Robust/H_∞ Filtering for Nonlinear Systems

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November 30, 1996. Revised: November 7, 1997.

Abstract

A robust (or H_∞) approach to filtering for nonlinear systems is considered. A bound on the estimate error as a function of the disturbance energy is obtained. The corresponding dynamic programming equation is a first–order PDE. This has computational ramifications. The case where the measurements are discrete–time is considered also. A numerical method is discussed.

Key Words: nonlinear filtering, H_∞ control, robust control, viscosity solutions

1. Introduction

We present a robust approach to the filtering problem. The term robust is used here to indicate the natural generalizations of the original H_∞ control problem. The generalizations include nonlinear H_∞ control, robust escape problems, robust control under partial information, and robust filtering. These approaches can be formulated in terms of deterministic, differential games where the disturbances represent antagonistic players (see for instance [1], [13], [14] among many notable others). Further, it has recently been shown that the nonlinear H_∞ problem is equivalent to the risk–averse limit of a risk–sensitive stochastic formulation of the problem ([5], [6]) thus yielding a stochastic interpretation.

The fact that the disturbances are chosen by an antagonistic player (trying to degrade one's filter estimate quality) rather than being random leads naturally to a worst–case based strategy. This is especially appropriate for systems where one desires to track a non–cooperative target [4]. In such a case the state would not be expected to follow a nominal

* Research partially supported by ONR grant N0014-96-1-0267 and AFOSR grant F49620-95-1-0296. wmm@math.ncsu.edu

path with diffusion–type disturbances around that trajectory.

The robust/H_∞ approach uses a dynamical model of the form

$$\frac{dX}{dt} = f(X) + \sigma(X)w$$

where X is the state, f represents the nominal dynamics, w is a deterministic, but (a priori) unknown, L_2 process, and σ is some multiplier on the disturbance. This model is in contrast to the diffusion model

$$dX_t = f(X)\,dt + \sigma(X)\,dW_t$$

where $W_\cdot$ is a Brownian motion. The fact that the disturbance in the robust model is an L_2 process rather than a Brownian motion has important numerical ramifications. In particular, the information is carried along characteristics which propagate at a finite rate, thus allowing one to use numerical approaches such as the one described below.

We will use an analogous measurement model of the form

$$z(t) = g(X(t)) + \rho(X(t))v(t)$$

where z is the measurement process, g is the nominal measurement model, v is a finite–energy noise signal, and ρ is some multiplier on the noise. One can view w and v together as antagonistic to our goal, and attempt to attenuate their effect. This will be called Robust (or H_∞) Filtering.

In Section 2, we will use dynamic programming to obtain a value function (or information state). We will demonstrate that the viscosity solution of the dynamic programming equation is this value function. We will also discuss two related filter state estimates: the Mortensen [16] estimator and the robust filter. For the robust filter, we will obtain a bound on the error in the estimate in terms of the energy of the disturbances. We will consider the case of discrete–time measurements in Section 3. In Section 4, we will discuss a numerical method appropriate to this problem. Note that the dynamic programming equations for these problems will be first–order equations rather than the second–order equations associated with the standard filtering formulation. For most nonlinear systems, the computational costs of optimal (stochastic) filtering are prohibitive for real–time computation for all but the lowest dimensional systems. With robust filtering however, it appears that the computational load will be reduced substantially.

Finally, we note that some of this work harks back to Mortensen [16] and Hijab [8], but is now set in the framework of nonlinear H_∞ control and viscosity solutions. In particular, the information state is quite close. However, this information state is interpreted in an H_∞ framework which then leads to a different estimate. This work is also closely related to the study of H_∞ control under partial observations [10], [11]. However, in this case, the chosen criterion leads to a finite–dimensional dynamic programming equation.

2. Basic Development

As noted in the introduction, for the robust filter we model the state dynamics with an ODE of the form

$$\begin{aligned} \frac{dX}{dt} &= f(X) + \sigma(X)w \\ X(0) &= x_0 \end{aligned} \tag{2.1}$$

where X is the state taking values in $\Re^n$ and $w \in L_2$ is the disturbance taking values in $\Re^m$. σ is $n \times m$. We assume that filtering begins at time $t = 0$.

We suppose that the measurements take the form

$$z(t) = g(X(t)) + \rho(X(t))v(t) \tag{2.2}$$

where z takes values in $\Re^l$, $g(\cdot)$ represents the disturbance–free measurement model, and $v \in L_2$ is the measurement disturbance term taking values in $\Re^k$ (where $k \geq l$). We have included a, possibly state–dependent, $l \times k$ multiplier ρ on the disturbance.

We will assume that f, σ, g and ρ are all continuously differentiable. We will assume that f, g and σ are globally Lipschitz in x, that σ is bounded, and that $a \doteq \sigma\sigma^T$ is uniformly non–degenerate, i.e. that there exists $\delta > 0$ such that

$$\xi^T \sigma(x)\sigma^T(x)\xi \geq \delta|\xi|^2 \qquad \forall\, x \in \Re^n,\ \ \forall\, \xi \in \Re^n. \tag{2.3}$$

We will also assume that $\mathrm{Range}(\rho(x)) = \Re^l$ for all $x \in \Re^n$ which guarantees that for any z, x there exists some v satisfying (2.2) which should certainly be the case if our measurement model is properly constructed. Finally, we define ρ^{-1} by

$$\rho^{-1}(x)b = \mathrm{argmin}\{|v| : \rho(x)v = b\}. \tag{2.4}$$

Assume that ρ^{-1} is uniformly bounded, that is, there exists $C_\rho < \infty$ such that $|\rho^{-1}(x)b| \leq C_\rho|b|$ for all $x \in \Re^n$ and $b \in \Re^l$. Note also that these assumptions imply that if we view the integral version of (2.1)

$$x_T \doteq X(T) = x_0 + \int_0^T f(X(t)) + \sigma(X(t))w(t)\, dt \tag{2.5}$$

as a mapping from x_0 to x_T then this mapping is one–to–one and onto for any $w \in L_2$, thereby ensuring that we may invert it.

Let $p_0(x_0)$ be a measure of our uncertainty about the initial state x_0, and let it have at most quadratic growth, so that $0 \leq p_0(x) \leq C(1 + |x|^2)$ for all $x \in \Re^n$ for some $C < \infty$. Suppose we wish to estimate the state at time T. Consider a cost criterion of the form

$$J(T,x_T,w(\cdot)) = -p_0(x_0) - \int_0^T |w(t)|^2 + |v(t)|^2\, dt \tag{2.6}$$

$$= -p_0(x_0) - \int_0^T |w(t)|^2 + |\rho^{-1}(X(t))[z(t) - g(X(t))]|^2\, dt \tag{2.7}$$

where x_0 is given by (2.5) for any particular w. Note that v no longer appears in (2.7) since it is fixed for a given choice of disturbance w, measurement time–history z and specific terminal state x_T. Similarly, x_0 is fixed given x_T and w as well. Therefore we define the value function by

$$P(T,x_T) = \sup_{w\in L_2} J(T,x_T,w); \tag{2.8}$$

this is also known as the information state (see, for instance, [10], [11]).

The Hamilton–Jacobi equation corresponding to (2.1), (2.7), (2.8) is

$$\begin{aligned} 0 &= V_t - H(t,x,\nabla_x V) \qquad && t>0 \\ V(0,x) &= -p_0(x) && t=0 \end{aligned} \tag{2.9}$$

where ∇_x represents the gradient with respect to the space variable and

$$H(t,x,q) = \sup_{w\in\Re^m} \left\{ -[f(x)+\sigma(x)w]^T q - |w|^2 - |\rho^{-1}(x)(z(t)-g(x))|^2 \right\} \tag{2.10a}$$

$$= -f^T(x)q + \frac{1}{4} q^T a(x) q - |\rho^{-1}(x)(z(t)-g(x))|^2. \tag{2.10b}$$

If we can obtain an L_2–bound on the size of ϵ–optimal disturbances w, then it can be shown that P is a continuous viscosity solution of (2.9).

Lemma 2.1: There exists $M<\infty$ such that for $\epsilon \le 1$, ϵ–optimal w satisfy $\|w\|^2 \le M(|x_T|^2+1)$ where M may depend on T.

Proof: For comparison, first consider the case where $w\equiv 0$. We will obtain a lower bound on $J(T,x_T,0)$. By the global Lipschitz property of f, there exists $K<\infty$ such that $|f(x)|\le K(|x|+1)$ for all $x\in\Re^n$. Consequently,

$$\frac{d}{dt}|X|^2 \ge -2K|X|^2 - 2K|X| \ge -4K|X|^2 - K/2$$

which implies

$$|X(t)|^2 \le e^{4KT}|x_T|^2 + \frac{1}{8}e^{4KT} \qquad \forall\, t\in[0,T]. \tag{2.11}$$

By the assumptions, there exist $m_z, C_g<\infty$ such that $\|z\|_{L_2[0,T]}\le m_z$ and $|g(x)|\le C_g(1+|x|)$ for all $x\in\Re^n$. By these bounds, the assumptions on p_0 and ρ^{-1}, and (2.7), there exists $\overline{M}$ such that

$$J(T,x_T,0) \ge -C(1+|x_0|^2) - \overline{M}\int_0^T (1+|X(t)|^2)\, dt.$$

Combining this with (2.11) implies there exists $\widehat{M} < \infty$ such that

$$J(T, x_T, 0) \geq -\widehat{M}(1 + |x_T|^2)$$

where $\widehat{M}$ may depend on T. On the other hand,

$$J(T, x_T, w) \leq -\|w\|^2.$$

Combining these last two inequalities yields the result. $\square$

In the case where $z(\cdot)$ is continuous, standard techniques (see for instance [14, Th. 2.10]) yield the following theorem. (Extensions of viscosity solutions to cases where the time–dependency of the Hamiltonian is only measurable are well–known but more technical [3], [9], [12], [17].)

Theorem 2.2: P is a continuous viscosity solution of (2.9).

If it can be shown that the value P satisfies a quadratic growth condition of the form

$$|P(t, x_1) - P(t, x_2)| \leq K(1 + R)|x_1 - x_2| \tag{2.12}$$

for any x_1, x_2 such that $|x_1| \leq R$ and $|x_2| \leq R$ for any $R < \infty$ and any $t \in [0, T]$, then P is the unique viscosity solution to (2.9) among the class of quadratically growing solutions [15]. Under the additional assumption that σ is constant, condition (2.12) has been obtained [14, Prop 2.12]. A recent result [2] suggests that this uniqueness may be obtained in the absence of a bound of the form (2.12).

We will now indicate how the solution to (2.9) may be used to obtain a robust state estimate. First we will indicate a direct approach which was originally proposed by Mortensen ([8], [16]), and then we will combine the above with a quadratic estimation criterion to obtain the robust filter. For the direct Mortensen approach, note that $-P(T, x)$ represents, in some sense, the minimum disturbance energy needed for the target state at time T to be x for a given measurement path $z(\cdot)$. The Mortensen estimate is any

$$\widehat{e}_T \in \operatorname{argmax}_{x \in \Re^n} P(T, x). \tag{2.13}$$

We can assert the existence of the argmax in (2.13) under reasonable assumptions as follows.

Theorem 2.3: Suppose there exists $C_p > 0, \overline{x}_0 \in \Re^n$ such that $p_0(x) \geq C_p|x - \overline{x}_0|^2$ for all $x \in \Re^n$. Then $\widehat{e}_T$ exists.

Proof: Let the global Lipschitz constant on f be K, and let the bound on σ be m_σ. Let $\overline{x}_T \in \Re^n$ be such that $\overline{X}(0) = \overline{x}_0$ when $\frac{d\overline{X}}{dt} = f(\overline{X})$ and $\overline{X}(T) = \overline{x}_T$ (the existence of which is guaranteed by the assumptions). Let X satisfy the dynamics of (2.1) with $X(T) = x_T$ for any $w \in L_2[0,T]$ and any $x_T \in \Re^n$. Let $x_0 \doteq X(0)$. Then

$$|X(t) - \overline{X}(t)| = |(x_0 - \overline{x}_0) + \int_0^t [f(X) - f(\overline{X}) + \sigma(X)w]\,dr|$$
$$\leq |x_0 - \overline{x}_0| + \int_0^t [K|X - \overline{X}| + m_\sigma|w|]\,dr$$

which by Cauchy-Schwarz

$$\leq |x_0 - \overline{x}_0| + m_\sigma\sqrt{T}\|w\|_{L_2} + \int_0^t K|X - \overline{X}|\,dr. \tag{2.14}$$

Employing (2.14) and Gronwall's inequality yields

$$|x_T - \overline{x}_T| \leq (|x_0 - \overline{x}_0| + m_\sigma\sqrt{T}\|w\|_{L_2})C_T$$

where

$$C_T = \left(1 + KTe^{KT}\right).$$

Consequently,

$$C_p|x_0 - \overline{x}_0|^2 + \|w\|_{L_2}^2 \geq C_p\left[\frac{|x_T - \overline{x}_T|}{C_T} - m_\sigma\sqrt{T}\|w\|_{L_2}\right]^2 + \|w\|_{L_2}^2$$

which after some calculations one finds

$$\geq \frac{C_p}{C_T^2(1 + m_\sigma^2 T)}|x_T - \overline{x}_T|^2.$$

Therefore, by (2.6) and the assumption

$$J(T, x_T, w) \leq -\frac{C_p}{C_T^2(1 + m_\sigma^2 T)}|x_T - \overline{x}_T|^2.$$

Since this is true for any $w \in L_2$, one has

$$P(T, x_T) \leq -\frac{C_p}{C_T^2(1 + m_\sigma^2 T)}|x_T - \overline{x}_T|^2 \tag{2.15}$$

which yields the result. $\square$

Although the direct Mortensen estimate is of some interest, the following robust estimate will have desirable properties (analogous to H_∞ control) as discussed below. Specifically, the information state can be combined with a quadratic estimate error criterion to obtain the robust filter. Let the dynamics and measurement models be those given above in (2.1) and (2.2), and make the same assumptions. However, now let the cost criterion take the form

$$I(T, e, x_T, w) = -\gamma^2\left[p_0(x_0) + \int_0^T |w(t)|^2 + |v(t)|^2\,dt\right] + |x_T - e|^2$$

where we again note that x_0 and $v(\cdot)$ are given by x_T and $w(\cdot)$. Let

$$W(T,e) = \sup_{x_0 \in \Re^n} \sup_{w \in L_2} I(T,e,x_T,w) = \sup_{x_T \in \Re^n} \sup_{w \in L_2} I(T,e,x_T,w). \tag{2.16}$$

Note that

$$W(T,e) = \sup_{x_T \in \Re^n} \left[\gamma^2 P(T,x_T) + |x_T - e|^2\right]. \tag{2.17}$$

It is clear here that γ must be large enough so that this supremum will be finite and achieved at some x_T. By (2.15), one obtains the following.

Theorem 2.4: There exists $\gamma < \infty$ such that the supremum in (2.17) is finite.

This lower bound on such γ is directly analogous to the optimal disturbance attenuation parameter in H_∞ control below which the supremum in that problem becomes unbounded. Note that in a computational system, given a P, one can choose a γ corresponding to this P such that the supremum is finite.

Now note that since $W(T,e)$ is a supremum of functions which are strictly convex in e, it is also strictly convex in e. Further, $W(T,e) \to \infty$ as $|e| \to \infty$. Consequently, the minimum over e is obtained at some point, and we define the robust filter estimate uniquely at time T as:

$$e_T \doteq \operatorname{argmin}_{e \in \Re^n} W(T,e). \tag{2.18}$$

Further, by (2.15) one may choose γ large enough so that $\min_{e \in \Re^n} W(T,e) \leq 0$, in which case one obtains the error estimate

$$|x_T - e_T|^2 \leq \gamma^2 \left[p_0(x_0) + \|w\|^2 + \|v\|^2\right] \tag{2.19}$$

for all $x_0 \in \Re^n$ and all $v, w \in L_2$ where $\|\cdot\|$ represents the L_2–norm over $[0,T]$. This is the robust bound on the estimate error in terms of the energy of the disturbance.

Note that the information state computations are recursive. That is, having obtained an estimate at time T, one can obtain an estimate at time $\widehat{T} > T$ by extending the solution of (2.9) from T to $\widehat{T}$.

There is an interesting special case for the robust filter. Let $x_T^M = \operatorname{argmax}_x P(T,x)$. Suppose that one may choose γ large enough so that

$$\gamma^2 P(T,x_T^M) \geq \gamma^2 P(T,x) + |x - x_T^M|^2 \qquad \forall x \in \Re^n,$$

i.e. $\gamma^2 P$ decays faster than $|x - x_T^M|^2$ increases as one moves away from the maximum. Then

$$W(T,x_T^M) = \gamma^2 P(T,x_T^M) \leq \sup_{x \in \Re^n} \left[\gamma^2 P(T,x) + |x-e|^2\right] = W(T,e) \qquad \forall\, e \in \Re^n.$$

That is, the robust filter estimate is identical to the estimate from the direct approach.

Lastly, one might also note that in the case where f, g are linear, σ, ρ are constant, and $p_0(x) = -(x - \overline{x}_0)^T \widetilde{P}_0(x - \overline{x}_0)$ for some $\overline{x}_0$ and positive definite symmetric $\widetilde{P}_0$, one finds that $P(t,\cdot)$ is quadratic. Specifically, the information state may be written in the form $P(t,x) = -(x - \overline{x}(t))^T \widetilde{P}(t)(x - \overline{x}(t)) + r(t)$ for proper choice of $\widetilde{P}, \overline{x}, r$. In this case, the Mortensen estimate (2.13) and the robust estimate (2.18) are identical. Further this estimate is identical to that generated by the (now linear) Kalman filter for the analogous (linear) stochastic model with initial covariance $\widetilde{P}_0^{-1}$, and in fact, $\widetilde{P}^{-1}(t)$ is the covariance matrix for $t > 0$.

3. Discrete–Time Measurements

For the robust approach with discrete–time measurements, we assume that at each measurement time t_i, we receive a measurement which we model as

$$z_i = g(X(t_i)) + \rho(X(t_i))v_i. \tag{3.1}$$

Suppose that we wish to estimate the state at time T, and that there have been N measurements in $(0,T]$. Then we may adapt the cost criterion (2.6) and value function (2.8) as follows. Let our cost criterion be

$$J(T, x_T, w(\cdot)) = -p_0(x_0) - \int_0^T |w(t)|^2\, dt - \sum_{i=1}^N |v_i|^2, \tag{3.2}$$

and let the value be given by

$$P(T, x_T) = \sup_{w \in L_2} J(T, x_T, w). \tag{3.3}$$

Due to the discrete nature of the measurements, it is helpful recall the form of the dynamic programming principle for such a system. Let $P(t_j^+, x)$ be the value at time t_j just after measurement j, and $P(t_j^-, x)$ be the value at time t_j just before the measurement. If T is not a measurement time, and $t \in (t_{k-1}, t_k)$, we have

$$P(T,x) = \sup_{w \in L_2([t,T];\Re^m)} \left\{ P(t, X(t)) - \int_t^T |w(r)|^2\, dr - \sum_{j=k}^N |\rho^{-1}(X(t_j))[z_j - g(X(t_j))]|^2 \right\},$$

and when T occurs at the time of measurement j, we have

$$P(T^+, x) = P(T^-, x) - |\rho^{-1}(x)[z_j - g(x)]|^2.$$

The corresponding dynamic programming equations are

$$V(0,x) = -p_0(x) \qquad t = 0 \tag{3.4a}$$

$$\begin{aligned} 0 &= V_t - \sup_{w \in \Re^m} \{-[f(x) + \sigma(x)w]^T \nabla_x V - |w|^2\} && t \in (t_j, t_{j+1}) \\ &= V_t + f^T(x)\nabla_x V - \frac{1}{4}(\nabla_x V)^T a(x) \nabla_x V && t \in (t_j, t_{j+1}) \end{aligned} \tag{3.4b}$$

$$V(t_j^+, x) = V(t_j^-, x) - |\rho^{-1}(x)[z_j - g(x)]|^2 \qquad t = t_j \text{ for some } j \tag{3.4c}$$

The subsequent analysis is quite similar to that of the continuous–time measurement case, and we do not repeat it here. In this case, however, there are no difficulties associated with the possible lack of continuity of the measurement process since the measurements are discrete–time.

4. Numerical Methods

In this section, we will discuss an approach to the numerical implementation of the robust filter. We will discuss only the continuous–time measurements case; the discrete–time measurement case is similar. The main effort in obtaining the robust estimates of the target state will be expended in the solution of (2.9) at relevant points in the state space. Since (2.9) is a first–order PDE, the characteristic approach is one technique which may be employed. Let us discuss the relationship between characteristics and the information state/viscosity solution of (2.9). We will assume throughout this section that the Hamiltonian (2.10) is differentiable. In particular, we assume that $z(\cdot)$ is differentiable. (As noted above, such an additional condition is, of course, not needed in the discrete–time measurement case.)

The system of characteristic equation corresponding to (2.9), (2.10) is

$$\frac{dX}{dt} = -H_q(t, X, Q) \tag{4.1a}$$
$$\frac{dR}{dt} = H_t(t, X, Q) \tag{4.1b}$$
$$\frac{dQ}{dt} = H_x(t, X, Q) \tag{4.1c}$$

where R corresponds to V_t and Q corresponds to $\nabla_x V$. The proper initial conditions at some point $(0, x_0)$ on the hyperplane $t = 0$ may be obtained from (2.9) and are given by

$$\begin{aligned} X(0) &= x_0 \\ Q(0) &= -\frac{dp_0}{dx}(x_0) \\ R(0) &= H(0, x_0, Q(0)). \end{aligned} \tag{4.2}$$

The "value" of the solution along any such trajectory at any point $t < 0$ is given by

$$U(t) = -p_0(x_0) + \int_0^t Q^T(s)\frac{dX}{dt}(s) + R(s)\,ds \tag{4.3}$$

for a trajectory starting at $(0, x_0)$. We can index these solutions by the initial value of the space variable as $U(t; x_0) = U(t)$ and $X(t; x_0) = X(t)$ for a trajectory starting at $(0, x_0)$. When the Hamiltonian is sufficiently smooth, and these characteristic trajectories cover the entire region $G \doteq (0, \infty) \times \Re^n$ without having their state–space projections cross, then it is well known that the solution to (2.9) at any point $(t, x) \in G$ is given by $U(t; x_0)$ for the particular characteristic trajectory such that $X(t;x_0)=x$. However, for most systems of interest, there is no guarantee that the state–space components, $X(\cdot;x_0)$, of these characteristics will not cross one another. In this case, we define

$$\widehat{V}(t,x)=\max\{U(t;x_0):\,X(t;x_0)=x\}. \tag{4.4}$$

We show that this algorithm does indeed yield the information state. This is similar to a result in [18].

Theorem 4.1: $\widehat{V}(T,x)=P(T,x)$ for all $(T,x)\in[0,\infty)\times\Re^n$.

Proof: The bulk of the proof is essentially Pontryagin's Maximum Principle. Fix $T<\infty$. Given (T,x_T), by Lemma 2.1 and [6, Th 3.4.1], there exists an optimal w^* and corresponding trajectory X^* for problem (2.8). Consequently,

$$P(T,x_T)=-p_0(X^*(0))-\int_0^T |w^*(t)|^2+|\rho^{-1}(X^*(t))[z(t)-g(X^*(t))]|^2\,dt. \tag{4.5}$$

Let

$$L(t,x,w)\doteq -|w|^2-|\rho^{-1}(x)[z(t)-g(x)]|^2.$$

Define the costate Q^* by

$$\begin{aligned}\dot{Q}^*&=-[f_x(X^*)+(\sigma(X^*)w^*)_x]Q^*+L_x(t,X^*,w^*)\\ Q^*(0)&=-\nabla_x p_0(X^*(0)).\end{aligned}$$

Let

$$h(t,x,q,w)=L(t,x,w)-q\cdot[f(x)+\sigma(x)w].$$

Then (see for instance [7]), $h(t,X^*(t),Q^*(t),w)$ is maximized at $w=w^*(t)$ almost everywhere, and in particular,

$$h_w(t,X^*(t),Q^*(t),w^*(t))=0 \qquad \text{a.e.} \tag{4.6}$$

But (4.6) is simply

$$w^*(t)=-\frac{1}{2}\sigma^T(X^*(t))Q^*(t) \qquad \text{a.e.} \tag{4.7}$$

Noting that X^*,Q^* are continuous, we redefine w^* on a set of measure zero so that (4.7) holds everywhere. This in turn implies $X^*,Q^*\in C^1(0,T)$.

Further,

$$\begin{aligned}\dot{X}^*&=f(X^*)+\sigma(X^*)w^*=f(X^*)-\frac{1}{2}\sigma(X^*)\sigma^T(X^*)Q^*=-H_q(t,X^*,Q^*)\\ \dot{Q}^*&=-[f_x(X^*)+(\sigma(X^*)w^*)_x]Q^*+L_x(t,X^*,w^*)=H_x(t,X^*,Q^*).\end{aligned}$$

Finally, by (4.5)

$$
\begin{aligned}
P(T,x_T) &= -p_0(X^*(0)) - \int_0^T \left[\frac{1}{4}Q^{*T}\sigma(X^*)\sigma^T(X^*)Q^* + |\rho^{-1}(X^*(t))[z(t)-g(X^*(t))]|^2\right] dt \\
&= -p_0(X^*(0)) + \int_0^T Q^{*T}\left[f(X^*) - \frac{1}{2}\sigma(X^*)\sigma^T(X^*)Q^*\right] dt \\
&\quad + \int_0^T \left[Q^{*T}\left(\frac{1}{4}\sigma(X^*)\sigma^T(X^*)Q^* - f(X^*)\right) - |\rho^{-1}(X^*(t))[z(t)-g(X^*(t))]|^2\right] dt \\
&= -p_0(X^*(0)) + \int_0^T \left[Q^{*T}\frac{dX^*}{dt} + R^*\right] dt
\end{aligned}
$$

where R^* is given by (4.1b), (4.2) with X^*, Q^*. Since this last equation is (4.3), it implies there is an x_0^* (i.e. $X^*(0)$) and corresponding solution of (4.1)–(4.3) such that $P(T,x_T) = U(T;x_0^*)$.

For any other solution, we have as above

$$
-p_0(X(0)) + \int_0^T \left[Q^T\frac{dX}{dt} + R\right] dt \;= -p_0(X(0)) - \int_0^T \left[|w|^2 + |\rho^{-1}(X(t))[z(t)-g(X(t))]|^2\right] dt
$$

with $w \doteq -\frac{1}{2}\sigma^T Q$ and $\frac{dX}{dt} = f(X) + \sigma w$. Consequently, $P(T,x_T) \geq U(T;X(0))$. □

Lastly, we indicate a computation of the information state by this method for a simple, one–dimensional state example. Once one computes the information state to the desired accuracy, the computation of the robust filter estimate (2.18) is quite straight forward. The purpose of this example is simply to indicate that this method can provide an efficient means for obtaining robust filter estimates for a highly nonlinear system. Let the dynamics and (discrete–time) measurement models be given by

$$
\begin{aligned}
\dot{X} &= 4\sin(X) + \sigma w \\
z_i &= \frac{X(t_i)}{2} + [X(t_i)]^3 + \rho v_i
\end{aligned}
$$

with a measurement sample period of 0.025 units. The solution is propagated forward 1 time unit (i.e. 40 measurement updates). The noise multipliers were taken to be $\sigma = 1$ and $\rho = 5$. The (true) initial state is $X(0) = 0$. The initial information is described by $p_0(x) = x^2$ which corresponds to a correct initial state estimate. Note that the dynamics are unstable in a neighborhood of the origin. The disturbances were chosen to be

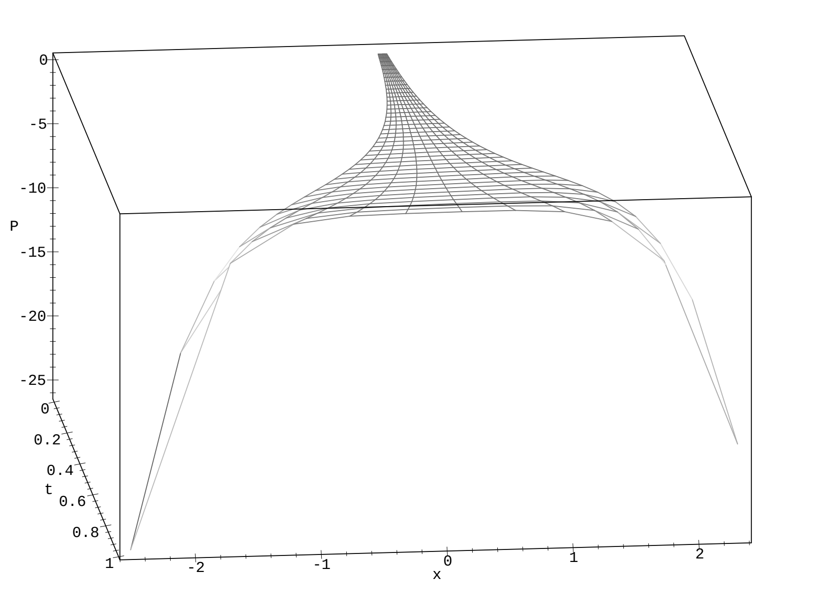
$$
w(t) = \frac{1}{4} + \sin\left(\frac{-\pi}{2}t\right) \qquad \text{and} \qquad v_i = \frac{3}{20}\sin\left(\frac{\pi}{2}t_i\right)
$$

so that these disturbances represent a slowly varying, biased noise signal. (Note that the robust filter does not require a statistical interpretation of these noise signals.) The resulting information state appears in Figure 1.

The true state at $T = 1$ is $X(1) = -1.012$. This example is designed so that the accumulation of biased measurements and locally unstable state dynamics lead to substantial differences between the estimators. The Mortensen estimate (2.13) is $\hat{e}_1 = 0.28$. The robust estimate (2.18) with $\gamma^2 = 2.5$ is $e_1 = 0.01$. Perhaps more importantly, the error bound for the robust estimate with $\gamma^2 = 2.5$ and the noise inputs given above is $|x(1)-e_1| < \sqrt{2.5(0.705)} = 1.33$, and of course, robust estimate e_1 satisfies this bound. The lowest possible value of γ^2 was approximately 1.9, and this lead to an estimate of $e_1 = -0.05$. As one increases γ^2, the robust estimate approaches the Mortensen estimate as it obviously must given their definitions. (Although it is not appropriate to apply a Kalman filter without some probabilistic model, as a point of reference it might be of interest to note that the discrete–time, extended Kalman filter with assumption of zero–mean, uncorrelated noise and assumed variances of 0.25 and 0.011 for w_i and v_i respectively at a time step–size of 0.025, leads to an estimate of $e_1^K = 0.6$.)

The author would like to thank the reviewers for their constructive comments.

Figure 1



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