption 2.6, via (3.31) and (3.36).

4.2. Step 2 – Obtaining $\mathcal{B}_{\hat{t}}^\oplus$ from $\mathcal{B}_\tau^\oplus$. The max-plus dual space fundamental solution semigroup (3.45) of Theorem 3.7 and Corollary 3.8 provides a iterative mechanism for constructing $\mathcal{B}_{\hat{t}}^\oplus$ from $\mathcal{B}_\tau^\oplus$. With a view to evaluating $\mathcal{B}_{\hat{t}}^\oplus\,\tilde{a} = \mathcal{B}_\tau^\oplus\,\mathcal{B}_\tau^\oplus\,\cdots\,\mathcal{B}_\tau^\oplus\,\tilde{a}$ (κ times) given $\tilde{a} \doteq \mathcal{D}_\psi\,\widetilde{\psi}$, select $t = (k-1)\,\tau$ in (3.49), (3.50) and (3.51), with $k = 2, \cdots, \kappa$. This yields a linear iteration of a triple $(\widehat{\mathcal{B}}_k^{1,1}, \widehat{\mathcal{B}}_k^{1,2}, \widehat{\mathcal{B}}_k^{2,2})$ of bounded linear operators, given by

$$\widehat{\mathcal{B}}_k^{1,1} \doteq \mathcal{B}_\tau^{1,1} - (\mathcal{B}_\tau^{1,2})'\,\left(\mathcal{B}_\tau^{2,2} + \widehat{\mathcal{B}}_{k-1}^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\,, \tag{4.1}$$

$$\widehat{\mathcal{B}}_k^{1,2} \doteq -\widehat{\mathcal{B}}_{k-1}^{1,2}\,\left(\mathcal{B}_\tau^{2,2} + \widehat{\mathcal{B}}_{k-1}^{1,1}\right)^+\,\mathcal{B}_\tau^{1,2}\,, \tag{4.2}$$

$$\widehat{\mathcal{B}}_k^{2,2} \doteq \widehat{\mathcal{B}}_{k-1}^{2,2} - \widehat{\mathcal{B}}_{k-1}^{1,2}\,\left(\mathcal{B}_\tau^{2,2} + \widehat{\mathcal{B}}_{k-1}^{1,1}\right)^+\,(\widehat{\mathcal{B}}_{k-1}^{1,2})'\,, \tag{4.3}$$

for $k = 2, \cdots, \kappa$, initialized with

$$(\widehat{\mathcal{B}}_1^{1,1}, \widehat{\mathcal{B}}_1^{1,2}, \widehat{\mathcal{B}}_1^{2,2}) = (\mathcal{B}_\tau^{1,1}, \mathcal{B}_\tau^{1,2}, \mathcal{B}_\tau^{2,2}) \tag{4.4}$$

via (3.32), (3.33) and (3.34). These operators completely describe (via (3.47) and (3.48)) the max-plus integral operator $\mathcal{B}_\tau^\oplus\,\mathcal{B}_{(k-1)\,\tau}^\oplus$ for $k = 2, \cdots \kappa$. The desired max-plus integral operator $\mathcal{B}_{\hat{t}}^\oplus$ follows from the $k = \kappa^{\text{th}}$ iterate

$$(\mathcal{B}_{\hat{t}}^{1,1}, \mathcal{B}_{\hat{t}}^{1,2}, \mathcal{B}_{\hat{t}}^{2,2}) = (\widehat{\mathcal{B}}_\kappa^{1,1}, \widehat{\mathcal{B}}_\kappa^{1,2}, \widehat{\mathcal{B}}_\kappa^{2,2})\,. \tag{4.5}$$

For all iterates, Theorem 3.4 states that $(\mathcal{S}_{k\,\tau}\,\widetilde{\psi})(x) = (\mathcal{D}_\psi^{-1}\,\mathcal{B}_\tau^\oplus\,\mathcal{B}_{(k-1)\,\tau}^\oplus\,\tilde{a})(x) = \frac{1}{2}\,\langle x, \widetilde{\mathcal{P}}(k\,\tau)\,x\rangle_{\mathscr{X}}$, where $\tilde{a} \doteq \mathcal{D}_\psi\,\widetilde{\psi}$. Hence, applying an argument analogous to that used in the proof of Corollary 3.8, existence of $\widetilde{\mathcal{P}}(k\,\tau)$ for each $k = 1, \cdots, \kappa$ as provided by Assumption 2.6 guarantees that the operator iteration (4.1), (4.2), (4.3) remains well-defined, subject to the aforementioned initialization (4.4). In particular, the pseudo-inverses employed there must exist, and the operators must remain bounded and linear.

Other iterations are also possible. For example, as an alternative to a linear iteration, a time-step doubling iteration may be used to construct $\mathcal{B}_{\hat{t}}^{\oplus}$. Such a scheme requires fewer iterations (than the linear scheme illustrated above) to reach $\hat{t}$. The details of such iterations are omitted for brevity.

4.3. Step 3 – Obtaining $\widetilde{\mathcal{P}}(\hat{t})$ from $\mathcal{B}_{\hat{t}}^{\oplus}$ and $\widetilde{\mathcal{M}}$. Theorem 3.4 and the commutation diagram (3.39) provide the mechanism for evaluating the solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ at time $\hat{t} \in (0, \tau^*]$ for any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ via the max-plus integral operator $\mathcal{B}_{\hat{t}}^{\oplus}$ obtained in the previous step. In particular, (3.38) states that

$$\tfrac{1}{2}\langle x, \widetilde{\mathcal{P}}(\hat{t})\, x\rangle_{\mathscr{X}} = (\mathcal{D}_{\psi}^{-1}\, \mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(x)\,, \qquad \tilde{a} \doteq \mathcal{D}_{\psi}\, \widetilde{\psi}\,. \tag{4.6}$$

Recall that Lemma 3.5, $\tilde{a} = \mathcal{D}_{\psi}\, \widetilde{\psi} = -\frac{1}{2}\langle z, \widetilde{\mathcal{N}}\, z\rangle$, with $\widetilde{\mathcal{N}} \in \Sigma(\mathscr{X})$ is as per (3.41). Applying $\mathcal{B}_{\hat{t}}^{\oplus}$ of the previous step to $\tilde{a}$ yields

$$\begin{aligned}
(\mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(y) &= \int_{\mathscr{X}}^{\oplus} B_{\hat{t}}(y,z) \otimes \tilde{a}(z)\, dz \\
&= \tfrac{1}{2}\langle y, \mathcal{B}_{\hat{t}}^{1,1}\, y\rangle + \int_{\mathscr{X}}^{\oplus} \tfrac{1}{2}\langle z, (\mathcal{B}_{\hat{t}}^{2,2} - \widetilde{\mathcal{N}})\, z\rangle + \langle z, \mathcal{B}_{\hat{t}}^{1,2}\, y\rangle\, dz
\end{aligned}$$

for all $y \in \mathscr{X}$. An analogous argument to the proof of Corollary 3.8 requires the existence of a finite right-hand side supremum. Consequently, Lemma B.2 implies that a bounded linear operator $\widetilde{\mathcal{E}}_{\hat{t}}$ exists such that

$$(\mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(y) = \tfrac{1}{2}\langle y, \widetilde{\mathcal{E}}_{\hat{t}}\, y\rangle_{\mathscr{X}}\,, \qquad \widetilde{\mathcal{E}}_{\hat{t}} \doteq \mathcal{B}_{\hat{t}}^{1,1} - (\mathcal{B}_{\hat{t}}^{1,2})' \left(\mathcal{B}_{\hat{t}}^{2,2} - \widetilde{\mathcal{N}}\right)^{+} \mathcal{B}_{\hat{t}}^{1,2}\,.$$

Applying the inverse semiconvex dual operator $\mathcal{D}_{\psi}^{-1}$ of (3.26) to obtain (4.6),

$$\begin{aligned}
(\mathcal{D}_{\psi}^{-1}\, \mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(x) &= \int_{\mathscr{X}}^{\oplus} \psi(x,y) \otimes (\mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(y)\, dy \\
&= \tfrac{1}{2}\langle x, \mathcal{M}\, x\rangle + \int_{\mathscr{X}}^{\oplus} \tfrac{1}{2}\langle y, (\widetilde{\mathcal{E}}_{\hat{t}} + \mathcal{M})\, y\rangle + \langle y, -\mathcal{M}\, x\rangle\, dy\,.
\end{aligned}$$

Again, an analogous argument to the proof of Corollary 3.8 requires the existence of a finite right-hand side supremum. Consequently, Lemma B.2 implies that a bounded linear operator $\widetilde{\mathcal{O}}_{\hat{t}}$ exists such that

$$(\mathcal{D}_{\psi}^{-1}\, \mathcal{B}_{\hat{t}}^{\oplus}\, \tilde{a})(x) = \tfrac{1}{2}\langle x, \widetilde{\mathcal{O}}_{\hat{t}}\, x\rangle_{\mathscr{X}}\,, \qquad \widetilde{\mathcal{O}}_{\hat{t}} \doteq \mathcal{M} - \mathcal{M}\left(\widetilde{\mathcal{E}}_{\hat{t}} + \mathcal{M}\right)^{+} \mathcal{M}\,. \tag{4.7}$$

Combining (4.6) and (4.7) implies that $\frac{1}{2}\langle x, \widetilde{\mathcal{P}}_{\hat{t}}\, x\rangle = \frac{1}{2}\langle x, \widetilde{\mathcal{O}}_{\hat{t}}\, x\rangle$ for all $x \in \mathscr{X}$, or

$$\widetilde{\mathcal{P}}(\hat{t}) = \mathcal{M} - \mathcal{M}\left(\widetilde{\mathcal{E}}_{\hat{t}} + \mathcal{M}\right)^{+} \mathcal{M}\,, \tag{4.8}$$

in which $\widetilde{\mathcal{E}}_{\hat{t}} \doteq \mathcal{B}_{\hat{t}}^{1,1} - (\mathcal{B}_{\hat{t}}^{1,2})' \left(\mathcal{B}_{\hat{t}}^{2,2} - \widetilde{\mathcal{N}}\right)^{+} \mathcal{B}_{\hat{t}}^{1,2}$, with $\widetilde{\mathcal{N}} \doteq \mathcal{M} + \mathcal{M}\,(\widetilde{\mathcal{M}} - \mathcal{M})^{-1}\, \mathcal{M}$.

4.4. Recipe. The solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) initialized with $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ for any $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ can be evaluated at any time within an interval of existence $(0, \tau^*]$ using the particular solution $\mathcal{P}$ of the same equation initialized with $\mathcal{P}(0) = \mathcal{M} \in \Sigma(\mathscr{X})$ as specified in Assumption 2.6 via the following recipe:

❶ Select a time $\hat{t} \in (0, \tau^*]$, $\tau^* \in \mathbb{R}_{>0}$ as per (2.26), at which evaluation of the solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation (1.1) satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ for some $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ is required. Fix iteration integer $\kappa \in \mathbb{Z}_{>1}$ and time $\tau \doteq \hat{t}/\kappa$. Construct the bounded linear operators $\mathcal{B}_\tau^{1,1}$, $\mathcal{B}_\tau^{1,2}$ and $\mathcal{B}_\tau^{2,2}$ from the evaluation of the known particular solution $\mathcal{P}(\tau)$ according to (3.32), (3.33) and (3.34).

❷ Iterate the operator triple $(\widehat{\mathcal{B}}_k^{1,1}, \widehat{\mathcal{B}}_k^{1,2}, \widehat{\mathcal{B}}_k^{2,2})$ as per (4.1), (4.2), (4.3) for $k = 2, \cdots, \kappa$, subject to the initialization (4.4), to obtain the final operator triple $(\mathcal{B}_{\hat{t}}^{1,1}, \mathcal{B}_{\hat{t}}^{1,2}, \mathcal{B}_{\hat{t}}^{2,2})$ at time $\hat{t}$ as per (4.5).

❸ Select any initial condition $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$. Evaluate the solution $\widetilde{\mathcal{P}}$ of the operator differential Riccati equation satisfying $\widetilde{\mathcal{P}}(0) = \widetilde{\mathcal{M}}$ at time $\hat{t}$ via (4.8).

4.5. An illustrative example. A brief example is provided to illustrate an application of the recipe of Section 4.4 to the numerical evaluation of solutions of a specific operator differential Riccati equation of the form (1.1). With ∂ denoting differentiation, select

$$\begin{aligned}
&\mathscr{X} \doteq \mathscr{W} \doteq \mathcal{L}_2(\Lambda; \mathbb{R}), \quad \Lambda \doteq (0, 2), \\
&\mathcal{A}\, x \doteq -(2 + \partial)\, x, \ x \in \mathsf{dom}(\mathcal{A}) \doteq \left\{ x \in \mathscr{X} \;\middle|\; \begin{array}{c} x \text{ absolutely continuouson } \Lambda \cup \{0\}, \\ x(0) = 0, \ \partial x \in \mathscr{X} \end{array} \right\}, \\
&\sigma\, x \doteq \tfrac{1}{\sqrt{2}}\, x, \quad \mathcal{C}\, x \doteq \tfrac{1}{3} \int_\Lambda x(\zeta)\, d\zeta, \quad x \in \mathsf{dom}(\sigma) = \mathsf{dom}(\mathcal{C}) = \mathscr{X}.
\end{aligned}$$

Attention is restricted to initializations $\mathcal{M} \in \Sigma(\mathscr{X})$ and $\widetilde{\mathcal{M}} \in \Sigma_{\mathcal{M}}(\mathscr{X})$ that assume an integral representation of the form

$$\mathcal{M}\, x = (\mathcal{M}\, x)(\cdot) = \int_\Lambda M(\cdot, \zeta)\, x(\zeta)\, d\zeta, \tag{4.9}$$

in which $M \in \mathcal{L}_2(\Lambda^2; \mathbb{R})$ denotes a kernel. Under this restriction, respective solutions of the operator differential Riccati equation (1.1) enjoy the same integral representation, with time-indexed kernels denoted respectively by $P(t), \widetilde{P}(t) \in \mathcal{L}_2(\Lambda^2; \mathbb{R})$, $t \in \mathbb{R}_{\ge 0}$. Consequently, the operator differential Riccati equation (1.1) may be equivalently represented via an integro-differential equation of the form

$$\frac{\partial P}{\partial t}(t, \eta, \zeta) = -4\, P(t, \eta, \zeta) + \frac{\partial P}{\partial \eta}(t, \eta, \zeta) + \frac{\partial P}{\partial \zeta}(t, \eta, \zeta) + \tfrac{1}{2} \int_\Lambda P(t, \eta, \rho)\, P(t, \rho, \zeta)\, d\rho + \tfrac{1}{3} \tag{4.10}$$

subject to the boundary and initial conditions

$$P(t, 0, \zeta) = 0 = P(t, \eta, 0), \quad P(0, \eta, \zeta) = M(0, \eta, \zeta), \tag{4.11}$$

for all $t \in \mathbb{R}_{\geq 0}$, $\eta, \zeta \in \Lambda$. Equations (4.10) and (4.11) are solved via a textbook application of Runge-Kutta (RK45) on a fine grid to provide a benchmark for application of the recipe of Section 4.4 via representation (4.9). Approximation errors generated by the dual-space propagation of Theorem 3.7 relative to the RK45 solution are illustrated in Figure 4.1 (scaled via the natural logarithm). The computational advantage illustrated there is due to the application of the time-step doubling iteration alluded to in Section 4.2 in computing $\mathcal{B}_{\hat{t}}^{\oplus}$. For reasons of brevity, further details are omitted.

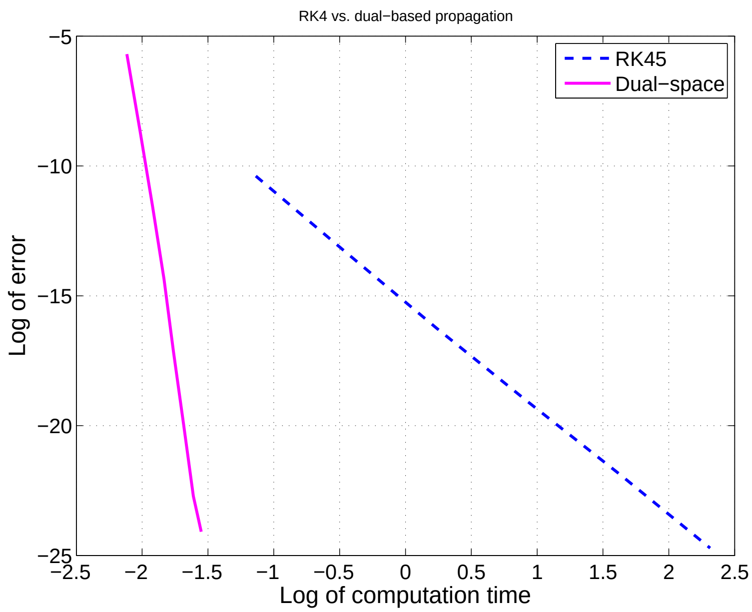

FIG. 4.1. *Approximation error versus computation time for standard (RK45) and dual-space propagation methods.*

5. Conclusion. By exploiting a connection between an operator differential Riccati equation and a specific infinite dimensional optimal control problem, dynamic programming is employed to develop an evolution operator for propagating solutions of this equation. Examination of this evolution in a dual space, defined via semiconvexity and the Legendre-Fenchel transform, reveals the existence of a time indexed dual space operator that can be used to propagate the solution of the operator differential Riccati equation from any initial condition in a particular class. By demonstrating that these time indexed dual space operators inherit a semigroup property from dynamic programming, the set of such time indexed operators is shown to define a fundamental solution semigroup for the operator differential Riccati equation.

REFERENCES

[1] G. Bachman and L. Narici, *Functional Analysis*, Dover (orig. Acad. Press, 1966), 2000.
[2] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Application, Birkhauser, 1997.
[3] R. Bellman, *Dynamic programming*, Princeton University Press, 1957.
[4] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter, *Representation and control of infinite dimensional systems*, Birkhaüser, second ed., 2007.

[5] R.F. Curtain and H.J. Zwart, *An introduction to infinite-dimensional linear systems theory*, vol. 21 of Texts in Applied Mathematics, Springer-Verlag, New York, 1995.
[6] P.M. Dower and W.M. McEneaney, *A max-plus based fundamental solution for a class of infinite dimensional Riccati equations*, in Proc. Joint 50^{th} IEEE Conference on Decision and Control and European Control Conference (Orlando FL, USA), 2011, pp. 615–620.
[7] ———, *A max-plus method for the optimal control of a diffusion equation*, in proc. 51^{st} IEEE Conference on Decision & Control (Maui HI, USA), 2012, pp. 618–623.
[8] ———, *A max-plus dual space fundamental solution semigroup for operator differential Riccati equations*, in review, MTNS'14, 2014.
[9] R. Hagen, S. Roch, and B. Silbermann, *C^*-algebras and numerical analysis*, vol. 236 of Pure and Applied Math., Marcel Dekker, 2001.
[10] E. Kreyszig, *Introductory functional analysis with applications*, Wiley, 1978.
[11] W.M. McEneaney, *Max-plus methods for nonlinear control and estimation*, Systems & Control: Foundations & Application, Birkhauser, 2006.
[12] ———, *A new fundamental solution for differential Riccati equations arising in control*, Automatica, 44 (2008), pp. 920–936.
[13] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, 1983.
[14] W.O. Ray, *Real Analysis*, Prentice-Hall, 1988.
[15] R.T. Rockafellar, *Conjugate duality and optimization*, SIAM Regional Conf. Series in Applied Math., 16 (1974).
[16] A.E. Taylor and D.C. Lay, *Introduction to functional analysis*, John Wiley, second ed., 1980.

Appendix A. Continuous and strongly continuous operators.

An operator-valued function $\mathcal{F} : \mathbb{R} \to \mathcal{L}(\mathscr{X})$ is continuous at $t_0 \in \mathbb{R}$ if given $\epsilon \in \mathbb{R}_{>0}$ there exists an $\delta \in \mathbb{R}_{>0}$ such that $|t - t_0| < \delta \Longrightarrow \|\mathcal{F}(t) - \mathcal{F}(t_0)\|_{\mathcal{L}(\mathscr{X})} < \epsilon$, in which $\|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} \doteq \sup_{\|x\|=1} \|\mathcal{F}(t)\, x\|$ denotes the induced operator norm of $\mathcal{F}(t) \in \mathcal{L}(\mathscr{X})$, and $\|\cdot\|$ is the norm on $\mathscr{X}$. An operator-valued function $\mathcal{F} : \mathbb{R} \to \mathcal{L}(\mathscr{X})$ is continuous on an open set $I \subset \mathbb{R}$ if it is continuous at every $t_0 \in I$. The space of operator-valued functions $C(I; \mathcal{L}(\mathscr{X}))$ is defined as the space of all such continuous operator-valued functions defined on $I \subset \mathbb{R}$.

Similarly, an operator-valued function $\mathcal{F} : \mathbb{R} \to \mathcal{L}(\mathscr{X})$ is strongly continuous on an open set $I \subset \mathbb{R}$ if, for every $x \in \mathscr{X}$, the function $\mathcal{F}(\cdot)\, x : I \to \mathscr{X}$ is continuous. That is, for any $t_0 \in I$ and $x \in \mathscr{X}$, given $\epsilon \in \mathbb{R}_{>0}$, there exists a $\delta \in \mathbb{R}_{>0}$ such that $|t - t_0| < \delta \Longrightarrow \|\mathcal{F}(t)\, x - \mathcal{F}(t_0)\, x\| < \epsilon$, in which $\|\cdot\|$ denotes the norm on $\mathscr{X}$. The space of strongly continuous operator-value functions $C_0(I; \mathcal{L}(\mathscr{X}))$ is defined as the space of all such strongly continuous operator-valued functions defined on $I \subset \mathbb{R}$. These definitions are summarized by (2.3) and (2.4).

Lemma A.1. *For any compact interval $I \subset \mathbb{R}$,*

$$C(I; \mathcal{L}(\mathscr{X})) \subset C_0(I; \mathcal{L}(\mathscr{X})) \equiv \mathcal{L}(\mathscr{X}; C(I; \mathscr{X}))\,, \tag{A.1}$$

in which $\|f\|_{C(I;\mathscr{X})} \doteq \sup_{t \in I} \|f(t)\|$.

Proof. Fix any $\mathcal{F} \in C(I; \mathcal{L}(\mathscr{X}))$, $\epsilon \in \mathbb{R}_{>0}$, and $x \in \mathscr{X}$, $\|x\| \neq 0$. Set $\epsilon_1 \doteq \epsilon/\|x\|$. As $\mathcal{F}$ is continuous by definition, there exists a $\delta(\epsilon_1) \in \mathbb{R}_{>0}$ such that $|t - t_0| < \delta(\epsilon_1) \Longrightarrow \|\mathcal{F}(t) - \mathcal{F}(t_0)\|_{\mathcal{L}(\mathscr{X})} < \epsilon_1$. However, as $\mathcal{F}(t)$, $\mathcal{F}(t_0) \in \mathcal{L}(\mathscr{X})$, $\|\mathcal{F}(t)\, x - \mathcal{F}(t_0)\, x\| \le \|\mathcal{F}(t) - \mathcal{F}(t_0)\|_{\mathscr{X}}\, \|x\| < \epsilon_1 \|x\| = \epsilon$. That is, $|t - t_0| \le \delta(\epsilon/\|x\|) \Rightarrow \|\mathcal{F}(t)\, x - \mathcal{F}(t_0)\, x\| < \epsilon$. As $\mathcal{F} \in C(I; \mathcal{L}(\mathscr{X}))$, $\epsilon \in \mathbb{R}_{>0}$, and $x \in \mathscr{X}$, $\|x\| \neq 0$, are arbitrary, it follows by definition that $\mathcal{F}$ is strongly continuous. That is, $\mathcal{F} \in C_0(I; \mathcal{L}(\mathscr{X}))$, and the left-hand relation in (A.1) holds.

The right-hand equivalence may be proved by showing that $C_0(I; \mathcal{L}(\mathscr{X})) \subset \mathcal{L}(\mathscr{X}; C(I; \mathscr{X}))$ and $\mathcal{L}(\mathscr{X}; C(I; \mathscr{X})) \subset C_0(I; \mathcal{L}(\mathscr{X}))$. To this end, first fix $\mathcal{F} \in C_0(I; \mathcal{L}(\mathscr{X}))$. As per [4], p.387, define $f(x)(t) \doteq \mathcal{F}(t)\, x$, and note that $\mathcal{F}$ is strongly continuous. Hence, given any $x \in \mathscr{X}$, $t_0 \in I$, $\epsilon \in \mathbb{R}_{>0}$, there exists a $\delta \in \mathbb{R}_{>0}$ such that $|t - t_0| < \delta \Longrightarrow \|f(x)(t) - f(x)(t_0)\| < \epsilon$. That is, $f(x) \in C(I; \mathscr{X})$ for each

$x \in \mathscr{X}$. Consequently, for each $x \in \mathscr{X}$, the function $\|f(x)(\cdot)\| : I \to \mathbb{R}_{\geq 0}$ must achieve a finite maximum $M_x \in \mathbb{R}_{\geq 0}$ on I by the Extreme Value Theorem. That is, for each $x \in \mathscr{X}$, $\|f(x)(t)\| = \|\mathcal{F}(t)\, x\| \leq M_x < \infty$ for all $t \in I$. Hence, the Uniform Boundedness Theorem (e.g. Theorem A.3.19 on p.586 of [5]) implies that there exists an $M \in \mathbb{R}_{\geq 0}$ such that $\sup_{t\in I} \|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} \leq M < \infty$. Consequently,

$$\begin{aligned}\|f(x)\|_{C(I;\mathscr{X})} &= \sup_{t\in I} \|f(x)(t)\| = \sup_{t\in I} \|\mathcal{F}(t)\, x\| \\ &\leq \sup_{t\in I} \|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} \|x\| \leq M \|x\| \,.\end{aligned}$$

That is $f : \mathscr{X} \to C(I;\mathscr{X})$ is bounded. Furthermore, as $f : \mathscr{X} \to C(I;\mathscr{X})$ satisfies by definition $f(x) = \mathcal{F}(\cdot)\, x$ for all $x \in \mathscr{X}$, it is linear. Hence, $f \in \mathcal{L}(\mathscr{X}; C(I;\mathscr{X}))$.

Conversely, select an $f \in \mathcal{L}(\mathscr{X}; C(I;\mathscr{X}))$. By definition, there exists a $K \in \mathbb{R}_{\geq 0}$ such that for all $x \in \mathscr{X}$, $\sup_{t\in I} \|f(x)(t)\| = \|f(x)\|_{C(I;\mathscr{X})} \leq K \|x\|$. Define $\mathcal{F}(t)\, x \doteq f(x)(t)$, $t \in I$. Hence, $\|\mathcal{F}(t)\, x\| \leq K \|x\|$ for all $x \in \mathscr{X}$, so that

$$\mathcal{F}(t) \in \mathcal{L}(\mathscr{X}) \qquad \forall\, t \in I\,. \tag{A.2}$$

Furthermore, as $f(x) \in C(I;\mathscr{X})$ for every $x \in \mathscr{X}$, given any $x \in \mathscr{X}$, $t_0 \in I$ and $\epsilon \in \mathbb{R}_{>0}$, there exists a $\delta \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}|t - t_0| < \delta \quad \Longrightarrow \quad & |f(x)(t) - f(x)(t_0)| < \epsilon \\ \Longleftrightarrow \quad & |\mathcal{F}(t)\, x - \mathcal{F}(t_0)\, x| < \epsilon\,.\end{aligned}$$

That is, $\mathcal{F} : I \to \mathcal{L}(\mathscr{X})$ is strongly continuous. Recalling (2.4) and (A.2), it follows that $\mathcal{F} \in C_0(I;\mathcal{L}(\mathscr{X}))$. ∎

In view of Lemma A.1, spaces C, C_0 of (2.3), (2.4) may be equipped respectively with the norms

$$\begin{aligned}\|\mathcal{F}\|_{C\{I\}} &\doteq \sup_{t\in I} \|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})}\,, && \mathcal{F} \in C(I;\mathcal{L}(\mathscr{X}))\,, \\ \|\mathcal{F}\|_{C_0\{I\}} &\doteq \sup_{\|x\|=1} \|\mathcal{F}(\cdot)\, x\|_{C(I;\mathscr{X})}\,, && \mathcal{F} \in C_0(I;\mathcal{L}(\mathscr{X}))\,.\end{aligned} \tag{A.3}$$

Lemma A.2. *$\|\cdot\|_C$ may be extended to $C_0(I;\mathcal{L}(\mathscr{X}))$, whereupon it is equivalent to $\|\cdot\|_{C_0}$.*

Proof. Fix $\mathcal{F} \in C_0(I;\mathcal{L}(\mathscr{X}))$. By definition (A.3),

$$\begin{aligned}\|\mathcal{F}\|_{C_0\{I\}} &= \sup_{\|x\|=1} \|\mathcal{F}(\cdot)\, x\|_{C(I;\mathscr{X})} = \sup_{\|x\|=1} \sup_{t\in I} \|\mathcal{F}(t)\, x\| \\ &= \sup_{t\in I} \sup_{\|x\|=1} \|\mathcal{F}(t)\, x\| = \sup_{t\in I} \|\mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} \equiv \|\mathcal{F}\|_{C\{I\}}\,.\end{aligned}$$

∎

Lemma A.3. *Given a compact $I \subset \mathbb{R}$, the normed spaces $(C(I;\mathcal{L}(\mathscr{X})), \|\cdot\|_{C\{I\}})$ and $(C_0(I;\mathcal{L}(\mathscr{X})), \|\cdot\|_{C_0\{I\}})$, defined via (2.3), (2.4), and (A.3), are Banach spaces.*

Proof. The proof that $(C(I;\mathcal{L}(\mathscr{X})), \|\cdot\|_{C\{I\}})$ is a Banach space follows an argument from the proof of Theorem 4.3.2 of [14], generalized to this setting. In particular, let $\{\mathcal{F}_n\}$ denote a Cauchy sequence in $C(I;\mathcal{L}(\mathscr{X}))$. By inspection of (A.3), $\|\mathcal{F}_n(t) - \mathcal{F}_m(t)\|_{\mathcal{L}(\mathscr{X})} \leq \|\mathcal{F}_n - \mathcal{F}_m\|_{C\{I\}}$ for any $t \in I$, and $n, m \in \mathbb{N}$. Thus, $\{\mathcal{F}_n(t)\}$ defines a Cauchy sequence in $(\mathcal{L}(\mathscr{X}), \|\cdot\|_{\mathcal{L}(\mathscr{X})})$, which is a Banach space (see for example Theorem 2.10-1 on p.118 of [10]). Hence, there exists a $\mathcal{F}(t) \in \mathcal{L}(\mathscr{X})$ such that $\mathcal{F}(t) = \lim_{n\to\infty} \mathcal{F}_n(t)$. Let $\{t_n\}$ denote a sequence in I such that $\lim_{n\to\infty} t_n = t$.

Fix $\epsilon \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ sufficiently large such that $\|\mathcal{F}_N - \mathcal{F}\|_{C\{I\}} < \frac{\epsilon}{3}$. Hence, applying (A.3),

$$\|\mathcal{F}_N(t_n) - \mathcal{F}(t_n)\| \le \tfrac{\epsilon}{3} \qquad \forall\, n \in \mathbb{N}. \tag{A.4}$$

As $\mathcal{F}_N \in C(I; \mathcal{L}(\mathscr{X}))$, there exists a $\delta \in \mathbb{R}_{>0}$ such that

$$\left.\begin{array}{r} |s - t| < \delta \\ s \in I \end{array}\right\} \implies \|\mathcal{F}_N(s) - \mathcal{F}_N(t)\|_{\mathcal{L}(\mathscr{X})} < \tfrac{\epsilon}{3}\,. \tag{A.5}$$

Furthermore, by definition of $\{t_n\}$, there exists an $M \in \mathbb{N}$ sufficiently large such that

$$n \ge M \quad \implies \quad |t_n - t| < \delta\,. \tag{A.6}$$

Hence, for all $n \ge M$, the triangle inequality combined with (A.4), (A.5). and (A.6) implies that

$$\begin{aligned} \|\mathcal{F}(t_n) - \mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} &\le \|\mathcal{F}(t_n) - \mathcal{F}_N(t_n)\|_{\mathcal{L}(\mathscr{X})} \\ &\quad + \|\mathcal{F}_N(t_n) - \mathcal{F}_N(t)\|_{\mathcal{L}(\mathscr{X})} + \|\mathcal{F}_N(t) - \mathcal{F}(t)\|_{\mathcal{L}(\mathscr{X})} < \epsilon\,. \end{aligned}$$

Hence, as $\epsilon \in \mathbb{R}{>}\, 0$ is arbitrary, the limit $\mathcal{F}$ must be continuous. That is, $\mathcal{F} \in C(I; \mathcal{L}(\mathscr{X}))$, which implies that $(C(I; \mathcal{L}(\mathscr{X})), \|\cdot\|_{C\{I\}})$ is complete (and hence is a Banach space).

In order to prove that $C_0(I; \mathcal{L}(\mathscr{X}))$ is a Banach space, recall by the right-hand equivalence of Lemma A.1 that $C_0(I; \mathcal{L}(\mathscr{X})) \equiv \mathcal{L}(\mathscr{X}; \mathcal{L}(I; \mathscr{X}))$, where $\mathcal{L}(I; \mathscr{X})$ is equipped with the norm $\|f\|_{C(I;\mathscr{X})} \doteq \sup_{t \in I} \|f(t)\|$. Using the same argument as above, it may be shown that $\mathscr{Y} \doteq (C(I; \mathscr{X}), \|\cdot\|_{C(I;\mathscr{X})})$ is a Banach space. Hence, applying Theorem 2.10-2 of [10], $\mathcal{L}(\mathscr{X}; \mathscr{Y})$ is also a Banach space. ☐

Appendix B. Quadratic functionals and max-plus integral operators.

LEMMA B.1. *Let $f : \mathscr{X} \mapsto \mathbb{R}$ denote a quadratic functional $f(x) = \frac{1}{2}\langle x,\, \mathcal{F}\, x\rangle$ in which $\mathcal{F} \in \Sigma(\mathscr{X})$ denotes a self-adjoint bounded linear operator.*

(i) f is closed;

(ii) f is convex if and only if f is nonnegative.

Proof. *(i)* Fix any $x \in \mathscr{X}$, $\delta \in (0, 1]$, $h \in B_{\mathscr{X}}(x; \delta)$.

$$\begin{aligned} |f(x + h) - f(x)| &\le \tfrac{1}{2}\, |\langle h,\, (\mathcal{F} + \mathcal{F}')\, x\rangle| + \tfrac{1}{2}\, |\langle h,\, \mathcal{F}\, h\rangle| \\ &\le \tfrac{1}{2}\|h\|\, (\|(\mathcal{F} + \mathcal{F}')\, x\| + \|\mathcal{F}\, h\|) \\ &\le K\, \|h\| \end{aligned}$$

for some $K \in \mathbb{R}_{>0}$, where the inequalities follow respectively by the triangle inequality, Cauchy-Schwartz, and by hypothesis (boundedness of the integral operator $\mathcal{F}$). That is, f is continuous (and so lower semicontinuous). A similar argument yields that f is always finite. Hence, f is closed.

(ii) Given $\alpha \in [0, 1]$ and applying the definition of the quadratic functional f, define the functional $\Delta_\alpha : \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ by

$$\begin{aligned} \Delta_\alpha(x, \xi) &\doteq \alpha f(x) + (1 - \alpha) f(\xi) - f\left(\alpha\, x + (1 - \alpha)\, \xi\right)\,. \\ &= \alpha\, \langle x,\, \mathcal{F}\, x\rangle + (1 - \alpha)\, \langle \xi,\, \mathcal{F}\, \xi\rangle - \langle \alpha\, x + (1 - \alpha)\, \xi,\, \mathcal{F}\, (\alpha\, x + (1 - \alpha)\, \xi)\rangle \\ &= \alpha(1 - \alpha)\, (\langle x,\, \mathcal{F}\, x\rangle + \langle \xi,\, \mathcal{F}\, \xi\rangle - \langle x,\, \mathcal{F}\, \xi\rangle - \langle \xi,\, \mathcal{F}\, x\rangle) \\ &= \alpha(1 - \alpha)\, \langle x - \xi,\, \mathcal{F}\, (x - \xi)\rangle \\ &= \alpha(1 - \alpha)\, f(x - \xi)\,, \end{aligned}$$

where linearity of $\mathcal{F}$ and properties of the inner product have been used. Supposing that f is nonnegative, $\Delta_\alpha(x,\xi) \ge 0$ for all $x,\,\xi\in\mathscr{X}$. That is, f is convex. Conversely, if f is convex, then it follows by inspection that f must be nonnegative. ☐

LEMMA B.2. *Let $f:\mathscr{X}\mapsto\mathbb{R}$ denote a quadratic functional $f(x)=\frac{1}{2}\langle x,\,\mathcal{F}\,x\rangle_{\mathscr{X}}+\langle x,\,\xi\rangle_{\mathscr{X}}$ with $\mathcal{F}:\mathscr{X}\mapsto\mathscr{X}$ a self-adjoint, bounded linear operator and $\xi\in\mathscr{X}$. Then,*

$$\sup_{x\in\mathscr{X}} f(x)<\infty \quad\Longrightarrow\quad \begin{cases}\mathcal{F}\ \textit{is non-positive}\,,\\ \mathcal{F}^+\ \textit{exists}\,,\end{cases} \tag{B.1}$$

where $\mathcal{F}^+$ denotes the Moore-Penrose pseudo-inverse of $\mathcal{F}$. Furthermore, there exists $x^*\in\mathscr{X}$ such that

$$f(x^*)=\sup_{x\in\mathscr{X}} f(x)=-\tfrac{1}{2}\langle\xi,\,\mathcal{F}^+\,\xi\rangle_{\mathscr{X}}\,,\quad \textit{where}\quad x^*=-\mathcal{F}^+\,\xi\,. \tag{B.2}$$

Proof. Assume that $\sup_{x\in\mathscr{X}} f(x)<\infty$. Suppose that $\mathcal{F}$ is positive, so that given $\epsilon\in\mathbb{R}_{>0}$, there exists an $\bar{x}\in\mathscr{X}$ such that $\min\{\frac{1}{2}\langle\bar{x},\,\mathcal{F}\,\bar{x}\rangle,\,\langle\bar{x},\,\xi\rangle_{\mathscr{X}}\}>\frac{\epsilon}{2}$. With $k\in\mathbb{R}$, $k>1$,

$$f(k\,\bar{x})=\tfrac{k^2}{2}\langle\bar{x},\,\mathcal{F}\,\bar{x}\rangle_{\mathscr{X}}+k\langle\bar{x},\,\xi\rangle_{\mathscr{X}}\ge k\left[\tfrac{1}{2}\langle\bar{x},\,\mathcal{F}\,\bar{x}\rangle_{\mathscr{X}}+\langle\bar{x},\,\xi\rangle_{\mathscr{X}}\right]>k\,\epsilon\,.$$

Hence, $\sup_{x\in\mathscr{X}} f(x)\ge\sup_{k>1} f(k\,\bar{x})\ge\epsilon\,\sup_{k>1} k=\infty$, which contradicts the assertion of the left-hand side of (B.1). That is, $\mathcal{F}$ must be a non-positive operator. So, as $-\mathcal{F}$ is a non-negative, self-adjoint, bounded linear operator, a square-root operator $\widehat{\mathcal{F}}$ exists [1] such that $-\mathcal{F}=\widehat{\mathcal{F}}'\,\widehat{\mathcal{F}}=\widehat{\mathcal{F}}\,\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is also non-negative, self-adjoint, bounded and linear. Hence, $\langle x,\,\mathcal{F}\,x\rangle_{\mathscr{X}}=-\langle x,\,\widehat{\mathcal{F}}'\,\widehat{\mathcal{F}}\,x\rangle_{\mathscr{X}}=-\langle\widehat{\mathcal{F}}\,x,\,\widehat{\mathcal{F}}\,x\rangle_{\mathscr{X}}$, and

$$f(x)=-\tfrac{1}{2}\langle\widehat{\mathcal{F}}\,x,\,\widehat{\mathcal{F}}\,x\rangle_{\mathscr{X}}+\langle x,\,\xi\rangle_{\mathscr{X}}\,. \tag{B.3}$$

Let $\mathscr{N}(\widehat{\mathcal{F}})$ and $\mathscr{R}(\widehat{\mathcal{F}})$ denote the null and range spaces of $\widehat{\mathcal{F}}$ respectively. As $\widehat{\mathcal{F}}$ is self-adjoint, $\mathscr{N}^\perp(\widehat{\mathcal{F}})=\mathscr{R}(\widehat{\mathcal{F}})$ and $\mathscr{R}(\widehat{\mathcal{F}})$ is closed (c.f. [16], Theorem 4.10.1). Furthermore, $\mathscr{X}=\mathscr{N}(\widehat{\mathcal{F}})\,\hat{\oplus}\,\mathscr{N}^\perp(\widehat{\mathcal{F}})$, where $\hat{\oplus}$ denotes the direct sum (c.f. [16], Theorem 2.7.4). In particular, with $\xi\in\mathscr{X}$ as per the lemma statement,

$$\xi=\xi^N+\xi^R\,,\quad\text{where}\quad \xi^N\in\mathscr{N}(\widehat{\mathcal{F}})\,,\ \xi^R\in\mathscr{R}(\widehat{\mathcal{F}})\,,\ \langle\xi^N,\,\xi^R\rangle=0\,.$$

Suppose $\xi^N\ne0$, and define $\chi\doteq k\,\xi^N$ for some $k\in\mathbb{R}_{>0}$. Then,

$$\begin{aligned} f(\chi)=-\tfrac{1}{2}\langle\widehat{\mathcal{F}}\,\chi,\,\widehat{\mathcal{F}}\,\chi\rangle_{\mathscr{X}}+\langle\chi,\,\xi\rangle_{\mathscr{X}}&=-\tfrac{k^2}{2}\langle\widehat{\mathcal{F}}\,\xi^N,\,\widehat{\mathcal{F}}\,\xi^N\rangle_{\mathscr{X}}+k\langle\xi^N,\,\xi^N+\xi^R\rangle_{\mathscr{X}}\\ &=k\left\|\xi^N\right\|_{\mathscr{X}}^2\,.\end{aligned}$$

So, $\sup_{x\in\mathscr{X}} f(x)\ge\left\|\xi^N\right\|_{\mathscr{X}}^2\sup_{k\in\mathbb{R}_{>0}} k=\infty$, which contradicts the assertion of the left-hand side of (B.1). That is, $\xi^N=0$, so that $\xi\in\mathscr{R}(\widehat{\mathcal{F}})$. Hence, there exists a $y\in\mathscr{X}$ such that $\xi=\widehat{\mathcal{F}}\,y$. So, returning to (B.3),

$$\begin{aligned} f(x)=-\tfrac{1}{2}\langle\widehat{\mathcal{F}}\,x,\,\widehat{\mathcal{F}}\,x\rangle_{\mathscr{X}}+\langle x,\,\widehat{\mathcal{F}}\,y\rangle_{\mathscr{X}}&=\tfrac{1}{2}\langle y,\,y\rangle_{\mathscr{X}}-\tfrac{1}{2}\langle\widehat{\mathcal{F}}\,x-y,\,\widehat{\mathcal{F}}\,x-y\rangle_{\mathscr{X}}\\ &=\tfrac{1}{2}\langle y,\,y\rangle_{\mathscr{X}}-\tfrac{1}{2}\left\|\widehat{\mathcal{F}}\,x-y\right\|_{\mathscr{X}}^2\,.\end{aligned} \tag{B.4}$$

Recalling from above that $\mathscr{R}(\widehat{\mathcal{F}})$ is closed, $\widehat{\mathcal{F}}$ has a pseudo inverse [9], denoted by $\widehat{\mathcal{F}}^+$. Consequently, the supremum over $\mathscr{X}$ in (B.2) is attained at $x^*=\widehat{\mathcal{F}}^+\,y=\widehat{\mathcal{F}}^+\widehat{\mathcal{F}}^+\,\xi$.

As $\mathcal{F}$ is self-adjoint, and $-\mathcal{F} = \widehat{\mathcal{F}}\,\widehat{\mathcal{F}}$, it follows that the pseudo-inverse of $\mathcal{F}$ also exists and is given by $\mathcal{F}^+ = -\widehat{\mathcal{F}}^+\,\widehat{\mathcal{F}}^+$. That is, the maximizer may be rewritten as $x^* = -\mathcal{F}^+\,\xi$, as per (B.2). Substituting in (B.4) yields

$$f(x^*) = \tfrac{1}{2}\langle y,\, y\rangle_{\mathscr{X}} = \tfrac{1}{2}\langle \widehat{\mathcal{F}}^+\,\xi,\, \widehat{\mathcal{F}}^+\,\xi\rangle_{\mathscr{X}} = \tfrac{1}{2}\langle \xi,\, \widehat{\mathcal{F}}^+\,\widehat{\mathcal{F}}^+\,\xi\rangle_{\mathscr{X}} = -\tfrac{1}{2}\langle \xi,\, \mathcal{F}^+\,\xi\rangle_{\mathscr{X}}\,,$$

as per (B.2). ◻

LEMMA B.3. *Consider any max-plus integral operator $\mathcal{O}^\oplus$ of the form (3.36) with*

$$\mathcal{O}^\oplus\,a = \left(\mathcal{O}^\oplus\,a\right)(\cdot) \doteq \int_{\mathscr{X}}^\oplus O(\cdot, z) \otimes a(z)\,dz\,, \tag{B.5}$$

defined with respect to kernel functional $O : \mathscr{X} \times \mathscr{X} \mapsto \mathbb{R}$ and functionals $a : \mathscr{X} \mapsto \mathbb{R}$. Suppose there exists a functional $a : \mathscr{X} \mapsto \mathbb{R}$ such that $a, (\mathcal{O}^\oplus\,a) : \mathscr{X} \mapsto \mathbb{R}$ are finite-valued everywhere on $\mathscr{X}$. Then, the kernel functional O is also finite-valued everywhere on $\mathscr{X} \times \mathscr{X}$. That is, $O(y,z) < \infty$ for all $y, z \in \mathscr{X}$.

Proof. Fix any $y, z \in \mathscr{X}$. From (B.5), $O(y,z) \le (\mathcal{O}^\oplus\,a)(y) - a(z) < \infty$ by finiteness of $(\mathcal{O}^\oplus\,a)(y)$ and $a(z)$. ◻