A Robust Control Framework for Option Pricing

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June 10, 1996

Abstract

A new approach is taken to the problem of option pricing. In the standard framework, the option pricing problem involves determining a price such that the option writer can guarantee a certain bound on the cost almost surely. Due to this form, the problem may be reformulated in terms of deterministic differential games of the type employed in robust and H_∞ control. Different models yield different prices. The standard model yields the Black and Scholes price. Both a deterministic model and the standard model with the Ito integral replaced by the Stratonovich integral yield the price corresponding to a stop–loss hedging technique. With these methods, it can also easily be shown that for the standard model with a bounded, stochastic volatility, the Black and Scholes price corresponding to the upper bound for volatility is sufficient to hedge the option.

Key Words: option pricing, Black–Scholes formula, robust control, viscosity solutions

AMS Subject Classifications: 35B37, 49J35, 49L25, 60G35, 90A09, 90D25, 90D40, 93B36, 93E20

1. Introduction

The term robust control may be used to refer to a class of control problems where one attempts to attenuate the effects of a worst–case scenario; see Basar and Bernhard (1991), Ball, Helton and Walker (1993), Fleming and McEneaney (1995), Isidori (1992), James (1993), McEneaney (1995), McEneaney and Dupuis (1994), van der Schaft (1993). This class includes H_∞ control. The option pricing problem as modeled in the seminal work of Black and Scholes (1973) is generally formulated in a now standard, stochastic framework.

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For a great many stochastic control problems, one considers a cost criterion which is some sort of expectation of a functional of the process. However, for the option pricing problem, the option writer would like to guarantee a certain bound on the cost *almost surely.* That is, for almost every path of the noise process, the option writer would like to ensure that they will not incur a loss. Thus, even in this stochastic framework, the option pricing problem is very nearly formulated as a robust control problem.

This paper will present several results related to the application of methods of robust control to the option pricing problem. First, robust control will be applied to obtain a simple option price independent of volatility. This price corresponds to a hedging strategy often referred to as the stop–loss strategy; see Hull (1993). Then it will be shown that robust methods applied to the standard stochastic model yield the Black and Scholes price in a rather easy manner. Along the way, it will be shown that the Black and Scholes price is also the correct price for L^1 driving processes (for a particular model) as well as Brownian motion. Then the robust approach will be used to obtain a result demonstrating the validity of the Black and Scholes price for systems with stochastic volatility in a rather easy manner. We will now elaborate on this.

There are two major approaches to the problem of dealing with disturbances (noise processes) in systems. The first is to model the disturbances as stochastic processes with associated notions of filtrations and probability measures. The second approach is to view these disturbances as *unknown*, but deterministic, processes of finite energy. This approach is embodied in H_∞ and, more generally, robust control methodologies. The second approach has become increasingly popular in recent years. This popularity has been fueled to a large extent by the reduced computational and analytical complexity of the latter approach over the former. Given the above, it seems natural to reformulate the option pricing problem in terms of robust control. As will be seen below, this reformulation leads to new insights and methods for the option pricing problem.

The two different approaches imply different assumptions regarding the price process of the underlying asset. The Black and Scholes option price is often derived through assuming that the asset price process is a continuous semi–martingale driven by a Brownian motion process such as given by the stochastic differential equation (SDE)

$$dP_t = b_t P_t \, dt + \sigma_t P_t \, dB_t \tag{1.1}$$

where P is the price of the underlying asset, b and σ are the mean rate of return and volatility, and B is a Brownian motion process relative to some probability space and filtration. As with all models, these assumptions differ from reality in some respects. For instance, the actual asset price process may not be continuous.

Another model of the asset price process could be used in a direct robust control approach. Here, it is assumed that the driving noise process is not a Brownian motion, but is instead an unknown, deterministic process of finite variation; typically a process in $L^1[0,T]$ for all $T < \infty$. That is, the asset price process would be given by an ordinary differential equation (ODE) such as

$$\frac{dP}{dt} = b(t)P(t) + \sigma(t)P(t)w(t) \tag{1.2}$$

where w is the unknown noise process of finite variation. This model also differs from reality. First of all, this model implies a continuous asset price process which is again at odds with the actual process. (Note that a robust approach with a discrete time model has been considered by Howe, Rustem and Selby (1994).) On the other hand, the driving process, w, now has finite (albeit arbitrarily high) variation. (It should perhaps be noted here that given any absolutely continuous prices process, mean rate of return and volatility such that $\sigma(t)P(t)$ is strictly bounded away from zero, there exists a process $w \in L^1$ such that (1.2) is satisfied.)

These two models will be used to derive two different option prices. The first model may be used with methods of stochastic control to obtain the Black and Scholes price. Although the result is standard, the method is interesting in that the game approach can be used to obtain not only the Black and Scholes price but other results as well. In particular, the second model will be used with the methods of dynamic programming and game theory to obtain a simple robust option price (stop–loss) which corresponds to some first–order option price models; see Hull (1993). Of course this simple robust option price is not of great practical value for well–known reasons. Yet, both of these prices are incorrect not only because of the model discrepancies discussed above, but perhaps more importantly (Figlewski (1989)), because of the transaction costs which are not included in the models used to derive the prices. The question of transaction costs is beyond the scope of what can be included in this paper. However, it will become clear that the following will provide a substantial framework for analysis of that matter.

What is probably of more interest than the robust option price above, is that the robust method can be applied with the stochastic model (1.1) to obtain and expand upon the Black and Scholes results. We now turn to these issues.

Consider (1.1) again; this definition is meaningful only as a shorthand for the integral equation

$$P_t = P_0 + \int_0^t b_u P_u \, du + \int_0^t \sigma_u P_u \, dB_u \tag{1.3}$$

where this last integral is a stochastic integral. There are several definitions for the stochastic integral, and the correct definition even for simple integrands is not so clear as in the case of deterministic calculus. The two most common definitions are the Ito integral and the Stratonovich integral (see for instance, Karatzas and Shreve (1988) for definitions). The interpretation commonly used in finance is the Ito integral. The choice of definition essentially reflects what we believe about the process. Consider first the Stratonovich integral. This is the definition that one would choose if one believed that (1.3) represented the limit of an ODE (i.e. (1.2)) as the noise term converged from a process of bounded variation to a Brownian motion. In this case, the robust analysis for the ODE model would lead one to the *same* option price as derived for the robust model above. Now consider instead the Ito integral. One would use the Ito definition of the stochastic integral in (1.3) if one believed that this integral term must fundamentally represent a martingale. Let us take this martingale approach. There is a relationship between the Ito and Stratonovich integrals. If one interprets the stochastic integral in (1.3) as an Ito integral, then P_t can be written in terms of a Stratonovich integral as

$$P_t = P_0 + \int_0^t \left(b_u P_u - \frac{1}{2}\sigma_u^2 P_u \right) du + \int_0^t \sigma_u P_u \circ dB_u \tag{1.4}$$

where we use the $\circ$ notation in the last term of (1.4) to indicate that this last integral is to be interpreted as a Stratonovich integral. Since this last integral is a Stratonovich integral, (1.4) can be interpreted as the limit of an ODE as the noise process converges to a Brownian motion. This ODE is

$$\frac{dP}{dt} = b(t)P(t) - \frac{1}{2}\sigma^2(t)P(t) + \sigma(t)P(t)w(t). \tag{1.5}$$

We will also show that if one applies the robust approach with model (1.5), one again obtains the Black and Scholes price. That is, the robust approach with asset price model (1.5) rather than (1.2) yields the Black and Scholes option price rather than the simple robust option price. This has the secondary ramification of implying that the Black and Scholes price is valid for L^1 disturbances as well as Brownian motions when model (1.5) is used for the L^1 disturbances. Thus, the robust approach subsumes two previous models. That is, with asset price model (1.2), the robust approach yields a simple option price corresponding to a stop–loss strategy (see Hull (1993)), and with asset price model (1.5), it yields the Black and Scholes price. Both prices are valid *relative* to their respective models. Further, by demonstrating the convergence of (1.5) to (1.4) (with correct choice of w), the Black and Scholes price for the standard stochastic model is obtained by the robust analysis. This may be of significant practical interest as the robust analysis is quite simple.

Although transaction costs are beyond what can be considered in this paper, one can approach the question of stochastic volatility. For the robust approach with model (1.2), the volatility is irrelevant, and the price is independent of volatility. For the Black and Scholes option price however, one can use the robust approach with asset price model (1.5). In particular, this approach will be used to show that the case of bounded stochastic volatility where the volatility depends on the asset price and another, independent, stochastic process can be hedged by the Black and Scholes price corresponding to the upper bound of the volatility.

The outline of the remainder of the paper is as follows. In Section 2, the analysis for the robust approach with model (1.2) will be discussed, and the corresponding simple option price will be derived. In Section 3, the Black and Scholes price will be obtained via the robust approach applied to the prelimit model (1.5) of Stratonovich SDE (1.4) corresponding to Ito SDE (1.3). Finally, in Section 4, the stochastic volatility problem will be analyzed, and the validity of a Black and Scholes price for the problem will be demonstrated. Note that only the European option will be considered.

2. A Stop–Loss Price

In this section, we consider a model of the asset price given by

$$\begin{aligned} \frac{dP}{dt} &= bP + \sigma P w \\ P(s) &= p \end{aligned} \tag{2.1}$$

where b and σ are the (constant) mean rate of return and volatility, and $w(\cdot)$ is the disturbance process. We assume $w \in L^1[s,T]$.

We model the wealth of the contingent claim writer by

$$\begin{aligned} \frac{dX}{dt} &= rX + (b-r)\lambda + \sigma\lambda w \\ X(s) &= x \end{aligned} \tag{2.2}$$

where r is the (constant) riskless rate of return and λ is the amount of wealth invested in the underlying asset at any given time. Note that λ is unrestricted (although the for the optimal hedge below, $\lambda(t) \geq 0$ for all t).

Let q be the strike price, and consider a payoff of the form

$$J(s,x,p,\lambda,w) = [P(T) - q]^{+} - X(T). \tag{2.3}$$

As mentioned in the introduction, the robust control approach involves modeling the problem as a game where the disturbance acts as an antagonistic player, and the control (hedging strategy) is formulated so as to guarantee a certain result for any action of the antagonistic player. The Elliott–Kalton formulation (see Elliott and Kalton (1972), Evans and Souganidis (1984)) of the game will be used. Consequently, we must define non–anticipative strategies.

We say that $\theta : L^1[s,T] \to L^1[s,T]$ is a non–anticipative strategy for the hedging portfolio if the following holds: if $w, \hat{w} \in L^1[s,T]$ are such that $w(u) = \hat{w}(u)$ for all $u \in [s,t]$, then $\theta[w](u) = \theta[\hat{w}](u)$ for all $u \in [s,t]$. Let the set of non–anticipative strategies for the hedging portfolio be denoted by Θ. Similarly, let the strategies for the disturbance be mappings $\xi : L^1[s,T] \to L^1[s,T]$ which are non–anticipative in the analogous sense. Let the set of non–anticipative strategies for the disturbance be denoted by Ξ. The lower value of the game is

$$\underline{W}(t,x,p) = \inf_{\theta \in \Theta} \sup_{w \in L^1} J(t,x,p,\theta[w],w) \tag{2.4a}$$

and the upper value is

$$\overline{W}(t,x,p) = \sup_{\xi \in \Xi} \inf_{\lambda \in L^1} J(t,x,p,\lambda,\xi[\lambda]). \tag{2.4b}$$

It will be shown that $\underline{W} = \overline{W}$, and we denote the common value by $\widetilde{W}$. It is desired that one find the smallest x for given t,p,T such that $\widetilde{W}(t,x,p) \le 0$. The robust option price is defined to be this smallest x. We note here that the fair price of an option is often defined as the smallest x such that there exists a hedging strategy guaranteeing that $\widetilde{W}(t,x,p) \le 0$. (See for instance Karatzas and Shreve (1988), p. 377 or Soner, Shreve and Cvitanic (1995) equation (2.11).) Also note that this is equivalent to a no–arbitrage condition as discussed, for instance, in Davis, Panas and Zariphopoulou (1993). In particular, if the price is either lower or higher than this value, then this presents an arbitrage opportunity for either an option buyer or seller, respectively. One might note that we ask here not for the actual existence of a hedging strategy guaranteeing this result, but only for hedging policies which come arbitrarily close. However, in the analysis below we find that there does, in fact, exist such a hedge.

We will use the PDE (partial differential equation) approach to the problem. Let $\Re^{2+} \doteq \Re \times (0,\infty)$ and $G \doteq (-\infty, T) \times \Re^{2+}$. The Isaacs equation corresponding to the upper value is

$$\begin{aligned} 0 &= W_t + \inf_{\lambda \in \Re} \sup_{w \in \Re} \{[rx + (b-r)\lambda + \sigma \lambda w]W_x + [bp + \sigma p w]W_p\} \\ &= W_t + rxW_x + bpW_p + \inf_{\lambda \in \Re} \sup_{w \in \Re} \{(b-r)\lambda W_x + \sigma[\lambda W_x + pW_p]w\} \qquad \forall\, (t,x,p) \in G \\ W(T,x,p) &= (p-q)^+ - x \qquad \forall\, (x,p) \in \Re^{2+} \end{aligned} \tag{2.5}$$

where we are abusing notation by using λ and w to indicate both the functions in the dynamics and the variables over which the inf and sup are taken in the Isaacs equation. It is clear that the only choice for λ such that the supremum over w in (2.5) would be bounded is

$$\lambda = \frac{-pW_p}{W_x}. \tag{2.6}$$

Therefore, (2.5) becomes

$$\begin{aligned} 0 &= W_t + rxW_x + rpW_p \qquad \forall\, (t,x,p) \in G \\ W(T,x,p) &= (p-q)^+ - x \qquad \forall\, (x,p) \in \Re^{2+}. \end{aligned} \tag{2.7}$$

On the other hand, the Isaacs equation for the lower value is

$$\begin{aligned}
0 &= W_t + \sup_{w\in\Re}\inf_{\lambda\in\Re}\{[rx+(b-r)\lambda+\sigma\lambda w]W_x+[bp+\sigma pw]W_p\} \\
&= W_t + rxW_x + bpW_p + \inf_{\lambda\in\Re}\sup_{w\in\Re}\{(b-r)\lambda W_x+\sigma[\lambda W_x+pW_p]w\} \qquad \forall\,(t,x,p)\in G \qquad (2.8)\\
W(T,x,p) &= (p-q)^+ - x \qquad \forall\,(x,p)\in\Re^{2+}.
\end{aligned}$$

The only choice for w such that the infimum over λ is bounded is

$$w=\frac{r-b}{\sigma},$$

and with this choice of w, one again obtains (2.7). This implies that any solution of (2.7) such that (2.6) is well–defined will be both a solution for (2.5) and a solution for (2.8).

In order to put the following in the standard format, rewrite (2.7) in the form

$$0=-W_t-rxW_x-rpW_p \qquad \forall\,(t,x,p)\in G \qquad (2.9a)$$

$$W(T,x,p)=(p-q)^+-x \qquad \forall\,(x,p)\in\Re^{2+}. \qquad (2.9b)$$

Consider

$$\widetilde{W}(t,x,p)\doteq e^{r(T-t)}\left\{\left[p-qe^{-r(T-t)}\right]^+-x\right\}. \qquad (2.10)$$

Lemma 2.1: $\widetilde{W}$ is a viscosity solution (see, for instance, Fleming and Soner (1992)) of (2.9).

Proof: First note that $\widetilde{W}$ satisfies (2.9b). Also, $\widetilde{W}$ is in $C^\infty(G\setminus S)$ where $S\doteq\{(t,x,p)\in G: pe^{r(T-t)}=q\}$, and it satisfies (2.9a) classically there. On S, the supergradient, $D^+\widetilde{W}$, is empty, and the subgradient is

$$D^-\widetilde{W}=\left\{a\in\Re^3:\, a=\begin{pmatrix}(rx-\alpha rp)e^{r(T-t)}\\ -e^{r(T-t)}\\ \alpha e^{r(T-t)}\end{pmatrix},\ \alpha\in[0,1]\right\}.$$

For $a\in D^-\widetilde{W}$, one has

$$-a_1-rxa_2-rpa_3=-(rx-\alpha rp)e^{r(T-t)}+rxe^{r(T-t)}+\alpha rpe^{r(T-t)}=0.$$

Therefore, $\widetilde{W}$ is a viscosity solution of (2.9). □

From this lemma, one sees immediately that $\widetilde{W}$ is a viscosity solution of (2.8), and it is not difficult to verify that it is also a viscosity solution of (2.5).

Now we turn to the main task of this section. It will be shown that $\widetilde{W}$ is equal to both the upper and lower values of the game (2.4), i.e. $\overline{W}=\widetilde{W}=\underline{W}$. Note that (2.6) combined with (2.10) suggests the following optimal feedback hedging policy

$$\overline{\lambda}(t,x,p)\doteq\begin{cases} p & \text{if } p>qe^{-r(T-t)}\\ 0 & \text{if } p\le qe^{-r(T-t)}.\end{cases} \qquad (2.11)$$

Let $\overline{V} \doteq \sup_{w\in L^1} J(t,x,p,\overline{\lambda},w)$, and note that

$$\underline{W} \leq \overline{V} \tag{2.12}$$

It will now be shown that $\widetilde{W} = \overline{V}$. Let

$$\begin{aligned} R_1 &\doteq G \cap \{(t,x,p) : \, p < qe^{-r(T-t)}\} \\ R_2 &\doteq G \cap \{(t,x,p) : \, p > qe^{-r(T-t)}\} \\ S &\doteq G \cap \{(t,x,p) : \, p = qe^{-r(T-t)}\}. \end{aligned} \tag{2.13}$$

Let

$$F(t,x,p) \doteq \left(p - qe^{-r(T-t)}\right)^+ - x,$$

and note that

$$J(t,x,p,\overline{\lambda},w) = F(T,X(T),P(T)) \tag{2.14}$$

when $X(t) = x$, $P(t) = p$, and X, P evolve with controls $\overline{\lambda}$ and w. Let

$$H(\cdot) \doteq F(\cdot, X(\cdot), P(\cdot)) \tag{2.15}$$

and note that $H \in C[s,T]$. Also note that by taking logs in (2.1) it is easily shown that if $P(s) > 0$, then $P(t) > 0$ for all $t \geq s$, which implies that a trajectory starting in G stays in G for all t.

By (2.15), (2.6), and (2.2), for $t \in [H^{-1}(R_1 \cup S)]^\circ$, one has

$$\frac{dH}{dt} = F_t + F_x\frac{dX}{dt} + F_p\frac{dP}{dt} = -rX = rH, \tag{2.16}$$

and for $t \in H^{-1}(R_2)$,

$$\frac{dH}{dt} = qre^{-r(T-t)} + \frac{dP}{dt} - \frac{dX}{dt} = rH. \tag{2.17}$$

From (2.16) and (2.17), it can be seen that

$$H(s) = e^{-r(T-s)}H(T)$$

which by (2.15)

$$= e^{-r(T-s)}J(s,x,p,\overline{\lambda},w),$$

or in other words,

$$\begin{aligned} J(s,x,p,\overline{\lambda},w) &= e^{r(T-s)}H(s) \\ &= e^{r(T-s)}\left[\left(p - qe^{-r(T-s)}\right)^+ - x\right] \\ &= \widetilde{W}(s,x,p). \end{aligned}$$

Since this is true for all $w \in L^1$, we have

$$\overline{V}(s,x,p) = \widetilde{W}(s,x,p)$$

which by (2.12) yields

$$\underline{W}(s,x,p) \leq \widetilde{W}(s,x,p). \tag{2.18}$$

Now the reverse inequality will be obtained. Let $w_0(t) = \frac{r-b}{\sigma}$ for all t, and note that

$$\underline{W}(s,x,p) \geq \inf_{\theta\in\Theta} J(s,x,p,\theta[w_0],w_0). \tag{2.19}$$

But with this disturbance, w_0,

$$\frac{dP}{dt} = rP$$

and

$$\frac{dX}{dt} = rX$$

independent of θ. Consequently, for initial conditions (s,x,p),

$$\begin{aligned} P(t) &= e^{r(t-s)}p, \\ X(t) &= e^{r(t-s)}x, \end{aligned}$$

and

$$\begin{aligned} J(s,x,p,\theta[w_0],w_0) &= \left(e^{r(T-s)}p - q\right)^+ - e^{r(T-s)}x \\ &= \widetilde{W}(s,x,p). \end{aligned} \tag{2.20}$$

By (2.19) and (2.20)

$$\underline{W}(s,x,p) \geq \widetilde{W}(s,x,p). \tag{2.21}$$

Then, combining (2.18) and (2.21) yields

$$\underline{W}(s,x,p) = \widetilde{W}(s,x,p).$$

A similar argument yields the same result for $\overline{W}$, and we have

$$\underline{W} = \widetilde{W} = \overline{W}.$$

In other words, we have the following.

Theorem 2.2: The game given by (2.1)–(2.4) has value, and that value is the unique viscosity solution of (2.9).

Further, this implies that given $s < T$ and $p > 0$, the fair price of a call option on the underlying asset is the value of x such that

$$\widetilde{W}(s,x,p) = 0.$$

By (2.10), the fair price is

$$x = \left(p - qe^{-r(T-s)}\right)^{+}. \tag{2.22}$$

More exactly, when one models the assest price by (2.1) with an unknown disturbance, $w \in L^1$, one obtains the fair price (2.22). This is the well–known stop–loss hedge price. Although this is theoretically correct for L^1 disturbances, in reality it is not feasible. In particular, note that the hedge λ given by (2.6), (2.10) is

$$\overline{\lambda} = \begin{cases} p & \text{if } p > qe^{-r(T-t)} \\ 0 & \text{if } p \leq e^{-r(T-t)}. \end{cases}$$

Since one cannot trade at exactly the switching price, small losses may occur on each passing of that level. This may lead to arbitrarily large losses (see Hull (1993)).

3. The Black and Scholes Formula

Robust analysis can also be used to obtain the Black and Scholes formula if one uses the proper model. With this in mind, let the asset price and wealth processes satisfy the following stochastic differential equations (SDE's)

$$dP_t = bP_t\,dt + \sigma P_t\,dB_t \tag{3.1a}$$

and

$$dX_t = [rX_t + (b - r)\lambda_t]\,dt + \sigma\lambda_t\,dB_t \tag{3.1b}$$

respectively. Here, we let $B.$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and adapted to some filtration $\mathcal{F}_t$. It is assumed that $\lambda.$ will be progressively measurable (see Fleming and Soner (1992)) with respect to this filtration. (As before, λ_t is not restricted although the optimal value will be nonnegative.) Recall that these SDE's are shorthand for their integral counterparts where the Ito definition of the stochastic integral is used. Let the initial conditions be $P_s = p$ and $X_s = x$. Again, b, r and σ are constant. One might want to note that some authors choose to use Girsanov's Theorem to change probability measures to another measure, $\widehat{\mathcal{P}}$, such that the drift coefficient b is replaced by r in (3.1). This measure, $\widehat{\mathcal{P}}$, is absolutely continuous with respect to $\mathcal{P}$. Since, in the Black and Scholes framework, one desires a result almost surely, one obtains the exact same price after this change of measure. $\widehat{\mathcal{P}}$ is often referred to as the risk–neutral measure. Since we are interested in results using game fomulations (such as in the previous section) as well as the Black and Scholes model, we will use model (3.1) to maintain the formal correspondence with model (2.1)–(2.2) rather than transform to the risk–neutral measure.

Consider a feedback hedging policy of the form

$$\lambda_t = L(t, X_t, P_t).$$

With such a policy, one may rewrite (3.1) in terms of Stratonovich (i.e. Fisk–Stratonovich) SDE's of the form (see, for instance, Karatzas and Shreve, Remark 5.2.22 (1988) or Kloeden, Platen and Schurz, Sec. 2.2.D (1991))

$$dP_t = \left[bP_t - \frac{1}{2}\sigma^2 P_t\right] dt + \sigma P_t \circ dB_t \tag{3.2a}$$

and

$$\begin{aligned} dX_t = &\left[rX_t + (b-r)L(t,X_t,P_t) - \frac{1}{2}\sigma^2\left(L_x(t,X_t,P_t)L(t,X_t,P_t) + L_p(t,X_t,P_t)P_t\right)\right] dt \\ &+ \sigma L(t,X_t,P_t) \circ dB_t \end{aligned} \tag{3.2b}$$

where the $\circ$ notation indicates the Stratonovich integral.

Now let $\epsilon > 0$, and let

$$v^\epsilon(t) \doteq \left[B_{s+\epsilon(i+1)} - B_{s+\epsilon i}\right] \frac{1}{\epsilon}$$

for all $t \in [s+\epsilon i, s+\epsilon(i+1))$, $i \in \{0,1,2,...\}$, and $\omega \in \Omega$. Let $B^\epsilon(t) \doteq \int_s^t v^\epsilon(r)\,dr$ for all $t \geq s$ (where s is the initial time) and for all $\omega \in \Omega$. Then $B^\epsilon(\cdot)$ is a piecewise linear interpolation of $B.$ which converges to $B.$ for each $\omega \in \Omega$. For each $\omega \in \Omega$, consider the ordinary differential equations (ODE's)

$$\frac{dP^\epsilon}{dt} = \left[bP^\epsilon - \frac{1}{2}\sigma^2 P^\epsilon\right] + \sigma P^\epsilon v^\epsilon \tag{3.3a}$$

and

$$\begin{aligned} \frac{dX\epsilon}{dt} = &\left[rX^\epsilon + (b-r)L(t,X^\epsilon,P^\epsilon) - \frac{1}{2}\sigma^2\left(L_x(t,X^\epsilon,P^\epsilon)L(t,X^\epsilon,P^\epsilon) + L_p(t,X^\epsilon,P^\epsilon)P^\epsilon\right)\right] \\ &+ \sigma L(t,X^\epsilon,P^\epsilon)v^\epsilon. \end{aligned} \tag{3.3b}$$

Let

$$C_b^4 = \left\{ f : \Re^3 \to \Re : \sup_{(t,x,p)} \sum_{1\leq|\alpha|\leq 4} D^\alpha f < \infty \text{ and } \sup_{(t,x,p)} \frac{|f(t,x,p)|}{1+|(t,x,p)|} < \infty \right\}$$

where multi–index notation is being used for the partial derivatives. If $L \in C_b^4$, then by Kunita, Th. 5.7.3 and Ex. 5.7.4 (1990), $(X^\epsilon, P^\epsilon) \to (X, P)$ as C^1–flows as $\epsilon \downarrow 0$. In particular,

$$E\left[\sup\{|X^\epsilon(t) - X_t| : t \in [s,T], |x| \leq N\}\right] \to 0$$

as $\epsilon \downarrow 0$ for all $N < \infty$, and similarly for $|P^\epsilon - P|$. This implies that

$$\left|(P^\epsilon(T) - q)^+ - X^\epsilon(T) - \left[(P_T - q)^+ - X_T\right]\right| \to 0 \qquad \text{a.s.} \tag{3.4}$$

for any initial condition.

With this in mind, consider the following deterministic game. Let the dynamics be given by

$$\frac{dP}{dt} = bP - \frac{1}{2}\sigma^2 P + \sigma P w \tag{3.5a}$$
$$\frac{dX}{dt} = rX + (b-r)L - \frac{1}{2}\sigma^2[L_x L + L_p P] + \sigma L w \tag{3.5b}$$

where $w \in L^1[s,T]$, $L(t,x,p)$ is a feedback hedging policy, and the initial conditions are $P(s) = p$ and $X(s) = x$. Let the payoff be given by

$$J(s,x,p,w) = (P(T)-q)^+ - X(T) \tag{3.6}$$

where we suppress the dependence on L since that will be chosen a priori. In particular, we will look for a feedback policy L such that, for proper choice of x,

$$\underline{W}(s,x,p) \doteq \sup_{w \in L^1} J(s,x,p,w) \le 0. \tag{3.7}$$

Although this game can be defined without reference to the above stochastic limit, note that by (3.4), if such a feedback hedging policy exists (and $L \in C_b^4$), then the stochastic system (3.1) would be such that

$$(P_T - q)^+ - X_T \le 0 \qquad \text{a.s.}$$

As in Section 2, let $\Re^{2+} \doteq \Re \times (0,\infty)$ and $G \doteq (-\infty, T) \times \Re^{2+}$. The dynamic programming equation corresponding to (3.5)–(3.7) is

$$\begin{aligned} 0 = W_t + \sup_{w \in \Re} \bigg\{ &\left[rx + (b-r)L - \frac{1}{2}\sigma^2(L_x L + L_p p) + \sigma L w \right] W_x \\ &+ \left[bp - \frac{1}{2}\sigma^2 p + \sigma p w \right] W_p \bigg\} \qquad (t,x,p) \in G \\ W(T,x,p) = (p-q)^+ - x \qquad &(x,p) \in \Re^{2+}. \end{aligned} \tag{3.8}$$

In order for the supremum in (3.8) to be bounded, we choose the feedback policy

$$L(t,x,p) = \frac{-pW_p(t,x,p)}{W_x(t,x,p)}. \tag{3.9}$$

With this choice, (3.8) becomes

$$\begin{aligned} &0 = W_t + rxW_x + rpW_p + \frac{1}{2}\sigma^2 p^2 \left[W_{pp} - 2\frac{W_p}{W_x} W_{px} + \left(\frac{W_p}{W_x}\right)^2 W_{xx} \right] \qquad (t,x,p) \in G \\ &W(T,x,p) = (p-q)^+ - x \qquad (x,p) \in \Re^{2+}. \end{aligned} \tag{3.10}$$

Recall that the Black and Scholes price is $\hat{x} = V(s,p)$ where V satisfies

$$\begin{aligned} &0 = -V_t + rV - rpV_p - \frac{1}{2}\sigma^2 p^2 V_{pp} \qquad (t,p) \in (-\infty, T) \times (0,\infty) \\ &V(T,p) = (p-q)^+ \qquad p \in (0,\infty). \end{aligned} \tag{3.11}$$

If it can be shown that the solution to (3.10) exists and is the value of (3.5)–(3.7), then it will have been shown that the Black and Scholes price is sufficient to hedge model (3.5)–(3.7). Further, if it can be shown that the corresponding feedback policy $L \in C_b^4$, then the Black and Scholes price for the stochastic model (3.1) will have been obtained via the robust approach applied to the approximating systems. These results follow.

Let $\overline{V}$ be the solution to (3.11) (i.e. the Black and Scholes price), and let

$$\widetilde{W}(s,x,p) = e^{r(T-s)}\left[\overline{V}(s,p) - x\right]. \tag{3.12}$$

Theorem 3.1: $\widetilde{W}$ given by (3.12) is in $C^\infty(G) \cap C((-\infty, T] \times \Re^{2+})$, and is a solution to (3.10).

Proof: The smoothness follows from the smoothness of $\overline{V}$ as given by the Black and Scholes formula. The fact that it is a solution is easily verified by substitution. □

Remark: Notice the similarity of (3.12) to (2.10) particularly when the explicit Black and Scholes formula is substituted for $\overline{V}$.

Remark: With W given by (3.12), hedging portfolio (3.9) is given by

$$L(t,x,p) = p\overline{V}_p(t,p). \tag{3.13}$$

Now that we have obtained the existence of a smooth solution to (3.10), we will employ a verification argument to show that this solution is the value of the deterministic problem (3.5)–(3.7) with feedback control (3.9) (i.e. (3.13)).

Let $(s,x,p) \in G$, and let P and X be given by (3.5) with hedging policy (3.9) and initial conditions $P(s) = p$ and $X(s) = x$. It is easily verified that $(X(t), P(t))$ stays in $\Re^{2+}$ by letting $Q \doteq \log P$, and noting that Q satisfies

$$\frac{dQ}{dt} = b - \frac{1}{2}\sigma^2 + \sigma w.$$

Since $w \in L^1$, $P(t) > 0$ for all $t \geq s$. Integrating along the trajectory one has

$$\widetilde{W}(T, X(T), P(T)) = \widetilde{W}(s,x,p) + \int_s^T \widetilde{W}_t + \widetilde{W}_x \frac{dX}{dt} + \widetilde{W}_p \frac{dP}{dt}\, dt.$$

Using (3.5) to substitute for $\frac{dX}{dt}$ and $\frac{dP}{dt}$ and cancelling terms yields

$$\begin{aligned}\widetilde{W}(T, X(T), P(T)) = \widetilde{W}(s,x,p) + \int_s^T \Bigg\{ & \widetilde{W}_t + rX\widetilde{W}_x + rP\widetilde{W}_p \\ & + \frac{1}{2}\sigma^2 P^2 \left[\widetilde{W}_{pp} - 2\frac{\widetilde{W}_p}{\widetilde{W}_x}\widetilde{W}_{px} + \left(\frac{\widetilde{W}_p}{\widetilde{W}_x}\right)^2 \widetilde{W}_{xx} \right] \Bigg\}\, dt \qquad \forall\, w \in L^1\end{aligned}$$

which by (3.10)

$$= \widetilde{W}(s,x,p) \qquad \forall\, w \in L^1.$$

Applying the terminal condition of (3.10) to the left hand side of this yields

$$\widetilde{W}(s,x,p) = (P(T) - q)^+ - X(T) \qquad \forall\, w \in L^1$$

which, by (3.6), implies

$$\widetilde{W}(s,x,p) = J(s,x,p,w) \qquad \forall\, w \in L^1 \tag{3.14}$$

when L is given by (3.9). This yields

Theorem 3.2: $\widetilde{W}$ is the value of the game (3.5)–(3.7) with feedback (3.9).

Remark: This also implies that $\widetilde{W}$ is the unique smooth solution of (3.10).

Note that since $\widetilde{W}(s,x,p) = 0$ for $x = \overline{V}(t,p)$, we have established that the Black and Scholes price is correct price for robust model (3.5) (as well as the standard stochastic model). As a corollary, recall from (3.13) that $L(t,x,p) = p\overline{V}_p(t,p)$. Using the explicit Black and Scholes formula (see, for instance, Karatzas and Shreve (1988)), for $\overline{V}$, it is easy to see that $L \in C_b^4$. Since $\overline{V}$ is the correct value for game (3.5)–(3.7), this implies by our earlier remarks that it is also the correct price for model (3.1). That is, the Black and Scholes result is obtained through robust analysis applied to the pre–limit processes. Of course, this yields something more since the Black and Scholes price has been shown to be valid for all L^1 disturbances rather than just those piecewise linear disturbances converging to Brownian motion.

Also note that by (3.14), the payoff is independent of the choice of $w \in L^1$. Therefore, with initial wealth $\overline{x} = \overline{V}$ and hedging policy (3.13), $J(s,\overline{x},p,w) = \widetilde{W}(s,\overline{x},p) = 0$ for all $w \in L^1$; the option writer neither profits nor loses for any w.

4. Stochastic Volatility and the Black and Scholes Price

The Black and Scholes price is typically derived under the assumption of constant volatility (or volatility which is a deterministic function of time). In this section, we consider the case where the volatility is stochastic and bounded, and show that a Black and Scholes price is still valid. There are many possibilities for the volatility model. One such model is to have the volatility depend on the price of the underlying asset and another, independent, stochastic process. This is the model which will be discussed immediately below. The results can be generalized in a number of ways, and some of these appear in remarks that will follow later. It should be mentioned that a related case where the volatility is adapted to the Brownian motion driving the price process (but not also another process) was recently examined in Jeanblanc–Picque, El Karoui and Viswanathan (1992).

Consider the model

$$dP_t = bP_t\,dt + \sigma(P_t, Y_t)P_t\,dB_t \tag{4.1a}$$

and

$$dX_t = [rX_t + (b-r)\lambda_t]\,dt + \sigma(P_t, Y_t)\lambda_t\,dB_t \tag{4.1b}$$

$$dY_t = f_1(Y_t) + f_2(Y_t)\,dB_t^{(2)} \tag{4.1c}$$

where $B_{\cdot}^{(2)}$ is another Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ independent of $B_{\cdot}$. As usual, the initial conditions are $P_s = p$, $X_s = x$ and $Y_s = y$. The parameters b and r are constant. It will be assumed that $\sigma(\cdot,\cdot) \in C_b^4$, that all the derivatives of σ up to order four are bounded above by $\frac{K}{1+|p|}$ for some $K < \infty$ (but not necessarily the function itself), and that $\sigma(p,y) \in [0, \sigma_2]$ where $\sigma_2 < \infty$ for all p, y. Also, assume that f_1 is Lipschitz, and $f_2 \in C_b^4$. The smoothness assumptions on σ_2 and f_2 will later be weakened. It will be shown that the Black and Scholes price for the stochastic model with constant volatility σ_2 is sufficient to hedge an option with model (4.1). This is what would be expected intuitively. However, it is not as obvious as one would expect in the light of the counterexample of Jeanblanc–Picque, El Karoui and Viswanathan (1992).

As in the previous section, we rewrite (4.1) in terms of Stratonovich integrals rather than Ito integrals. For this purpose, let λ_t be given in feedback form as

$$\lambda_t = L(t, X_t, P_t, Y_t). \tag{4.2}$$

Now consider equation (4.1a). Note that

$$\begin{aligned}\sigma(P_t, Y_t)P_t = A_t &+ \int_s^t [P_u\sigma_p(P_u, Y_u) + \sigma(P_u, Y_u)]\,\sigma(P_u, Y_u)P_u\,dB_u \\ &+ \int_s^t \sigma_y(P_u, Y_u)P_u f_2(Y_u)\,dB_u^{(2)}\end{aligned} \tag{4.3}$$

where A_t is an adapted process of bounded variation. By (4.3), the cross–variation of σP with B is

$$\langle \sigma P_t, B_t\rangle = \left\langle \int_s^t [P_u\sigma_p + \sigma]\sigma P_u\,dB_u + \int_s^t \sigma_y P_u f_2\,dB_u^{(2)}, B_t \right\rangle$$

which by Karatzas and Shreve, (3.2.26) or Prop. 3.2.17 (1988)

$$= \int_s^t [P_u\sigma_p + \sigma]\sigma P_u\,d\langle B_u, B_u\rangle + \int_s^t \sigma_y P_u f_2\,d\langle B_u^{(2)}, B_u\rangle$$

which since B and $B^{(2)}$ are independent Brownian motions

$$= \int_s^t [P_u\sigma_p + \sigma]\sigma P_u\,du.$$

This implies that

$$\int_s^t \sigma(P_u, Y_u) P_u \, dB_u = \int_s^t \sigma(P_u, Y_u) P_u \circ dB_u - \frac{1}{2}\int_s^t \left[P_u^2 \sigma(P_u, Y_u)\sigma_p(P_u, Y_u) + \sigma^2(P_u, Y_u) P_u\right] dt. \tag{4.4}$$

Similarly, one has

$$\begin{aligned} \int_s^t \sigma(P_u, Y_u) L(u, X_u, P_u, Y_u) \, dB_u &= \int_s^t \sigma(P_u, Y_u) L(u, X_u, P_u, Y_u) \circ dB_u \\ &\quad - \frac{1}{2}\int_s^t \left[\sigma^2 L_x L + \sigma^2 L_p P_u + \sigma\sigma_p P_u L\right] du \end{aligned} \tag{4.5}$$

where, for simplicity, we have suppressed the arguments in the last integral. Employing (4.4) and (4.5), we may rewrite (4.1) with hedging policy (4.2) in terms of Statonovich integrals as follows.

$$dP_t = \left\{bP_t - \frac{1}{2}\sigma P_t[\sigma + \sigma_p P_t]\right\} dt + \sigma P_t \circ dB_t \tag{4.6a}$$

$$\begin{aligned} dX_t &= \left\{rX_t + (b-r)L - \frac{1}{2}[\sigma^2 L_x L + \sigma^2 L_p P_t + \sigma\sigma_p P_t L]\right\} dt \\ &\quad + \sigma L \circ dB_t \end{aligned} \tag{4.6b}$$

$$\begin{aligned} dY_t &= [f_1 - \frac{1}{2} f_2 f_{2y}] \, dt + f_2 \circ dB_t^{(2)} \\ &\doteq \hat{f}_1 \, dt + f_2 \circ dB_t^{(2)}. \end{aligned} \tag{4.6c}$$

As before, one would like to find the smallest initial value x such that

$$(P_T - q)^+ - X_T \leq 0 \qquad \text{a.s.} \tag{4.7}$$

Again, as in Section 3, we consider the piecewise linear approximations to B and $B^{(2)}$ which we denote by $B_t^\epsilon = \int_s^t v^\epsilon(u)\, du$ and $B_t^{(2),\epsilon} = \int_s^t v^{(2),\epsilon}(u)\, du$. Then consider the ODE's (for each $\omega \in \Omega$)

$$\frac{dP^\epsilon}{dt} = \left\{bP^\epsilon - \frac{1}{2}\sigma P^\epsilon[\sigma + \sigma_p P^\epsilon]\right\} + \sigma P^\epsilon v^\epsilon \tag{4.8a}$$

$$\begin{aligned} \frac{dX^\epsilon}{dt} &= \left\{rX^\epsilon + (b-r)L - \frac{1}{2}[\sigma^2 L_x L + \sigma^2 L_p P^\epsilon + \sigma\sigma_p P^\epsilon L]\right\} \\ &\quad + \sigma L v^\epsilon \end{aligned} \tag{4.8b}$$

$$\frac{dY^\epsilon}{dt} = \hat{f}_1 + f_2 v^{(2),\epsilon}. \tag{4.8c}$$

If $L \in C_b^4$, then by the assumptions on σ, $\sigma L \in C_b^4$ and

$$\left|(P^\epsilon(T) - q)^+ - X^\epsilon(T) - \left[(P_T - q)^+ - X_T\right]\right| \to 0 \qquad \text{a.s.} \tag{4.9}$$

This leads us to consider the following deterministic game. Let the dynamics be given by

$$\frac{dP}{dt} = \left[bP - \frac{1}{2}\sigma P(\sigma + \sigma_p P)\right] + \sigma P w \tag{4.10a}$$

$$\frac{dX}{dt} = rX + (b-r)L - \frac{1}{2}[\sigma^2 L_x L + \sigma^2 L_p P + \sigma\sigma_p P L] + \sigma L w \tag{4.10b}$$

$$\frac{dY}{dt} = \hat{f}_1 + f_2 w^{(2)} \tag{4.10c}$$

where $w, w^{(2)} \in L^1$ and $L = L(t,x,p,y)$ is a feedback policy as above. Let the payoff be given by

$$J(s,x,p,y,w,w^{(2)}) = (P(T)-q)^+ - X(T) \tag{4.11}$$

where again the dependence on L is suppressed since it is to be chosen a priori. In particular, we will look for a feedback policy L such that, for proper choice of x,

$$\underline{W}(s,x,p,y) \doteq \sup_{w,w^{(2)}\in L^1} J(s,x,p,y,w,w^{(2)}) \le 0. \tag{4.12}$$

Note that if there exists a hedge $L \in C_b^4$ such that $\underline{W}(s,x,p,y) = 0$, then condition (4.7) will be met for the original stochastic system (4.1) as well (see Section 3). That is, L would be a hedge for the stochastic system guaranteeing (4.7), as well as being a hedge for the deterministic game such that (4.12) is guaranteed.

The dynamic programming equation corresponding to (4.10)–(4.12) is

$$\begin{aligned} 0 = W_t + \sup_{w\in\Re}\sup_{w^{(2)}\in\Re} \Big\{ &\Big[bp - \frac{1}{2}\sigma^2 p(\sigma+\sigma_p p) + \sigma p w\Big] W_p \\ &+ \Big[rx + (b-r)L - \frac{1}{2}(\sigma^2 L_x L + \sigma^2 L_p p + \sigma\sigma_p pL) + \sigma L w\Big] W_x \qquad (4.13a)\\ &+ \Big[\hat f_1 + f_2 w^{(2)}\Big] W_y \Big\} \qquad (t,x,p,y)\in G\times\Re \\ W(T,x,p,y) = (p-q)^+ - x \qquad &(x,p,y)\in\Re^{2+}\times\Re. \qquad (4.13b) \end{aligned}$$

In order that the supremum in (4.13a) be bounded, we choose the feedback hedging policy

$$L(t,x,p,y) = \frac{-pW_p(t,x,p,y)}{W_x(t,x,p,y)}. \tag{4.14}$$

With this choice, the dynamic programming equation becomes

$$\begin{aligned} 0 = W_t + \sup_{w^{(2)}\in\Re} \Big\{ &\Big[rp - \frac{1}{2}\sigma^2 p\Big] W_p \\ &+ \Big[rx - \frac{1}{2}(\sigma^2 L_x L + \sigma^2 L_p p)\Big] W_x + \Big[\hat f_1 + f_2 w^{(2)}\Big] W_y \Big\} \qquad (t,x,p,y)\in G\times\Re \qquad (4.15a)\\ W(T,x,p,y) = (p-q)^+ - x \qquad &(x,p,y)\in\Re^{2+}\times\Re. \qquad (4.15b) \end{aligned}$$

The following long remark is not essential to the proof of the results to follow, but provides a formal rationale for the results.

Remark: The problem corresponding to (4.15) has dynamics

$$\begin{aligned} \frac{dP}{dt} &= rP - \frac{1}{2}\sigma^2 P \\ \frac{dX}{dt} &= rX - \frac{1}{2}\sigma^2[L_x L + L_p P] \\ \frac{dY}{dt} &= \hat f_1 + f_2 w^{(2)} \end{aligned}$$

and value

$$\underline{W}'(s,x,p,y) = \sup_{w^{(2)}\in L^1} [P(T)-q]^+ - X(T).$$

Since X and P are affected by $w^{(2)}$ only through σ, formally one would expect this to be equivalent to the game with dynamics

$$\begin{aligned}\frac{dP}{dt} &= rP - \frac{1}{2}\sigma^2 P\\ \frac{dX}{dt} &= rX - \frac{1}{2}\sigma^2[L_x L + L_p P]\end{aligned}$$

and value

$$\underline{W}''(s,x,p,y) = \sup_{\sigma\in\Sigma}[P(T)-q]^+ - X(T)$$

where Σ would be some appropriate family of functions. The dynamic programming equation corresponding to this game would be

$$0 = W_t + rpW_p + rxW_x + \max_{\sigma\in[0,\sigma_2]} \frac{1}{2}\sigma^2 \left[W_{pp} - 2\frac{W_p}{W_x}W_{px} + \left(\frac{W_p}{W_x}\right)^2 W_{xx}\right] \tag{4.16a}$$

$$W(T,x,p,y) = (p-q)^+ - x. \tag{4.16b}$$

That is, we formally obtain a PDE similar to that of Section 3, but with the addition of a maximization over σ. This concludes the formal remark.

Let $\overline{V}_{\sigma_2}$ be the Black and Scholes price given constant volatility σ_2. Then it is not difficult to show that

$$\widetilde{W}_{\sigma_2}(s,x,p,y) \doteq e^{r(T-s)}\left[\overline{V}_{\sigma_2}(s,p) - x\right] \tag{4.17}$$

is a classical solution to (4.16) since, with $W=\widetilde{W}$, the last term in brackets on the right of (4.16a) is just $-[\widetilde{W}_{\sigma_2}]_x[\overline{V}_{\sigma_2}]_{pp}$ which is non–negative by the well–known convexity of $\overline{V}_{\sigma_2}$ in p. Note that the solution is independent of y.

Note that $\widetilde{W}_{\sigma_2}$ satisfies (4.13), (4.14) with an inequality; that is $\widetilde{W}_{\sigma_2}$ satisfies partial differential inequality (PDI)

$$\begin{aligned}0 \geq W_t + rpW_p + rxW_x + \sup_{w^{(2)}\in\Re}&\left\{\frac{1}{2}\sigma^2\left[W_{pp} - 2\frac{W_p}{W_x}W_{px} + \left(\frac{W_p}{W_x}\right)^2 W_{xx}\right]\right. \qquad (4.18a)\\ &\left.+\left[\hat{f}_1 + f_2 w^{(2)}\right]W_y\right\} \qquad (t,x,p,y)\in G\times\Re\end{aligned}$$

$$W(T,x,p,y) = (p-q)^+ - x \qquad (x,p,y)\in\Re^{2+}\times\Re. \tag{4.18b}$$

Further, recall from Section 3 that $\widetilde{W}_{\sigma_2}$ is smooth, and that any trajectory starting with $(p,x,y)\in\Re^{2+}\times\Re$ will stay in $\Re^{2+}\times\Re$ for all $t\geq s$.

Therefore, we may integrate along the trajectory to obtain

$$\widetilde{W}_{\sigma_2}(T,X(T),P(T),Y(T)) = \widetilde{W}_{\sigma_2}(s,x,p,y) + \int_s^T [\widetilde{W}_{\sigma_2}]_t + [\widetilde{W}_{\sigma_2}]_x\frac{dX}{dt} + [\widetilde{W}_{\sigma_2}]_p\frac{dP}{dt} + [\widetilde{W}_{\sigma_2}]_y\frac{dY}{dt}\,dt.$$

Using (4.10) and (4.14) to substitute for $\frac{dP}{dt}$, $\frac{dX}{dt}$, noting that $[\widetilde{W}_{\sigma_2}]_y = 0$, and cancelling terms yields

$$\widetilde{W}_{\sigma_2}(T, X(T), P(T), Y(T)) = \widetilde{W}_{\sigma_2}(s, x, p, y) + \int_s^T \Bigg\{ [\widetilde{W}_{\sigma_2}]_t + rP[\widetilde{W}_{\sigma_2}]_p + rX[\widetilde{W}_{\sigma_2}]_x$$

$$+ \frac{1}{2}\sigma^2 P^2 \left[[\widetilde{W}_{\sigma_2}]_{pp} - 2\frac{[\widetilde{W}_{\sigma_2}]_p}{[\widetilde{W}_{\sigma_2}]_x}[\widetilde{W}_{\sigma_2}]_{px} + \left(\frac{[\widetilde{W}_{\sigma_2}]_p}{[\widetilde{W}_{\sigma_2}]_x}\right)^2 [\widetilde{W}_{\sigma_2}]_{xx} \right] \Bigg\} dt \qquad \forall\, w \in L^1$$

which by (4.18a)

$$\leq \widetilde{W}(s, x, p, y) \qquad \forall\, w, w^{(2)} \in L^1.$$

Therefore, by (4.18b),

$$\widetilde{W}_{\sigma_2}(s, x, p, y) \geq \widetilde{W}_{\sigma_2}(T, X(T), P(T), Y(T)) = [P(T) - q]^+ - X(T) \qquad \forall\, w, w^{(2)} \in L^1. \tag{4.19}$$

Recall that $\overline{V}_{\sigma_2}$ is the Black and Scholes price with constant volatility σ_2, and that

$$\widetilde{W}_{\sigma_2}(s, \overline{V}_{\sigma_2}(s, p), p, y) = 0. \tag{4.20}$$

Combining (4.19) and (4.20) one obtains

$$0 \geq (P(T) - q)^+ - X(T) \qquad \forall\, w, w^{(2)} \in L^1 \tag{4.21}$$

if $x = \overline{V}_{\sigma_2}(s, p)$. In other words, $\overline{V}_{\sigma_2}(s, p)$ provides sufficient wealth such that with hedging policy (4.14),(4.17), $X(T) \geq (P(T) - q)^+$ for all $w, w^{(2)} \in L^1$. We now have

Theorem 4.1: The Black and Scholes price for constant volatility σ_2 provides sufficient initial wealth to hedge robust model (4.10)–(4.12).

Corollary 4.2: The Black and Scholes price for constant volatility σ_2 provides sufficient initial wealth to hedge stochastic model (4.1).

Proof: As noted in Section 3, hedging policy L given by (4.14), (4.17) is $L = p[\overline{V}_{\sigma_2}]_p \in C_b^4$. The Corollary then follows from the Theorem and (4.9). □

Remark: We may now weaken the assumptions that f_2 and σ are four–times continuously differentiable. Instead, assume that f_2 and σp are uniformly continuous on $G \times \Re$. Then, employing standard mollifications, one finds that given $\delta > 0$ there exist f_2^δ and σ^δ such that $f_2^\delta, \sigma^\delta \in C^\infty$ and

$$|f_2^\delta(y) - f_2(y)| < \delta$$

$$|\sigma^\delta(p, y)p - \sigma(p, y)p| < \delta$$

and

$$|\sigma^\delta(p,y)L - \sigma(p,y)L| = |\sigma^\delta(p,y)p[\overline{V}_{\sigma_2}]_p(s,p) - \sigma(p,y)p[\overline{V}_{\sigma_2}]_p(s,p)| < \delta$$

for all $(s,x,p,y) \in G \times \Re$.

Suppose that these mollifications satisfy the original assumptions on f_2 and σ (i.e. that all derivatives up to order four of σ^δ are bounded above by $\frac{K}{1+|p|}$ and all derivatives up to order four of f_2^δ are bounded). For example, this will be true if σ and f_2 satisfied the assumptions (without the continuous differentiability) and were piecewise linear.

Let P^δ, X^δ and Y^δ be the corresponding stochastic processes given by (4.1) with σ^δ and f_2^δ. Then, by Krylov, Cor 2.5.5 (1980), $E|P_t^\delta - P_t| \to 0$, $E|X_t^\delta - X_t| \to 0$, and $E|Y_t^\delta - Y_t| \to 0$ for all $t \in [s,T]$. This implies that

$$\begin{aligned} |P_t^\delta - P_t| &\to 0 \qquad \text{a.s.} \\ |X_t^\delta - X_t| &\to 0 \qquad \text{a.s.} \\ |Y_t^\delta - Y_t| &\to 0 \qquad \text{a.s.} \end{aligned} \tag{4.22}$$

But by Corollary 4.2,

$$\left(P_T^\delta - q\right)^+ - X_T^\delta \le 0 \qquad \text{a.s.}$$

if $x = \overline{V}_{\sigma_2+\delta}(s,p)$. Combining this with (4.22) yields

$$(P_T - q)^+ - X_T \le 0 \qquad \text{a.s.}$$

That is, the Black and Scholes price for constant volatility σ_2 is still sufficient to hedge the option.

Remark: The result can also be generalized to

$$\sigma(t,p,y) \in [0, \sigma_2(t)] \qquad \forall\, (t,p,y) \in (-\infty) \times (0,\infty) \times \Re$$

where $0 < \sigma_2(t) \le \overline{\sigma}_2 < \infty$ for all t. See Jeanblanc–Picque, El Karoui and Viswanathan (1992) for the corresponding Black and Scholes formula.

Remark: We can generalize the dependence of σ to incorporate other factors as well. For instance, one could suppose that σ might depend on the appearance of a "head–and–shoulders" pattern in the price of the asset. This can be accomplished by introducing another process, say $I_\cdot$, given by $I_t \doteq \int_s^t g(u)P_u\,du$ for a particular function g, and letting $\sigma = \sigma(P_t, Y_t, I_t)$ in (4.1). This would not cause any new terms to appear indirectly in the Stratonovich form (4.6), and the subsequent analysis would be unchanged.

The author would like to thank Wendell Fleming and Steven Shreve for helpful discussions.

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