A New Fundamental Solution for Differential Riccati Equations Arising in Control

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Abstract

The matrix differential Riccati equation (DRE) is ubiquitous in control and systems theory. The presence of the quadratic term implies that a simple linear-systems fundamental solution does not exist. Of course it is well-known that the Bernoulli substitution may be applied to obtain a linear system of doubled size. Here however, tools from max-plus analysis and semiconvex duality are brought to bear on the DRE. We consider the DRE as a finite-dimensional solution to a deterministic linear/quadratic control problem. Taking the semiconvex dual of the associated semigroup, one obtains the solution operator as a max-plus integral operator with quadratic kernel. The kernel is equivalently represented as a matrix. Using the semigroup property of the dual operator, one obtains a matrix operation whereby the kernel matrix propagates as a semigroup. The propagation forward is through some simple matrix operations. This time-indexed family of matrices forms a new fundamental solution for the DRE. Solution for any initial condition is obtained by a few matrix operations on the fundamental solution and the initial condition. In analogy with standard-algebra linear systems, the fundamental solution can be viewed as an exponential form over a certain idempotent semiring. This fundamental solution has a particularly nice control interpretation, and might lead to improved DRE solution speeds.

Key words: Riccati, differential equation, numerical methods, max-plus algebra, Legendre transform, semiconvexity, Hamilton-Jacobi-Bellman equations, idempotent analysis.

AMS subject classifications. 34A34, 47D99, 49LXX, 65L20, 93C05, 93C10, 93C15

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1 Introduction

In recent years, max-plus analysis and semiconvexity/semiconcavity-based methods have expanded greatly, finding wide application in control (c.f., [3], [5], [11], [13], [17], [14]). Much of the work on applications of max-plus analysis in control has focused on discrete-event systems. However, there were recent breakthroughs in the solution of Hamilton-Jacobi-Bellman (HJB) PDEs (c.f., [1], [2], [10], [17], [18], [21]). Now, surprisingly, we see that this theory is yielding a fundamental new result in the area of Riccati equations.

The matrix differential Riccati equation (DRE) is ubiquitous in control and systems theory. We consider time-invariant DREs of the form

$$\dot{P}_t = F(P_t) \doteq A'P_t + P_tA + C + P_t\Sigma P_t \tag{1}$$

where C is symmetric and $\Sigma = \sigma\sigma'$ is symmetric, nonnegative definite with at least one positive eigenvalue. (The non-positive definite case is equivalent.) Throughout, we assume that all of the matrices are $n \times n$. We suppose one has initial condition, $P_0 = p_0$ where p_0 is also symmetric. This DRE has an interpretation as the matrix defining a control value function. Numerous numerical methods have been used to solve such problems, including direct Runge-Kutta methods, the Chandrasekhar decomposition approach, and variations of the Davison-Maki approach (c.f., [12]). The Daivson-Maki approach uses the Bernoulli substitution to create a linear system of two matrices, each of the same size as P_t. We denote these matrices as V_t^1 and V_t^2. The linear system is

$$\begin{pmatrix} \dot{V}_t^1 \\ \dot{V}_t^2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} V_t^1 \\ V_t^2 \end{pmatrix}$$

where $\mathcal{A} = \mathcal{A}(A, C, \Sigma)$ is $2n \times 2n$. The solution of the DRE is recovered by letting

$$P_t = V_t^2 (V_t^1)^{-1}, \tag{2}$$

where in particular, one uses initial conditions $V_0^1 = I$, $V_0^2 = p_0$. So, one obtains the solution of the Riccati equation from fundamental solution $\exp(\mathcal{A}t)$. That is, P_t is obtained from (2) where

$$\begin{pmatrix} V_t^1 \\ V_t^2 \end{pmatrix} = \exp(\mathcal{A}t) \begin{pmatrix} I \\ p_0 \end{pmatrix}.$$

Although the Davison-Maki approach is quite nice, we will obtain another fundamental solution, and this fundamental solution has a particularly clear control-theoretic motivation. This new approach will also generate a $2n \times 2n$ object as the fundamental solution. In this approach, the matrix object is separated into four blocks, and the operations are actually more naturally viewed as operations on the four $n \times n$ blocks, which are not obviously equivalent to simple operations on the overall matrix. The new approach will be constructed through a finite-dimensional semigroup defined by this fundamental solution. The forward propagation of the fundamental solution is naturally defined by this operation with the semigroup property.

We now give some sense of the tools which will be applied in the technical development. First, we will consider linear/quadratic control problems parameterized by $z \in I\!R^n$, and the value functions associated with these control problems take the form

$$V^z(t,x) = \tfrac{1}{2}(x - \Lambda_t z)' P_t (x - \Lambda_t z) + r_t.$$

We note that $V^z(t+\tau, x) = S_\tau[V^z(t,\cdot)](x)$ where S_τ is a max-plus linear semigroup. Semiconvex duality, introduced in [10], [17], is a small perturbation of convex duality. However, semiconvexity is a typical property for value functions [5], [17]. Further, the space of semiconvex functions is a max-plus vector space [17], [10], [3], [6], [15]. Working in the semiconvex-dual space, S_τ has a semiconvex-dual operator, $\mathcal{B}_\tau$ which takes the form of a max-plus integral operator with kernel, $B_\tau = B_\tau(x,y)$, taking the form of a quadratic function. The matrix, β_τ, defining this quadratic kernel function will be the fundamental solution of the DRE. We will define a multiplication operation ($\circledast$-multiplication) with the semigroup property, specifically

$$\beta_{t+\tau} = \beta_t \circledast \beta_\tau$$

where the $\circledast$ operation involves inverse, multiplication and addition $n \times n$-matrix operations (in the standard algebra). The definition of $\circledast$ appears just below (28). We will also define an exponentiation operation ($\circledast$-exponentiation) such that $\beta_t = \beta_1^{\circledast t}$. The solution of (1) will be obtained by

$$P_t = D_\psi^{-1} \beta_t D_\psi p_0$$

where the D_ψ and D_ψ^{-1} operators are descended from the semiconvex dual and its inverse. These operations are also implemented through some $n \times n$ (standard algebra) matrix operations. It is important to note that the fundamental solution approach has the benefit that one only solves once for β_t, even if one wishes to solve the DRE for a variety of initial conditions.

Although the fundamental solution is of theoretical interest, it can also be useful in numerical solution of differential equations. For example, in [12], the Davison-Maki fundamental solution is used in two numerical methods which are investigated there for numerical speed. There are many issues to be considered in construction of numerical methods, and these are beyond the scope of this paper. Nevertheless, we do indicate some initial steps in the direction of computational solution of DREs using the new fundamental solution.

In summary, the structure of the paper is as follows. First, we obtain the solution of the DRE for a particular initial condition in terms of the fundamental solution and the initial condition (Theorems 3.3 and 3.5). Next, we show how the fundamental solution may be obtained from an arbitrarily small segment of a particular solution of the DRE (Theorem 4.7). Next we develop a natural exponential representation of the fundamental solution, and briefly discuss an associated semiring (Sections 5 and 6). Lastly, an exponential-order numerical method based on the fundamental solution is developed and applied in an example.

2 The linear-quadratic control problem

As indicated above, the fundamental solution to the DRE will be obtained through an associated optimal control problem. Recall that we are considering the DRE given by (1). Since we will be employing semiconvex duality (see below and [10, 17, 18]), we will choose some (duality-parametrizing) symmetric matrix, Q, such that

$$F(Q) > 0, \tag{3}$$

where we note that, for any square matrix D, we will use the notation $D > 0$ to indicate that matrix D is positive definite throughout. We will need to consider the specific solution of DRE (1) with initial condition

$$\widetilde{P}_0 = Q. \tag{4}$$

We assume:

There exists a solution of DRE (1), $\widetilde{P}_t$, with initial condition (4), satisfying $\widetilde{P}_t > Q$ (i.e., $\widetilde{P}_t - Q$ positive-definite) for $t \in (0, \overline{T})$ with $\overline{T} > 0$, and we note specifically, that we may have $\overline{T} = +\infty$. $\quad (A.e)$

This is the last assumption until Section 7, where numerical methods are discussed.

We will be obtaining the fundamental solution β_t for solutions with initial conditions, $P_0 = p_0 > Q$. Note that we do not assume stability of the DRE, and finite-time blow-up is possible. We will let

$$\widetilde{T} = \widetilde{T}(p_0) = \sup\{t \geq 0 \mid P_t \text{ exists, and } P_t > Q\}, \tag{5}$$

and we let

$$\widehat{T} = \widehat{T}(p_0) \doteq \overline{T} \wedge \widetilde{T} \tag{6}$$

where $\wedge$ indicates the minimum operation.

Remark 2.1 Note that with $\Sigma > 0$, we may take $Q = -kI$ for arbitrarily large k, so that one can ensure (3) will hold for such Q (as well as for any $p_0 > Q$).

We will be using a control value function to motivate and develop the fundamental solution. Consider the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) problems on $[0, \overline{T}) \times I\!R^n$, indexed by $z \in I\!R^n$, given by

$$V^z_t = H(x, \nabla V^z) \tag{7}$$

$$V^z(0, x) = \psi(z, x) \tag{8}$$

where ∇ indicates the gradient with respect to x, and

$$H(x, p) = (Ax)'p + \tfrac{1}{2}x'Cx + \tfrac{1}{2}p'\Sigma p \tag{9}$$

$$\psi(x, z) = \tfrac{1}{2}(x - z)'Q(x - z). \tag{10}$$

Theorem 2.2 *For any $z \in I\!R^n$, there exists a solution to (7),(8) in $C^\infty((0,\overline{T}) \times I\!R^n) \cap C([0,\overline{T}) \times I\!R^n)$, and this is given by*

$$V^z = \tfrac{1}{2}(x - \Lambda_t z)' \widetilde{P}_t (x - \Lambda_t z) + z' R_t z \tag{11}$$

where $\widetilde{P}$ satisfies (1),(4), and Λ, r satisfy $\Lambda_0 = I$, $R_0 = 0$,

$$\dot{\Lambda} = -\left[\widetilde{P}^{-1} C + A\right] \Lambda \qquad \text{and} \qquad \dot{R} = \Lambda' C \Lambda. \tag{12}$$

The proof is immediate by substitution into (1),(4). Next we need a verification theorem in order to connect the HJB PDE to the control value function and semigroup. For any $z \in I\!R^n$, let

$$W^z(-t, x) \doteq V^z(t, x) \qquad \forall\, (t, x) \in [0, \overline{T}) \times I\!R^n, \tag{13}$$

Note that $W^z(0, x) = \psi(z, x)$ and $W^z_r = -H(r, \nabla W^z)$ on $(-\overline{T}, 0) \times I\!R^n$. For $x, z \in I\!R^n$, $r \in (-\overline{T}, 0]$ and $w \in L_2(r, 0)$, let

$$J^z(r, x, w) \doteq \int_r^0 \tfrac{1}{2} \xi_\rho' C \xi_\rho - \tfrac{1}{2} |w_\rho|^2 \, d\rho + \psi(z, \xi_0) \tag{14}$$

where ξ satisfies

$$\dot{\xi}_\rho = A \xi_\rho + \sigma w_\rho \tag{15}$$

$$\xi_r = x. \tag{16}$$

The optimal control problem value function is defined to be

$$\overline{W}^z(r, x) = \sup_{w \in L_2(r,0)} J^z(r, x, w) \tag{17}$$

for all $x, z \in I\!R^n$ and $r \in (-\overline{T}, 0]$. By standard methods, one obtains the following verification theorem.

Theorem 2.3 *Let $x, z \in I\!R^n$ and $r \in (-\overline{T}, 0]$. One has*

$$W^z(r, x) \geq J^z(r, x, w) \qquad \forall\, w \in L_2(r, 0)$$

and

$$W^z(r, x) = J^z(r, x, \tilde{w})$$

where $\tilde{w}_\rho = \bar{w}(\rho, \xi_\rho) \doteq \sigma' \nabla W^z(\rho, \xi_\rho) = \sigma' \widetilde{P}_{-\rho}(\xi_\rho - \Lambda_{-\rho} z)$, which implies $\overline{W}^z = W^z$.

Proof. For completeness, we include a sketch of the proof. Let $w \in L_2(r, 0)$.

$$\begin{aligned} J^z(r, x, w) = & \int_r^0 \tfrac{1}{2} \xi_\rho' C \xi_\rho - \tfrac{1}{2} |w_\rho|^2 + (A\xi_\rho + \sigma w)' \nabla W^z(\rho, \xi_\rho) \, d\rho + \psi(z, \xi_0) \\ & - \int_r^0 (A\xi_\rho + \sigma w)' \nabla W^z(\rho, \xi_\rho) \, d\rho \end{aligned}$$

which by definition of H,

$$\leq \int_r^0 H(\xi_\rho, \nabla W^z(\rho,\xi_\rho))\,d\rho + \psi(z,\xi_0) - \int_r^0 (A\xi_\rho + \sigma w)'\nabla W^z(\rho,\xi_\rho)\,d\rho$$

which by (7), (13) and (15),

$$= \int_r^0 -W^z_\rho(\rho,\xi_\rho) - \dot{\xi}_\rho'\nabla W^z(\rho,\xi_\rho)\,d\rho + \psi(z,\xi_0)$$

$$= -\int_r^0 \frac{d}{d\rho}[W^z(\rho,\xi_\rho)]\,d\rho + \psi(z,\xi_0)$$

$$= W^z(r,x) - W^z(0,\xi_0) + \psi(z,\xi_0) = W^z(r,z).$$

The second assertion follows by the choice of $\bar{w}$ and a similar argument. □

For $\phi : I\!R^n \to I\!R$ given by $\phi(x) = \frac{1}{2}(x-z)'p_0(x-z)$ (and actually for a much larger set of functions), we define the semigroup, S_τ, by

$$S_\tau[\phi](x) = V^z(\tau,x) \tag{18}$$

$$= \tfrac{1}{2}(x - \Lambda_\tau z)'\widetilde{P}_\tau(x - \Lambda_\tau z) + z'R_\tau z \tag{19}$$

where $\Lambda_0 = I$ and $R_0 = 0$. Recall that S_τ is a max-plus linear operator. (This is discussed in more detail in [10, 16, 17], among others.)

3 Solution via the semiconvex dual semigroup

We let $\oplus, \otimes$ denote the max-plus addition and multiplication operations. We say that $\phi : I\!R^n \to I\!R^- \doteq I\!R \cup \{-\infty\}$ is semiconvex if given $R < \infty$, there exists finite, symmetric $C_R > 0$ such that $\phi(x) + \frac{1}{2}x'C_R x$ is convex on $B_R(0)$. We say that ϕ is uniformly semiconvex with (symmetric matrix) constant K if $\phi(x) + \frac{1}{2}x'Kx$ is convex on $I\!R^n$, and we denote this space as $\mathcal{S}^K(I\!R^n)$. Recall that $\mathcal{S}^K(I\!R^n)$ is a max-plus vector space (c.f., [17]).

Semiconvex duality is parameterized by quadratic functions. We will use the quadratic ψ given in (10) to define our semiconvex duality. The main duality result (c.f., [17], [10], where proofs may be found) is

Theorem 3.1 *Let $\phi \in \mathcal{S}^K(I\!R^n)$ where $-K > Q$. Then, for all $x \in I\!R^n$,*

$$\phi(x) = \max_{z\in I\!R^n}[\psi(x,z) + a(z)]$$

$$\doteq \int_{I\!R^n}^{\oplus} \psi(x,z)\otimes a(z)\,dz \doteq \psi(x,\cdot)\odot a(\cdot) \doteq \mathcal{D}_\psi^{-1}[a] \tag{20}$$

where for all $z \in I\!R^n$

$$a(z) = -\max_{x\in I\!R^n}[\psi(x,z) - \phi(x)]$$

$$= -\int_{I\!R^n}^{\oplus} \psi(x,z)\otimes[-\phi(x)]\,dx = -\{\psi(\cdot,z)\odot[-\phi(\cdot)]\} \tag{21}$$

which using the notation of [6]

$$= \left\{\psi(\cdot, z) \odot [\phi^-(\cdot)]\right\}^- \doteq \mathcal{D}_\psi[\phi]. \tag{22}$$

Recall that $\widetilde{P}_t > Q$ for all $t \in (0, \overline{T})$, and consequently, for any $t \in (0, \overline{T})$, there exists K_t such that $\widetilde{P}_t > -K_t > Q$ (i.e., such that $\widetilde{P}_t + K_t > 0$ and $-Q - K_t > 0$), and such that (noting (19))

$$S_t[\psi(\cdot, z)](\cdot) \in \mathcal{S}^{K_t} \qquad \forall\, z \in I\!\!R^n.$$

Therefore, by Theorem 3.1, for all $t \in (0, \overline{T})$ and all $x, z \in I\!\!R^n$

$$S_t[\psi(\cdot, z)](x) = \int_{I\!\!R^n}^{\oplus} \psi(x, y) \otimes B_t(y, z)\, dy = \psi(x, \cdot) \odot B_t(\cdot, z) \tag{23}$$

where for all $y \in I\!\!R^n$

$$B_t(y, z) = -\int_{I\!\!R^n}^{\oplus} \psi(x, y) \otimes \{-S_t[\psi(\cdot, z)](x)\}\, dx = \{\psi(\cdot, y) \odot [S_t[\psi(\cdot, z)](\cdot)]^-\}^-. \tag{24}$$

Lemma 3.2 *There exists symmetric $d_t < -Q$ such that $B_t(y, z) - \frac{1}{2}y'd_t y$ is strictly concave for all $z \in I\!\!R^n$.*

Proof. Note that since $\widetilde{P}_t > Q$, there exists $\delta_t > 0$ such that $S_t[\psi(\cdot, z)](x) \in \mathcal{S}^{-(Q+\delta_t)}$ (as a function of x) for any $z \in I\!\!R^n$. The proof then follows from [17], Theorem 7.12. $\square$

In analogy to the spaces of uniformly semiconvex functions, we say that $\phi : I\!\!R^n \to I\!\!R^+ \doteq I\!\!R \cup \{+\infty\}$ is uniformly semiconcave with (symmetric matrix) constant d if $\phi(x) - \frac{1}{2}x'dx$ is concave on $I\!\!R^n$, and we denote this space as $\mathcal{S}^d_-(I\!\!R^n)$. We define the time-indexed max-plus linear operators $\mathcal{B}_t$ by

$$\mathcal{B}_t[a](z) \doteq B_t(\cdot, z) \odot a(\cdot) = \int_{I\!\!R^n}^{\oplus} B_t(y, z) \otimes a(y)\, dy. \tag{25}$$

In Section 4, we will see that the $\mathcal{B}_t$ satisfy the semigroup property. (Below, we will also see that we may use a space of uniformly semiconcave functions as the domain.) We say that B_t is the kernel of max-plus integral operator $\mathcal{B}_t$.

Theorem 3.3 *Let $\phi(x) \doteq \frac{1}{2}x'p_0 x$ and $a(z) = \mathcal{D}_\psi[\phi]$. Then, for $t \in (0, \widehat{T})$,*

$$S_t[\phi](x) = \psi(x, \cdot) \odot \mathcal{B}_t[a](\cdot) = \mathcal{D}_\psi^{-1}\mathcal{B}_t[a](x) = \mathcal{D}_\psi^{-1}\mathcal{B}_t\mathcal{D}_\psi[\phi](x) \tag{26}$$

for all $x \in I\!\!R^n$.

Proof. By assumption, $S_t[\phi] \in \mathcal{S}^{K_t}$ with $Q < -K_t$. The proof then follows by the proof of Proposition 7.17 in [17]. $\square$

Now, note that by (19) and (24),

$$B_t(y,z) = -\max_{x\in I\!R^n}\left\{\tfrac{1}{2}(x-y)'Q(x-y) - \left[\tfrac{1}{2}(x-\Lambda_t z)'\widetilde{P}_t(x-\Lambda z) + \tfrac{1}{2}z'R_t z\right]\right\} \tag{27}$$

where $t < \overline{T}$ guarantees strict concavity of the argument of the maximum. Note also, that $B_t(x,y)$ is a quadratic function; this supports the above assertion regarding the domain of $\mathcal{B}_t$.

Prior to computing B_t from (27), we introduce the $\circledast$-multiplication operation and a result regarding a more general version of (27), which we will use later as well. Let η and α be $2n \times 2n$ matrices with $n \times n$ block structure denoted as

$$\eta = \begin{bmatrix} \eta^{1,1} & \eta^{1,2} \\ \eta^{1,2'} & \eta^{2,2} \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} \alpha^{1,1} & \alpha^{1,2} \\ \alpha^{1,2'} & \alpha^{2,2} \end{bmatrix}. \tag{28}$$

We define the $\circledast$ multiplication operation by

$$\eta \circledast \alpha = \begin{bmatrix} \gamma^{1,1} & \gamma^{1,2} \\ \gamma^{1,2'} & \gamma^{2,2} \end{bmatrix}$$

where

$$\begin{aligned} \gamma^{1,1} &= \eta^{1,1} - \eta^{1,2}S^{-1}\eta^{1,2'}, \\ \gamma^{1,2} &= -\eta^{1,2}S^{-1}\alpha^{1,2}, \\ \gamma^{2,1} &= \gamma^{1,2'}, \\ \gamma^{2,2} &= \alpha^{2,2} - \alpha^{1,2'}S^{-1}\alpha^{1,2}, \end{aligned}$$

and $S \doteq \eta^{2,2} + \alpha^{1,1}$.

Lemma 3.4 *Let η and α be $2n \times 2n$ matrices with block structure given in (28). Let*

$$F(x,y) \doteq \max_{z\in I\!R^n}\left\{\tfrac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}'\eta\begin{pmatrix} x \\ z \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} z \\ y \end{pmatrix}'\alpha\begin{pmatrix} z \\ y \end{pmatrix}\right\}.$$

Then,

$$F(x,y) = \tfrac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}'\gamma\begin{pmatrix} x \\ y \end{pmatrix}$$

where $\gamma = \eta \circledast \alpha$.

Combining (27) and Lemma 3.4, one obtains the following.

Theorem 3.5

$$B_t(x,y) = \tfrac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}'\beta_t\begin{pmatrix} x \\ y \end{pmatrix} \tag{29}$$

where β_t has the same block structure as η above, and in particular,

$$\begin{aligned}
\beta_t^{1,1} &= Q\Delta_t^{-1}Q - Q = Q\Delta_t^{-1}\widetilde{P}_t, \\
\beta_t^{1,2} &= -\widetilde{P}_t\Delta_t^{-1}Q\Lambda_t = -Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t, \\
\beta_t^{2,1} &= \beta_t^{1,2\prime}, \\
\beta_t^{2,2} &= \Lambda_t'\widetilde{P}_t\Lambda_t + R_t + \Lambda_t'\widetilde{P}_t\Delta_t^{-1}\widetilde{P}_t\Lambda_t = R_t + \Lambda_t'Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t,
\end{aligned} \tag{30}$$

and $\Delta_t \doteq Q - \widetilde{P}_t$.

4 The DRE fundamental solution semigroup

Now we will use the semigroup nature of the S_t operators to obtain the semigroup nature of the $\mathcal{B}_t$ operators, and consequently the propagation of the B_t and β_t. The propagation of $\beta_t = (\beta_1)^{\circledast t}$ will be the dynamics of the fundamental solution of the DRE.

The next two lemmas are straightforward, and proofs are not included. The first is a statement about continuity of solutions with respect to initial conditions.

Lemma 4.1 *Let P_t satisfy (1) with initial condition $P_0 = p_0$. In the case where $\overline{T} < \infty$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|p_0 - Q| < \delta$ implies $\widehat{T} > \overline{T} - \varepsilon$. In the case where $\overline{T} = +\infty$, given $M < \infty$, there exists $\delta > 0$ such that $|p_0 - Q| < \delta$ implies $\widehat{T} > M$.*

Lemma 4.2 *Let $a(z) = \frac{1}{2}(z - \overline{z})'q_a(z - \overline{z}) + r_a$ with $q_a < -Q$, and let $\phi = \mathcal{D}_\psi^{-1}a$. Then,*

$$\phi(x) = \tfrac{1}{2}(x - \overline{z})'\left[QU^{-1}q_a\right](x - \overline{z}) + r_a \tag{31}$$

where $U = Q + q_a$. Alternatively, let $\phi(x) = \frac{1}{2}(x - \overline{x})'q_p(x - \overline{x}) + r_p$ with $q_p > Q$, and let $a = \mathcal{D}_\psi\phi$. Then,

$$a(z) = \tfrac{1}{2}(z - \overline{x})'\left[Q\Delta^{-1}q_p\right](z - \overline{x}) + r_p \tag{32}$$

where $\Delta = Q - q_p$.

Based on this lemma, it is natural to make the following definitions, which inherit notation from $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$. For symmetric $q_p > Q$, define $D_\psi[q_p] \doteq Q(Q - q_p)^{-1}q_p$, and for symmetric $q_a < -Q$, define $D_\psi^{-1}[q_a] = Q(Q + q_a)^{-1}q_a$.

Lemma 4.3 *Let $a(z) = \frac{1}{2}(z - \overline{z})'q_a(z - \overline{z}) + r_a$ with $q_a < -kI$. Then, $D_\psi^{-1}[q_a] \to Q$ as $k \to \infty$ (i.e., given $\delta > 0$, there exists $\bar{k} < \infty$ such that $k > \bar{k}$ implies $|D_\psi^{-1}[q_a] - Q| < \varepsilon$).*

PROOF. The proof is a straightforward application of linear algebra, and so only the main steps are included. Note that

$$(Q+q_a)^{-1} = (I+q_a^{-1}Q)^{-1}q_a^{-1} = \left[\sum_{j=0}^{\infty}(-1)^j(q_a^{-1}Q)^j\right]q_a^{-1}.$$

Consequently,

$$Q(Q+q_a)^{-1}q_a = Q + Q\sum_{j=1}^{\infty}(-1)^j(q_a^{-1}Q)^j.$$

This implies, where the $|\cdot|$ notation indicates induced norm, that

$$|Q - Q(Q+q_a)^{-1}q_a| \le |Q|\sum_{j=1}^{\infty}|(q_a^{-1}Q)^j|$$

$$\le |Q|\sum_{j=1}^{\infty}(|Q|\,|q_a^{-1}|)^j \le \frac{|Q|^2|q_a^{-1}|}{1+|Q||q_a^{-1}|}. \square$$

It will be handy to define the following informal closeness notion. For $T=+\infty$ and $t\in(0,\infty)$, let $\mu(t,T)=1/t$, and for $T,t\in(0,\infty)$, let $\mu(t,T)=|T-t|$. We now obtain a partial semigroup property for the $\mathcal{B}_t$ operators. It is likely the operator-domain over which this is obtained can be expanded, but that is not required for attainment of our goals here.

Lemma 4.4 *Given $\varepsilon>0$ and $T_\varepsilon\in(0,\overline{T})$ such that $\mu(T_\varepsilon,\overline{T})<\varepsilon$, there exists $k_\varepsilon<\infty$ such that for all $a\in\mathcal{S}_-^{-k_\varepsilon I}$,*

$$\mathcal{B}_{t_1+t_2}[a] = \mathcal{B}_{t_1}\mathcal{B}_{t_2}[a],$$

and equivalently,

$$B_{t_1+t_2}(\zeta,\cdot)\odot a(\cdot) = \left[\int_{I\!R^n}^{\oplus} B_{t_1}(\zeta,z)\otimes B_{t_2}(z,\cdot)\,dz\right]\odot a(\cdot) \qquad \forall\,\zeta\in I\!R^n$$

for all $t_1,t_2\ge 0$ such that $t_1+t_2<T_\varepsilon$.

PROOF. Given such $\varepsilon, T_\varepsilon$, Lemma 4.1 implies that there exists $\delta>0$ such that if $|p_0-Q|<\delta$, then with $\widehat{T}$ given by (6), $\widehat{T}>T_\varepsilon$.

By Lemma 4.3, there exists $k_\varepsilon<\infty$ such that for $\tilde{q}_a<-Q$, $\tilde{q}_a<-k_\varepsilon I$, one has $|D_\psi^{-1}\tilde{q}_a-Q|<\delta$, and so $\widehat{T}>T_\varepsilon$ (with $p_0=D_\psi^{-1}\tilde{q}_a$). On the other hand, $a\in\mathcal{S}_-^{-k_\varepsilon I}$ (with specific constants given by $a(z)=\frac{1}{2}(z-\overline{z})'q_a(z-\overline{z})+r_a$) implies that $q_a<-k_\varepsilon I$. Also, for k_ε sufficiently large, $a\in\mathcal{S}_-^{-k_\varepsilon I}$ implies $q_a<-Q$. Combining these, one sees that $a\in\mathcal{S}_-^{-k_\varepsilon I}$ for k_ε sufficiently large implies $\widehat{T}>T_\varepsilon$ (with $p_0=D_\psi^{-1}\tilde{q}_a$).

Let $t_1, t_2 > 0$ with $t_1 + t_2 < \widehat{T}$. Let $a \in \mathcal{S}_-^{-k_\varepsilon I}$ and $\phi = \mathcal{D}_\psi^{-1} a$. By the semigroup property of S_t,

$$S_{t_1+t_2}[\phi](x) = S_{t_1} S_{t_2}[\phi] \tag{33}$$

where existence is guaranteed by the above.

Note that

$$S_{t_1+t_2}[\phi](x) = S_{t_1+t_2}\left[\int_{I\!R^n}^{\oplus} \psi(\cdot, y) \otimes a(y)\, dy\right](x)$$

which by max-plus linearity (where the supremum interchange is less problematic than the standard-algebra integral intechange due to less measurability issues)

$$= \int_{I\!R^n}^{\oplus} S_{t_1+t_2}[\psi(\cdot, y)](x) \otimes (y)\, dy$$

which by (23)

$$\begin{aligned}
&= \int_{I\!R^n}^{\oplus} \int_{I\!R^n}^{\oplus} \psi(x, z) \otimes B_{t_1+t_2}(z, y)\, dz \otimes a(y)\, dy \\
&= \int_{I\!R^n}^{\oplus} \psi(x, z) \otimes \int_{I\!R^n}^{\oplus} B_{t_1+t_2}(z, y) \otimes a(y)\, dz\, dy \\
&= \int_{I\!R^n}^{\oplus} \psi(x, z) \otimes \left[\int_{I\!R^n}^{\oplus} B_{t_1+t_2}(z, y) \otimes a(y)\, dy\right] dz \qquad (34) \\
&= \int_{I\!R^n}^{\oplus} \psi(x, z) \otimes \mathcal{B}_{t_1+t_2}[a](z)\, dz. \qquad (35)
\end{aligned}$$

Also,

$$S_{t_1} S_{t_2}[\phi](x) = S_{t_1} S_{t_2}\left[\int_{I\!R^n}^{\oplus} \psi(\cdot, y) \otimes a(y)\, dy\right](x)$$

which by max-plus linearity

$$= \int_{I\!R^n}^{\oplus} S_{t_1} S_{t_2}[\psi(\cdot, y)](x) \otimes a(y)\, dy. \tag{36}$$

However, by (23),

$$S_{t_2}[\psi(\cdot, y)](x) = \int_{I\!R^n}^{\oplus} \psi(x, z) \otimes B_{t_2}(z, y)\, dz. \tag{37}$$

Combining (36) and (37),

$$S_{t_1} S_{t_2}[\phi](x) = \int_{I\!R^n}^{\oplus} S_{t_1}\left[\int_{I\!R^n}^{\oplus} \psi(\cdot, z) \otimes B_{t_2}(z, y)\, dz\right](x) \otimes a(y)\, dy$$

which again by max-plus linearity

$$
\begin{aligned}
&= \int_{\mathbb{R}^n}^{\oplus} \int_{\mathbb{R}^n}^{\oplus} S_{t_1}[\psi(\cdot,z)](x) \otimes B_{t_2}(z,y) \otimes a(y)\, dz\, dy \\
&= \int_{\mathbb{R}^n}^{\oplus} \int_{\mathbb{R}^n}^{\oplus} \int_{\mathbb{R}^n}^{\oplus} \psi(x,\zeta) \otimes B_{t_1}(\zeta,z) \otimes B_{t_2}(z,y) \otimes a(y)\, dy\, dz\, d\zeta \\
&= \int_{\mathbb{R}^n}^{\oplus} \int_{\mathbb{R}^n}^{\oplus} \psi(x,\zeta) \otimes \left\{ \left[\int_{\mathbb{R}^n}^{\oplus} B_{t_1}(\zeta,z) \otimes B_{t_2}(z,y)\, dz \right] \otimes a(y)\, dy \right\} d\zeta \quad (38) \\
&= \int_{\mathbb{R}^n}^{\oplus} \psi(x,\zeta) \otimes \mathcal{B}_{t_1}\mathcal{B}_{t_2}[a](\zeta)\, d\zeta \\
&= \mathcal{D}_{\psi}^{-1} \mathcal{B}_{t_1}\mathcal{B}_{t_2}[a]. \quad (39)
\end{aligned}
$$

Combining (33), (35), (39) and Lemma 4.2 yields the first assertion. Combining (33), (34), (38) and Lemma 4.2 yields the second. □

Lemma 4.5 *Let $G, \widehat{G} \in C_B^2(\mathbb{R}^n)$ (continuous, uniformly bounded second derivatives). Suppose*

$$
G(x,\cdot) \odot a(\cdot) = \widehat{G}(x,\cdot) \odot a(\cdot) \qquad \forall\, x \in \mathbb{R}^n
$$

for all $a \in \mathcal{S}_-^d$ for some finite, symmetric d. Then, $G = \widehat{G}$.

Proof. Suppose there exists $(\overline{x},\overline{y})$ such that $G(\overline{x},\overline{y}) \neq \widehat{G}(\overline{x},\overline{y})$. Then, there exist $\delta, \varepsilon > 0$ such that

$$
|G(x,y) - \widehat{G}(x,y)| > \varepsilon \qquad \forall\, (x,y) \in B_\delta(\overline{x},\overline{y}). \tag{40}
$$

Also, since $G, \widehat{G} \in C_B^2$, there exist $M_1, M_2 < \infty$ such that

$$
\begin{aligned}
G(x,y) &\leq G(\overline{x},\overline{y}) + M_1|(x,y)-(\overline{x},\overline{y})| + M_2|(x,y)-(\overline{x},\overline{y})|^2 \\
\widehat{G}(x,y) &\leq \widehat{G}(\overline{x},\overline{y}) + M_1|(x,y)-(\overline{x},\overline{y})| + M_2|(x,y)-(\overline{x},\overline{y})|^2
\end{aligned}
$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Consequently, with $M_3 = M_2 + M_1/\delta$,

$$
G(x,y) \leq G(\overline{x},\overline{y}) + M_3|(x,y)-(\overline{x},\overline{y})|^2, \quad \widehat{G}(x,y) \leq \widehat{G}(\overline{x},\overline{y}) + M_3|(x,y)-(\overline{x},\overline{y})|^2
$$

for all $(x,y) \notin B_\delta(\overline{x},\overline{y})$.

Let $\overline{\mathcal{M}} \doteq \{\lceil M_3, \lceil M_3 + 1, \ldots\}$. For $M \in \overline{\mathcal{M}}$, let $a_M(y) = -M|y-\overline{y}|^2$. Let

$$
\begin{aligned}
z_M = G(\overline{x},\cdot) \odot a_M(\cdot) &= \max_y [G(\overline{x},y) + a_M(y)] \\
&\geq G(\overline{x},\overline{y}),
\end{aligned} \tag{41}
$$

and let $y_M \in \operatorname{argmax}_y\{G(\overline{x},y)+a_M(y)\}$. Suppose $y_M \not\to \overline{y}$. Then, there exists $\bar{\delta} > 0$ and a subsequence such that $|y_{M_k} - \overline{y}| \geq \bar{\delta}$ for all k, which implies $G(\overline{x}, y_{M_k}) + a_{M_k}(y_{M_k}) \to -\infty$ which contradicts (41). Consequently,

$$
y_M \to \overline{y}. \tag{42}
$$

Now, using the fact that $a_M \leq 0$,

$$G(\overline{x},\cdot) \odot a_M(\cdot) = [G \otimes a_M](\overline{x}, y_M) \leq G(\overline{x}, y_M)$$

which by (42)

$$\to G(\overline{x},\overline{y}). \tag{43}$$

By (41) and (43),

$$G(\overline{x},\cdot) \odot a_M(\cdot) \to G(\overline{x},\overline{y}).$$

Similarly,

$$\widehat{G}(\overline{x},\cdot) \odot a_M(\cdot) \to \widehat{G}(\overline{x},\overline{y}).$$

Conequently, there exists $\bar{M} < \infty$ such that for all $M \geq \bar{M}$,

$$|G(\overline{x},\cdot) \odot a_M(\cdot) - G(\overline{x},\overline{y})| < \varepsilon/4, \tag{44}$$
$$|\widehat{G}(\overline{x},\cdot) \odot a_M(\cdot) - \widehat{G}(\overline{x},\overline{y})| < \varepsilon/4. \tag{45}$$

By (40), (44) and (45),

$$G(\overline{x},\cdot) \odot a_M(\cdot) \neq \widehat{G}(\overline{x},\cdot) \odot a_M(\cdot)$$

which contradicts the assumption. □

Combining Lemmas 4.4 and 4.5, we have:

Theorem 4.6 *For all $t_1, t_2 \geq 0$ such that $t_1 + t_2 < \overline{T}$,*

$$B_{t_1+t_2}(\zeta, x) = \int_{I\!R^n}^{\oplus} B_{t_1}(\zeta, z) \otimes B_{t_2}(z, x)\, dz \qquad \forall\, x, \zeta \in I\!R^n.$$

Combining Theorem 4.6 with Lemma 3.4 and Theorem 3.5, one obtains the semigroup propagation of the fundamental solution of the DRE, and this is:

Theorem 4.7 *The forward propagation of semigroup β_t is given by*

$$\beta_{t_1+t_2} = \beta_{t_1} \circledast \beta_{t_2} \tag{46}$$

where as above,

$$\begin{aligned}
[\beta_{t_1} \circledast \beta_{t_2}]^{1,1} &= \beta_{t_1}^{1,1} - \beta_{t_1}^{1,2} U_{t_1,t_2}^{-1} \beta_{t_1}^{1,2\prime} \\
[\beta_{t_1} \circledast \beta_{t_2}]^{1,2} &= -\beta_{t_1}^{1,2} U_{t_1,t_2}^{-1} \beta_{t_2}^{1,2} \\
[\beta_{t_1} \circledast \beta_{t_2}]^{2,1} &= -\beta_{t_2}^{1,2\prime} U_{t_1,t_2}^{-1} \beta_{t_1}^{1,2\prime} \\
[\beta_{t_1} \circledast \beta_{t_2}]^{2,2} &= \beta_{t_2}^{2,2} - \beta_{t_2}^{1,2\prime} U_{t_1,t_2}^{-1} \beta_{t_2}^{1,2}
\end{aligned}$$

where $U_{t_1,t_2} \doteq \beta_{t_1}^{2,2} + \beta_{t_2}^{1,1}$.

In the next section, we will indicate how the forward propagation of the fundamental solution of the DRE, (46), can be viewed in a sense analogous to an exponential. Prior to that, let us recap how one uses the fundamental solution to obtain a solution for any initial condition.

We suppose one wishes to obtain the solution of (1) at time t with initial condition $P_0 = p_0$. We suppose that one wishes to use β_t to obtain P_t. Recall that

$$S_t[\phi] = \mathcal{D}_\psi^{-1}\mathcal{B}_t\mathcal{D}_\psi\phi. \tag{47}$$

Let $\phi(x) = \frac{1}{2}x'p_0x$. By Lemma 4.2, $a(z) \doteq [\mathcal{D}_\psi\phi](z) = \frac{1}{2}z'q_0z$ where

$$q_0 = D_\psi p_0 = Q(Q-p_0)^{-1}p_0. \tag{48}$$

Then, by (25) and Lemma 3.4,

$$\begin{aligned}\mathcal{B}_t[\mathcal{D}_\psi\phi](y) = B_t(\cdot,y)\odot q_0(\cdot) &= \max_{z\in I\!R^n}\left\{\tfrac{1}{2}\begin{pmatrix} y \\ z\end{pmatrix}'\beta_t\begin{pmatrix} y \\ z\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} z \\ 0\end{pmatrix}'\begin{bmatrix} q_0 & 0 \\ 0 & 0\end{bmatrix}\begin{pmatrix} z \\ 0\end{pmatrix}\right\} \\ &= \tfrac{1}{2}\begin{pmatrix} y \\ 0\end{pmatrix}'\begin{bmatrix} q_t & 0 \\ 0 & 0\end{bmatrix}\begin{pmatrix} y \\ 0\end{pmatrix}\end{aligned} \tag{49}$$

where

$$q_t = \beta_t^{1,1} - \beta_t^{1,2}\left(\beta_t^{2,2}+q_0\right)^{-1}\beta_t^{1,2'} = \left\{\beta_t \circledast \begin{bmatrix} q_0 & 0 \\ 0 & 0\end{bmatrix}\right\}^{1,1} \doteq \beta_t\circledast' q_0 \tag{50}$$

where this defines the $\circledast'$ operation.

Recall, from (19), that we may represent $S_t[\phi]$ as

$$S_t[\phi] = \tfrac{1}{2}x'P_tx. \tag{51}$$

However,

$$S_t[\phi] = \mathcal{D}_\psi^{-1}\mathcal{B}_t\mathcal{D}_\psi\phi \tag{52}$$

where, by (49), (50)

$$[\mathcal{B}_t\mathcal{D}_\psi\phi](z) = \tfrac{1}{2}z'q_tz = \tfrac{1}{2}z'\left\{\beta_t\circledast' D_\psi p_0\right\}z. \tag{53}$$

Combining (51)–(53), we see

$$P_t = D_\psi^{-1}q_t = D_\psi^{-1}[\beta_t\circledast' D_\psi p_0] \tag{54}$$

where, using Lemma 4.2,

$$D_\psi^{-1}q_t = Q(Q+q_t)^{-1}q_t. \tag{55}$$

Equation (54) indicates how one obtains the solution P_t from the fundamental solution β_t and initial condition p_0. In particular, one performs the following steps:

- Obtain q_0 from p_0 via (48).

- Obtain q_t from β_t and q_0 via (50).
- Obtain P_t from from q_t via (55).

This may be repeated for any number of initial conditions, p_0. The choice of symmetric Q (which parametrizes semiconvex duality) is partially free, only needing to be sufficient to ensure existence of the semiconvex duals.

Remark 4.8 It is worth noting that if one worked in the dual space, one simply has $q_t = \beta_t \circledast' q_0$ as the solution of the (dual of the) DRE.

In the next section, in analogy with standard-algebra linear systems, we discuss the interpretation of β_t as an exponential. Then, in Section 6, we indicate the associated semiring. Lastly, in Sections 7 and 8, we make some tentative comments about numerical issues, and provide an example.

5 Propagation as exponentiation

We have seen that this fundamental solution propagates according to the matrix operation $\beta_{t_1+t_2} = \beta_{t_1} \circledast \beta_{t_2}$. There are two issue here: the fundamental solution concept, and the issue of numerical solution. With regard to the latter, we note that one can obtain β_τ for some very small τ by a single step of a Runge-Kutta method. Then one obtains $\beta_{n\tau}$ by repeated $\circledast$-multiplication, or better yet, by $\beta_{2\tau} = \beta_\tau \circledast \beta_\tau$, $\beta_{4\tau} = \beta_{2\tau} \circledast \beta_{2\tau}$, and so on. We will discuss this further in the next section. The former issue regards the notion of fundamental solution. Recall that for a standard-algebra linear system, one views the fundamental solution as $e^{At} = (e^A)^t$, and so $(e^A)^{t_1+t_2} = (e^A)^{t_1} \cdot (e^A)^{t_2}$. Thus, β_t is analogous to $(e^A)^t$, and we would like some similar exponential-type representation here.

Naturally, we define $\circledast$-exponentiation for positive integer powers through $\beta^{\circledast 2} = \beta \circledast \beta$, $\beta^{\circledast 3} = [\beta^{\circledast 2}] \circledast \beta$, et cetera. Using Theorem 4.7, this immediately yields $\beta_{nt} = \beta_t^{\circledast n}$. However, this only works for integer powers. We will extend this to positive real powers so that we may simply write $\beta_t = (\beta_1)^{\circledast t}$ for any $t > 0$. Then, propagation of solutions in the dual space is given by

$$q_t = \beta_1^{\circledast t} \circledast' q_0,$$

and is given, in the original space, by

$$P_t = D_\psi^{-1} \beta_1^{\circledast t} \circledast' D_\psi p_0.$$

Let $\mathcal{Q}$ denote the set of rationals. Given any $t \in (0, \infty)$, let $e_t \doteq \{s \in (0, \infty) \,|\, \exists p \in \mathcal{Q}$ such that $s = pt\}$. As is well-known, the collection of such e_t forms an uncountable set of equivalence classes covering $(0, \infty)$.

Suppose $s \in e_t$. Then, there exists $p = m/n$ with $m, n \in \mathcal{N}$ such that $s = pt$. Let $\tau = t/n$. Then, $t = n\tau$ and $s = m\tau$. Consequently, by Theorem 4.7, $\beta_s = \beta_\tau^{\circledast m}$ and $\beta_t = \beta_\tau^{\circledast n}$. With this in mind, we make the following extension of $\circledast$-exponentiation to rationals, and the fact that this extension is well-defined will be proved immediately below.

Definition 5.1 *Let $s = pt$ with $p = m/n$, $m, n \in \mathcal{N}$. We define $\beta_t^{\circledast p} \doteq \beta_\tau^{\circledast m}$ where $\tau = t/n$.*

We need to demonstrate that the definition is independent of the choice of $m, n \in \mathcal{N}$. That is, suppose $p = m_0/n_0 = m_1/n_1$. Let $\tau_0 = t/n_0$ and $\tau_1 = t/n_1$. We must show $\beta_{\tau_0}^{\circledast m_0} = \beta_{\tau_1}^{\circledast m_1}$. We will use the following, trivially-verified result.

Lemma 5.2 $\left[\beta_t^{\circledast n}\right]^{\circledast m} = \beta_t^{\circledast (nm)}$.

Let $\bar{\tau} = t/(n_0 n_1)$. Then, $\tau_0 = n_0 \bar{\tau}$ and $\tau_1 = n_1 \bar{\tau}$. Consequently,

$$\beta_{\tau_0}^{\circledast m_0} = \left(\beta_{\bar{\tau}}^{\circledast n_1}\right)^{\circledast m_0}$$

which by Lemma 5.2,

$$= \beta_{\bar{\tau}}^{\circledast (n_1 m_0)}. \tag{56}$$

Similarly,

$$\beta_{\tau_1}^{\circledast m_1} = \left(\beta_{\bar{\tau}}^{\circledast n_0}\right)^{\circledast m_1} = \beta_{\bar{\tau}}^{\circledast (n_0 m_1)}. \tag{57}$$

However, $n_1 m_0 = n_0 m_1$, and so by (56) and (57),

$$\beta_{\tau_0}^{\circledast m_0} = \beta_{\tau_1}^{\circledast m_1}.$$

In other words, the definition is independent of the choice of $m, n \in \mathcal{N}$ such that $m/n = p$.

Next we extend the $\circledast$-exponentiation definition to exponents which may not be rational. Let $t \in (0, \infty)$. From (30), the continuity of $\widetilde{P}_t$, and the fact that $\widetilde{P}_t > Q$ on $(0, \overline{T})$, we see that β_t is continuous on $(0, \overline{T})$. Consequently, we may define

$$\beta_1^{\circledast t} = \lim_{p_n \in \mathcal{Q},\, p_n \to t} \beta_{p_n} = \lim_{p_n \in \mathcal{Q},\, p_n \to t} \beta_1^{\circledast p_n}.$$

We have now obtained the fundamental solution as an $\circledast$-exponential,

$$\beta_t = \beta_1^{\circledast t},$$

and solutions for any initial values as

$$q_t = \beta_1^{\circledast t} \circledast' q_0, \ \text{ and } P_t = D_\psi^{-1} \beta_1^{\circledast t} \circledast' D_\psi p_0.$$

6 $\langle \oplus, \circledast \rangle$-Semirings

There are underlying semirings with the $\oplus, \circledast$ operations, and this seems to be quite interesting. These semirings are related to the convolution semiring of [13]. We only touch on the matter here. Consider the case where $\bar{a}$ and $\bar{b}$ are 2×2 matrices of the form

$$\bar{a} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}, \qquad \bar{b} = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix} \tag{58}$$

with $a, b \in [0, +\infty) \cup \{+\infty\} \doteq \mathcal{W}^+$.

Define the mapping from $\mathcal{W}^+$ onto 2×2 matrices of the form given in (58) as

$$\mathcal{M}(a) \doteq \bar{a} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}.$$

Let the space of such matrices be denoted by $\mathcal{L}_2^{\mathcal{W}^+}$. Then $\bar{c} = \bar{a} \circledast \bar{b}$ if and only if $\bar{c} = \mathcal{M}(c)$ where

$$c = \frac{ab}{a+b} \doteq a \circledast b$$

which defines the $\circledast$ operation on $\mathcal{W}^+$. Also, define $\oplus$ on $\mathcal{W}^+$ by $a \oplus b = \max\{a, b\}$. Then define $\oplus$ on $\mathcal{L}_2^{\mathcal{W}^+}$ by

$$\bar{a} \oplus \bar{b} = \mathcal{M}(a \oplus b).$$

We define $a \oplus +\infty$ in the natural way, and let

$$a \circledast +\infty \doteq \lim_{b \to +\infty} \frac{ab}{a+b} = a.$$

It is obvious that $a \oplus b,\ a \circledast b \in \mathcal{W}^+$ for all $a, b \in \mathcal{W}^+$.

Theorem 6.1 $\langle \mathcal{W}^+, \oplus, \circledast \rangle$ *is a commutative idempotent semiring.*

Proof. Note that idempotency is immediate as $a \oplus b = a$ if $a \geq b$ (c.f., [13]). Note that 0 is the $\oplus$ identity, and $+\infty$ is the $\circledast$ identity. One needs to show that $\oplus$ and $\circledast$ are commutative and associative, and that the distributive property holds. We will skip the special cases involving $+\infty$. The commutative and associative properties of $\oplus$ are immediate, as is the commutative property of $\circledast$.

Let $a, b, c \in \mathcal{W}^+$. Then,

$$a \circledast (b \circledast c) = a \circledast \frac{bc}{b+c} = \frac{abc}{ab+bc+ca},$$

and similarly one finds $(a \circledast b) \circledast c = abc/(ab+bc+ca)$ which proves associativity.

For the distributive property, again let $a,b,c \in \mathcal{W}^+$, and without loss of generality, suppose $b \ge c$. Then,

$$a \circledast (b \oplus c) = a \circledast b. \tag{59}$$

However, $b \ge c$ and $a,b,c \in \mathcal{W}^+$ imply

$$b(a+c) \ge c(a+b)$$

and again using $a,b,c \in \mathcal{W}^+$, this implies

$$\frac{b}{a+b} \ge \frac{c}{a+c}$$

and, upon multiplying by a on both sides,

$$a \circledast b \ge a \circledast c,$$

which implies

$$(a \circledast b) \oplus (a \circledast c) = a \circledast b. \tag{60}$$

Combining (59) and (60), we see that the distributive property holds. □

The following is immediate by the bijective and order-preserving properties of $\mathcal{M}$.

Corollary 6.2 $\langle \mathcal{L}_2^{\mathcal{W}^+}, \oplus, \circledast \rangle$ *is a commutative idempotent semiring.*

7 Numerical method

The existence of this fundamental solution is certainly of interest as a mathematical object. However, it can also be used to numerically obtain solutions of DREs. The fact that the word numerically appears in the previous statement is due to the fact that one cannot analytically obtain β_τ for some $\tau > 0$. One obtains β_τ for some very small value of $\tau > 0$, and then propagates the fundamental solution forward analytically via $\circledast$-multiplication (equivalently, $\circledast$-exponentiation for integer exponents). In contrast to say, Runge-Kutta methods, which are of polynomial order, this yields a numerical method which is of *exponential order.* Solutions can be obtained extremely quickly. However, there is an instability in the propagation, similar to that in say the Kalman filter, and so small round-off errors can grow rapidly even if the solution is stable (c.f., [4] for discussion and solution of this problem in the Kalman filter case). We do not attempt to remove, or attenuate, the instability in this fundamental solution of the DRE, as that significantly exceeds the scope of what should be examined in a first excursion into this domain. Nonetheless, we will indicate how this fundamental solution can be used to obtain an exponential-order method for solution of the DRE.

With a Runge-Kutta algorithm, one obtains a solution with polynomial-order error. With the standard fourth-order Runge-Kutta method, the error in the solution over the fixed interval $[0,T]$ drops like δ^4 where δ is the time-step size. Consequently, the error decreases like $(1/M)^4$ where M is the computational effort. Using this fundamental solution, we can instead obtain the solution at T where the error drops like γ^M for some $\gamma \in (0,1)$; the method is of exponential order. An example will also be included, demonstrating the computational efficiency.

Fix some $T < \infty$. We choose integer N, and let $\tau = \frac{T}{2^N}$. We will obtain β_τ. From this, we compute $\beta_{2\tau} = \beta_\tau \circledast \beta_\tau$, and $\beta_{2^n\tau} = \beta_{2^{n-1}\tau} \circledast \beta_{2^{n-1}\tau}$ for $n \in \{2, 3 \ldots N\}$, yielding $\beta_T = \beta_{2^N\tau}$. Finally, given any initial condition, p_0, we obtain P_T from $P_T = D_\psi^{-1}\beta_T \circledast' D_\psi p_0$. These computations are all approximation-error free, neglecting machine roundoff. The only (approximation) error source is through the computation of β_τ. Of course given another initial condition, p_0', the solution is $P_T' = D_\psi^{-1}\beta_T \circledast' D_\psi p_0'$, and so only negligible additional computational effort is required to produce P_T'.

We will compute β_τ through a single step of a Runge-Kutta method of order $\bar{\alpha}$, thus producing an error in β_τ on the order of $\tau^{\bar{\alpha}+1}$. (Recall the local one-step error for a Runge-Kutta method is one order greater than the fixed-horizon error.) For the fourth-order Runge-Kutta algorithm, $\bar{\alpha} = 4$, one has $\bar{\alpha}+1 = 5$. As noted above, this requires four function evaluations per step. To obtain $\bar{\alpha} = 5$ (i.e., $\bar{\alpha}+1 = 6$), one needs to perform six function evaluations per step. Alternatively, one could employ multiple steps of a Runge-Kutta method with step-size δ, and obtain $\beta_\tau = \beta_{m\delta}$ for some integer m. One would balance m against N to produce the minimal total error as a function of computational effort. We will not examine such an optimization here.

We will determine the error in β_τ from the use of one step in an $\bar{\alpha}$-order Runge-Kutta method. This will then be mapped directly into an error in P_t. Recall that β_τ is obtained from $\widetilde{P}_\tau$ through equations (30). Consequently, we will note the error order in $\widetilde{P}_\tau$, and then determine the resulting order of the error in β_τ induced by this error in $\widetilde{P}_\tau$. From this, we obtain the exponential order of the error in P_T. As indicated above, we use an $\bar{\alpha}$-order Runge-Kutta algorithm with resultant error in $\widetilde{P}_\tau$ of order $\tau^{\bar{\alpha}+1}$. With a fourth-order Runge-Kutta method, we have (one-step) error order τ^5. Now we begin the main task of the section: mapping the error in $\widetilde{P}_\tau$ into the error in β_τ.

Let $\widetilde{P}_\tau$ be the exact solution at time τ with initial condition $\widetilde{P}_0 = Q$. Let $\widehat{P}_\tau$ be the solution computed by one-step of the Runge-Kutta method (again with initial condition $\widehat{P}_0 = Q$ of course). Let $\mathcal{E}_\tau \doteq \widehat{P}_\tau - \widetilde{P}_\tau$, and note

$$|\mathcal{E}_\tau| \le k_r \tau^{\bar{\alpha}+1} \tag{61}$$

for some k_r dependent on the specific problem coefficients. Similarly, letting $\widehat{\Lambda}_\tau$ and $\widehat{R}_\tau$ be the one-step Runge-Kutta solutions for Λ_τ and R_τ, we have (where we use the same constant for simplicity)

$$|\widehat{\Lambda}_\tau - \Lambda_\tau| \le k_r \tau^{\bar{\alpha}+1} \text{ and } |\widehat{R}_\tau - R_\tau| \le k_r \tau^{\bar{\alpha}+1}. \tag{62}$$

By examining (30), we see that there will be errors induced directly by multiplication by $\widehat{P}_\tau$, and also errors induced by the error in $\Delta_\tau = Q - \widetilde{P}_\tau$, where our approximation will be denoted by $\widehat{\Delta}_\tau \doteq Q - \widehat{P}_\tau$. We will need several estimates. First we collect some observations and an assumption.

Let $F(P)$ denote the right-hand side of (1), i.e., $F(P) = A'P + PA + C + P\Sigma P$. Note that given $R < \infty$, there exists $K_L < \infty$ such that

$$|F(P_1) - F(P_2)| \leq K_L |P_1 - P_2| \qquad \forall\, P_1, P_2 \in B_R(Q) \tag{63}$$

where $|\cdot|$ will denote the induced norm. Throughout the remainder, we assume there exists $c > 0$ such that

$$|F(Q)x| \geq c|x| \qquad \forall\, x \in I\!R^n \tag{A.l}$$

which of course implies $|F(Q)| \geq c$.

The first step will be to obtain a lower bound on $|\Delta_\tau^{-1}|$. By (63) and standard results, there exist $\bar{\tau}_0 > 0$ and $D < \infty$ such that

$$|\widetilde{P}_t - Q| \leq Dt \qquad \forall\, t \in [0, \bar{\tau}_0]. \tag{64}$$

Let $\tau_0 \doteq \min\{\bar{\tau}_0, R/D\}$. Now, fix any $x \in I\!R^n$, and note

$$\left| \frac{d}{dt} \left[(\widetilde{P}_t - Q)x - F(Q)xt \right] \right| = \left| \left[F(\widetilde{P}_t) - Q \right] x \right|$$

which by (63)

$$\leq K_L |\widetilde{P}_t - Q| |x|$$

which by (64)

$$\leq K_L D |x| t \tag{65}$$

for all $t \in [0, \tau_0]$. This immediately implies

$$\left| (\widetilde{P}_t - Q)x - F(Q)xt \right| \leq \frac{K_L D |x|}{2} t^2 \qquad \forall\, t \in [0, \tau_0]. \tag{66}$$

Now, for $t \leq \tau_2 \doteq c/(K_L D)$, one has

$$K_L D |x| t \leq c|x|$$

which by Assumption $(A.l)$

$$\leq |F(Q)x|,$$

which implies

$$\frac{K_L D |x|}{2} t^2 \leq \left| \frac{F(Q)xt}{2} \right| \qquad \forall\, t \in [0, \tau_2].$$

Combining this observation with (66) yields

$$\left| (\widetilde{P}_t - Q)x - F(Q)xt \right| \leq \left| \frac{F(Q)xt}{2} \right| \qquad \forall\, t \in [0, \tau_3]$$

where $\tau_3 = \tau_0 \wedge \tau_2$, and so by the triangle inequality,

$$|\Delta_t x| = \left|(\widetilde{P}_t - Q)x\right| \geq \left|\frac{F(Q)x}{2}\right| t,$$

which by $(A.l)$,

$$\geq \frac{c}{2}|x|t \qquad \forall\, t \in [0, \tau_3]. \tag{67}$$

This, of course implies the existence of Δ_t^{-1}, and so we may rewrite (67) as

$$|\Delta_t^{-1}\Delta_t x| \leq \frac{1}{K_i t}|\Delta_t x| \qquad \forall\, x \in I\!\!R^n,\ \forall t \in [0, \tau_3] \tag{68}$$

where $K_i = c/2$. However, (67) also implies that Δ_t maps onto $I\!\!R^n$, and so (68) implies

$$|\Delta_t^{-1} y| \leq \frac{|y|}{K_i t} \qquad \forall\, y \in I\!\!R^n,\ \forall t \in [0, \tau_3],$$

and we have:

Lemma 7.1 *There exist $\tau_3 > 0$ and $K_i \in (0, \infty)$ such that*

$$|\Delta_t^{-1}| \leq \frac{1}{K_i t} \qquad \forall\, t \in [0, \tau_3].$$

We will also need the following.

Lemma 7.2 *There exist $k_1 > 0$ and $\tau_4 > 0$ such that*

$$|\Delta_\tau^{-1} - \widehat{\Delta}_\tau^{-1}| \leq k_1 \tau^{\bar{\alpha}-1} \qquad \forall\, \tau \in [0, \tau_4].$$

Proof. Note that

$$\begin{aligned} 0 &= \widehat{\Delta}_\tau^{-1}\widehat{\Delta}_\tau - \Delta_\tau^{-1}\Delta_\tau \\ &= (\widehat{\Delta}_\tau^{-1} - \Delta_\tau^{-1})\Delta_\tau + \widehat{\Delta}_\tau^{-1}(\widehat{\Delta}_\tau - \Delta_\tau) \end{aligned}$$

which implies

$$\Delta_\tau^{-1} - \widehat{\Delta}_\tau^{-1} = (\widehat{\Delta}_\tau^{-1} - \Delta_\tau^{-1})(\widehat{\Delta}_\tau - \Delta_\tau)\Delta_\tau^{-1} + \Delta_\tau^{-1}(\widehat{\Delta}_\tau - \Delta_\tau)\Delta_\tau^{-1},$$

which upon rearrangement (and noting $\widehat{\Delta}_\tau - \Delta_\tau = \widehat{P}_\tau - \widetilde{P}_\tau$), yields

$$\Delta_\tau^{-1} - \widehat{\Delta}_\tau^{-1} = \Delta_\tau^{-1}(\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\left[I + (\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\right]^{-1}. \tag{69}$$

Also, from (61) and Lemma 7.1,

$$\begin{aligned}|(\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}| &\le |\widehat{P}_\tau - \widetilde{P}_\tau||\Delta_\tau^{-1}| \\ &\le \frac{k_r}{K_i}\tau^{\bar{\alpha}} \qquad \forall\, \tau \in [0, \tau_3].\end{aligned}$$

This implies

$$\left|\left[I + (\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\right] x\right| \ge \left(1 - \frac{k_r}{K_i}\tau^{\bar{\alpha}}\right)|x| \ge \tfrac{1}{2}|x| \qquad \forall\, x \in I\!R^n$$

for all $\tau \in [0, \tau_4]$ for some $\tau_4 \in (0, \tau_3]$. This implies $\left[I + (\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\right]^{-1}$ exists and

$$\left|\left[I + (\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\right]^{-1}\right| \le 2. \tag{70}$$

Now, from (69),

$$|\Delta_\tau^{-1} - \widehat{\Delta}_\tau^{-1}| \le |\Delta_\tau^{-1}||\widehat{P}_\tau - \widetilde{P}_\tau||\Delta_\tau^{-1}| \left|\left[I + (\widehat{P}_\tau - \widetilde{P}_\tau)\Delta_\tau^{-1}\right]^{-1}\right|,$$

which by (61), (70) and Lemma 7.1,

$$\le \frac{2k_r}{K_i^2}\tau^{\bar{\alpha}-1}. \ \square$$

Let $\widehat{\beta}_\tau$ be obtained from $\widehat{P}_\tau$ (i.e., from the Runge-Kutta-generated $\widetilde{P}_\tau$ approximation).

Theorem 7.3 *There exist $\bar{k} > 0$ and $\tau_5 > 0$ such that*

$$|\hat{\beta}_\tau^{1,1} - \beta_\tau^{1,1}|, |\hat{\beta}_\tau^{1,2} - \beta_\tau^{1,2}|, |\hat{\beta}_\tau^{2,1} - \beta_\tau^{2,2}|, |\hat{\beta}_\tau^{2,2} - \beta_\tau^{2,2}| < \bar{k}\tau^{\bar{\alpha}-1}$$

for all $\tau \in [0, \tau_5]$.

Proof. From (30), we see that

$$\begin{aligned}|\hat{\beta}_\tau^{1,1} - \beta_\tau^{1,1}| &= \left|Q\widehat{\Delta}_\tau^{-1}\widehat{P}_\tau - Q\Delta_\tau^{-1}\widetilde{P}_\tau\right| \\ &\le |Q| \left\{|\widehat{\Delta}_\tau^{-1} - \Delta_\tau^{-1}||\widehat{P}_\tau - \widetilde{P}_\tau| + |\widehat{\Delta}_\tau^{-1} - \Delta_\tau^{-1}||\widetilde{P}_\tau| + |\Delta_\tau^{-1}||\widehat{P}_\tau - \widetilde{P}_\tau|\right\}\end{aligned}$$

which by (61) and Lemmas 7.1 and 7.2,

$$\le |Q| \left\{k_1 k_r \tau^{\bar{\alpha}-1}\tau^{\bar{\alpha}+1} + k_1\tau^{\bar{\alpha}-1}(|Q| + k_r\tau^{\bar{\alpha}+1}) + \frac{k_r}{K_i\tau}\tau^{\bar{\alpha}+1}\right\}$$

which for proper choice of $\bar{k}_1$,

$$\le \bar{k}_1 \tau^{\bar{\alpha}-1}. \tag{71}$$

Next, also using (30), we note that

$$\begin{aligned}
|\hat{\beta}_\tau^{1,2} - \beta_\tau^{1,2}| &= \left| Q\widehat{\Delta}_\tau^{-1}\widehat{P}_\tau\widehat{\Lambda}_\tau - Q\Delta_\tau^{-1}\widetilde{P}_\tau\Lambda_\tau \right| \\
&\le |Q\widehat{\Delta}_\tau^{-1}\widehat{P}_\tau - Q\Delta_\tau^{-1}\widetilde{P}_\tau||\Lambda_\tau| + |Q\widehat{\Delta}_\tau^{-1}\widehat{P}_\tau - Q\Delta_\tau^{-1}\widetilde{P}_\tau||\widehat{\Lambda}_\tau - \Lambda_\tau| \\
&\quad + Q\Delta_\tau^{-1}\widetilde{P}_\tau||\widehat{\Lambda}_\tau - \Lambda_\tau|
\end{aligned}$$

which by Lemma 7.1, (62) and (71),

$$\begin{aligned}
&\le \bar{k}_1\tau^{\bar{\alpha}-1}(|\Lambda_\tau - I| + 1) + \bar{k}_1\tau^{\bar{\alpha}-1}k_r\tau^{\bar{\alpha}+1} \\
&\quad + \frac{|Q|}{K_i\tau}\left(|\widetilde{P}_\tau - Q| + |Q|\right)k_r\tau^{\bar{\alpha}+1}.
\end{aligned} \tag{72}$$

Again, by (64) and standard results, there exists $\tau_5 \in (0, \tau_4]$ such that

$$|\widetilde{P}_\tau - Q| \le D\tau \ \text{ and } \ |\Lambda_\tau - I| \le D_2\tau$$

for all $\tau \in [0, \tau_5]$. Combining these with (72) implies there exists $\bar{k}_2 > 0$ such that

$$|\hat{\beta}_\tau^{1,2} - \beta_\tau^{1,2}| \le \bar{k}_2\tau^{\bar{\alpha}-1} \tag{73}$$

for all $\tau \in [0, \tau_5]$.

A similar analysis yields the required result for $|\hat{\beta}_\tau^{2,2} - \beta_\tau^{2,2}|$, and we do not include the details. □

Recall that we will be computing the fundamental solution at time T as $\beta_T = \beta_{2^N\tau}$ from the iteration $\beta_{2^{k+1}\tau} = \beta_{2^k\tau} \circledast \beta_{2^k\tau}$ for $k \in \{1, 2 \ldots N-1\}$, that is, via $(N-1)$ $\circledast$-multiplications. Note that there are no approximations in this sequence of operations. The error bounds at time $T = 2^N\tau$ will be obtained from the error bounds at time τ, which were obtained in Theorem 7.3. (We consider only the error sources introduced by the approximation in the method – not machine round-off.)

The following result is a very coarse bound on the error propagation. We make no attempt here to achieve more than a statement of the exponential order of the numerical method under rather strong conditions. The reader should note that we did not assume that we were considering only DRE/initial condition combinations that yielded stable solutions, and in fact, we allow for problems with finite-time blow-up of the solutions. For such problems, the sensitivity of the solution near the time of blow-up to errors at initialization can be extremely high. (Our assumption that $T \le T_\varepsilon$ for fixed positive ε guarantees that we stay at least ε away from the "vertical" asymptote.) Consequently, our error bounds allow for solutions with geometric growth. Refinements for specific classes of problems with well-behaved solutions are clearly beyond the scope of this introductory study.

We indicate an easy lemma prior to proceeding to the main result on error propagation in the $\circledast$-multiplications.

Lemma 7.4

$$P_M \doteq \sup_{t\in[0,T_\varepsilon]} |\widetilde{P}_t| < \infty, \qquad \Lambda_M \doteq \sup_{t\in[0,T_\varepsilon]} |\Lambda_t| < \infty,$$

and

$$R_M \doteq \sup_{t\in[0,T_\varepsilon]} |R_t| < \infty.$$

PROOF. The result for $\widetilde{P}_t$ follows from the definition of T_ε. The result for Λ_t then follows from the fact that it is the solution of a linear system with bounded coefficient (see (12)). Then one notes from (12), that R_t is an integral with bounded integrand. □

For the remaining results we will assume

$$Q \text{ is either positive definite or negative definite,} \tag{A.Q}$$

and

$$\exists \bar\delta > 0 \text{ such that } \beta_t^{1,1} - \bar\delta I > 0, \quad \beta_t^{2,2} - \bar\delta I > 0 \qquad \forall\, t \in [0, T_\varepsilon]. \tag{A.\beta}$$

We then let $\sigma_M = \min\{|\lambda| \,|\, \lambda \text{ is an eigenvalue of } Q\}$.

For $i,j \in \{1,2\}$, let $\delta^{i,j}_{2^k\tau} \doteq \hat\beta^{i,j}_{2^k\tau} - \beta^{i,j}_{2^k\tau}$ where $\hat\beta^{i,j}_{2^k\tau} = \hat\beta^{i,j}_{2^{k-1}\tau} \circledast \hat\beta^{i,j}_{2^{k-1}\tau}$.

Theorem 7.5 *Suppose $|\delta^{1,2}_{2^k\tau}| \le 1$, $|\delta^{1,1}_{2^k\tau}| \le \bar\delta$ and $|\delta^{2,2}_{2^k\tau}| \le \bar\delta$ where $k \in \{1, 2 \ldots N-1\}$. There exists $\overline{\tau} < \infty$ such that, if τ is sufficiently small, then*

$$\begin{aligned}
|\delta^{1,1}_{2^{k+1}\tau}| &\le (1+\bar c_1)|\delta^{1,1}_{2^k\tau}| + \bar c_2 |\delta^{1,2}_{2^k\tau}| + \bar c_1 |\delta^{2,2}_{2^k\tau}| \\
|\delta^{1,2}_{2^{k+1}\tau}| &\le \bar c_1|\delta^{1,1}_{2^k\tau}| + \bar c_2 |\delta^{1,2}_{2^k\tau}| + \bar c_1 |\delta^{2,2}_{2^k\tau}| \\
|\delta^{2,2}_{2^{k+1}\tau}| &\le \bar c_1|\delta^{1,1}_{2^k\tau}| + \bar c_2 |\delta^{1,2}_{2^k\tau}| + (1+\bar c_1) |\delta^{2,2}_{2^k\tau}|
\end{aligned}$$

where

$$\begin{aligned}
&\bar c_1 = 2\bar c^2, \quad \bar c_2 = 2\bar c, \\
&\bar c = \max_{t\in[0,\infty)} \left(1 + \frac{P_M \Lambda_M |Q|}{K_i t}\right) g_S(t),
\end{aligned}$$

and

$$g_S(t) = \begin{cases} \frac{2K_i t}{\sigma_M^2} & \text{if } t \in (0, \overline{\tau}] \\ \frac{1}{2\bar\delta} & \text{if } t \in (\overline{\tau}, \infty). \end{cases}$$

As it is rather technical, the proof of Theorem 7.5 is delayed to the appendix.

We now continue with the coarse estimates; the goal is simply to prove exponential order under some, perhaps overly strong, conditions. Let $\bar{\bar{c}} \doteq 3\max\{1+\bar{c}_1, \bar{c}_2\}$. We see that for all $i,j \in \{1,2\}$ and all $k \in \{1,2\dots N-1\}$,

$$|\delta^{i,j}_{2^{k+1}\tau}| \le \bar{\bar{c}} \max_{\hat{i},\hat{j}\in\{1,2\}} |\delta^{\hat{i},\hat{j}}_{2^k\tau}|.$$

More succinctly, with $\tilde{\delta}_{2^k\tau} \doteq \max_{\hat{i},\hat{j}\in\{1,2\}} |\delta^{\hat{i},\hat{j}}_{2^k\tau}|$,

$$\tilde{\delta}_{2^{k+1}\tau} \le \bar{\bar{c}}\tilde{\delta}_{2^k\tau}. \tag{74}$$

Recall from Theorem 7.3 that $\tilde{\delta}_\tau \le \bar{k}\tau^{\bar{\alpha}-1}$. Suppose that $\bar{\alpha}$ is sufficiently large such that $\bar{k}\tau^{\bar{\alpha}-1} < \min\{1,\bar{\delta}\}$. Then, by Theorem 7.5 and (74),

$$\tilde{\delta}_{2\tau} < \bar{\bar{c}}\tilde{\delta}_\tau < \bar{\bar{c}}\bar{k}\tau^{\bar{\alpha}-1}.$$

Next, if we also have $\bar{\bar{c}}\bar{k}\tau^{\bar{\alpha}-1} < \min\{1,\bar{\delta}\}$, then again by Theorem 7.5 and (74), we obtain

$$\tilde{\delta}_{2^2\tau} < \bar{\bar{c}}\tilde{\delta}_{2\tau} < \bar{\bar{c}}^2\bar{k}\tau^{\bar{\alpha}-1}.$$

By induction, we see that if

$$\bar{\bar{c}}^{N-1}\bar{k}\tau^{\bar{\alpha}-1} < \min\{1,\bar{\delta}\},$$

then

$$\tilde{\delta}_{2^N\tau} < \bar{\bar{c}}^{N-1}\bar{k}\tau^{\bar{\alpha}-1}.$$

However, note that $N = \log_2(T/\tau)$ and

$$\bar{\bar{c}}^{\log_2(T/\tau)-1} = (T/\tau)^{\ln(\bar{\bar{c}})}\,\frac{1}{2\bar{\bar{c}}}.$$

Consequently, we have:

Lemma 7.6 *If τ is sufficiently small, if $\bar{k}\tau^{\bar{\alpha}-1} < \min\{1,\bar{\delta}\}$, and if*

$$\left[\frac{\bar{k}T^{\ln(\bar{\bar{c}})}}{2\bar{\bar{c}}}\right]\tau^{\bar{\alpha}-(1+\ln(\bar{\bar{c}}))} < \min\{1,\bar{\delta}\},$$

then

$$|\hat{\beta}^{i,j}_T - \beta^{i,j}_T| \le \tilde{\delta}_{2^N\tau} < \bar{\bar{c}}^{N-1}\bar{k}\tau^{\bar{\alpha}-1} \qquad \forall\, i,j \in \{1,2\}. \tag{75}$$

The computational effort consists of two components: The effort for the single Runge-Kutta step of order $\bar{\alpha}$, $e_{\bar{\alpha}}$, and the effort to perform the $N-1$ $\circledast$-multiplications, each requiring effort e_m. The total computational effort is $\bar{e} = e_{\bar{\alpha}} + (N-1)e_m$. Rearranging this, we have

$$N-1 = \frac{\bar{e}-e_{\bar{\alpha}}}{e_m}. \tag{76}$$

Substituting (76) into (75) (and noting $\tau = T/(2^N)$) yields

$$|\hat{\beta}_T^{i,j} - \beta_T^{i,j}| < \bar{k}T^{\bar{\alpha}-1}\left(\frac{\bar{\bar{c}}}{2^{\bar{\alpha}-1}}\right)^{\frac{\bar{e}-e_{\bar{\alpha}}}{e_m}}$$
$$= \bar{k}T^{\bar{\alpha}-1}\left(\frac{\bar{\bar{c}}}{2^{\bar{\alpha}-1}}\right)^{\frac{-e_{\bar{\alpha}}}{e_m}}\left[\left(\frac{\bar{\bar{c}}}{2^{\bar{\alpha}-1}}\right)^{\frac{1}{e_m}}\right]^{\bar{e}}.$$

We have:

Theorem 7.7 *Suppose τ is sufficiently small, and that $\bar{\alpha}$ is sufficiently large so that both the conditions of Lemma 7.6 and the inequality $\bar{\bar{c}}/(2^{\bar{\alpha}-1}) < 1$ are satisfied. Then, the method displays an exponential order of convergence with respect to total computational effort, $\bar{e}$, and this is given by*

$$|\hat{\beta}_T^{i,j} - \beta_T^{i,j}| < \overline{K}_1\left[\overline{K}_2\right]^{\bar{e}}$$

where

$$\overline{K}_1 = \bar{k}T^{\bar{\alpha}-1}\left(\frac{\bar{\bar{c}}}{2^{\bar{\alpha}-1}}\right)^{\frac{-e_{\bar{\alpha}}}{e_m}} \quad \text{and} \quad \overline{K}_2 = \left[\left(\frac{\bar{\bar{c}}}{2^{\bar{\alpha}-1}}\right)^{\frac{1}{e_m}}\right]. \tag{77}$$

8 Example

There are two components to the above results. The first is the new, and intrinsically interesting, fundamental solution of the DRE. The second is the application of this fundamental solution in a numerical method. We include a very simple example as an indication that the numerical method does indeed function as indicated, with exponential convergence as a function of computational effort. The example also indicates the unresolved issue of stability of a numerical method based on this new fundamental solution.

As a first example, we consider DRE (1) with matrices of size 2×2, and coefficients

$$A = \begin{bmatrix} -2 & 1.6 \\ -1.6 & -0.4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.5 & 0.2 \\ 0.2 & 1.6 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.216 & -0.008 \\ -0.008 & 0.216 \end{bmatrix},$$

with semiconvexity matrix given by

$$Q = \begin{bmatrix} -1 & -0.2 \\ -0.2 & -1 \end{bmatrix},$$

and initial condition

$$p_0 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

The problem was to compute the solution at time $T = 4$. The matrix used in the semiconvex duality was

$$Q = \begin{bmatrix} -1 & -0.2 \\ -0.2 & -1 \end{bmatrix}.$$

A basic fourth-order Runge-Kutta method was applied with 3200 steps, to generate what we used as the "true" solution. Fourth-order Runge-Kutta approximations were computed with numbers of steps ranging from 8 to 800. The new fundamental solution method was also applied with N ranging from 4 to 13 (i.e., τ ranging from $\tau = T/2^4 = 1/16$ to $\tau = T/2^{13} = 1/2048$. The errors in each are computed by comparison with the "true" solution indicated above as the Runge-Kutta fourth-order solution with 3200 steps. In particular, we take the error to be $e = \sum_{i=1}^{n} \sum_{j=1}^{n} |P_{i,j}^a - P_{i,j}|$ with P being the true solution at time T and P^a being the approximate.

We will examine the errors as a function of computational cost. The measure computational cost used here is the number of floating point multiplications. We assume that an $n \times n$ matrix inverse requires approximately $4n^3/3$ matrix multiplications. The number of multiplications required by the basic fourth-order Runge-Kutta method on this problem is approximately $4n^3 N_{RK}$ where N_{RK} is the number of Runge-Kutta steps (with four function evaluations per step). For the new fundamental solution approach, we assume that only a single-step of the same fourth-order Runge-Kutta method is used to initialize β_τ, and then approach discussed in the previous section is used to obtain $\beta_T = \beta_{2^N \tau}$. Including also the D_ψ and D_ψ^{-1} operations, the number of multiplications for the new fundamental solution approach is approximately $(19N/3 + 16)n^3 + 4n^2$.

In Figure 1, we plot the log of the solution error as a function of the log of the computational cost (i.e., of the number of multiplications) for both approaches on the above problem. Note that at an error size of $e^{-22`} \simeq 3 \times 10^{-10}$, the computational effort required by the new fundamental solution method is lower by nearly a factor of $e^{3.5} \simeq 30$. Also note that at the very bottom of the curve for the new fundamental solution method, there is a sudden halt in the improvement as a function of effort. This appears to be due to some instability in the method, where it is likely that round-off error in the computation of β_τ is exploding at time T. If this stability issue can be resolved without tremendously affecting the convergence rate, then the computational effort ratio at higher approximation levels would be much greater than that at e^{-22}. However, as noted earlier, a deeper study of numerical issues is beyond the scope of this paper, which is introducing a new fundamental solution for Riccati equations.

Recall that the fundamental solution approach allows us to compute solutions for multiple initial conditions from a single fundamental solution by some relatively simple manipulations. With this in mind, we also plot the log of the solution error as a function of the log of the computational cost for the case where 10 initial conditions are considered. In this case the cost of the Runge-Kutta approach grows by a factor of 10 while that of the new fundamental solution approach grows much more slowly, and this can be seen in Figure 2. For example, at an error of e^{-22}, the computational effort required by the new fundamental solution method is lower by over a factor of $e^{5.5} \simeq 240$.

We can also use this example to verify the exponential convergence rate of the new fundamental solution method. Recalling Theorem 7.7, we expect that the error in the particular solution, δ_T^P, should satisfy $|\delta_T^P| \propto (\widetilde{K})^{\bar{e}}$ where $\bar{e}$ is the computational effort, and $\widetilde{K}$ is some coefficient which must be less than one for convergence. Consequently,

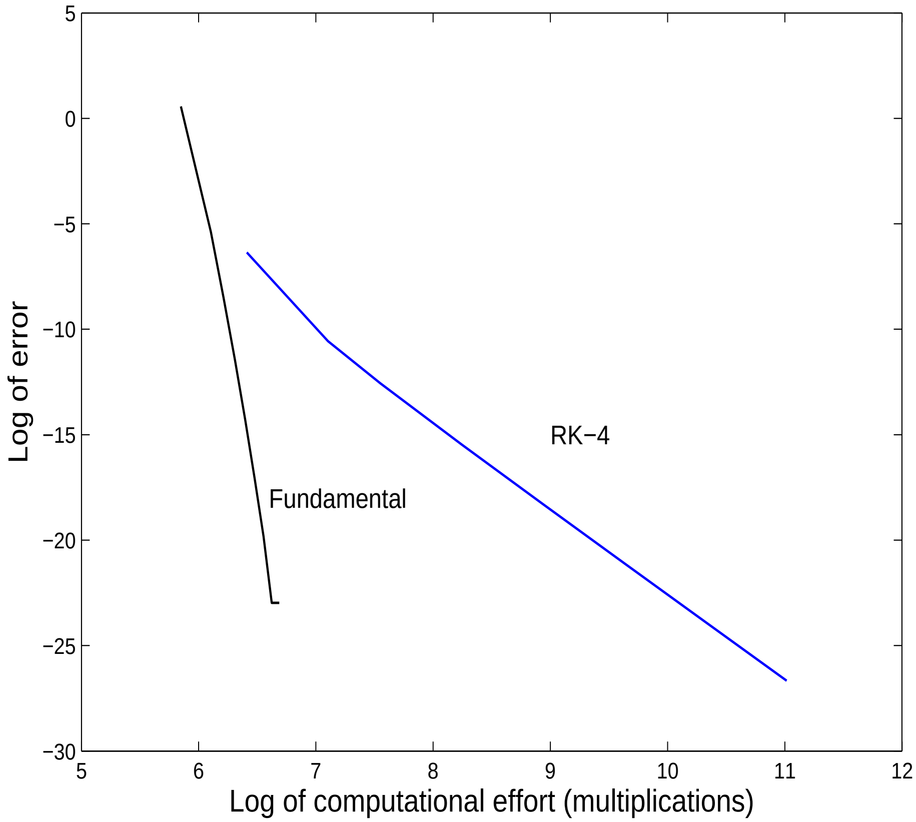

Figure 1: Log-log plot of error versus effort

plotting $\log(|\delta_T^P|)$ against $\bar{e}$ should yield a straight line with slope $\log(\widetilde{K})$. In Figure 3, we see this staight-line behavior (up until the instability point). The slope is approximately $-24/400$, leading to an estimate for $\widetilde{K}$ of roughly 0.94.

9 Summary

A new fundamental solution for the DRE has been obtained. It is intimately connected with the control interpretation of Riccati equations. At another level, this fundamental solution seems representative of a deep connection between quadratic systems and the max-plus semiring (and/or the newly-introduced $\langle\oplus,\circledast\rangle$ semiring), which is analogous to the connection between linear systems and the standard field. In fact, the whole notion of fundamental solutions in linear systems as exponentials (i.e., e^{At}) is echoed in the max-plus/quadratic systems arena.

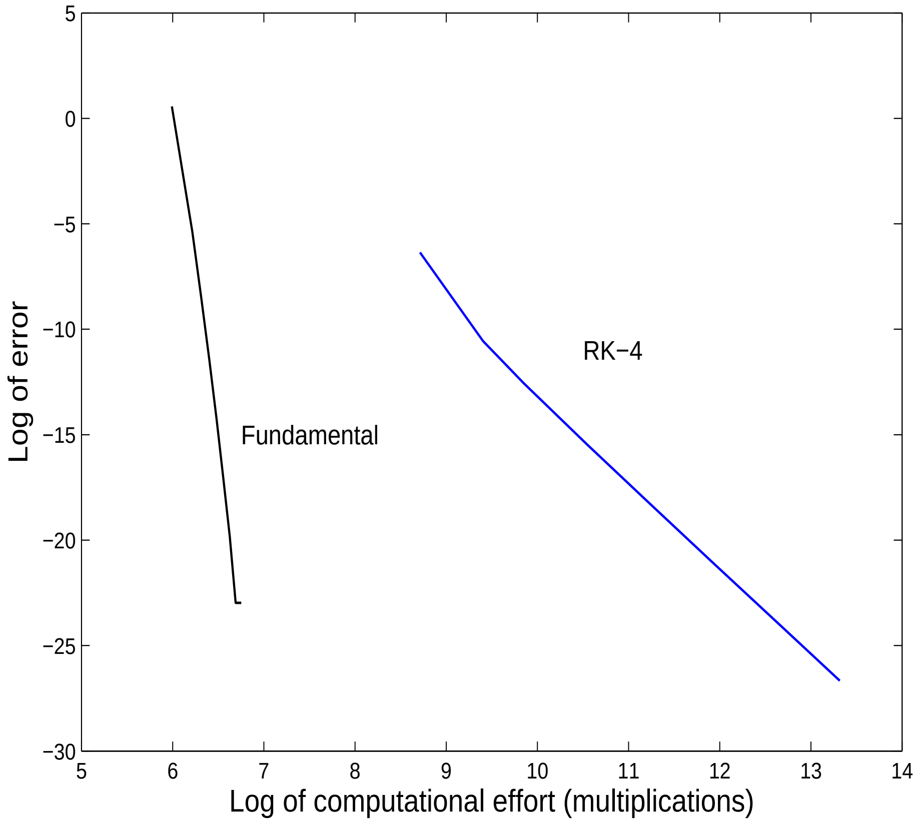

Figure 2: Log-log plot of error versus effort, 10 initial conditions

There are natural extensions of this work to DREs with time-dependent coefficients. Further, this approach is clearly extendable to infinite-dimensional systems, including first-order Riccati PDEs. It also opens up new avenues for solution of more general differential equations with quadratic nonlinearities. Extension of the semiring to moduloids ("vector spaces" loosely speaking) over spaces of matrices may also be of theoretical value.

Not unexpectedly, the fundamental solution allows for exponentially fast numerical schemes, and these can be especially useful when one wishes to solve a system for multiple initial conditions. The author encountered numerical instability as the error tolerance became very small, and it would clearly be of interest to determine how this could be eliminated. Such development might require separation of the problem domain into those systems possessing solutions for all (forward) time, and those whose the solutions exist only on finite time-intervals. Both the theoretical and numerical aspects are of independent interest.

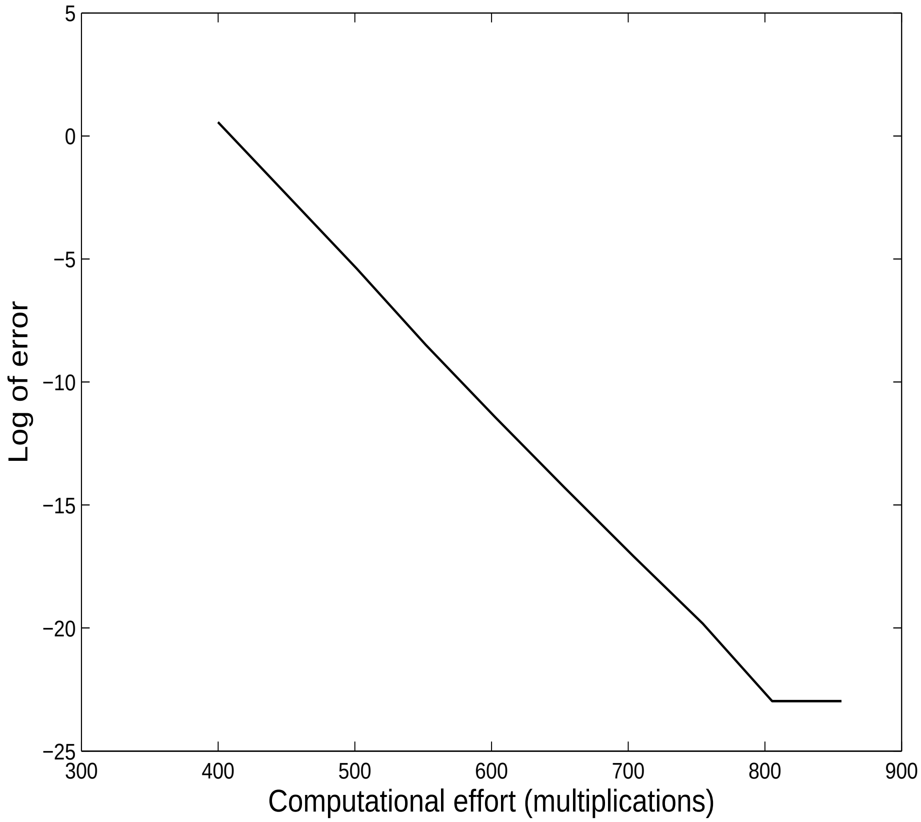

Figure 3: Log of error versus effort

10 Acknowledgments

The author thanks the referees for helpful suggestions.

11 Appendix (Proof of Theorem 7.5)

First we will obtain bounds on three objects. Once these are obtained, the result will follow easily. Along the way, several quite coarse estimates are made. As noted earlier, one might be able to improve quite a bit on these bounds. However, we are considering a rather wide class of problems, where in particular, the fundamental solution must suffice for use with a very wide class of initial conditions. Under such broad conditions, and time-wise global error estimates (as opposed to local one-step error estimates), it is not clear whether sharpening the coarse estimates will be fruitful. In any case, such an endeavor must be left to future investigators.

Recall the β_t $\circledast$-multiplication propagation given in Theorem 4.7; we will now take $t_1 = t_2 = t$, and let U_t denote $U_{t,t}$. Now, note that by Assumption $(A.\beta)$,

$$\begin{aligned}|U_t x| &\doteq |(\beta_t^{1,1} + \beta_t^{2,2})x| \geq 2\bar{\delta}|x| \\ &= 2\bar{\delta}|U_t^{-1}U_t x| \qquad \forall\, x \in I\!R^n,\ \forall t \in [0, T_\varepsilon]\end{aligned}$$

which implies $|U_t^{-1}y| \leq \frac{1}{2\bar{\delta}}|y|$ for all $y \in I\!R^n$. Consequently,

$$|U_t^{-1}| \leq \frac{1}{2\bar{\delta}} \qquad \forall\, t \in [0, T_\varepsilon]. \tag{78}$$

Lemma 11.1 *There exists $\bar{\tau} > 0$ such that for all $t \in (0, \bar{\tau}]$,*

$$|U_t^{-1}| \leq \frac{2K_i t}{\sigma_M^2}. \tag{79}$$

Proof. Because Λ_t is a solution of (12) with initial condition $\Lambda_0 = I$, there exists $D_\Lambda < \infty$ such that for all $t \in [0, T_\varepsilon]$,

$$|I - \Lambda_t| \leq D_\Lambda t. \tag{80}$$

For any $n \times n$ $\overline{A}$,

$$|\Lambda_t' \overline{A} \Lambda_t - \overline{A}| \leq |(\Lambda_t' - I)\overline{A}\Lambda_t| + |I\overline{A}(\Lambda_t - I)|$$

which, using (80),

$$\leq |\overline{A}|(|\Lambda_t| + 1)D_\Lambda t, \tag{81}$$

and using Lemma 7.4,

$$\leq |\overline{A}|(\Lambda_M + 1)D_\Lambda t \doteq |\overline{A}|k_\Lambda t \qquad \forall\, t \in [0, T_\varepsilon]. \tag{82}$$

Letting $\overline{A} = Q\Delta_t^{-1}\widetilde{P}_t$ in (82) yields

$$|\Lambda_t' Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t - Q\Delta_t^{-1}\widetilde{P}_t| \leq |Q\Delta_t^{-1}\widetilde{P}_t|k_\Lambda t. \tag{83}$$

Using the triangle inequality, one finds that for any $x \in I\!R^n$,

$$|(\Lambda_t' Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t + Q\Delta_t^{-1}\widetilde{P}_t)x| \geq |2Q\Delta_t^{-1}\widetilde{P}_t x| - |(\Lambda_t' Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t - Q\Delta_t^{-1}\widetilde{P}_t)x|$$

which by (83)

$$\geq 2|Q\Delta_t^{-1}\widetilde{P}_t x| - |Q\Delta_t^{-1}\widetilde{P}_t|k_\Lambda t|x|. \tag{84}$$

Now, by (30),

$$\begin{aligned}|U_t x| = |(\beta_t^{1,1} + \beta_t^{2,2})x| &= |(Q\Delta_t^{-1}\widetilde{P}_t + \Lambda_t' Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t + R_t)x| \\ &\geq |(\Lambda_t' Q\Delta_t^{-1}\widetilde{P}_t\Lambda_t + Q\Delta_t^{-1}\widetilde{P}_t)x| - |R_t x|\end{aligned}$$

which by (84)

$$\geq 2|Q\Delta_t^{-1}\widetilde{P}_t x| - |Q\Delta_t^{-1}\widetilde{P}_t x|k_\Lambda t - |R_t x|. \tag{85}$$

However, R_t is an integral over $[0,t]$ with bounded integrand. Therefore, there exists $k_R < \infty$ such that $|R_t| \leq k_R t$ for all $t \in [0, T_\varepsilon]$. Using this in (85) yields

$$|U_t x| \geq |Q\Delta_t^{-1}\widetilde{P}_t x|(2 - k_\Lambda t) - k_R t|x| \qquad \forall\, x \in I\!R^n,\ \forall t \in [0, T_\varepsilon]. \tag{86}$$

By Assumption $(A.Q)$ and the definition of σ_M,

$$|Q\Delta_t^{-1}\widetilde{P}_t x| \geq \sigma_M |\Delta_t^{-1}\widetilde{P}_t x| \qquad \forall\, x \in I\!R^n$$

which by Lemma 7.1

$$\geq \frac{\sigma_M}{K_i t}|\widetilde{P}_t x| \qquad \forall\, x \in I\!R^n. \tag{87}$$

Also,

$$|\widetilde{P}_t x| \geq |Qx| - |(\widetilde{P}_t - Q)x|$$

which by (64)

$$\geq \sigma_M |x| - Dt|x| \qquad \forall\, t \in [0, \tau_0]$$

which, with $\bar{\tau}_2 \doteq \tau_0 \wedge (\sigma_M/(2D))$,

$$\geq \frac{\sigma_M}{2}|x| \qquad \forall\, x \in I\!R^n,\ \forall t \in [0, \bar{\tau}_2].$$

Substituting this into (87) yields

$$|Q\Delta_t^{-1}\widetilde{P}_t x| \geq \frac{\sigma_M^2}{2K_i t}|x| \doteq \frac{\tilde{l}}{t}|x| \qquad \forall\, x \in I\!R^n,\ \forall t \in [0, \bar{\tau}_2]. \tag{88}$$

Combining (86) and (88), one sees that for t sufficiently small,

$$|U_t x| \geq \left[\frac{\tilde{l}}{t}(2 - k_\Lambda t) - k_R t\right]|x|$$

which for t sufficiently small,

$$\geq \frac{\tilde{l}}{t}|x| \qquad \forall\, x \in I\!R^n.$$

Consequently, $|U_t^{-1}U_t x| = |x| \leq (t/\tilde{l})|U_t x|$ for all $x \in I\!R^n$, and this implies $|U_t^{-1}y| \leq (t/\tilde{l})|y|$ for all $y \in I\!R^n$. □

Lemma 11.2 *There exists $\overline{\tau} > 0$ such that*

$$|U_t^{-1}| \leq g_S(t) = \begin{cases} \frac{2K_i t}{\sigma_M^2} & \text{if } t \in (0, \overline{\tau}] \\ \frac{1}{2\delta} & \text{if } t \in (\overline{\tau}, \infty). \end{cases} \qquad \forall\, t \in [0, T_\varepsilon]. \tag{89}$$

Proof. The result follows directly from (78) and Lemma 11.1. $\square$

Let $\widehat{U}_t \doteq \hat{\beta}_t^{1,1} + \hat{\beta}_t^{2,2}$ and $\delta_t^U \doteq \widehat{U}_t^{-1} - U_t^{-1}$. We now proceed to bound δ_t^U. We will use the same approach as at the top of the proof of Lemma 7.2. In particular, one has

$$0 = \widehat{U}_t^{-1}\widehat{U}_t - U_t^{-1}U_t = (\widehat{U}_t^{-1} - U_t^{-1})U_t + \widehat{U}_t^{-1}(\widehat{U}_t - U_t),$$

and this implies

$$\widehat{U}_t^{-1} - U_t^{-1} = -(\widehat{U}_t^{-1} - U_t^{-1})(\widehat{U}_t - U_t)U_t^{-1} - U_t^{-1}(\widehat{U}_t - U_t)U_t^{-1}$$

which yields

$$\delta_t^U = -U_t^{-1}(\widehat{U}_t - U_t)U_t^{-1}\left[I + (\widehat{U}_t - U_t)U_t^{-1}\right]^{-1}. \tag{90}$$

Note that $\widehat{U}_t - U_t = \delta_t^{1,1} + \delta_t^{2,2}$, and so, using Lemma 11.2, one obtains

$$\left|(\widehat{U}_t - U_t)U_t^{-1}\right| \le (|\delta_t^{1,1}| + |\delta_t^{2,2}|)/(2\bar{\delta}),$$

which by the assumptions of Theorem 7.5,

$$\le 1 \qquad \forall\, t \in [0, T_\varepsilon],$$

and this implies

$$\left|I + (\widehat{U}_t - U_t)U_t^{-1}\right|^{-1} \le 2 \qquad \forall\, t \in [0, T_\varepsilon]. \tag{91}$$

Combining (90) and (91), one obtains

$$|\delta_t^U| \le 2|U_t^{-1}|^2|\widehat{U}_t - U_t| \le 2|U_t^{-1}|^2(|\delta_t^{1,1}| + |\delta_t^{2,2}|). \tag{92}$$

Finally, combining (92) and Lemma 11.2 yields:

Lemma 11.3

$$|\delta_t^U| \le 2[g_S(t)]^2\left(|\delta_t^{1,1}| + |\delta_t^{2,2}|\right) \qquad \forall\, t \in [0, T_\varepsilon]. \tag{93}$$

Lemma 11.4

$$|\beta_t^{1,2}| \le \frac{P_M \Lambda_M |Q|}{K_i t} \wedge \bar{b} \qquad \forall\, t \in (0, T_\varepsilon]. \tag{94}$$

Proof. From (30), we have

$$|\beta_t^{1,2}| \le |Q||\Delta_t^{-1}||\widetilde{P}_t||\Lambda_t|$$

which by Lemma 7.4

$$\le P_M \Lambda_M |Q||\Delta_t^{-1}|$$

which by Lemma 7.1

$$\leq \frac{P_M \Lambda_M |Q|}{K_i t} \qquad \forall\, t \in (0, \tau_3]. \square \tag{95}$$

We have now obtained all the constituent estimates, and proceed directly to the bounds asserted in the theorem statement. We work mainly with $\delta^{1,1}_{2^{k+1}\tau}$; a quick examination of the assertion of Theorem 4.7 shows that the other terms will follow easily by nearly identical steps.

Let $k \in \{1, 2 \ldots N-1\}$. Note that

$$\begin{aligned}
|\delta^{1,1}_{2^{k+1}\tau}| &= |\hat{\beta}^{1,1}_{2^{k+1}\tau} - \beta^{1,1}_{2^{k+1}\tau}| \\
&= \left| [\hat{\beta}_{2^k\tau} \circledast \hat{\beta}_{2^k\tau}]^{1,1} - [\beta_{2^k\tau} \circledast \beta_{2^k\tau}]^{1,1} \right|
\end{aligned}$$

which by Theorem 4.7

$$\begin{aligned}
&\leq |\hat{\beta}^{1,1}_{2^k\tau} - \beta^{1,1}_{2^k\tau}| + |\hat{\beta}^{1,2}_{2^k\tau} \widehat{U}^{-1}_{2^k\tau} (\hat{\beta}^{1,2}_{2^k\tau})' - \beta^{1,2}_{2^k\tau} U^{-1}_{2^k\tau} (\beta^{1,2}_{2^k\tau})'| \\
&= |\delta^{1,1}_{2^k\tau}| + |\delta^{1,2}_{2^k\tau} \widehat{U}^{-1}_{2^k\tau} (\hat{\beta}^{1,2}_{2^k\tau})' + \beta^{1,2}_{2^k\tau} \delta^U_{2^k\tau} (\hat{\beta}^{1,2}_{2^k\tau})' + \beta^{1,2}_{2^k\tau} U^{-1}_{2^k\tau} (\delta^{1,2}_{2^k\tau})'| \\
&\leq |\delta^{1,1}_{2^k\tau}| + |\delta^{1,2}_{2^k\tau} \delta^U_{2^k\tau} + \delta^{1,2}_{2^k\tau} U^{-1}_{2^k\tau} + \beta^{1,2}_{2^k\tau} \delta^U_{2^k\tau}|\, |\beta^{1,2}_{2^k\tau} + \delta^{1,2}_{2^k\tau}| + |\beta^{1,2}_{2^k\tau} U^{-1}_{2^k\tau} (\delta^{1,2}_{2^k\tau})'| \\
&\leq |\delta^{1,1}_{2^k\tau}| + 2|\delta^{1,2}_{2^k\tau}||\delta^U_{2^k\tau}||\beta^{1,2}_{2^k\tau}| + |\delta^{1,2}_{2^k\tau}|^2 |\delta^U_{2^k\tau}| + 2|\delta^{1,2}_{2^k\tau}||U^{-1}_{2^k\tau}||\beta^{1,2}_{2^k\tau}| \\
&\quad + |\delta^{1,2}_{2^k\tau}|^2 |U^{-1}_{2^k\tau}| + |\beta^{1,2}_{2^k\tau}|^2 |\delta^U_{2^k\tau}|.
\end{aligned}$$

We will be making very coarse estimates which can obviously be improved. Noting that we assume $|\delta^{1,2}_{2^k\tau}| \leq 1$ in the theorem statement, the last inequality yields

$$\begin{aligned}
|\delta^{1,1}_{2^{k+1}\tau}| &\leq |\delta^{1,1}_{2^k\tau}| + \left[|\beta^{1,2}_{2^k\tau}|^2 + 2|\beta^{1,2}_{2^k\tau}| + 1 \right] |\delta^U_{2^k\tau}| + \left[2|\beta^{1,2}_{2^k\tau}| + 1 \right] |U^{-1}_{2^k\tau}||\delta^{1,2}_{2^k\tau}| \\
&\leq |\delta^{1,1}_{2^k\tau}| + (1 + |\beta^{1,2}_{2^k\tau}|)^2 |\delta^U_{2^k\tau}| + 2(1 + |\beta^{1,2}_{2^k\tau}|)|U^{-1}_{2^k\tau}||\delta^{1,2}_{2^k\tau}|.
\end{aligned}$$

Applying Lemmas 11.2, 11.3 and 11.4, to the terms in this estimate yields

$$\begin{aligned}
|\delta^{1,1}_{2^{k+1}\tau}| &\leq |\delta^{1,1}_{2^k\tau}| + 2\left[1 + \frac{P_M \Lambda_M |Q|}{K_i 2^k \tau}\right]^2 \left[g_S(2^k\tau)\right]^2 \left(|\delta^{1,1}_{2^k\tau}| + |\delta^{2,2}_{2^k\tau}|\right) \\
&\quad + 2\left[1 + \frac{P_M \Lambda_M |Q|}{K_i 2^k \tau}\right] \left[g_S(2^k\tau)\right] |\delta^{1,2}_{2^k\tau}| \\
&\leq (1 + \bar{c}_1)|\delta^{1,1}_{2^k\tau}| + \bar{c}_2 |\delta^{1,2}_{2^k\tau}| + \bar{c}_1 |\delta^{2,2}_{2^k\tau}|
\end{aligned}$$

where

$$\bar{c}_1 = 2\bar{c}^2, \quad \bar{c}_2 = 2\bar{c},$$

and

$$\bar{c} = \max_{t \in [0,\infty)} \left[1 + \frac{P_M \Lambda_M |Q|}{K_i t}\right] [g_S(t)] < \infty.$$

We have now obtained the bound on $|\delta^{1,1}_{2^{k+1}\tau}|$. Examining the right-hand sides of the four equations describing the components of $\beta_{t_1} \circledast \beta_{t_2}$ (where $t_1 = t_2 = 2^k\tau$ in our case here), one sees that the right-hand sides are quite similar. In particular, the product terms are identical to the product term which we just bounded in the case of $\delta^{1,1}_{2^{k+1}\tau}$. The only other term is the trivial $\beta^{2,2}_{2^k\tau}$ term. Consequently, the bounds on $|\delta^{1,2}_{2^{k+1}\tau}|$ and $|\delta^{2,2}_{2^{k+1}\tau}|$ are immediate given the bounds we obtained just above. This completes the proof of Theorem 7.5. $\square$

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