Finite Time–Horizon Risk Sensitive Control and the Robust Limit under a Quadratic Growth Assumption

Francesca Da Lio* William M. McEneaney†

June 8, 2001

Abstract

The finite time–horizon risk sensitive limit problem for continuous, nonlinear systems is considered. Previous results are extended to cover more typical examples. In particular, the cost may grow quadratically, and the diffusion coefficient may depend on the state. It is shown that the risk sensitive value function is the solution of the corresponding dynamic programming equation. It is also shown that this value converges to the value of the Robust control problem as the cost becomes infinitely risk averse with corresponding scaling of the diffusion coefficient.

Key Words: risk-sensitive control, robust, H_∞ , viscosity solutions, nonlinear HJB equations, nonlinear Isaacs equations

AMS subject classifications: 35B37, 49L25, 90D25, 93B36, 93C10, 93E05, 93E20

1 Introduction

The nonlinear, finite time–horizon risk sensitive limit problem is considered. It is, by now, well known that the value functions of risk–sensitive stochastic control problems tend to converge to the value functions of the corresponding Robust/H_∞ control problems as one approaches infinite risk aversion. This was addressed first in the LEQG (linear–exponential–quadratic–Gaussian) case where, in fact, one does not need to take the risk averse limit [27], [51], [5],[11]. In the nonlinear case, results were first developed for finite time–horizon control problems [52],[28], [16], [41], [42]. Further studies considered nonlinear, infinite time–horizon problems (in which case one gets the H_∞ limit) [17], [42], [18], [15], [43], [22], [47], [37], [30], [4] and nonlinear escape problems [36], [8]. Other studies have involved discrete systems [48], [12], [14], and the partial observations case [6], [29].

The results for both the finite time–horizon problem and the infinite time–horizon problem have mainly been obtained under assumptions which preclude quadratic cost criteria. In other words, the LEQG results were not subsumed by the nonlinear results. Since control systems for nonlinear systems are often designed by analogy with the linear–quadratic theory, it would be most desirable for the nonlinear theory to subsume the LEQG theory. For the infinite time-horizon problem, this is addressed in a preliminary fashion in [26] and [37] (with some results taken from [38], [39] in this latter case). The infinite time–horizon case presents certain technical difficulties which are not present in the finite time horizon case. In particular, there is a certain lack of uniqueness of solutions to the dynamic programming equation

*Dept. of Mathematics, University of Torino, via Carlo Alberto, 10 10123 Torino, ITALY, dalio@dm.unito.it

†Dept. of Mathematics, North Carolina State University, Raleigh, NC 27695–8205, USA, wmm@math.ncsu.edu.
Research partially supported by AFOSR grant F49620-95-1-0296.

(DPE) in the infinite time–horizon case which is not a problem in the finite time–horizon case. That is, there may be an infinite number of viscosity solutions and multiple classical solutions (when they exist) to the DPE. (This is in addition to the multiplicity incurred by scaling with an additive constant.) This question is addressed in [38], [39].

One approach to the finite time horizon problem is to prove that the value function is a viscosity (or classical) solution to the associated DPE for the risk sensitive problem. This is the approach taken by most of the work to date, and is the approach taken here as well. More specifically, let the parameter associated with risk–sensitivity be ε (described more accurately below). This parameter will also be used scale the diffusion term. One wishes to show that as $\varepsilon \downarrow 0$ (i.e. as the problem becomes infinitely risk averse), the value function of the risk sensitive problem converges to that of a Robust (H_∞) control problem. It is easily seen that the DPE for the risk sensitive problem formally converges to the DPE for the Robust control problem as $\varepsilon \downarrow 0$. One may also show that the value of the Robust control problem is a viscosity solution of this limit DPE. If one proves uniform boundedness and equicontinuity results for the pre–limit value functions, then by the stability of viscosity solutions, these converge (subsequentially) to viscosity solutions of the limit DPE. A key step is to prove a uniqueness result for viscosity solutions of this limit DPE. Then, one can assert that the solutions of the pre–limit DPE (the risk sensitive values) converge to the unique solution of the limit DPE which must also then be the value of the limit Robust control problem.

One of the key difficulties encountered in earlier attempts (cf. [16], [42]) employing the plan outlined in the previous paragraph was that the viscosity solution uniqueness results did not cover the limit DPE's except under rather strict conditions such as globally Lipschitz conditions on the cost and dynamics. In [41], viscosity solution uniqueness results were extended to cover certain cases where the cost could grow quadratically. However, it was still required that one prove a certain Lipschitz condition with Lipschitz constant growing linearly with the norm of the state, uniformly for the pre–limit solutions as well as the limit. Some recent results (obtained independently and concurrently) [2], [25] have extended the uniqueness results in ways that do not require this Lipschitz condition. These results allow one to extend the previous risk sensitive limit results to a larger class which subsumes a large variety of LEQG problems. (However, it does not subsume all, and in particular, the control set must be bounded.) This extension is the subject of this paper. Consequently, the uniqueness result 4.1 is a central topic of the paper. Note that the uniqueness results in [2], [25] do not cover the particular limit PDE encountered here. In the last Section 4 of the present paper, the results of [2] are extended to yield the uniqueness result needed here.

Some other technical difficulties arise in this case as well. In particular, there is some difficulty with a Novikov condition that was not present previously (see discussion prior to Lemma 2.2). Some difficulties also arise in the proof of the equicontinuity of the value functions. These last technical difficulties were overcome in this paper by consideration of the risk sensitive value function as the value function for a stochastic game, and the use of known results showing that this game value is the limit of the value of a discrete game as the time–step size goes to zero. The delicateness of the required estimates is not surprising given the possible finite–time blow–up property of the Riccati equations associated with linear–quadratic problems.

In Sections 2 and 3, a representation result will be obtained that states that any classical solution of the risk sensitive DPE must be the value function of the risk sensitive problem. The assumptions will also be listed in that section. Further, in Section 3, the existence of the above classical solution will be proved, and the uniform boundedness and equicontinuity results will be obtained along the way. Section 4 contains the limit result. Of course these results require the uniqueness result discussed above. This uniqueness result is stated and proved there. At the same time, it is shown that this unique solution is the value of the Robust control problem, thus making the connection between the risk averse limit of the risk sensitive problem and the Robust control problem.

2 Risk Sensitive Representation Result

In the stochastic (pre–limit) risk sensitive problem, we consider a system of the form

$$
\begin{aligned}
dy_t^\varepsilon &= f(y_t^\varepsilon, u_t)\,dt + \sqrt{\frac{\varepsilon}{\gamma^2}}\sigma(y_t^\varepsilon)\,dB_t \\
y_s^\varepsilon &= x
\end{aligned}
\tag{1}
$$

where y_t^ε is the state at time t taking values in $\Re^n$, x is the (known) initial state at time $s \geq 0$, f represents the nominal dynamics with control u taking values in $U \in \Re^l$, $\{B_\cdot, \mathcal{F}_\cdot\}$ is an m–dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_0$ contains all the P–negligible elements of $\mathcal{F}$ and σ is an $n \times m$–valued diffusion coefficient. The role of the parameters $\varepsilon, \gamma > 0$ will become more clear subsequently. However, perhaps it should be remarked here that ε will be a measure of the risk sensitivity, and scales the diffusion term in (1) above so that the cost below will remain bounded (for each x, as a function of ε).

We consider a cost criterion of the form

$$
J^\varepsilon(s, x, u_\cdot) = \mathrm{E}\exp\left\{\frac{1}{\varepsilon}\left[\int_s^T \ell(y_t^\varepsilon, u_t)\,dt + \psi(y_T^\varepsilon)\right]\right\}
\tag{2}
$$

where $0 \leq s \leq T < \infty$, T is a fixed terminal time, ℓ is the running cost, and ψ is the terminal cost. The value function is

$$
\begin{aligned}
V^\varepsilon(s,x) &= \inf_{u \in \mathcal{U}_s} \varepsilon \log J^\varepsilon(s, x, u_\cdot) \\
&= \varepsilon \log \inf_{u \in \mathcal{U}_s} J^\varepsilon(s, x, u_\cdot)
\end{aligned}
\tag{3}
$$

where $\mathcal{U}_s$ is the set of U–valued, $\mathcal{F}_t$–progressively measurable controls such that there exists a strong solution to (1).

We make the following assumptions throughout the paper.

$$
(A0) \qquad U \text{ is a compact subset of } \Re^l;
$$

$$
(A1) \qquad \begin{cases}
i) & f \in C^2(\Re^n \times U, \Re^n); \\
ii) & |f_x(x,u)| \leq K, \quad \forall x, \in \Re^n,\ \forall u \in U; \\
iii) & |f(x,u)| \leq K(1+|x|), \quad \forall x \in \Re^n,\ \forall u \in U; \\
iv) & |f_{xx}(x,u)| \leq K/(1+|x|), \quad \forall x \in \Re^n,\ \forall u \in U;
\end{cases}
$$

$$
(A2) \qquad \begin{cases}
i) & \sigma \in C^2(\Re^n, \mathcal{L}(\Re^m, \Re^n)); \\
ii) & |\sigma(x)| \leq M, \quad \forall x \in \Re^n; \\
iii) & \xi^T \sigma(x) \sigma^T(x) \xi \geq \eta |\xi|^2 \quad \forall \xi, x \in \Re^n; \\
iv) & |\sigma_x(x)| \leq L_\sigma/(1+|x|), \quad \forall x \in \Re^n; \\
v) & |\sigma_{xx}(x)| \leq L_\sigma/(1+|x|^2), \quad \forall x \in \Re^n;
\end{cases}
$$

$$
(A3) \qquad \begin{cases}
i) & \ell \in C^\infty(\Re^n \times U, \Re), \quad |\ell_{xx}(x,u)| \leq \overline{C}, \quad \forall (x,u) \in \Re^n \times U; \\
ii) & \ell(x,u) \geq 0, \quad \forall (x,u) \in \Re^n \times U; \\
iii) & |\ell(x,u)| \leq C(1+|x|^2), \quad \forall (x,u) \in \Re^n \times U; \\
iv) & |\ell(x,u) - \ell(y,u)| \leq C(1+|x|+|y|)|x-y|, \quad \forall x, y \in \Re^n, \forall u \in U;
\end{cases}
$$

$$
(A4) \qquad \begin{cases} i) & \psi \in C^\infty(\Re^n, \Re), \qquad |\psi_{xx}(x)| \le \overline{C}, & \forall x \in \Re^n; \\ ii) & \psi(x) \ge 0, & \forall x \in \Re^n; \\ iii) & |\psi(x)| \le C(1+|x|^2), & \forall x \in \Re^n; \\ iv) & |\psi(x) - \psi(y)| \le C(1+|x|+|y|)|x-y|, & \forall x, y \in \Re^n; \\ v) & \text{higher (above second) derivatives of } \psi \text{ uniformly bounded;} \end{cases}
$$

where $L_\sigma \in \Re$, $C, K, M, \eta \in (0, \infty)$, and $\mathcal{L}(\Re^m, \Re^n)$ appearing in $(A2.i)$ represents the space of $n \times m$ matrices. The is some intended redundancy in the above assumptions; for instance $(A1.iii)$ is included to indicate that the same constant, K, will be used as in $(A1.ii)$ even though the existence of some constant for which $(A1.iii)$ holds follows from $(A0)$, $(A1.i)$, $(A1.ii)$. Note that many of the assumptions are needed only for the results of Section 3, and in particular, are not needed for the uniqueness result of Section 4. Specifically in Section 4 we use $(A1.iii), (A2.ii), (A3.ii) - (A3.iv), (A4.ii) - (A4.iv)$; while $(A1.iv)$, $(A2.v)$ and $(A4.v)$ are used only to obtain the results in Section 3. This is also true of the requirement of continuous, bounded second–derivatives in $(A3.i)$ and $(A4.i)$. It seems that many of the above smoothness assumptions which are required in the pre–limit analysis but not in the limit uniqueness result might be removable, however the pre–limit analysis is already quite technical, and so no attempt is made to reduce the assumptions further. There will be one additional assumption in Section 3.

In Section 3, it will be shown that there exists a classical solution, $\widetilde{V}^\varepsilon \in C^{1,2}$ to partial differential equation (PDE)

$$
\begin{aligned} 0 &= V_s + \frac{\varepsilon}{2\gamma^2} \sum_{i,j=1}^n a_{i,j}(x) V_{x_i x_j} + \frac{1}{2\gamma^2} \nabla V^T a(x) \nabla V + \min_{u \in U} [f(x,u) \cdot \nabla V + \ell(x,u)], \quad 0 \le s \le T \\ V(T,x) &= \psi(x) \end{aligned} \tag{4}
$$

where $a \doteq \sigma\sigma^T$. For the purposes of this section, assume that the classical solution exists, and we postpone the proof of this fact to the next section. Note that

$$
\frac{1}{2\gamma^2} \nabla V^T a \nabla V = \max_{w \in \Re^m} \left[(\sigma w) \cdot \nabla V - \frac{\gamma^2}{2} \|w\|^2 \right]. \tag{5}
$$

Define a measurable (see for instance [20]) feedback control

$$
\overline{u}(t,x) \in \operatorname{argmin}[f(x,u) \cdot \nabla \widetilde{V}^\varepsilon(t,x) + \ell(x,u)]. \tag{6}
$$

Using [50], one proves the existence of a strong solution, $\overline{y}^\varepsilon$, to (1) with feedback $\overline{u}$ (see [42]). (Also see [3] for a quicker proof of a weak solution.) Then define the $\mathcal{F}_t$–progressively measurable control $\widetilde{u}_\cdot$ given by $\widetilde{u}_t(\omega) \doteq \overline{u}(t, \overline{y}_t^\varepsilon(\omega))$ for all $\omega \in \Omega$.

The first lemma is rather easily proved.

Lemma 2.1

$$
\varepsilon \log J^\varepsilon(s,x,\widetilde{u}) \le \widetilde{V}^\varepsilon(s,x) \qquad \forall\, s \in [0,T], \ \forall\, x \in \Re^n.
$$

Proof. By Ito's rule and (1),

$$
\widetilde{V}^\varepsilon(T, y_T^\varepsilon) - \widetilde{V}^\varepsilon(s,x) = \int_s^T \left[f(\overline{y}_t^\varepsilon, \widetilde{u}_t) \cdot \nabla \widetilde{V}^\varepsilon + \widetilde{V}_t^\varepsilon + \frac{\varepsilon}{2\gamma^2} \sum_{i,j=1}^n a_{i,j}(\overline{y}_t^\varepsilon) \widetilde{V}^\varepsilon_{x_i x_j} \right] dt + \int_s^T \sqrt{\frac{\varepsilon}{\gamma^2}} (\nabla \widetilde{V}^\varepsilon)^T \sigma \cdot dB_t.
$$

Using (4) and (6) one finds

$$
\int_s^T \ell(\overline{y}_t^\varepsilon, \widetilde{u}_t)\, dt + \psi(\overline{y}_T^\varepsilon) = \widetilde{V}^\varepsilon(x) + \sqrt{\frac{\varepsilon}{\gamma^2}} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T \sigma\, dB_t - \frac{1}{2\gamma^2} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T a \nabla \widetilde{V}^\varepsilon\, dt
$$

which yields

$$\mathrm{E}\exp\frac{1}{\varepsilon}\left[\int_s^T \ell(\overline{y}_t^\varepsilon,\widetilde{u}_t)\,dt+\psi(\overline{y}_T^\varepsilon)\right] = e^{\frac{1}{\varepsilon}\widetilde{V}^\varepsilon(x)}$$

$$\cdot\mathrm{E}\exp\left[\frac{1}{\sqrt{\gamma^2\varepsilon}}\int_s^T(\nabla\widetilde{V}^\varepsilon)^T\sigma\,dB_t-\frac{1}{2\gamma^2\varepsilon}\int_s^T(\nabla\widetilde{V}^\varepsilon)^T a\nabla\widetilde{V}^\varepsilon\,dt\right].$$

By [24] (see also [45]), this last term is a supermartingale, and so one has

$$\mathrm{E}\exp\frac{1}{\varepsilon}\left[\int_s^T \ell(\overline{y}_t^\varepsilon,\widetilde{u}_t)\,dt+\psi(\overline{y}_T^\varepsilon)\right]\le e^{\frac{1}{\varepsilon}\widetilde{V}^\varepsilon(x)}.$$

□

For the approach that we will use to prove a reverse inequality for (not necessarily optimal) controls, one needs to use Girsanov's Theorem. Consequently, one is interested in controls for which a weak form of the Novikov condition holds. More specifically, we will be interested in controls for which there exists $\tau>0$, $m<\infty$ such that

$$\mathrm{E}\exp\frac{1}{2\gamma^2\varepsilon}\left[\int_t^{t+\tau}(\nabla\widetilde{V}^\varepsilon(t,y_t^\varepsilon))^T a(y_t^\varepsilon)\nabla\widetilde{V}^\varepsilon(t,y_t^\varepsilon)\,dt\right]\le m \qquad \forall\, 0\le s\le t\le t+\tau\le T. \tag{7}$$

The following lemma applies only to controls satisfying (7), but this lemma will also be useful in a generalization to appear at a later point in the paper.

Lemma 2.2 *Let $0\le s\le T$ and $x\in\Re^n$. Let $u\in\mathcal{U}_s$ be a control such that (7) holds. Then*

$$\varepsilon\log J^\varepsilon(s,x,u)\ge\widetilde{V}^\varepsilon(s,x).$$

Proof. Let y^ε be the solution to (1) corresponding to the initial conditions and control in the lemma statement. Consider the change of drift given by

$$y_t^\varepsilon = x+\int_s^t f(y_r^\varepsilon,u_r)\,dr+\sqrt{\frac{\varepsilon}{\gamma^2}}\int_s^t\sigma(y_r^\varepsilon)\,dB_r \tag{8}$$

$$= x+\int_s^t[f(y_r^\varepsilon,u_r)+\sigma(y_r^\varepsilon)w_r]\,dr+\sqrt{\frac{\varepsilon}{\gamma^2}}\int_s^t\sigma(y_r^\varepsilon)\,d\widehat{B}_r \tag{9}$$

$$\tag{10}$$

where w is an adapted process to be given below. By (7) (see, for instance, [31]), $\widehat{B}$ is a Brownian motion under new probability measure, $\widehat{P}$ where

$$P(d\omega)=\exp\left[-\frac{1}{\sqrt{\gamma^2\varepsilon}}\int_s^t(\nabla\widetilde{V}^\varepsilon(r,y_r^\varepsilon))^T\sigma(y_r^\varepsilon)\,d\widehat{B}_r-\frac{1}{2\gamma^2\varepsilon}\int_s^t(\nabla\widetilde{V}^\varepsilon(r,y_r^\varepsilon))^T a(y_r^\varepsilon)\nabla\widetilde{V}^\varepsilon(r,y_r^\varepsilon)\,dr\right]\widehat{P}(d\omega).$$

Consequently,

$$\mathrm{E}\exp\frac{1}{\varepsilon}\left[\int_s^T\ell(y_t^\varepsilon,u_t)\,dt+\psi(y_T^\varepsilon)\right] = \widehat{\mathrm{E}}\exp\frac{1}{\varepsilon}\left[\int_s^T\left(\ell(y_t^\varepsilon,u_t)-\frac{1}{2\gamma^2}(\nabla\widetilde{V}^\varepsilon(t,y_t^\varepsilon))^T a(y_t^\varepsilon)\nabla\widetilde{V}^\varepsilon(t,y_t^\varepsilon)\right)dt\right.$$

$$\left.-\sqrt{\frac{\varepsilon}{\gamma^2}}\int_s^T(\nabla\widetilde{V}^\varepsilon(t,y_t^\varepsilon))^T\sigma(y_t^\varepsilon)\,d\widehat{B}_t+\psi(y_T^\varepsilon)\right]. \tag{11}$$

By Ito's rule with dynamics (9),

$$\widetilde{V}^\varepsilon(y^\varepsilon_T) - \widetilde{V}^\varepsilon(x) \;=\; \int_s^T [\widetilde{V}^\varepsilon_t + (f(y^\varepsilon_t, u_t) + \sigma(y^\varepsilon_t) w_t) \cdot \nabla \widetilde{V}^\varepsilon + \frac{\varepsilon}{2\gamma^2} \sum_{i,j=1}^n a_{i,j}(y^\varepsilon_t) \widetilde{V}^\varepsilon_{x_i x_j}]\, dt$$
$$+\sqrt{\frac{\varepsilon}{\gamma^2}} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T \sigma(y^\varepsilon_t)\, d\widehat{B}_t.$$

Taking $w_t \doteq \frac{1}{\gamma^2}\sigma^T(y^\varepsilon_t)\nabla\widetilde{V}^\varepsilon(y^\varepsilon_t)$ yields

$$\begin{aligned}\psi(y^\varepsilon_T) - \widetilde{V}^\varepsilon(x) \;=\; &\int_s^T [\widetilde{V}^\varepsilon_t + f(y^\varepsilon_t, u_t) \cdot \nabla \widetilde{V}^\varepsilon + \frac{1}{2\gamma^2} (\nabla \widetilde{V}^\varepsilon)^T a(y^\varepsilon_t) \nabla \widetilde{V}^\varepsilon \\ &+ \frac{\varepsilon}{2\gamma^2} \sum_{i,j=1}^n a_{i,j}(y^\varepsilon_t) \widetilde{V}^\varepsilon_{x_i x_j}]\, dt \\ &+ \frac{1}{2\gamma^2} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T a(y^\varepsilon_t) \nabla \widetilde{V}^\varepsilon\, dt + \sqrt{\frac{\varepsilon}{\gamma^2}} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T \sigma(y^\varepsilon_t)\, d\widehat{B}.\end{aligned} \tag{12}$$

Now, since this choice of u is not necessarily optimal in (4), one has at any time

$$0 \le \widetilde{V}^\varepsilon_t + \frac{\varepsilon}{2\gamma^2} \sum_{i,j=1}^n a_{i,j}(y^\varepsilon_t) \widetilde{V}^\varepsilon_{x_i x_j} + \frac{1}{2\gamma^2} (\nabla \widetilde{V}^\varepsilon)^T a(y^\varepsilon_t) \nabla \widetilde{V}^\varepsilon + f(y^\varepsilon_t, u_t) \cdot \nabla \widetilde{V}^\varepsilon + \ell(y^\varepsilon_t, u_t). \tag{13}$$

Combining (12), (13) yields

$$\begin{aligned}\exp \frac{1}{\varepsilon} \widetilde{V}^\varepsilon(x) \;\le\; &\exp \frac{1}{\varepsilon} \Big[\int_s^T \ell(y^\varepsilon_t, u_t)\, dt + \psi(y^\varepsilon_T) - \sqrt{\frac{\varepsilon}{\gamma^2}} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T \sigma(y^\varepsilon_t)\, d\widehat{B} \\ &- \frac{1}{2\gamma^2} \int_s^T (\nabla \widetilde{V}^\varepsilon)^T a(y^\varepsilon_t) \nabla \widetilde{V}^\varepsilon\, dt \Big]\end{aligned}$$

which, by (11), implies

$$\exp \frac{1}{\varepsilon} \widetilde{V}^\varepsilon(x) \le \mathrm{E} \exp \frac{1}{\varepsilon} \left[\int_s^T \ell(y^\varepsilon_t, u_t)\, dt + \psi(y^\varepsilon_T) \right].$$

□

Summarizing the results in this section, we have

Lemma 2.3 *Let $0 \le s \le T$ and $x \in \Re^n$. For any $u \in \mathcal{U}_s$ such that (7) holds, one has*

$$\varepsilon \log J^\varepsilon(s, x, u) \ge \widetilde{V}^\varepsilon(s, x),$$

while with optimal $\widetilde{u}$ given by (6) one has

$$\varepsilon \log J^\varepsilon(s, x, \widetilde{u}) \le \widetilde{V}^\varepsilon(s, x).$$

Lemma 2.3 is not quite what we will obtain for the risk sensitive pre–limit result. The lemma is lacking due to the requirement that (7) holds in order to obtain the first assertion. Without that condition, the lemma would show that $\widetilde{V}^\varepsilon$ is the value of the risk sensitive control problem. In the next section, while proving the existence of a classical solution to (4), we will obtain results that allow one to improve Lemma 2.3 in this way; specifically, that the value, V^ε, given by (3) is this classical solution. The improved result will be Theorem 3.11.

3 Existence and Uniformity

The representation results of the previous section required the existence of a $C^{1,2}$ solution to (4). That will be obtained in this section. Also, in order to prove the risk–averse limit result to follow, we will need continuity and local boundedness estimates on $\widetilde{V}^{\varepsilon}$ independent of $\varepsilon > 0$. These will be obtained at the same time as the existence result. Lastly, we will obtain the promised improvement over Lemma 2.3. These results will all be obtained through the stochastic game representation.

Before proceeding with the technical details to follow, we give some formal indications of the general concepts.

Using convex duality, one may rewrite DPE (4) as

$$\begin{aligned} &0 = V_s + \frac{\varepsilon}{2\gamma^2}\sum_{i,j=1}^{n} a_{i,j}(x)V_{x_i x_j} + \min_{u\in U}\max_{w\in\Re^m}\left\{[f(x,u)+\sigma(x)w]\cdot\nabla V + \ell(x,u) - \frac{\gamma^2}{2}|w|^2\right\} \\ &V(T,x) = \psi(x). \end{aligned} \tag{14}$$

In this form, it is intuitive that this PDE might be associated with the stochastic differential game with dynamics

$$\begin{aligned} &dy_t^{\varepsilon} = [f(y_t^{\varepsilon},u_t) + \sigma(y_t^{\varepsilon})w_t]\,dt + \sqrt{\frac{\varepsilon}{\gamma^2}}\sigma(y_t^{\varepsilon})\,dB_t \\ &y_s^{\varepsilon} = x \end{aligned} \tag{15}$$

and payoff

$$J_g^{\varepsilon}(s,x;u,w) = \mathrm{E}\left[\int_s^T \ell(y_t^{\varepsilon},u_t) - \frac{\gamma^2}{2}|w_t|^2\,dt + \psi(y_T^{\varepsilon})\right].$$

In this game, $w.$ is the control for the opposing (disturbance) player who is trying to maximize this payoff which we are trying to minimize. The definitions of value for such a game [23] are discussed further below. The connections between such games and their corresponding DPE's (now second–order Isaacs equations) are not known for the class of systems considered here, so it will be necessary to consider some more restrictive assumptions, and then to relax these assumptions to obtain the above game and DPE.

One technique that is sometimes useful for games associated with H_∞ control is to obtain an L_2–bound on ε–optimal controls, w, for the disturbance player [40], [38], [39]. This is then used to obtain continuity estimates on the value function. The simplest analogue is to prove a bound of the form $\|w\|_{L_2([s,T]\times\Omega)} \le M < \infty$ for ε–optimal w. Such a bound is not difficult to obtain in this case, but it does not yield the equicontinuity estimates for the value needed for the proofs to follow. Instead, one would ideally obtain an almost sure bound of some form. This bound is rather technical due to the presence of the Brownian motion with its unbounded variation. Instead, a discrete game is analyzed here in order to obtain an analogous bound, and consequent continuity estimates. A limit is then taken to obtain the continuity estimates for the original game. We now proceed with the analysis.

We first work with a system where the dynamics and cost functions are all bounded. Let

$$\widetilde{\ell}^R(x,u) = \begin{cases} \ell(x,u) & \text{if } \ell(x,u) \le R \\ R & \text{otherwise,} \end{cases} \tag{16}$$

$$\widetilde{\psi}^R(x) = \begin{cases} \psi(x) & \text{if } \psi(x) \le R \\ R & \text{otherwise} \end{cases} \tag{17}$$

$$\widetilde{f}^b(x,u) = \begin{cases} f(x,u) & \text{if } |f(x,u)| \le b \\ b\frac{f(x,u)}{|f(x,u)|} & \text{otherwise.} \end{cases} \tag{18}$$

The final assumption in the paper is that for all R sufficiently large, one can mollify $\widetilde{\ell}^R, \widetilde{\psi}^R, \widetilde{f}^b$ yielding ℓ^R, ψ^R, f^b such that

$$\left\{\begin{array}{ll} \ell^R \in C^2(\Re^n \times U) & \\ \ell^R(x,u) = \widetilde{\ell}^R(x,u) & \forall |x| \le D_R^\ell \\ \ell^R(x,u) \le R & \forall x \in \Re^n \\ |\nabla \ell^R(x,u)| \le 2C'|x| & \forall x \in \Re^n \\ |\ell^R_{xx}(x,u)| \le \overline{C}' & \forall x \in \Re^n \end{array}\right\} \qquad \left\{\begin{array}{ll} \psi^R \in C^2(\Re^n) & \\ \psi^R(x) = \widetilde{\psi}^R(x) & \forall |x| \le D_R^\psi \\ \psi^R(x) \le R & \forall x \in \Re^n \\ |\nabla \psi^R(x)| \le 2C'|x| & \forall x \in \Re^n \\ |\psi^R_{xx}(x)| \le \overline{C}' & \forall x \in \Re^n \end{array}\right\}$$

$$(A5) \qquad \left\{\begin{array}{ll} f^b \in C^2(\Re^n \times U) & \\ f^b(x,u) = \widetilde{f}^b(x,u) & \forall |x| \le D_b^f, u \in U \\ |f^b(x,u)| \le b & \forall x \in \Re^n, u \in U \\ |f^b_x(x,u)| \le K' & \forall x \in \Re^n, u \in U \\ |f^b_{xx}(x,u)| \le K'/(1+|x|) & \forall x \in \Re^n, u \in U \end{array}\right\}.$$

where $D_R^\ell, D_R^\psi \to \infty$ as $R \to \infty$, $D_b^f \to \infty$ as $b \to \infty$. We claim that this can be achieved by a relatively standard mollification, but do not want to expend the journal space with the technical details, and so make it an assumption. In particular, for the ψ case, we claim that one would use the following convolution for $R > C$: $\psi^R(x) = \int J_\varepsilon(x-y)\widetilde{\psi}^R(y)\,dy$ where $J_1(z) = k\exp\{-1/(1-|z|^2)\}$ on $|z| \le 1$ and 0 elsewhere where k is such that $\int_{\Re^n} J_1 = 1$, $J_\varepsilon(z) = \varepsilon^{-n}J_1(z/\varepsilon)$, $D_R^\psi = \frac{1}{2}\sqrt{R/C - 1}$, $\varepsilon = 0$ for $|x| \le D_R^\psi$, $\varepsilon = \frac{1}{2}D_R^\psi \exp\{1 - 1/[1 - |(|x| - 3D_R^\psi/2)/(D_R^\psi/2)|]\}$ for $D_R^\psi < |x| \le 3D_R^\psi/2$, and $\varepsilon = D_R^\psi/2$ for $3D_R^\psi/2 < |x|$. The bounds on the derivatives use the smoothness of ψ itself (as well as $\max_{x,R}|\varepsilon_x|$, $\max_{x,R}|\varepsilon_{xx}|$ bounded for sufficiently large R) on $|x| \le 3D_R^\psi/2$, and then rely on the fact that the derivatives can be translated onto J_ε itself in the convolution for $|x| > 3D_R^\psi/2$ (cf. [46],[53]). Again, by making $(A5)$ an assumption, we avoid the details, and leave them only to an interested reader.

Also, let

$$\mathcal{W}_s \doteq \{\mathcal{F}_t\text{–adapted, right–continuous } w. \text{ such that } \|w\|_{L_2([s,T]\times\Omega)} < \infty\} \tag{19}$$

$$\mathcal{W}_s^m \doteq \{w \in \mathcal{W}_s : |w_t(\omega)| \le m \quad \forall t \in [s,T], \omega \in \Omega\}. \tag{20}$$

Note that condition (19) implies that any $w \in \mathcal{W}_s$ is $\mathcal{F}_t$–progressively measurable ; see [21], [9]. Let the dynamics of the modified game be

$$\begin{aligned} dy_t^{\varepsilon,b} &= [f^b(y_t^{\varepsilon,b}, u_t) + \sigma(y_t^{\varepsilon,b})w_t]\,dt + \sqrt{\frac{\varepsilon}{\gamma^2}}\sigma(y_t^{\varepsilon,b})\,dB_t \\ y_s^{\varepsilon,b} &= x, \end{aligned} \tag{21}$$

and define the payoff by

$$J_g^{\varepsilon,b,R}(s,x;u,w) = \mathrm{E}\left\{\int_s^T \left[\ell^R(y_t^{\varepsilon,b}, u_t) - \frac{\gamma^2}{2}|w_t|^2\right]dt + \psi^R(y_T^{\varepsilon,b})\right\}. \tag{22}$$

An admissible strategy for the player with control u is a mapping $\theta : \mathcal{W}_s \to \mathcal{U}_s$ (alternatively, $\theta : \mathcal{W}_s^m \to \mathcal{U}_s$ when the disturbance set is $\mathcal{W}_s^m$) such that if, almost surely, $w_r = \widetilde{w}_r$ for a.e. $r \in [s,t]$, then, almost surely, $\theta[w]_r = \theta[\widetilde{w}]_r$ for a.e. $r \in [s,t]$, for any $t \in [s,T]$; see [23]. The set of all admissible strategies will be denoted by Θ_s. Define the set of admissible strategies for the player with control w analogously, and denote them by Λ_s in the case where the disturbance set is $\mathcal{W}_s$. (In the case where the disturbance set is $\mathcal{W}_s^m$, denote it by Λ_s^m.) Then (see [23]), the upper and lower values are

$$\underline{V}^{\varepsilon,b,R,m}(s,x) = \inf_{\theta\in\Theta_s}\sup_{w\in\mathcal{W}_s^m} J_g^{\varepsilon,b,R}(s,x,\theta[w],w) \tag{23}$$

$$\overline{V}^{\varepsilon,b,R,m}(s,x) = \sup_{\lambda\in\Lambda_s^m}\inf_{u\in\mathcal{U}_s} J_g^{\varepsilon,b,R}(s,x,u,\lambda[u]), \tag{24}$$

and we note that by (22), these are bounded. Then, by [23], $\underline{V}^{\varepsilon,b,R,m} = \overline{V}^{\varepsilon,b,R,m}$ is the unique, bounded, continuous viscosity solution to the Isaacs equation

$$
\begin{aligned}
&0 = V_s + \frac{\varepsilon}{2\gamma^2}\sum_{i,j=1}^{n} a_{i,j}(x)V_{x_i x_j} + \min_{u\in U}\max_{|w|\le m}\{[f^b(x,u)+\sigma(x)w]\cdot\nabla V + \ell^R(x,u) - \frac{\gamma^2}{2}|w|^2\} \\
&V(T,x) = \psi^R(x).
\end{aligned}
\tag{25}
$$

Lemma 3.1 $\underline{V}^{\varepsilon,b,R,m} = \overline{V}^{\varepsilon,b,R,m}$ *is the unique, bounded, $C^{1,2}$ solution to (25).*

PROOF. From above, we know it is the unique, bounded, continuous viscosity solution. Additionally, [23] tells us that it is Lipschitz in x and Holder continuous in s. (We will be interested in specific bounds independent of ε, b, R, m further below, but for simplicity we do not look for these here.) Under our assumptions, it is not difficult to show that it is also Lipschitz in s, and we do so now.

Fix $s, s+\delta \in [0,T]$, $\delta > 0$. We will bound $|\underline{V}^{\varepsilon,b,R,m}(s,x) - \underline{V}^{\varepsilon,b,R,m}(s+\delta,x)|$. Fix any $x \in \Re^n$, $u \in \mathcal{U}_s$ and $w \in \mathcal{W}_s^m$. Let the corresponding solution of (21) be denoted by $y^{0,\varepsilon,b}_\cdot$. Recall the probability space is $(\Omega, \{\mathcal{F}_t\}, P)$. Define a new Brownian motion $B^\delta_\cdot$ by $B^\delta_t = B_{t-\delta}$ on a new probability space with filtration $\mathcal{F}^0_t = \mathcal{F}_{t-\delta}$. Let $u^\delta_t = u_{t-\delta}$ and $w^\delta_t = w_{t-\delta}$. Let the solution of (21) corresponding to this new Brownian motion, controls u^δ, w^δ and initial condition x at time $s+\delta$ be denoted by $y^{\delta,\varepsilon,b}_\cdot$. Then

$$
y^{\delta,\varepsilon,b}_t = y^{0,\varepsilon,b}_{t-\delta} \qquad \forall\, t\in[\delta, T+\delta].
$$

Consequently, one has

$$
J^{\varepsilon,b,R}_g(s+\delta,x;u^\delta,w^\delta) = \mathrm{E}\left[\int_s^{T-\delta} \ell^R(y^{0,\varepsilon,b}_t,u_t) - \frac{\gamma^2}{2}\|w_t\|^2\,dt + \psi^R(y^{0,\varepsilon,b}_{T-\delta})\right].
$$

Therefore,

$$
\begin{aligned}
&J^{\varepsilon,b,R}_g(s+\delta,x;u^\delta,w^\delta) - J^{\varepsilon,b,R}_g(s,x;u,w) \\
&\quad = \mathrm{E}\left[\psi^R(y^{0,\varepsilon,b}_{T-\delta}) - \int_{T-\delta}^{T}\big(\ell^R(y^{0,\varepsilon,b}_t,u_t) - \frac{\gamma^2}{2}\|w_t\|^2\big)\,dt - \psi^R(y^{0,\varepsilon,b}_T)\right] \\
&\quad \le (R+\frac{\gamma^2}{2}m^2)\delta + \mathrm{E}\left[\psi^R(y^{0,\varepsilon,b}_{T-\delta}) - \psi^R(y^{0,\varepsilon,b}_T)\right]
\end{aligned}
$$

which by Ito's rule

$$
\begin{aligned}
&= (R+\frac{\gamma^2}{2}m^2)\delta + \mathrm{E}\bigg\{\int_{T-\delta}^{T}\nabla\psi^R(y^{0,\varepsilon,b}_t)\cdot\Big[f^b(y^{0,\varepsilon,b}_t,u_t) + \sigma(y^{0,\varepsilon,b}_t)w_t\Big]\,dt \\
&\qquad\qquad + \frac{\varepsilon}{2\gamma^2}\int_{T-\delta}^{T}\sum_{i=1,j=1}^{n}\psi^R_{x_i x_j}(y^{0,\varepsilon,b}_t)a_{ij}(y^{0,\varepsilon,b}_t)\,dt\bigg\} \\
&\le \left[R + \frac{\gamma^2}{2}m^2 + 2M_R(b+Mm) + \frac{\varepsilon\overline{C}M^2}{2\gamma^2}\right]\delta.
\end{aligned}
$$

From this (and an analogous bound on the other side using approximately optimal w), one easily finds that $\underline{V}^{\varepsilon,b,R,m}$ is Lipschitz in s.

On the other hand, (25) has a $C^{1,2}$ solution which is globally Lipschitz and bounded [44] (see also [34], [20]). Since classical solutions are viscosity solutions, uniqueness of viscosity solutions for (25) implies that $\underline{V}^{\varepsilon,b,R,m}$ is, in fact, this $C^{1,2}$ solution. $\square$

Lipschitz bounds dependent on ε, b, R, m were sufficient to obtain the above result. Below however, we will need bounds independent of these parameters. These bounds will require more technical arguments. We now present such arguments. The first bound is actually quite straight–forward.

Lemma 3.2 *There exists $\widetilde{\gamma} < \infty$ such that for all $\gamma > \widetilde{\gamma}$,*

$$\underline{V}^{\varepsilon,b,R,m}(s,x) \le M_T(1+|x|^2) \qquad \forall\, s \in [0,T],\ \forall\, x \in \Re^n,$$

and further, for $\delta \le 1$, any δ–optimal $w_\delta \in \mathcal{W}_s$ with respect to any $\theta \in \Theta_s$ (i.e. $J_g^{\varepsilon,b,R}(s,x;\theta[w_\delta],w_\delta) \ge \sup_{w\in\mathcal{W}_s} J_g^{\varepsilon,b,R}(s,x;\theta[w],w) - \delta$) satisfies

$$\|w_\delta\|^2_{L_2(\Omega\times[0,T])} \le M'_T(1+|x|^2). \tag{26}$$

We note that M_T, M'_T depend on T, but are independent of ε, b, R, m.

Remark 3.3 A value for $\widetilde{\gamma}$ is easily obtained from the proof. Since this $\widetilde{\gamma}$ is general for all systems satisfying the assumptions, it is quite conservative. If one has additional structure such as contractivity of f, a much smaller value for $\widetilde{\gamma}$ can be obtained [38], [39].

Proof. Suppressing certain arguments for notational simplicity, one has

$$|y_t^{\varepsilon,b}|^2 \le |x|^2 + 2\int_s^t (y_r^{\varepsilon,b})^T[f^b + \sigma w_r]\,dr + 2\sqrt{\frac{\varepsilon}{\gamma^2}}\int_s^t (y_r^{\varepsilon,b})^T \sigma\, dB_r + \frac{\varepsilon}{\gamma^2}\int_s^t \sum a_{ii}\,dr$$

where (see [35] for example) the stochastic integral is a square–integrable martingale. Then, using $(A1.iii), (A2.ii)$,

$$\begin{aligned} \mathrm{E}|y_t^{\varepsilon,b}|^2 &\le |x|^2 + 2K\mathrm{E}\int_s^t (|y_r^{\varepsilon,b}| + |y_r^{\varepsilon,b}|^2)\,dr + 2M\mathrm{E}\int_s^t |y_r^{\varepsilon,b}||w_r|\,dr + \frac{\varepsilon}{\gamma^2}M^2(t-s) \\ &\le |x|^2 + (3K+M)\int_s^t \mathrm{E}|y_r^{\varepsilon,b}|^2\,dr + \Big(\frac{\varepsilon}{\gamma^2}M^2 + K\Big)(t-s) + M\int_s^t \mathrm{E}|w_r|^2\,dr. \end{aligned}$$

By Gronwall's inequality, there exists $C_t < \infty$ such that

$$\mathrm{E}|y_t^{\varepsilon,b}|^2 \le C_t(1 + |x|^2 + \|w\|^2_{L_2(\Omega\times[0,T])}).$$

Consequently,

$$J_g^{\varepsilon,b,R}(s,x;u,w) \le C'_T(1+|x|^2) + (\widetilde{\gamma}^2 - \gamma^2)\|w\|^2_{L_2(\Omega\times[0,T])}$$

for proper choice of $C'_T, \widetilde{\gamma}$ which depend on T but not b, R and $\varepsilon \le 1$. This yields the first assertion. The second then follows by comparison with $\overline{w} \equiv 0$. □

The proof of continuity estimates uniform over b, R, m and $\varepsilon \le 1$ are considerably more technical than the above results. In the deterministic setting, an L_2–bound on δ–optimal w is sufficient to yield equicontinuity estimates. However, in the stochastic case, the type of L_2–bound obtained in the previous lemma is not sufficient. This is due to the fact that L_2 bounds over the sample space as well as the state space do not imply bounds on the expectation of the exponentiation of the state space L_2–norm of w, and this is exactly what appears when using Gronwall–type estimates. (Recall we are only assuming Lipschitz conditions on the dynamics - not stability). However, one expects that if a set of Brownian paths does not diverge much from the origin, then the near-optimal disturbance paths over that portion of the sample–space should not have "too large" L_2–norms. Measurability conditions make it difficult to refine that idea into a proof in the continuous–time case, so we work first with discrete–time approximations, and then use well–known limit results to obtain the continuity estimate.

Lemma 3.4 *For γ sufficiently large (i.e. $\gamma \geq \overline{\gamma}$), there exists $\overline{C}_{T,D} < \infty$ (depending on $T, D \in [0,\infty)$ but independent of b, R, m, ε) such that*

$$|\underline{V}^{\varepsilon,b,R,m}(s,x) - \underline{V}^{\varepsilon,b,R,m}(\hat{s},\hat{x})| \leq \overline{C}_{T,D}[|x-\hat{x}| + |s-\hat{s}|] \qquad \forall\, s,\hat{s} \in [0,T],\ x,\hat{x} \in B_D(0).$$

Actually $C_{T,D}$ is linear in D for the space estimate.

PROOF. We first obtain the Lipschitz estimate in the space variable. This requires the discrete game approximation. Once one has this, the estimate in the time variable follows without having to use discrete approximations. The proof is very technical, and a significant portion of it is moved to Appendix A.

Define the discrete approximation as follows. Let the number of time–steps be N, and the step–size is then $\Delta \doteq T/N$. We assume $\Delta \leq 1$ freely throughout the proof. Let B_n be a sequence of IID m–dimensional random variables with $\mathrm{E}[B_n] = 0$, $\mathrm{E}[B_n B_n^T] = I_m$ and finite range. In particular, we will actually take the B_n such that the components always consist entirely of the terms -1 and 1 in order to simplify the proof. Let the B_n be adapted to the sequence of σ–algebras $\mathcal{F}_n$ where $\mathcal{F}_0$ is trivial. Let the control and disturbance at step n be u_n and w_n which are measurable with respect to $\mathcal{F}_{n-1}$. We suppress the b and R superscripts on f, ℓ, ψ, and also use the original bounds (without the $'$ superscripts) of Assumption $(A5)$ to simplify matters – where we note that these coefficients were independent of b, R. The dynamics for the discrete game are then

$$\overline{y}_n = \overline{y}_{n-1} + [f(\overline{y}_{n-1}, u_n) + \sigma(\overline{y}_{n-1})w_n]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(\overline{y}_{n-1})B_n. \tag{27}$$

Let $\mathcal{W}^N$ be the set of sequences of L_2 disturbances, $\{w_n\}_{n=1}^N$, of length N where each w_n is $\mathcal{F}_{n-1}$–measurable. Let Θ^N be the set of admissible strategies for the player with control u, where admissibility is defined analogously to that in the continuous–time case above, and we do not repeat it here. For any $\theta \in \Theta^N$, $w \in \mathcal{W}^N$ and initial state, $y_0 = x$, define the payoff

$$J^N(x,\theta,w) = \mathrm{E}\left[\sum_{i=1}^N [\ell(\overline{y}_{n-1}, u_n) - \frac{\gamma^2}{2}|w_n|^2]\Delta + \psi(\overline{y}_N)\right]$$

and value

$$V^N(x) = \inf_{\theta \in \Theta^N} \overline{V}^N(x,\theta)$$
$$\overline{V}^N(x,\theta) = \sup_{w \in \mathcal{W}^N} J^N(x,\theta,w).$$

Note that if one obtains a Lipschitz bound in x on $\overline{V}^N(x,\theta)$ which is uniform over $\theta \in \Theta^N$, then one automatically obtains the desired Lipschitz bound in x on $V^N(x)$. Consequently, we will work only with $\overline{V}^N(x,\theta)$, and again to simplify the notation, drop the θ argument – simply referring to $\overline{V}^N(x)$.

We will use dynamic programming to work back from our assumed Lipschitz continuity estimate on ψ to obtain the estimate on V^N. Let

$$\overline{V}_N^N(x_N) = \psi(x_N)$$
$$\overline{J}_{n-1}^N(x_{n-1}, v_n) = \mathrm{E}\Big[\big(\ell(x_{n-1}) - \frac{\gamma^2}{2}|v_n|^2\big)\Delta + \overline{V}_n^N(y_n)\Big]$$
$$\overline{V}_{n-1}^N(x_{n-1}) = \sup_{v_n \in \Re^m} \overline{J}_{n-1}^N(x_{n-1}, v_n) \qquad \forall\, n = 1, 2, \ldots N$$

where we again drop θ and let y_n be given by

$$y_n = x_{n-1} + [f(x_{n-1}) + \sigma(x_{n-1})w_n]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(x_{n-1})B_n$$

and $w_n = v_n$ where x_{n-1}, v_n are deterministic. Then, of course,

$$\overline{V}^N(x) = \overline{V}_0^N(x). \tag{28}$$

We now proceed to obtain the Lipschitz bound in x on near–optimal $\overline{V}_0^N(x)$. First we obtain a bound on $|v_N|$ depending on $|x_{N-1}|$. Given v_N, let

$$y_N = x_{N-1} + [f(x_{N-1}) + \sigma(x_{N-1})v_N]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(x_{N-1})B_N \tag{29}$$

$$y_N^0 = x_{N-1} + [f(x_{N-1})]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(x_{N-1})B_N. \tag{30}$$

Then by Assumption $(A4)$,

$$\begin{aligned}
\psi(y_N) - \psi(y_N^0) &\leq C(3/2 + 2|y_N| + 2|y_N^0|)|\sigma(x_{N-1})v_N\Delta| \\
&\leq C\Big[(3/2 + 4K\Delta) + 2(1 + 2K\Delta)|x_{N-1}| + 2M|v_N|\Delta + 4\sqrt{\frac{\varepsilon\Delta}{\gamma^2}}M|B_N|\Big]M|v_N|\Delta \\
&\leq C\Big[(3/2 + 4K\Delta) + 2(1 + 2K\Delta)|x_{N-1}|\Big]M|v_N|\Delta + 3CM^2|v_N|^2\Delta^2 + 2CM^2\varepsilon\Delta|B_N|^2/\gamma^2.
\end{aligned}$$

(The weaker constants 3/2 and 2 appearing in the first term above are being used here in place of 1 merely for reasons of consistency with the later estimates; this will become clear upon a reading of Appendix A.) This implies that for $\delta\Delta$–optimal v_N,

$$\begin{aligned}
-\delta\Delta &\leq \overline{J}_{N-1}^N(x_{N-1}, v_n) - \overline{J}_{N-1}^N(x_{N-1}, 0) \\
&\leq -\left(\frac{\gamma^2}{2} - 3CM^2\Delta\right)|v_N|^2\Delta + CM\Big[(3/2 + 4K\Delta) + 2(1 + 2K\Delta)|x_{N-1}|\Big]|v_N|\Delta + 2CM^2\varepsilon\Delta/\gamma^2
\end{aligned}$$

or,

$$0 \leq -\left(\frac{\gamma^2}{2} - 3CM^2\Delta\right)|v_N|^2 + CM\Big[(3/2 + 4K\Delta) + 2(1 + 2K\Delta)|x_{N-1}|\Big]|v_N| + 2CM^2\varepsilon/\gamma^2 + \delta.$$

(One can use $a^2 + b^2 \geq 2ab$ to eliminate the cross–terms in $|v_N|, |x_{N-1}|$, leading to an additional term in the coefficient in the quadratic term, thus leading to a stricter requirement below on γ, but that is avoidable in this proof.) Consequently, with Δ sufficiently small so that the coefficient on the quadratic term is negative, one obtains

$$|v_N| \leq \frac{C}{\gamma}\left[C_1 + C_2|x_{N-1}|\right] \tag{31}$$

for proper choice of C_1, C_2 which may be chosen independently of Δ for sufficiently small Δ.

Next we propagate the Lipschitz property back one step. This is a rather technical step, and consequently we have moved the estimates to Appendix A. The resulting Lipschitz bound is

$$\begin{aligned}
\left|\overline{V}_{N-1}^N(x_{N-1}) - \overline{V}_{N-1}^N(\hat{x}_{N-1})\right| &\leq b(|\hat{x}_{N-1}|)\left[1 + c_1\Delta + c_2\frac{K_4}{\gamma}\Delta + c_3\frac{K_4^2}{\gamma^2}\Delta^2\right]|x_{N-1} - \hat{x}_{N-1}| \\
&\quad + \overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4}{\gamma}\Delta\right]|x_{N-1} - \hat{x}_{N-1}|
\end{aligned} \tag{32}$$

provided $\Delta \le \min\{1, d_1\}$ where the constants and restrictions are given in the appendix. Also, the function b is given in the appendix as

$$b(x) = \begin{cases} C(\frac{3}{2} + 2|x|^2) & \text{if } |x| \le 1 \\ C(\frac{3}{2} + 2|x|) & \text{if } |x| > 1. \end{cases} \tag{33}$$

This bound required a one-sided bound on ψ_{xx}. In fact, semi-convexity of ψ is sufficient to obtain this step. Consequently, the next step is to propagate the semi-convexity of ψ backward one step to obtain a semi-convexity bound on $\overline{V}^N_{N-1}$. This step also appears in Appendix A.

Next one works backwards with minor variants of (31), (32) and the semi-convexity bound. This appears in Appendix A, and leads of course to a Lipschitz bound on $\overline{V}^N_0(x)$. One must then let $N \to \infty$, and show that this bound does not blow up. The corresponding sequences are given in the appendix, and the limit bound is denoted simply as $M_\infty b(|x|)$. Since there are well–known results demonstrating the convergence of such discrete–time games to their continuous–time counterparts (see, for instance, [13],[23]), one obtains a Lipschitz bound on the limit continuous–time game. In other words (reinstating the superscripts), one has obtained the Lipschitz bound

$$|\underline{V}^{\varepsilon,b,R,m}(s,x) - \underline{V}^{\varepsilon,b,R,m}(s,\hat{x})| \le M_\infty b(|x|)|x-\hat{x}| \qquad \forall\, s \in [0,T],\ |\hat{x}| \le |x| < \infty.$$

Once one obtains the above Lipschitz bound in the space variable, one may obtain the bound in the time variable without resorting to the discrete–time setting. The details are relatively long and technical, but rather standard. Consequently, they will only be sketched. Perhaps it should be noted that they rely on the bounds on the second derivatives of ℓ and ψ of course. As a first step, let us note a result which is a slightly more specific bound on near–optimal disturbances than that of Lemma 3.2, and which is particularly oriented toward small time–horizons. It also allows for random initial state. It is stated as a lemma below, and the proof is sketched. Following the sketched proof of the lemma, the remaining steps in the proof of the Lipschitz bound in the time variable are presented.

Lemma 3.5 *Consider the above stochastic game, but now let the terminal time be denoted by τ so as to differentiate this from the main problem of the section (with terminal time denoted by T), let $\tau \le 1$, and let $s \in [0,\tau]$. Also, now let the initial state, $y^\varepsilon_s = x$ be random and independent of B_t for $t \ge s$. Then there exists $\widetilde{\gamma} < \infty$ such that for all $\gamma > \widetilde{\gamma}$; for $\delta \le \tau - s$, any δ–optimal $w_\delta \in \mathcal{W}_s$ with respect to any $\theta \in \Theta_s$ satisfies*

$$\|w_\delta\|_{L_2(\Omega\times[s,\tau])} \le \overline{M}' \sqrt{1 + \mathrm{E}\big[|x|^2\big]}\sqrt{\tau - s} \tag{34}$$

for properly chosen $\overline{M}'$ (independent of x, s, τ).

PROOF. The proof relies again on comparison of the near–optimal disturbance against the use of the disturbance process $w \equiv 0$. To simplify notation, we will work directly with f, ℓ, ψ rather than their cut-off's (f^b, ℓ^R, ψ^R); the structure of the proof is unchanged. Let $\theta \in \Theta_s$. Let y^ε satisfy (15) with disturbance w^δ which is δ–optimal with respect to θ (for the problem over $[s,\tau]$). Using standard techniques, one shows there exists $\widehat{C}_1 < \infty$ such that

$$\mathrm{E}\left[|y^\varepsilon_t - x|^2\right] \le \widehat{C}_1 \left[\Big(1 + \mathrm{E}\big[|x|^2\big]\Big)(t-s) + \int_s^t \mathrm{E}|w^\delta_r|^2\, dr\right] \qquad \forall\, t \in [s,\tau]. \tag{35}$$

Similarly,

$$\mathrm{E}|y^\varepsilon_t - x| \le \widehat{C}_2 \left[\Big(1 + \mathrm{E}|x|\Big)(t-s) + \int_s^t \mathrm{E}|w^\delta_r|\, dr\right] \qquad \forall\, t \in [s,\tau]. \tag{36}$$

Denote the payoff (with $u = \theta[w^\delta]$) as

$$
\begin{aligned}
J^\varepsilon_{g,\tau,\delta}(s,x;u,w^\delta) &\doteq \mathrm{E}\left[\int_s^\tau \ell(y^\varepsilon_t,u_t) - \frac{\gamma^2}{2}|w^\delta_t|^2\,dt + \psi(y^\varepsilon_\tau)\right] \\
&\le \mathrm{E}[\psi(x)] + \mathrm{E}[\ell(x)](\tau-s) + \mathrm{E}\Bigg\{\int_s^\tau \left[|\ell_x(x)||y^\varepsilon_t - x| + \frac{1}{2}|\ell_{xx}(\xi'_t)||y^\varepsilon_t - x|^2 - \frac{\gamma^2}{2}|w^\delta_t|^2\right] dt \\
&\qquad + |\psi_x(x)||y^\varepsilon_\tau - x| + \frac{1}{2}|\psi_{xx}(\xi'_\tau)||y^\varepsilon_\tau - x|^2\Bigg\}
\end{aligned}
$$

where ξ'_t is on the line from x to y^ε_t for each t. Then, using (35), (36) $(A3)$ and $(A4)$ on this yields

$$
\begin{aligned}
J^\varepsilon_{g,\tau,\delta}(s,x;u,w^\delta) &\le \mathrm{E}[\psi(x)] + \mathrm{E}[\ell(x)](\tau-s) + \widehat{C}_3\left[(1+\mathrm{E}\big[|x|^2\big])(\tau-s) + \int_s^\tau \mathrm{E}\big[(1+|x|)|w^\delta_t|\big]\,dt\right] \\
&\quad + \widehat{C}_4\left[(1+\mathrm{E}\big[|x|^2\big])(\tau-s) + \int_s^\tau \mathrm{E}|w^\delta_t|^2\,dt\right] - \frac{\gamma^2}{2}\int_s^\tau \mathrm{E}|w^\delta_t|^2\,dt \qquad (37)
\end{aligned}
$$

for proper choice of $\widehat{C}_3, \widehat{C}_4$.

On the other hand, let y^0 be the solution of (15) with the same θ, but with $w_0 \equiv 0$. Let the corresponding payoff be denoted by $J^\varepsilon_{g,\tau,0}(s,x;u_0,w_0)$. By similar techniques, one has

$$
\mathrm{E}|y^0_t - x|^2 \le \widehat{C}_5\big(1+\mathrm{E}\big[|x|^2\big]\big)(t-s) \quad \text{and} \quad \mathrm{E}|y^0_t - x| \le \widehat{C}_6\big(1+\mathrm{E}|x|\big)(t-s)
$$

for proper choice of $\widehat{C}_5, \widehat{C}_6$. This leads to

$$
J^\varepsilon_{g,\tau,0}(s,x;u_0,w_0) \ge \mathrm{E}[\psi(x)] + \mathrm{E}[\ell(x)](\tau-s) - \widehat{C}_7\big(1+\mathrm{E}\big[|x|^2\big]\big)(\tau-s) \qquad (38)
$$

for proper choice of $\widehat{C}_7$.

Now, using the δ–optimality of w^δ, and combining (37) and (38) yields

$$
\begin{aligned}
0 &\le \delta + \widehat{C}_8\big(1+\mathrm{E}\big[|x|^2\big]\big)(\tau-s) + \widehat{C}_3\int_s^\tau \mathrm{E}\big[(1+|x|)|w^\delta_t|\big]\,dt - \left(\frac{\gamma^2}{2} - \frac{\widetilde{\gamma}^2}{2}\right)\int_s^\tau \mathrm{E}|w^\delta_t|^2\,dt \\
&\le \delta + \widehat{C}_8\big(1+\mathrm{E}\big[|x|^2\big]\big)(\tau-s) + \widehat{C}_3\sqrt{2(1+\mathrm{E}|x|^2)}\sqrt{\tau-s}\left[\int_s^\tau \mathrm{E}|w^\delta_t|^2\,dt\right]^{\frac{1}{2}} - \left(\frac{\gamma^2}{2} - \frac{\widetilde{\gamma}^2}{2}\right)\int_s^\tau \mathrm{E}|w^\delta_t|^2\,dt
\end{aligned}
$$

where $\widehat{C}_8 \doteq \widehat{C}_3 + \widehat{C}_4 + \widehat{C}_7$ and $\frac{\widetilde{\gamma}^2}{2} \doteq \widehat{C}_4$. Letting $\delta \le \tau - s$, this yields the desired result by solving this last quadratic equation. □

Now we return to the sketch of the proof of the Lipschitz condition in the time variable. Again, to simplify matters, we work directly with f, ℓ, ψ rather than their cut-offs (f^b, ℓ^R, ψ^R). Consider $\underline{V}^\varepsilon(s,x)$ and $\overline{V}^\varepsilon(\hat{s},x)$ (where we are suppressing the cut-off superscripts) where $0 \le s \le \hat{s} \le T$ and $\tau \doteq \hat{s} - s$. Note that under the given conditions, one has strong solutions of (15). Let the probability triple under which $\underline{V}^\varepsilon(s,x)$ is computed be $(\Omega, P, \mathcal{F}_\cdot)$. As in the proof of Lemma 3.1, for the computation of $\underline{V}^\varepsilon(\hat{s},x)$, one may use instead $(\Omega, P, \widehat{\mathcal{F}}_\cdot)$ where $\widehat{\mathcal{F}}_t \doteq \mathcal{F}_{t-\tau}$ for all $t \ge \hat{s} = s + \tau$; the corresponding Brownian motion is $\widehat{B}_t = B_{t-\tau}$ for all $t \ge \hat{s}$.

Let $\mathcal{U}_{[r_1,r_2]}$ $(\widehat{\mathcal{U}}_{[r_1,r_2]})$ be the set of $\mathcal{F}_t$– $(\widehat{\mathcal{F}}_t$–) progressively measurable controls over time interval $[r_1, r_2]$. Let $\widetilde{\theta}$ be $\frac{\varepsilon}{2}$–optimal for problem $\underline{V}^\varepsilon(\hat{s},x)$, and let $\widehat{\theta}$ be any extension of $\widetilde{\theta}$ to time $T+\tau$. Noting that $\widehat{\theta} : L_2[s+\tau, T+\tau] \to \widehat{\mathcal{U}}_{[s+\tau,T+\tau]}$, let $\overline{\theta} : L_2[s,T] \to \mathcal{U}_{[s,T]}$ be given by $\overline{\theta}_t[w] = \widehat{\theta}_{t+\tau}[\widetilde{w}]$ where $\widetilde{w}_{t+\tau} = w_t$. Then let w be $\frac{\varepsilon}{2}$–optimal for $\underline{V}^\varepsilon(s,x)$ corresponding to $\overline{\theta}$. Let $\widehat{w}$ be given by $\widehat{w}_t = w_{t-\tau}$ for all $t \geq \hat{s}$. Let y^ε be the solution to (15) with the above w and $\overline{u} \doteq \overline{\theta}[w]$. Let $\widehat{y}^\varepsilon$ be the solution to (15) with new initial condition $\widehat{y}^\varepsilon_{\hat{s}} = x$ and the above $\widehat{w}$ and $\widehat{u} \doteq \widehat{\theta}[\widehat{w}]$. Then

$$\widehat{y}^\varepsilon_t = y^\varepsilon_{t-\tau} \qquad \forall\, t \in [\hat{s}, T].$$

Note that

$$\underline{V}^\varepsilon(s,x) - \underline{V}^\varepsilon(\hat{s},x) \;\leq J^\varepsilon_g(s,x,u,w) - J^\varepsilon_g(\hat{s},x\widehat{u},\widehat{w}) + \varepsilon,$$

and by the time shift

$$= \mathrm{E}\left[\int_{T-\tau}^{T} \ell(y^\varepsilon_t, \overline{u}_t) - \frac{\gamma^2}{2}|w_t|^2\,dt + \psi(y^\varepsilon_T) - \psi(\widehat{y}^\varepsilon_{T-\tau})\right] + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, one must show that this is bounded above by $\overline{C}_{T,D}\tau$ for any $|x| \leq D < \infty$ (for proper choice of $\overline{C}_{T,D}$). The bound from below is similar, and so is not discussed here. Also, since the bound on the integral term is somewhat simpler than the bound on the terminal cost difference, we consider only a bound on $\mathrm{E}\left[\psi(y^\varepsilon_T) - \psi(\widehat{y}^\varepsilon_{T-\tau})\right]$ of the desired form here. Note that by $(A4)$,

$$\mathrm{E}\left[\psi(y^\varepsilon_T)\right] - \mathrm{E}\left[\psi(\widehat{y}^\varepsilon_{T-\tau})\right] \leq \mathrm{E}\left[\psi_x(y^\varepsilon_{T-\tau})[y^\varepsilon_T - y^\varepsilon_{T-\tau}]\right] + \frac{1}{2}\overline{C}\,\mathrm{E}\left[|y^\varepsilon_T - y^\varepsilon_{T-\tau}|^2\right]. \tag{39}$$

Also, from Lemma 3.5 and the choice of w,

$$\|w\|_{L_2(\Omega\times[T-\tau,T])} \leq \overline{M}'\sqrt{1 + \mathrm{E}\left[|y^\varepsilon_{T-\tau}|^2\right]}\sqrt{\tau} \tag{40}$$

for ε sufficiently small. Using (39), techniques similar to those in the proof of Lemma 3.5, and employing (40) yields the desired bound. This completes the sketch of the proof of the Lipschitz bound in the time variable. □

Lemma 3.6 *Given any sequences $\{b_n\}, \{m_n\}$ such that $b_n, m_n \to \infty$, there exists a subsequence which we also subscript by n (to reduce notational complexity) such that*

$$\underline{V}^{\varepsilon,b_n,R,m_n} \to \widetilde{V}^{\varepsilon,R}$$

uniformly on compact sets where $\widetilde{V}^{\varepsilon,R} \in C^{1,2}$ (and the derivatives converge uniformly on compact sets as well). Further, $\widetilde{V}^{\varepsilon,R}$ is a solution to

$$\begin{aligned} &0 = V_s + \frac{\varepsilon}{2\gamma^2}\sum_{i,j=1}^{n} a_{i,j}(x)V_{x_ix_j} + \min_{u\in U}\max_{w\in\Re^m}\left\{[f(x,u) + \sigma(x)w]\cdot\nabla V + \ell^R(x,u) - \frac{\gamma^2}{2}|w|^2\right\} \\ &V(T,x) = \psi^R(x). \end{aligned} \tag{41}$$

PROOF. By Lemmas 3.1, 3.2 and 3.4, $\underline{V}^{\varepsilon,b,R,m}, \underline{V}^{\varepsilon,b_n,R,m_n}_s, \nabla\underline{V}^{\varepsilon,b_n,R,m_n}$ are all uniformly bounded on compact sets. Combining this with (25) implies that $\sum_{i,j=1}^n a_{i,j}\underline{V}^{\varepsilon,b_n,R,m_n}_{x_i,x_j}$ are also bounded uniformly on compact sets. Then following the method in [20], Appendix E (see also [42]), one obtains the result. □

Recall process (1) of Section 2. Define the modified risk–sensitive cost and value

$$J^{\varepsilon,R}(s,x,u_\cdot) = \mathrm{E}\exp\left\{\frac{1}{\varepsilon}\left[\int_s^T \ell^R(y^\varepsilon_t, u_t)\,dt + \psi^R(y^\varepsilon_T)\right]\right\} \tag{42}$$

and

$$V^{\varepsilon,R}(s,x) = \inf_{u\in\mathcal{U}_s} \varepsilon \log J^{\varepsilon,R}(s,x,u.). \tag{43}$$

Theorem 3.7 *Let $0 \le s \le T$ and $x \in \Re^n$. For any $u \in \mathcal{U}_s$, one has*

$$\varepsilon \log J^{\varepsilon,R}(s,x,u) \ge \widetilde{V}^{\varepsilon,R}(s,x),$$

while with optimal $\widetilde{u}^R$ given by (6) with $\widetilde{V}^{\varepsilon,R}, \ell^R$ replacing $\widetilde{V}^\varepsilon, \ell$, one has

$$\varepsilon \log J^{\varepsilon,R}(s,x,\widetilde{u}^R) = \widetilde{V}^{\varepsilon,R}(s,x).$$

Proof. By $\widetilde{V}^{\varepsilon,R} \in C^{1,2}$, assumptions $(A3.iv)$, $(A4.iv)$ and (42), one easily shows there exists $C_R < \infty$ such that

$$|\nabla \widetilde{V}^{\varepsilon,R}(s,x)| \le C_R \qquad \forall\, s \in [0,T],\ \forall\, x \in \Re^n.$$

In this case, (7) holds for any $u \in \mathcal{U}_s$. Consequently, the proof of Lemma 2.2 holds for $\widetilde{V}^{\varepsilon,R}$ for any control $u \in \mathcal{U}_s$, and so

$$\varepsilon \log J^{\varepsilon,R}(s,x,u) \ge \widetilde{V}^{\varepsilon,R}(s,x)$$

for any $u \in \mathcal{U}_s$. Lemma 2.1 also holds with $\widetilde{V}^{\varepsilon,R}$, $J^{\varepsilon,R}$ replacing $\widetilde{V}^\varepsilon$, J^ε. □

Corollary 3.8

$$\underline{V}^{\varepsilon,b,R,m} \to \widetilde{V}^{\varepsilon,R}$$

uniformly on compact sets (not just sequentially).

Proof. Combine the uniqueness implied by Theorem 3.7 with the subsequential convergence of Lemma (3.6). □

Corollary 3.9 *$\widetilde{V}^{\varepsilon,R}$ is monotonically increasing as a function of R, and is bounded above by*

$$\varepsilon \log J^\varepsilon(s,x,\widetilde{u}) \le \widetilde{V}^\varepsilon(s,x).$$

Proof. The first assertion follows by the definitions of ℓ^R, ψ^R. For the second assertion, note that by Theorem 3.7

$$\begin{aligned}\widetilde{V}^{\varepsilon,R}(s,x) &\le \varepsilon \log J^{\varepsilon,R}(s,x,\widetilde{u}) \\ &\le \varepsilon \log J^\varepsilon(s,x,\widetilde{u})\end{aligned}$$

which by Lemma 2.1

$$\le \widetilde{V}^\varepsilon(s,x).$$

□

We now obtain the promised existence result for (4).

Theorem 3.10 *$\widetilde{V}^{\varepsilon,R}(s,x) \uparrow \widetilde{V}^\varepsilon(s,x) \in C^{1,2}$ as $R \to \infty$ where $\widetilde{V}^\varepsilon$ is a solution to (4).*

PROOF. The result follows exactly as in Lemma 3.6 with the added monotonicity from Corollary 3.9. □

As an added bonus, we obtain the promised improvement over Lemma 2.3.

Theorem 3.11 *There exists a solution $\widetilde{V}^\varepsilon \in C^{1,2}$ of (4). Further, for any $0 \le s \le T$, $x \in \Re^n$ and $u \in \mathcal{U}_s$, one has*

$$\varepsilon \log J^\varepsilon(s,x,u) \ge \widetilde{V}^\varepsilon(s,x),$$

while with optimal $\widetilde{u}$ given by (6) one has

$$\varepsilon \log J^\varepsilon(s,x,\widetilde{u}) = \widetilde{V}^\varepsilon(s,x).$$

PROOF. Note that for any $u \in \mathcal{U}_s$,

$$\varepsilon \log J^\varepsilon(s,x,u) \ \ge\ \varepsilon \log J^{\varepsilon,R}(s,x,u)$$

which by Theorem 3.7

$$\ge\ \widetilde{V}^{\varepsilon,R}(s,x).$$

Combining this with Theorem 3.10 yields the first assertion. The second assertion follows by combining the first assertion with Lemma 2.1. □

4 Uniqueness and Risk Sensitive limit results

In this section we consider the following terminal value problem

$$\begin{cases} -W_t(t,x) + H(x, D_xW(t,x))) = 0 & \text{in } (0,T)\times\Re^n, \\ W(T,x) = \psi(x) & \text{in } \Re^n \end{cases} \tag{44}$$

where the Hamiltonian H is

$$\begin{aligned} H(x,p) \ &:= \ \max_{u\in U}\min_{\omega\in\Re^m}\{-(f(x,u)+\sigma(x)\omega)\cdot p - \ell(x,u) + \frac{\gamma^2}{2}|\omega|^2\} \\ &= \ \min_{\omega\in\Re^m}\{\frac{\gamma^2}{2}|\omega|^2 - \sigma(x)\omega\cdot p\} + \max_{u\in U}\{-f(x,u)\cdot p - \ell(x,u)\}. \end{aligned} \tag{45}$$

This is the DPE corresponding to a Robust/H_∞ control problem. This problem can be considered a *differential game* with the following cost functional

$$J(s,x,u,\omega) := \int_s^T \ell(y_x(t),u(t)) - \frac{\gamma^2}{2}|\omega(t)|^2 dt + \psi(y_x(T)), \tag{46}$$

where $u(\cdot) \in \mathcal{U} :=$ {measurable functions $[0,T] \to U$} is the control of the minimizing player, $\omega(\cdot) \in \mathcal{B} := L^2([0,T],\Re^m)$ is the control of the maximizing player and $y_x(\cdot)$ is the unique solution of the following dynamical system

$$\begin{cases} y'(t) = f(y(t),u(t)) + \sigma(y(t))\omega(t), \\ y(s) = x. \end{cases} \tag{S}$$

Note that we switch notation for the time argument here from y_t to $y(t)$ to emphasize that the y paths are now (deterministic) solutions of ODEs rather than (stochastic) solutions of SDEs.

We define the set of nonanticipating strategies for the minimizing player to be:

$$\begin{aligned}\Theta \;:=\; \{\theta\colon \mathcal{B} \to \mathcal{U} :\ & \text{given any } \tau \in [0,T],\ \omega(t) = \overline{\omega}(t) \\ & \text{for all } t \in [0,\tau] \text{ implies } \theta[\omega](t) = \theta[\overline{\omega}](t) \text{ for all } t \in [0,\tau]\}\,,\end{aligned}$$

and the set of nonanticipating strategies of the maximizing player to be:

$$\begin{aligned}\Lambda \;:=\; \{\lambda\colon \mathcal{U} \to \mathcal{B} :\ & \text{given any } \tau \in [0,T],\ u(t) = \overline{u}(t) \\ & \text{for all } t \in [0,\tau] \text{ implies } \lambda[u](t) = \lambda[\overline{u}](t) \text{ for all } t \in [0,\tau]\}\,.\end{aligned}$$

The *lower value* and the *upper value* of the game are given by (see, for instance, [10])

$$V(s,x) := \inf_{\theta\in\Theta} \sup_{\omega\in\mathcal{B}} J(s,x,\theta[\omega],\omega); \tag{47}$$

$$W(s,x) := \sup_{\lambda\in\Lambda} \inf_{u\in\mathcal{U}} J(s,x,u,\lambda[u]). \tag{48}$$

This section will be concerned with the *lower value*, analogous results will hold for the *upper value* as well.

In the following theorem we prove a uniqueness result for (44) in the set of nonnegative continuous functions with locally bounded superdifferential in the x variable and growing at most quadratically. More precisely, given $U\colon [0,T] \times \Re^n \to \Re$, we define

$$D_x^+ U(t,x) = \left\{ p \in \Re^n :\ \limsup_{|x-y|\to 0} \frac{U(t,y) - U(t,x) - (y-x)\cdot p}{|x-y|} \le 0 \right\},$$

$$||U||_R := \sup\{|U(t,x)| + |p|,\ (t,x) \in [0,T] \times \overline{B}_R(0)\,,\, p \in D_x^+ U(t,x)\},$$

and

$$\begin{aligned}\mathcal{K} \;:=\; \{U \in C([0,T]\times\Re^n) :\ & U(t,x) \ge 0,\ ||U||_R < +\infty, \qquad \forall\, R > 0, \\ \text{and} \ & \sup_{(t,x)\in[0,T]\times\Re^n} \frac{|U(t,x)|}{1+|x|^2} < +\infty\}.\end{aligned}$$

For the uniqueness result we use, besides some hypotheses stated in Section 2, the following less restrictive assumptions

$(H1)$ $f \in C(\Re^n \times U, \Re^n)$, $\sigma \in C(\Re^n, \mathcal{L}(\Re^m, \Re^n))$, $\ell \in C^{(}\Re^n \times U, \Re)$, $\psi \in C(\Re^n, \Re)$,

$(H2)$ There exists $M > 0$ such that

$$|\sigma(x)| \le M(1+|x|), \quad \forall x \in \Re; \tag{49}$$

$(H3)$ for any $R > 0$ there exists $L_R > 0$ such that

$$\begin{aligned}&|f(x,u) - f(y,u)| \le L_R|x-y|, \quad \forall x,y \in \overline{B}_R(0),\ \forall u \in U, \\ &|\sigma(x) - \sigma(y)| \le L_R|x-y|, \quad \forall x,y \in \overline{B}_R(0)\end{aligned}$$

Theorem 4.1 *Assume* $(A0)$, $(A1.iii)$, $(A3.ii)-(A3.iv)$, $(A4.ii)-(A4.iv)$ *and* $(H1)-(H3)$. *If* W_1, $W_2 \in \mathcal{K}$ *are viscosity solutions of (44), then* $W_1 = W_2$ *in* $[0,T]\times\Re^n$.

PROOF. We follow some arguments used in the proof of Theorem 3.1 in [2] for convex Hamiltonians. Let $W_1, W_2 \in \mathcal{K}$ be viscosity solutions of (44). We first note that, if (x,p) belongs to a compact $K \subseteq \Re^{2N}$, then there exists $\widetilde{R} > 0$, depending on K, such that

$$H(x,p) = H_{\widetilde{R}}(x,p) := \min_{|\omega| \leq \widetilde{R}} \{\frac{\gamma^2}{2}|\omega|^2 - \sigma(x)\omega \cdot p\} + \max_{u \in U}\{-f(x,u) \cdot p - \ell(x,u)\}. \tag{50}$$

In fact it can be easily verified that

$$\lim_{|\omega| \to \infty} \inf_{(x,p) \in K} \{\frac{\gamma^2}{2}|\omega|^2 - \sigma(x)\omega \cdot p\} = +\infty.$$

Thus, since $\gamma^2|0|^2 - \sigma(x)0 \cdot p = 0$, the min in (45) can be computed in a compact subset of $\Re^m$. Now fix $r > 0$ and choose $R > r$. First we suppose that

$$T \leq \delta(\gamma, r, R) := \min\left(1, \frac{R-r}{K(1+R)}, \frac{\gamma^2(R-r)^2}{2M^2(1+R)^2\overline{C}(1+R^2) + 2\gamma^2 K(R-r)(1+R)}\right), \tag{51}$$

where $\overline{C} = \max\left(C, \sup_{(t,x) \in [0,T] \times \Re^n} \frac{|W_1(t,x)|}{1+|x|^2}, \sup_{(t,x) \in [0,T] \times \Re^n} \frac{|W_2(t,x)|}{1+|x|^2}\right)$, C is the growth constant of ℓ and ψ.

Since W_1, $W_2 \in \mathcal{K}$ and (50) holds, there exists $\widetilde{R} > 0$ such that

$$H(x,p) = H_{\widetilde{R}}(x,p), \qquad \forall p \in D_x^+ W_1(t,x) \cup D_x^+ W_2(t,x), \forall (t,x) \in [0,T] \times \overline{B}_R(0).$$

Thus W_i $(i = 1,2)$ is a viscosity solution also of

$$\begin{cases} -W_t(t,x) + H_{\widetilde{R}}(x, D_x W(t,x)) = 0 & \text{in } (0,T) \times B_R(0), \\ W(x,T) = \psi(x) & \text{in } B_R(0). \end{cases} \tag{52}$$

Under the current assumptions the continuous viscosity solutions W of (52) satisfy the so-called *optimality principle* (see e.g. Propositions 2.1 and 2.2 in [33]), namely for all $s \in [0,T]$ and $0 \leq \rho \leq T - s$, the following estimate holds

$$W(s,x) = \inf_{\theta \in \Theta} \sup_{\substack{\omega \in \mathcal{B} \\ |\omega| \leq \widetilde{R}}} \left\{\mathcal{I}(s, (s+\rho) \wedge t_x^R(\theta[\omega], \omega)) + W((s+\rho) \wedge t_x^R(\theta[\omega], \omega), y_x((s+\rho) \wedge t_x^R(\theta[\omega], \omega)))\right\}, \tag{53}$$

where for all $s,t \in [0,T]$, $\mathcal{I}(s,t) := \int_s^t \ell(y(\tau), \theta[\omega](\tau)) - \frac{\gamma^2}{2}|\omega(\tau)|^2 d\tau$, $y_x(t) = y_x(t, \theta[\omega], \omega)$ is the unique solution of (S) corresponding to $\theta[\omega]$, ω (we drop the dependence on $\theta[\omega]$, ω for simplicity of notations) and $t_x^R(\theta[\omega], \omega)$ is the first exit time from $B_R(0)$, i.e.

$$t_x^R(\theta[\omega], \omega) := \inf\{t \geq s : |y_x(t)| \geq R\}.$$

In particular (53) is satisfied by W_i $(i = 1,2)$. We observe that, since $W_i \geq 0$, for any strategy θ the sup in (53), corresponding to $\rho = T - s$, may be confined to the controls ω such that $\mathcal{I}(s, T \wedge t_x^R(\theta[\omega], \omega)) + W_i((T \wedge t_x^R(\theta[\omega], \omega)), y(T \wedge t_x^R(\theta[\omega], \omega))) \geq 0$. For any strategy θ we define the set

$$\mathcal{A}_\theta := \{\omega \in \mathcal{B} : \ \mathcal{I}(s, T \wedge t_x^R(\theta[\omega], \omega)) + W_i((T \wedge t_x^R(\theta[\omega], \omega)), y_x(T \wedge t_x^R(\theta[\omega], \omega))) \geq 0\}.$$

We observe that $\mathcal{A}_\theta \neq \emptyset$, since it contains the control $\omega \equiv 0$. We claim that if $t_x^R(\theta[\omega], \omega)) < T$ then $\omega \notin \mathcal{A}_\theta$. In fact suppose that $y_x(\bar{t}) \in \partial B_R(0)$, for some $\bar{t} \in [s,T)$, and $y_x(t) \in B_R(0)$, $\forall t \in [s, \bar{t})$. Then we have the following estimate

$$\begin{aligned} R - |x| = |y_x(\bar{t})| - |x| \ &\leq \int_s^{\bar{t}} K(1 + |y_x(\tau)|) d\tau + \int_s^{\bar{t}} M(1 + |y_x(\tau)|)\omega(\tau)| d\tau \\ &\leq K(1+R)(\bar{t} - s) + M(1+R)||\omega||_{L^2(s,\bar{t})}(\bar{t} - s)^{1/2} \end{aligned}$$

Thus $||\omega||_{L^2(s,\bar{t})} \geq \chi[r,R](\bar{t})$, where

$$\chi[r,R](\bar{t}) := \frac{R - r - K(1+R)(\bar{t}-s)}{M(1+R)(\bar{t}-s)^{1/2}}.$$

We observe that, by the assumption (51), $\chi[r,R](t)$ is positive for all $t \in [s,T]$. Hence we have

$$\begin{aligned}
\mathcal{I}(s,\bar{t}) + W_i(\bar{t}, y_x(\bar{t})) &\leq \int_s^{\bar{t}} C(1+|y_x(t)|^2) - \frac{\gamma^2}{2}|\omega(t)|^2 \, dt + \widetilde{C}(1+|y_x(\bar{t})|^2) \\
&\leq C(1+R^2)(\bar{t}-s) - \frac{\gamma^2}{2}|\chi[r,R](\bar{t})|^2 + \widetilde{C}(1+R^2) \\
&\leq \overline{C}(1+R^2) + \gamma^2 \frac{K(R-r)(1+R)}{M^2(1+R)^2} - \frac{\gamma^2}{2}\frac{(R-r)^2}{M^2(1+R)^2(\bar{t}-s)} - \frac{\gamma^2}{2}\frac{K^2(\bar{t}-s)}{M^2} \\
&\leq \overline{C}(1+R^2) + \gamma^2 \frac{K(R-r)}{M^2(1+R)} - \frac{\gamma^2}{2}\frac{(R-r)^2}{M^2(1+R)^2(T-s)}
\end{aligned}$$

where $\widetilde{C} := \left(\sup_{(t,x)\in[0,T]\times\Re^n} \frac{|W_1(t,x)|}{1+|x|^2}, \sup_{(t,x)\in[0,T]\times\Re^n} \frac{|W_2(t,x)|}{1+|x|^2} \right)$.

By the condition (51) we have $\mathcal{I}(s,\bar{t}) + W(\bar{t}, y_x(\bar{t})) < 0$ and this proves the claim. Therefore for all $(s,x) \in [0,T] \times \overline{B}_r(0)$ each W_i $(i = 1,2)$ satisfies

$$W_i(s,x) = \inf_{\theta\in\Theta} \sup_{\substack{\omega\in\mathcal{A}_\theta \\ |\omega|\leq\widetilde{R}}} \left\{ \mathcal{I}(s,T) + \psi(y_x(T)) \right\}, \tag{54}$$

and we can conclude that $W_1 = W_2$ in $[0,T] \times \overline{B}_r(0)$. In the case that $T > \delta(\gamma,r,R)$ we can divide the interval $[0,T]$ in subintervals whose length is less than $\delta(\gamma,r,R)$. Let $0 = t_0, t_1 \ldots t_n = T$ be the points of such division and for any $k = 1 \ldots n$ let us consider the following Cauchy problem

$$\begin{cases} -W_t(t,x) + H_{\widetilde{R}}(x, D_x W(t,x)) = 0 & \text{in } (t_{k-1}, t_k) \times B_R(0), \\ W(x,t_k) = \psi_k(x) & \text{in } B_R(0). \end{cases} \tag{55}$$

where the terminal value $\psi_k(x)$ can concide either with $W_1(t_k,x)$ or with $W_2(t_k,x)$. We start with $k = n$. Since $|t_n - t_{n-1}| < \delta(\gamma,r,R)$ we can argue as above and obtain that the $W_i's$ concide in $[t_{n-1},t_n] \times \overline{B}_r(0)$. Then by proceeding backward in the time variable we obtain that for all $k \in \{0,\ldots,n\}$, $W_1 = W_2$ in $[t_{k-1},t_k] \times \overline{B}_r(0)$ and this completes the proof. □

In [40] it is proved that, under the hypotheses of Theorem 4.1 (with $(A2.ii)$ instead of $(H2)$), for sufficiently large γ the *lower value* V is a locally Lipschitz continuous function growing at most quadratically in the state variable (thus $V \in \mathcal{K}$) and it is a viscosity solution of (44). Thus we have the following corollary.

Corollary 4.2 *Assume the hypotheses of Theorem 4.1, with $(H2)$ replaced by $(A2.ii)$. Then for γ large enough the value V is the unique viscosity solution of (44) in $\mathcal{K}$.*

As a consequence of the uniqueness Theorem 4.1 we get the convergence of the value function $\widetilde{V}^\varepsilon$ of the risk-sensitive problem to the value V of the Robust/H_∞ control problem as $\varepsilon \to 0$. The proof of this result is standard (see e.g [1]), and it follows directly from the stability of the viscosity solutions with respect to the locally uniform convergence of the DPE for the risk-sensitive problem to the DPE for the corresponding Robust control problem and the local uniform boundedness and equicontinuity estimates obtained in Section 3.

Theorem 4.3 *Assume* $(A0)-(A5)$. *Then, for* γ *large enough,* $\widetilde{V}^\varepsilon \to V$ *uniformly on the compact subsets of* $[0,T] \times \Re^n$.

PROOF. From Lemmas 3.2, 3.4, 3.6 and Theorem 3.10 it follows that, for γ large enough and for any $D > 0$, there exist $C_{T,D}$, $M_{T,D} > 0$ (independent of ε) such that

$$|\widetilde{V}^\varepsilon(s_1, x) - \widetilde{V}^\varepsilon(s_2, z)| \leq C_{T,D}[|x - z| + |s_1 - s_2|] \qquad \forall\, s_1, s_2 \in [0,T],\, x, z \in \overline{B}_D(0) \tag{56}$$

and

$$|\widetilde{V}^\varepsilon(t,x)| \leq M_{T,D}, \qquad \forall\, t \in [0,T],\ \forall x \in \overline{B}_D(0). \tag{57}$$

By using the Ascoli-Arzelà Theorem and a standard *diagonal procedure* (see e.g. [1]), we get the existence of a subsequence of $\widetilde{V}^\varepsilon$ converging uniformly on the compact subsets of $[0,T] \times \Re^n$ to a function $U \in \mathcal{K}$. From the stability of viscosity solutions with respect to the locally uniform convergence of the DPE (4) for the risk-sensitive problem to the DPE (44) for the Robust control problem as $\varepsilon \to 0$, it follows that U is a viscosity solution of (44) and the uniqueness Theorem 4.1 implies that $U = V$. Thus we can conclude that the entire sequence $\widetilde{V}^\varepsilon$ converges to V as $\varepsilon \to 0$, uniformly on compact subsets of $[0,T] \times \Re^n$. □

5 Thanks

The authors wish to thank M. Bardi, W.H. Fleming, S.-J. Sheu, the Referee and the Associate Editor for helpful discussions and/or comments.

6 Appendix A

In this appendix, we first propagate the Lipschitz bound for $\overline{V}^N$ back one step from the terminal time, that is we obtain the Lipschitz bound for $\overline{V}^N_{N-1}$ given the corresponding bound for $\overline{V}^N_N = \psi$. We will then continue this procedure back to the initial time.

Let v^δ_N be δ–optimal at x_{N-1}. Then

$$\overline{V}^N_{N-1}(x_{N-1}) - \overline{V}^N_{N-1}(\hat{x}_{N-1}) \leq \mathrm{E}\big[\psi(y_N) - \psi(\hat{y}_N)\big] + C(1 + |x_{N-1}| + |\hat{x}_{N-1}|)|x_{N-1} - \hat{x}_{N-1}|\Delta + \delta \tag{58}$$

where y_N is given by (29) with disturbance v^δ_N, and

$$\hat{y}_N = \hat{x}_{N-1} + [f(\hat{x}_{N-1}) + \sigma(\hat{x}_{N-1})v^\delta_N]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(\hat{x}_{N-1})B_N. \tag{59}$$

The following rather technical estimates will lead to a Lipschitz bound on the $\mathrm{E}\big[\psi(y_N) - \psi(\hat{y}_N)\big]$ term in (58). In particular, a one-sided Lipschitz bound is obtained using a one-sided bound on ψ_{xx}. Symmetry then leads to the usual (two-sided) Lipschitz bound. This approach is employed due to the semi-convexity preserving nature of the maximizing control problem which has previously been applied for instance in [19].

First, we will replace the bound on ψ_x so as to smooth it near the origin. Note that by Assumption (A4), $|\psi_x(x)| \leq C(1 + 2|x|)$ for all x. A quick computation shows that $C(1+2|x|) \leq C(\frac{3}{2} + 2|x|^2)$ for all x. Consequently, we have

$$\psi_x(x) \leq b(x) \qquad \forall\, x \in \Re^n$$

where

$$b(x) = \begin{cases} C(\frac{3}{2} + 2|x|^2) & \text{if } |x| \le 1 \\ C(\frac{3}{2} + 2|x|) & \text{if } |x| > 1. \end{cases} \tag{60}$$

Now note that by the Mean Value Theorem

$$\psi(y_N) - \psi(\hat{y}_N) = \psi_x(\xi)(y_N - \hat{y}_N)$$

where

$$\xi \doteq \lambda y_N + (1-\lambda)\hat{y}_N,$$

and λ is a random variable with range $[0,1]$. Use the notation $(\delta f) \doteq f(x_{N-1}) - f(\hat{x}_{N-1})$ and $(\delta\sigma) \doteq \sigma(x_{N-1}) - \sigma(\hat{x}_{N-1})$, so that

$$y_N - \hat{y}_N = (x_{N-1} - \hat{x}_{N-1}) + ((\delta f) + (\delta\sigma)v_N^\delta)\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}(\delta\sigma)B_N.$$

(The δ in the notation (δf) is not intended to be confused with the δ superscript in v_N^δ; it is merely intended as a shorthand to reduce the size of the displayed equations.) Consequently, we have

$$\begin{aligned} \mathrm{E}[\psi(y_N) - \psi(\hat{y}_N)] &= \mathrm{E}[\psi_x(\xi)]\left[(x_{N-1} - \hat{x}_{N-1}) + ((\delta f) + (\delta\sigma)v_N^\delta)\Delta\right] \\ &\quad + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\mathrm{E}[\psi_x(\xi)(\delta\sigma)B_N]. \end{aligned} \tag{61}$$

We will work with each of the two terms on the right hand side of (61) separately. The second is easier, and we begin with that.

As mentioned in Section 3, we let each component of B_N, B_{Nj}, be independent with

$$P\Big(B_{Nj} = 1\Big) = \frac{1}{2}, \qquad P\Big(B_{Nj} = -1\Big) = \frac{1}{2} \qquad \text{for } j = 1, \ldots m.$$

Then, by the definition of y_N, (A1), (A2) and (31), there exists $K_1 < \infty$ such that

$$|y_N - x_{N-1}| \le K(1 + |x_{N-1}|)\Delta + K_1\sqrt{\Delta} + MD_v(x_{N-1})\Delta \qquad \forall\,\omega \in \Omega \tag{62}$$

where

$$K_1 = \sqrt{\frac{\varepsilon m}{\gamma^2}}M$$

and, as a shorthand,

$$D_v(x_{N-1}) = \frac{C}{\gamma}\left[C_1 + C_2|x_{N-1}|\right].$$

Similarly,

$$|\hat{y}_N - \hat{x}_{N-1}| \le K(1 + |\hat{x}_{N-1}|)\Delta + K_1\sqrt{\Delta} + MD_v(\hat{x}_{N-1})\Delta \qquad \forall\,\omega \in \Omega. \tag{63}$$

Let us take

$$|x_{N-1} - \hat{x}_{N-1}| \le \Delta \le 1 \tag{64}$$

in order to simplify matters below. (The bound on $|x_{N-1} - \hat{x}_{N-1}|$ is irrelevant to the Lipschitz bound; the bound on Δ is also not important since we will be taking $\Delta = T/N$ to zero.) Also, without loss of generality, let us suppose $|\hat{x}_{N-1}| \le |x_{N-1}|$. By the definition of ξ,

$$|\xi - x_{N-1}| \le \max\{|y_N - x_{N-1}|,\, |\hat{y}_N - \hat{x}_{N-1}| + |\hat{x}_{N-1} - x_{N-1}|\}$$

which by the (64)

$$\leq \max\{|y_N - x_{N-1}|, |\hat{y}_N - \hat{x}_{N-1}| + \Delta\}. \tag{65}$$

Then by (62), (63) and (65),

$$|\xi - x_{N-1}| \leq [K(2 + |x_{N-1}|) + K_1]\sqrt{\Delta} + MD_v(x_{N-1})\Delta \qquad \forall\, \omega \in \Omega. \tag{66}$$

Now,

$$\begin{aligned}
&\mathrm{E}\left[\psi_x(\xi)(\delta\sigma)B_N\right] \\
&= \mathrm{E}\left[\psi_x(x_{N-1})(\delta\sigma)B_N\right] + \mathrm{E}\left[(\psi_x(\xi) - \psi_x(x_{N-1}))(\delta\sigma)B_N\right] \\
&= \mathrm{E}\left[(\psi_x(\xi) - \psi_x(x_{N-1}))(\delta\sigma)B_N\right]
\end{aligned}$$

which for proper choice of ζ on the line from x_{N-1} to ξ

$$= \mathrm{E}\left[\psi_{xx}(\zeta)(\xi - x_{N-1})(\delta\sigma)B_N\right]$$

which by (A2) and (A4),

$$\geq -\overline{C}L_\sigma \mathrm{E}\left\{|\xi - x_{N-1}|\frac{|x_{N-1} - \hat{x}_{N-1}|}{1 + |\hat{x}_{N-1}|}|B_N|\right\}$$

which by (66) and (64)

$$\geq -\overline{C}\left[K_3 + M\frac{K_4}{\gamma}\right]|x_{N-1} - \hat{x}_{N-1}|\sqrt{\Delta} \qquad \forall\, \omega \in \Omega \tag{67}$$

where

$$K_3 \doteq L_\sigma \max_{z\in\Re^n}\left\{\frac{K(3 + |z|) + K_1}{1 + |z|}\right\} \quad \text{and} \quad K_4 \doteq CL_\sigma \max_{z\in\Re^n}\left\{\frac{C_1 + C_2(|z| + 1)}{1 + |z|}\right\}.$$

Employing (67) in (61) yields

$$\begin{aligned}
\mathrm{E}[\psi(y_N) - \psi(\hat{y}_N)] \geq\, &\mathrm{E}[\psi_x(\xi)]\left[(x_{N-1} - \hat{x}_{N-1}) + ((\delta f) + (\delta\sigma)v_N^\delta)\Delta\right] \\
&-\overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3 + M\frac{K_4}{\gamma}\right]\Delta|x_{N-1} - \hat{x}_{N-1}|.
\end{aligned} \tag{68}$$

We now turn to the first term. Employing (A1), (A2) and (31) in (68) yields

$$\begin{aligned}
\mathrm{E}[\psi(y_N) - \psi(\hat{y}_N)] \geq\, &-|\mathrm{E}[\psi_x(\xi)]|\left[1 + \left(K + L_\sigma M\frac{K_4}{\gamma}\right)\Delta\right]|x_{N-1} - \hat{x}_{N-1}| \\
&-\overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3 + M\frac{K_4}{\gamma}\right]\Delta|x_{N-1} - \hat{x}_{N-1}|.
\end{aligned} \tag{69}$$

A rather long argument will now be used to obtain a bound on

$$|\mathrm{E}[\psi_x(\xi)]|$$

This will involve the use of the bound b from (60). Recalling the form of b, we see that this will involve bounds on $\mathrm{E}|y_N|$, $\mathrm{E}|\hat{y}_N|$, $\mathrm{E}|y_N|^2$ and $\mathrm{E}|\hat{y}_N|^2$.

We work first with $\mathrm{E}|y_N|$ (and $\mathrm{E}|\hat{y}_N|$). As a shorthand, let

$$z_{x_{N-1}} \doteq x_{N-1} + (f(x_{N-1}) + \sigma(x_{N-1})v_N^\delta)\Delta.$$

Then

$$\begin{aligned}
\mathrm{E}|y_N| &= \mathrm{E}|z_{x_{N-1}} + \sqrt{\frac{\varepsilon}{\gamma^2}}\sigma(x)\sqrt{\Delta}B_N| \\
&= \frac{1}{2^m}\sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\cdots\sum_{j_m=1}^{2}\left|z_{x_{N-1}} + \sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta}\sigma(x)\big(-1^{j_1}, -1^{j_2}, \dots - 1^{j_m}\big)^T\right|. \qquad (70)
\end{aligned}$$

Define G, $\overline{G}$ by

$$\begin{aligned}
G(z_{x_{N-1}}, \Delta) &\doteq \max_{u\in\Re^m,\,|u|=1}\frac{1}{2}\left[\left|z_{x_{N-1}} + \sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta}\sigma(x)u\right| + \left|z_{x_{N-1}} - \sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta}\sigma(x)u\right|\right] \qquad (71)\\
&\le \max_{|v|\le M}\frac{1}{2}\left[\left|z_{x_{N-1}} + \sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta}v\right| + \left|z_{x_{N-1}} - \sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta}v\right|\right] \doteq \overline{G}(z_{x_{N-1}}, \Delta), \qquad (72)
\end{aligned}$$

and note that $\mathrm{E}|y_N| \le G(z_{x_{N-1}}, \Delta)$. It is an exercise in Lagrange multipliers (which we do not include since it is quite standard) to show that the maximum occurs when v is perpendicular to $z_{x_{N-1}}$. Consequently, (using Pythagorean Theorem)

$$G(z_{x_{N-1}}, \Delta) \le \overline{G}(z_{x_{N-1}}, \Delta) = \left[|z_{x_{N-1}}|^2 + \frac{\varepsilon}{\gamma^2}M^2\Delta\right]^{\frac{1}{2}}. \qquad (73)$$

Note that

$$\begin{aligned}
\overline{G}(z_{x_{N-1}}, 0) &= |z_{x_{N-1}}| \\
\frac{d}{d\Delta}\overline{G}(z_{x_{N-1}}, 0) &= \frac{\varepsilon}{2\gamma^2|z_{x_{N-1}}|}M^2
\end{aligned}$$

and

$$\frac{d^2}{d\Delta^2}\overline{G}(z_{x_{N-1}}, \Delta) = \frac{-1}{4\left[|z_{x_{N-1}}|^2 + \frac{\varepsilon}{\gamma^2}M^2\Delta\right]^{\frac{3}{2}}}\frac{\varepsilon^2}{\gamma^4}M^4 < 0.$$

Consequently,

$$\overline{G}(z_{x_{N-1}}, \Delta) \le |z_{x_{N-1}}| + \frac{\varepsilon}{2\gamma^2|z_{x_{N-1}}|}M^2\Delta. \qquad (74)$$

Combining (70), (72) and (74), one finds

$$\mathrm{E}|y_N| \le |z_{x_{N-1}}| + \frac{\varepsilon}{4\gamma^2|z_{x_{N-1}}|}M^2\Delta. \qquad (75)$$

Further, from the definition of $z_{x_{N-1}}$, (A1), (A2) and (31), $z_{x_{N-1}} \ge 1/4$ for $x_{N-1} \ge 1/2$ if

$$\Delta \le \frac{5}{4(2K + M(C_1 + C_2))} \doteq d_1.$$

Consequently, if $\Delta \le \min\{1, d_1\}$, then

$$\mathrm{E}|y_N| \le |z_{x_{N-1}}| + \frac{\varepsilon}{\gamma^2}M^2\Delta \qquad (76)$$

for any $x_{N-1} \ge 1/2$. Again using the definition of $z_{x_{N-1}}$, (A1), (A2) and (31), this implies

$$\mathrm{E}|y_N| \le |x_{N-1}| + \left[K(1 + |x_{N-1}|) + MD_v(x_{N-1}) + \frac{\varepsilon M^2}{\gamma^2}\right]\Delta \qquad \forall\, |x_{N-1}| \ge \frac{1}{2} \qquad (77)$$

provided $\Delta \le \min\{1, d_1\}$. A similar statement holds for $\hat{y}_N$.

Noting that $\psi_x(x) \le b(x) \le C(\frac{3}{2} + 2|x|)$ for all x, one finds using (77) and (A4) that

$$|\mathrm{E}\psi_x(\xi)| \le C\Big\{\frac{3}{2} + 2|x_{N-1}| + [2K_6(1+|x_{N-1}|) + 2MD_v(x_{N-1})]\,\Delta\Big\} \qquad \forall\,|x_{N-1}|, |\hat{x}_{N-1}| \ge \frac{1}{2} \tag{78}$$

where $K_6 = \max\{K, \varepsilon M^2/(\gamma^2)\}$, provided $\Delta \le \min\{1, d_1\}$.

We now turn to the quadratic portion of $b(x)$ which occurs for $|x| \le 1$. Using the fact that B_N has zero mean,

$$\begin{aligned}
\mathrm{E}|y_N|^2 &= \mathrm{E}\Big|z_{x_{N-1}} + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(x_{N-1})B_N\Big|^2 \\
&= |z_{x_{N-1}}|^2 + \frac{\varepsilon}{\gamma^2}\Delta \mathrm{E}\big[B_N^T\sigma^T\sigma B_N\big] \\
&\le |z_{x_{N-1}}|^2 + \frac{\varepsilon}{\gamma^2}M^2\Delta
\end{aligned}$$

which by the definition of $z_{x_{N-1}}$, (A1), (A2) and (31) again,

$$\le |x_{N-1}|^2 + K_7(1+|x_{N-1}|)^2\Delta + 2M^2D_v^2(x_{N-1})\Delta^2$$

where $K_7 = 2[K + K^2 + M(K_4/\gamma) + (\varepsilon/(2\gamma^2))M^2]$. In a similar manner to that above, this implies that

$$|\mathrm{E}\psi_x(\xi)| \le C\Big[\frac{3}{2} + 2|x_{N-1}|^2 + 2K_7(1+|x_{N-1}|)^2\Delta + 4M^2D_v^2(x_{N-1})\Delta^2\Big] \qquad \forall\, x_{N-1}, \hat{x}_{N-1} \in \Re^n. \tag{79}$$

Combining (78) and (79), one finds

$$|\mathrm{E}\psi_x(\xi)| \le b(|x_{N-1}|)\left[1 + K_8\Delta + 2M\frac{K_4}{\gamma}\Delta + 8M^2\frac{K_4^2}{\gamma^2}\Delta^2\right] \qquad \forall\,|x_{N-1} - \hat{x}_{N-1}| \le \frac{1}{2} \tag{80}$$

where $K_8 = \max\{2K_6, 4K_7\}$, provided $\Delta \le \min\{1, d_1\}$, which is the inequality indicated earlier.

Combining (69) and (80), one finds

$$\begin{aligned}
\mathrm{E}[\psi(y_N) - \psi(\hat{y}_N)] \ge &-b(|\hat{x}_{N-1}|)\left[1 + c_1\Delta + c_2\frac{K_4}{\gamma}\Delta + c_3\frac{K_4^2}{\gamma^2}\Delta^2\right]|x_{N-1} - \hat{x}_{N-1}| \\
&-\overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4}{\gamma}\Delta\right]|x_{N-1} - \hat{x}_{N-1}|
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
\overline{c}_1 &= K + K_8 + KK_8 \\
c_2 &= 2MK + K_8ML_\sigma \\
c_3 &= M^2[2L_\sigma + 8(1+K)].
\end{aligned}$$

provided $\Delta \le \min\{1, d_1\}$ and $|x_{N-1} - \hat{x}_{N-1}| \le \frac{1}{2}$. By symmetry, and by taking intermediary steps (to remove the $|x_{N-1} - \hat{x}_{N-1}| \le \frac{1}{2}$ bound), this implies

$$\begin{aligned}
\big|\mathrm{E}[\psi(y_N) - \psi(\hat{y}_N)]\big| \le\ &b(|\hat{x}_{N-1}|)\left[1 + c_1\Delta + c_2\frac{K_4}{\gamma}\Delta + c_3\frac{K_4^2}{\gamma^2}\Delta^2\right]|x_{N-1} - \hat{x}_{N-1}| \\
&+\overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4}{\gamma}\Delta\right]|x_{N-1} - \hat{x}_{N-1}|
\end{aligned} \tag{82}$$

provided $\Delta \le \min\{1, d_1\}$.

Finally, combining (82) with (58) (and letting $\delta \downarrow 0$) yields

$$\begin{aligned}\left|\overline{V}_{N-1}^N(x_{N-1}) - \overline{V}_{N-1}^N(\hat{x}_{N-1})\right| \le b(|\hat{x}_{N-1}|)\left[1 + c_1\Delta + c_2\frac{K_4}{\gamma}\Delta + c_3\frac{K_4^2}{\gamma^2}\Delta^2\right]|x_{N-1} - \hat{x}_{N-1}| \\ +\overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4}{\gamma}\Delta\right]|x_{N-1} - \hat{x}_{N-1}| \qquad (83)\end{aligned}$$

provided $\Delta \le \min\{1, d_1\}$ where $c_1 = \overline{c}_1 + 1$.

Lemma 6.1 $\overline{V}_{N-1}^N$ *is semi–convex with uniform constant* $C[2K\Delta + 4L_\sigma\frac{K_4}{\gamma}\Delta] + \overline{C}\{1 + c_4\Delta + c_5\frac{K_4}{\gamma}\Delta + c_6\frac{K_4^2}{\gamma^2}\Delta^2\}$ *where* c_4, c_5, c_6 *are given in the proof.*

Proof of Lemma: Recall that

$$\overline{V}_{N-1}^N(x_{N-1}) = \sup_{|v_N| \le C_1 + C_2|x_{N-1}|} \overline{J}_{N-1}^N(x_{N-1}, v_N)$$

where

$$\overline{J}_{N-1}^N(x_{N-1}, v_N) = \mathrm{E}\Big[\big(\ell(x_{N-1}) - \frac{\gamma^2}{2}|v_N|^2\big)\Delta + \psi(y_N)\Big].$$

Fix a v_N satisfying $|v_N| \le C_1 + C_2|x_{N-1}|$. First, a lower bound on

$$\frac{d^2}{dh^2}\overline{J}_{N-1}^N(x_{N-1} + h\eta, v_N)\Big|_{h=0} \qquad (84)$$

will be obtained for any $\eta \in \Re^n$ with $|\eta| = 1$.

One has

$$\frac{d^2}{dh^2}\overline{J}_{N-1}^N(x_{N-1} + h\eta, v_N)|_{h=0} = \eta^T l_{xx}(x_{N-1})\eta\Delta + \mathrm{E}\{{\zeta^1}^T\psi_{xx}(y_N)\zeta^1\} + \mathrm{E}\{\psi_x(y_N)\zeta^2\}$$

where

$$\begin{aligned}\zeta^1 &\doteq \frac{d}{dh}y_N^h\Big|_{h=0} \\ \zeta^2 &\doteq \frac{d^2}{dh^2}y_N^h\Big|_{h=0} \\ y_N^h &= (x_{N-1} + h\eta) + [f(x_{N-1} + h\eta) + \sigma(x_{N-1} + h\eta)v_N]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma(x_{N-1} + h\eta)B_N.\end{aligned}$$

Consequently, by Assumption (A4),

$$\frac{d^2}{dh^2}\overline{J}_{N-1}^N(x_{N-1} + h\eta, v_N)|_{h=0} \ge -\overline{C}\Delta + \mathrm{E}\{{\zeta^1}^T\psi_{xx}(y_N)\zeta^1\} + \mathrm{E}\{\psi_x(y_N)\zeta^2\}. \qquad (85)$$

Note that

$$\zeta^1 = \eta + [f_x(x_{N-1})\eta + \sigma_x(x_{N-1})\eta v_N]\,\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma_x(x_{N-1})\eta B_N.$$

Using the facts that $\mathrm{E}(B_N) = 0$ and $\mathrm{E}(B_N^2) = 1$, this yields

$$\mathrm{E}\{{\zeta^1}^T\psi_{xx}(y_N)\zeta^1\} \ge \quad -\overline{C}\mathrm{E}\Big|\eta + [f_x(x_{N-1})\eta + \sigma_x(x_{N-1})\eta v_N]\,\Delta$$

$$
\begin{aligned}
&\quad + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\sigma_x(x_{N-1})\eta B_N\Bigg|^2 \\
\geq \quad & -\overline{C}\Bigg[1 + \Big(2|f_x(x_{N-1})| + 2|\sigma_x(x_{N-1})v_N| + \frac{\varepsilon}{\gamma^2}|\sigma_x(x_{N-1})|^2\Big)\Delta \\
&\quad + \big(|f_x(x_{N-1})| + |\sigma_x(x_{N-1})v_N|\big)^2\Delta^2\Bigg]
\end{aligned}
$$

which by Assumptions (A1) and (A2), and (31)

$$
\geq \quad -\overline{C}\left[1 + K_9\Delta + 2\frac{K_4}{\gamma}\Delta + 2\frac{K_4^2}{\gamma^2}\Delta^2\right] \tag{86}
$$

where $K_9 = 2K + 2K^2 + \varepsilon L_\sigma^2/\gamma^2$.

Next, one finds

$$
\zeta^2 = \left[\eta^T f_{xx}(x_{N-1})\eta + \eta^T\sigma_{xx}(x_{N-1})\eta v_N\right]\Delta + \sqrt{\frac{\varepsilon\Delta}{\gamma^2}}\eta^T\sigma_{xx}(x_{N-1})\eta B_N.
$$

Then using (component-wise)

$$
(\psi_x)_i(y_N) = (\psi_x)_i(x_{N-1}) + [(\psi_{xx})_i(\xi_i)](y_N - x_{N-1})
$$

for proper choice of ξ_i on the line segment from x_{N-1} to y_N, one finds (leaving out the x_{N-1} argument where it is clear), and letting $H(\{\xi_i\})$ be the matrix composed of the rows $(\psi_{xx})_i(\xi_i)$

$$
\begin{aligned}
\mathrm{E}\{\psi_x(y_N)\zeta^2\} &= \psi_x(x_{N-1})\left[\eta^T f_{xx}(x_{N-1})\eta + \eta^T\sigma_{xx}(x_{N-1})\eta v_N\right]\Delta \\
&\quad + \mathrm{E}\Bigg\{H(\{\xi\}_i)\Bigg[\eta^T(f_{xx} + \sigma_{xx}v_N)\eta(f + \sigma v_N)]\Delta^2 \\
&\qquad + \Big[(f + \sigma v_N)\eta^T\sigma_{xx}\eta\sqrt{\frac{\varepsilon}{\gamma^2}} + \eta^T(f_{xx} + \sigma_{xx}v_N)\eta\sigma\sqrt{\frac{\varepsilon}{\gamma^2}}\Big]B_N\Delta^{3/2} \\
&\qquad + \eta^T\sigma_{xx}\eta\sigma\frac{\varepsilon}{\gamma^2}B_N^2\Delta\Bigg]\Bigg\} \qquad (87) \\
&\geq -|\psi_x(x_{N-1})|\left[|f_{xx}(x_{N-1})| + |\sigma_{xx}(x_{N-1})v_N|\right]\Delta \\
&\quad -\overline{C}\Bigg[\big(|f_{xx}| + |\sigma_{xx}v_N|\big)\big(|f| + |\sigma v_N|\big)\Delta^2 + \Big((|f| + |\sigma v_N|)|\sigma_{xx}| + (|f_{xx}| + |\sigma_{xx}v_N|)|\sigma|\Big)\sqrt{\frac{\varepsilon}{\gamma^2}}\Delta^{3/2} \\
&\qquad + |\sigma_{xx}|\,|\sigma|\frac{\varepsilon}{\gamma^2}\Delta
\end{aligned}
$$

which with a few bounds on rational functions of $|x|$ (such as $(3/2 + 2|x|)/(1 + |x|) \leq 2$) one has

$$
\begin{aligned}
&\geq -C\left[2K + 4L_\sigma\frac{K_4}{\gamma}\right]\Delta \\
&\quad -\overline{C}\Bigg\{\left[K^2 + ML_\sigma\frac{K_4^2}{\gamma^2} + KM\frac{K_4}{\gamma} + 2KL_\sigma\frac{K_4}{\gamma}\right]\Delta^2 \\
&\qquad + \left[K(L_\sigma + M) + 2ML_\sigma\frac{K_4}{\gamma}\right]\sqrt{\frac{\varepsilon}{\gamma^2}}\Delta^{3/2} + ML_\sigma\frac{\varepsilon}{\gamma^2}\Delta\Bigg\}
\end{aligned}
$$

$$\geq -C\left[2K\Delta + 4L_\sigma \frac{K_4}{\gamma}\Delta\right] - \overline{C}\left\{K_{11}\Delta + K_{12}\frac{K_4}{\gamma}\Delta + K_{13}\frac{K_4^2}{\gamma^2}\Delta^2\right\} \tag{88}$$

where

$$\begin{aligned} K_{11} &= K^2\Delta + K(L_\sigma + M)\sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta} + ML_\sigma\frac{\varepsilon}{\gamma^2} \\ K_{12} &= (KM + 2KL_\sigma)\Delta + 2ML_\sigma\sqrt{\frac{\varepsilon}{\gamma^2}}\sqrt{\Delta} \\ K_{13} &= ML_\sigma. \end{aligned}$$

Combining (85), (86) and (88) yields

$$\begin{aligned} \frac{d^2}{dh^2}\overline{J}^N_{N-1}(x_{N-1} + h\eta, v_N)|_{h=0} &\geq -C\left[2K\Delta + 4L_\sigma\frac{K_4}{\gamma}\Delta\right] \\ &\quad -\overline{C}\left\{1 + c_4\Delta + c_5\frac{K_4}{\gamma}\Delta + c_6\frac{K_4^2}{\gamma^2}\Delta^2\right\} \end{aligned} \tag{89}$$

where $c_4 = 1 + K_9 + K_{11}$, $c_5 = 2 + K_{12}$ and $c_6 = 2 + K_{13}$.

This implies that $\overline{J}^N_{N-1}(x, v_N)$ is semi–convex with uniform constant given by the right hand side of (89) over all of $\Re^n$ if $|v_N| \leq (C/\gamma)[C_1 + C_2|x|]$. By the definition of $\overline{V}^N_{N-1}$ as a supremum of these, $\overline{V}^N_{N-1}$ is semi–convex with this same constant. This completes the proof of the lemma. $\square$

We now proceed to step backwards. Note that we used the derivatives of ψ in the above argument. One can first smooth $\overline{V}^N$ via convolution with approximations to the identity, before proceeding backwards; since this is an obvious, but time-consuming, technicality, we do not include it. Instead, simply assuming differentiability, we now have from (31), (83) and the above lemma

$$|v_N(x)| \leq \frac{C}{\gamma}\left[C_1 + C_2|x|\right], \tag{90}$$

$$\left|\overline{V}^N_{N-1_x}(x)\right| \leq b(|x|)\left[1 + c_1\Delta + c_2\frac{K_4}{\gamma}\Delta + c_3\frac{K_4^2}{\gamma^2}\Delta^2\right] + \overline{C}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4}{\gamma}\Delta\right] \tag{91}$$

and that $\overline{V}^N_{N-1}$ is semiconvex with constant

$$\overline{C}_1 = C\left[2K\Delta + 4L_\sigma\frac{K_4}{\gamma}\Delta\right] + \overline{C}_0\left\{1 + c_4\Delta + c_5\frac{K_4}{\gamma}\Delta + c_6\frac{K_4^2}{\gamma^2}\Delta^2\right\} \tag{92}$$

where $\overline{C}_0 \doteq \overline{C}$.

Define

$$\begin{aligned} &K_4^{(0)} = K_4, \\ &M_0 = 1, \\ &M_1 = \left\{\left[1 + c_1\Delta + c_2\frac{K_4^{(0)}}{\gamma}\Delta + c_3\frac{{K_4^{(0)}}^2}{\gamma^2}\Delta^2\right] + \frac{\overline{C}_0}{b(0)}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4^{(0)}}{\gamma}\Delta\right]\right\} M_0. \end{aligned}$$

Note then that the above Lipschitz and semiconvexity bounds become

$$\begin{aligned} &\left|\overline{V}^N_{N-1_x}(x)\right| \leq b(|x|)M_1 \\ &\overline{C}_1 = \overline{C}_0\left\{1 + c_4\Delta + c_5\frac{K_4^{(0)}}{\gamma}\Delta + c_6\frac{{K_4^{(0)}}^2}{\gamma^2}\Delta^2\right\} + CM_0\left[2K\Delta + 4L_\sigma\frac{K_4^{(0)}}{\gamma}\Delta\right] \end{aligned}$$

Stepping backward, one finds

$$
\begin{aligned}
&|v_{N-1}(x)| \leq \frac{C}{\gamma}\left[C_1 + C_2|x|\right] M_1 \\
&\left|\overline{V}^N_{N-2_x}(x)\right| \leq b(|x|)M_2 \\
&\overline{C}_2 \doteq \overline{C}_1 \left\{ 1 + c_4\Delta + c_5\frac{K_4^{(1)}}{\gamma}\Delta + c_6\frac{K_4^{(1)^2}}{\gamma^2}\Delta^2 \right\} + CM_1\left[2K\Delta + 4L_\sigma\frac{K_4^{(1)}}{\gamma}\Delta\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&K_4^1 = M_1 K_4^{(0)} \\
&M_2 = \left\{ \left[1 + c_1\Delta + c_2\frac{K_4^{(1)}}{\gamma}\Delta + c_3\frac{K_4^{(1)^2}}{\gamma^2}\Delta^2 \right] + \frac{\overline{C}_1}{b(0)}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4^{(1)}}{\gamma}\Delta\right] \right\} M_1.
\end{aligned}
$$

Continuing this process, one has

$$
\begin{aligned}
&|v_{N-n}(x)| \leq \frac{C}{\gamma}\left[C_1 + C_2|x|\right] M_n \\
&\left|\overline{V}^N_{N-(n+1)_x}(x)\right| \leq b(|x|)M_{n+1} \\
&\overline{C}_{n+1} \doteq \overline{C}_n \left\{ 1 + c_4\Delta + c_5\frac{K_4^{(n)}}{\gamma}\Delta + c_6\frac{K_4^{(n)^2}}{\gamma^2}\Delta^2 \right\} + CM_n\left[2K\Delta + 4L_\sigma\frac{K_4^{(n)}}{\gamma}\Delta\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&K_4^{(n)} = M_n K_4^{(0)} \\
&M_{n+1} = \left\{ \left[1 + c_1\Delta + c_2\frac{K_4^{(n)}}{\gamma}\Delta + c_3\frac{K_4^{(n)^2}}{\gamma^2}\Delta^2 \right] + \frac{\overline{C}_n}{b(0)}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\Delta + M\frac{K_4^{(n)}}{\gamma}\Delta\right] \right\} M_n.
\end{aligned}
$$

Now, we note that one is interested here in the Lipschitz constant for $\overline{V}^N_0$, and in particular whether this stays bounded as $N \to \infty$. Noting in the above that $\Delta = T/N$, we see that this question reduces to consideration of the sequence

$$
\begin{aligned}
&M_{N,0} = 1 \\
&\overline{C}_{N,0} = \overline{C} \\
&M_{N,n+1} = \left\{ \left[1 + c_1\frac{T}{N} + c_2\frac{K_4^{(N,n)}}{\gamma}\frac{T}{N} + c_3\frac{K_4^{(N,n)^2}}{\gamma^2}\frac{T^2}{N^2} \right] + \frac{\overline{C}_{N,n}}{b(0)}\sqrt{\frac{\varepsilon}{\gamma^2}}\left[K_3\frac{T}{N} + M\frac{K_4^{(N,n)}}{\gamma}\frac{T}{N}\right] \right\} M_{N,n} \\
&\overline{C}_{N,n+1} \doteq \overline{C}_{N,n} \left\{ 1 + c_4\frac{T}{N} + c_5\frac{K_4^{(N,n)}}{\gamma}\frac{T}{N} + c_6\frac{K_4^{(N,n)^2}}{\gamma^2}\frac{T^2}{N^2} \right\} + CM_{N,n}\left[2K\frac{T}{N} + 4L_\sigma\frac{K_4^{(N,n)}}{\gamma}\frac{T}{N}\right]
\end{aligned}
$$

where

$$K_4^{(N,n)} = M_{N,n}K_4.$$

Note that $|\overline{V}^N_{0\ x}(x)| \leq b(|x|)M_{N,N}$.

Showing that $M_{N,N}$ stays bounded for sufficiently large γ can be done with some rather crude estimates. Again, we are not interested here in the minimal possible γ, but mainly in developing the machinery to handle risk sensitive control limit problems under quadratic growth assumptions. In that spirit, one may define

$$R_{N,n} \doteq \max\{M_{N,n}, \overline{C}_{N,n}\}.$$

Then, for $N > K_4 T$ one has

$$M_{N,n+1} \le R_{N,n}\left\{1 + c_1\frac{T}{N} + \frac{\overline{c}_2}{\gamma}R_{N,n}\frac{T}{N} + \frac{\overline{c}_3}{\gamma^2}R_{N,n}^2\frac{T}{N}\right\}$$
$$\overline{C}_{N,n+1} \le R_{N,n}\left\{1 + c_4\frac{T}{N} + \frac{\overline{c}_5}{\gamma}R_{N,n}\frac{T}{N} + \frac{\overline{c}_6}{\gamma^2}R_{N,n}^2\frac{T}{N}\right\}$$

where $\overline{c}_2 = c_2 + \sqrt{\varepsilon}K_3/b(0)$, $\overline{c}_3 = c_3 + \sqrt{\varepsilon}MK_4/b(0)$, $\overline{c}_4 = c_4 + 2CK$, $\overline{c}_5 = c_5K_4 + 4L_\sigma CK_4$ and $\overline{c}_6 = c_6K_4$. This implies

$$R_{N,n+1} \le R_{N,n}\left\{1 + c_7\frac{T}{N} + \frac{c_8}{\gamma}R_{N,n}\frac{T}{N} + \frac{c_9}{\gamma^2}R_{N,n}^2\frac{T}{N}\right\}$$

where $c_7 = c_1 + \overline{c}_4$, $c_8 = c_2 + \overline{c}_5$ and $c_7 = \overline{c}_3 + \overline{c}_6$. Letting $\overline{\rho}_{N,n} \doteq \ln(R_{N,n})$, one has

$$\overline{\rho}_{N,n+1} \le \overline{\rho}_{N,n} + \left[c_7 + \frac{c_8}{\gamma}e^{\overline{\rho}_{N,n}} + \frac{c_9}{\gamma^2}e^{2\overline{\rho}_{N,n}}\right]\frac{T}{N},$$

and perhaps one should note $\overline{\rho}_{N,0} = \max\{0, \ln(\overline{C})\}$. Then $\overline{\rho}_{N,n} \le \rho_{N,n}$ for all N, n where $\rho_{N,n}$ satisfies

$$\rho_{N,n+1} = \rho_{N,n} + \left[c_7 + \frac{c_{10}}{\gamma}e^{2\rho_{N,n}}\right]\frac{T}{N} \tag{93}$$
$$\rho_{N,0} = \max\{0, \ln(\overline{C})\} \tag{94}$$

and $c_{10} = c_8 + (c_9/\gamma)$. Note that then

$$M_{N,N} \le e^{\rho_{N,N}} \qquad \forall N.$$

Also, note that (93), (93) is simply the Euler numerical method (with N time–steps) for the ODE problem over $[0, T]$ given by

$$\dot{\widehat{\rho}} = c_7 + \frac{c_{10}}{\gamma}e^{2\rho}$$
$$\widehat{\rho}(0) = \max\{0, \ln(\overline{C})\}.$$

By elementary Calculus, one can show that given $T < \infty$, there is $\widetilde{\gamma} < \infty$ such that this ODE has a solution over $[0, T]$. Letting the maximum of this solution be ρ_M, one sees that $M_{N,N} \le e^{\rho_M + 1}$ for sufficiently large N. This behavior (increasingly large γ required for increasingly large T) is expected since the Riccati equation for the linear–quadratic case displays the finite–time blow–up property. (A reader curious about this point might note that letting $\widehat{\mathcal{M}} \doteq \exp(\widehat{\rho})$, $\widehat{\mathcal{M}}$ satisfies the ODE $\dot{\widehat{\mathcal{M}}} = c_7\widehat{\mathcal{M}} + (c_{10}/\gamma)\widehat{\mathcal{M}}^3$, which is somewhat worse than the Riccati equation which only has a squared term on the right–hand side; so we have lost a bit in all the bounding estimates.)

References

[1] M. Bardi and I. Capuzzo Dolcetta, **Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations**, Birkhäuser, Boston, 1997.

[2] M. Bardi and F. Da Lio, "On the Bellman equation for some unbounded control problems", Nonlinear differential eqs. appl. **4** (1997), 491–510.

[3] A. Bensoussan, J. Frehse and H. Nagai, “Some results on risk–sensitive control with full observation”, Appl. Math. Optim. **37** (1998), 1–42.

[4] A. Bensoussan and H. Nagai, “Conditions for no breakdown and Bellman equations of risk–sensitive control”, Appl. Math. Optim. (to appear).

[5] A. Bensoussan and J. H. van Schuppen, “Optimal control of partially observable stochastic systems with an exponential–of–integral performance index”, SIAM J. Control and Optim., (1985), 599–613.

[6] C.D. Charalambous and R.J. Elliott, “Remarks on the explicit solutions for nonlinear partially observable stochastic control problems and relations to H_∞ robust control”, Proc. 34^{th} IEEE CDC (1995) 2858–2863.

[7] F.H. Clarke, **Optimization and Nonsmooth Analysis**, Wiley, 1983.

[8] P. Dupuis and W.M. McEneaney, “Risk–sensitive and robust escape criteria”, SIAM J. Control and Optim., Vol. 35 (1997), 2021–2049.

[9] R. J. Elliott, **Stochastic Calculus and Applications**, Springer–Verlag, 1982.

[10] R. J. Elliott and N. J. Kalton, “The existence of value in differential games”, Memoirs of the Amer. Math. Society, **126** (1972).

[11] C.-h. Fan, J.L. Speyer, and C.R. Jaensch, “Centralized and decentralized solutions of the linear-exponential-gaussian problem”, IEEE Trans. Auto. Control ,**AC–39–10**, (1994) 1986-2003.

[12] E. Fernandez–Gaucherand and S.I. Marcus, “Risk–sensitive optimal control of hidden Markov models: a case study”, Proceedings 33rd IEEE CDC, Orlando, (1994) 1657–1662.

[13] W.H. Fleming, “The Cauchy problem for degenerate parabolic equations”, J. of Mathematics and Mechanics, **13** (1964) 987–1008.

[14] W.H. Fleming and D. Hernandez–Hernandez, “Risk–sensitive control of finite state machines on an infinite horizon I”, SIAM J. Control and Optim., **35** (1997) 1790–1810.

[15] W.H. Fleming and M.R. James, “The risk–sensitive index and the H_2 and H_∞ norms for nonlinear systems”, Math. Control Signals and Systems, **8** (1995) 199–221.

[16] W.H. Fleming and W.M. McEneaney, “Risk–sensitive optimal control and differential games”, (Proceedings of the Stochastic Theory and Adaptive Controls Workshop) Springer Lecture Notes in Control and Information Sciences 184, Springer–Verlag (1992).

[17] W.H. Fleming and W.M. McEneaney, “Risk–sensitive control with ergodic cost criteria”, Proc. 31st IEEE Conf. on Dec. and Control, (1992).

[18] W.H. Fleming and W.M. McEneaney, “Risk sensitive control on an infinite time horizon”, SIAM J. Control and Optim., Vol. 33, No. 6 (1995) 1881–1915.

[19] W.H. Fleming and W.M. McEneaney, “A max–plus based algorithm for an HJB equation of nonlinear filtering”, SIAM J. Control and Optim., 38 (2000), 683–710.

[20] W.H. Fleming and R.W. Rishel, **Deterministic and Stochastic Optimal Control**, Springer–Verlag (1975).

[21] W. H. Fleming and H. M. Soner, **Controlled Markov Processes and Viscosity Solutions**, Springer-Verlag, New York, 1992.

[22] W.H. Fleming and S.–J. Sheu, “Asymptotics for the first eigenvalue and eigenfunction of a nearly first order operator with large potential”, Annals of Probability, **25** (1997), 1953–1994.

[23] W.H. Fleming and P.E. Souganidis, "On the existence of value functions of two–player, zero–sum stochastic differential games", Indiana Univ. Math. Journal, **38** (1989) 293–314.

[24] I.V. Girsanov, "On transforming a certain class of stochastic processes by absolutely continuous substitution of measures", Th. of Prob. and its Applications, **5** (1960) 285–301.

[25] H. Ishii, "Comparison results for Hamilton–Jacobi equations without growth condition on solutions from above", Appl. Anal. 67 (1997), 357–372.

[26] H. Ishii, H. Nagai and F. Teramoto, "A singular limit on risk sensitive control and semi–classical analysis", Proc. of the 7th Japan-Russia Symp. on Probability Theory and Math. Stats., Ed. S. Watanabe et al., World Sicentific (1996), 164–173.

[27] D.H. Jacobson, "Optimal stochastic linear systems with exponential criteria and their relation to deterministic differential games", IEEE Trans. Auto. Control AC-18 (1973), 124–131.

[28] M.R. James, "Asymptotic analysis of non–linear stochastic risk–sensitive control and differential games", Math. Control, Signals and Systems 5 (1992), 401–417.

[29] M.R. James, J.S. Baras and R.J. Elliott, "Output feedback risk–sensitive control and differential games for continuous–time nonlinear systems", 32nd IEEE CDC, San Antonio, (1993).

[30] H. Kaise and H. Nagai, "Bellman–Isaacs equations of ergodic type related to risk–sensitive control and their singular limits", Asymptotic Analysis **16** (1998), 347–362.

[31] I. Karatzas and S. E. Shreve, **Brownian Motion and Stochastic Calculus**, Springer–Verlag (1988).

[32] G. Lebourg, "Valeur moyenne pour gradient généralisé", Co. R. Acad. Sci Paris 281 (1975), 795–797.

[33] P.L. Lions, P.E. Souganidis, " Differential games, optimal control and directional derivates of viscosity solutions of Bellman's and Isaacs' equations" SIAM J. Control Optim. 23 (1985), 566–583.

[34] O.A. Ladyzhenskaja, V.A. Solonnikov and N.N. Ural'ceva, **Linear and Quasi-linear Equations of Parabolic Tpye**, AMS Translations of Mathematical Mongraphs, (1968).

[35] R.S. Lipster and A.N. Shiryayev, **Statistics of Random Processes, I**, Springer–Verlag (1977).

[36] W.M. McEneaney and P. Dupuis, "A risk–sensitive escape criterion and robust limit", Proc. 33rd IEEE Conf. on Dec. and Control, (1994), 4195–4197.

[37] W.M. McEneaney and K. Ito, "Infinite time–horizon risk sensitive systems with quadratic growth", Proc. 36th IEEE Conf. on Decision and Control (1997).

[38] W.M. McEneaney, "Elimination of troublesome disturbances with application to representation results for H_∞ control DPEs", Proc. Seventh International Symposium on Dynamic Games and Applications, Shonan Village (1996), 662–671.

[39] W.M. McEneaney, "A Uniqueness Result for the Isaacs equation corresponding to Nonlinear H_∞ control", Math. Controls, Signals and Systems, Vol. 11 (1998), 303–334.

[40] W.M. McEneaney, "Robust control and differential games on a finite time horizon", Math. of Controls Signals and Systems, **8** (1995), 138–166.

[41] W.M. McEneaney, "Uniqueness for viscosity solutions of nonstationary HJB equations under some a priori conditions (with applications)", SIAM J. Control and Optim., Vol. 33, No. 5 (1995), 1560–1576.

[42] W.M. McEneaney, "Connections between risk–sensitive stochastic control, differential games and H_∞ control: the nonlinear case", doctoral thesis, Brown University (1993).

[43] H. Nagai, "Bellman equations of risk sensitive control", SIAM J. Control and Optim., **34** (1996), 74–101.

[44] O.A. Oleinik and S.N. Kruzhkov, "Quasi-linear parabolic equations of the second order with many independent variables", Uspekhi Matematicheskikh Nauk., **16**, No. 5 (1961), 115-155.

[45] Z. Pan and T. Basar, "Backstepping controller design for nonlinear stochastic systems under a risk–sensitive cost criterion", SIAM J. Control and Optim. **37** (1999), 957–995.

[46] R.A. Adams, **Sobolev Spaces**, Academic Press (1975).

[47] T. Runolfsson, "The equivalence between infinite–horizon optimal control of stochastic systems with exponential–of–integral performance index and stochastic differential games", John Hopkins University Technical Report JHU–ECE 91–07.

[48] T. Runolfsson, "Risk–sensitive control of Markov chains and differential games", Proceedings 32nd IEEE CDC, (1993).

[49] Andrzew Swiech, "Another approach to the existence of value functions of stochastic differential games", J. of Math. Analysis and Appls., **204** (1996) 884–897.

[50] A.J. Veretennikov, "On strong solutions and explicit formulas for solutions of stochastic integral equations", Math. USSR Sbornik, **39** (1981) 387–403.

[51] P. Whittle, "Risk–sensitive linear/quadratic/gaussian control", Adv. in Appl. Prob. 13 (1981), 764–777.

[52] P. Whittle, "A risk–sensitive maximum principle", Syst. Control Letters 15 (1990). 183–192.

[53] W.P. Ziemer, **Weakly Differentiable Functions**, Springer–Verlag (1989).