Infinite Time–Horizon Risk Sensitive Systems with Quadratic Growth

William M. McEneaney and Kazufumi Ito
Department of Mathematics, North Carolina State Univ., Raleigh, NC 27695-8205, USA
wmm@eos.ncsu.edu, kito@eos.ncsu.edu

Abstract

Previous work on nonlinear infinite time–horizon risk sensitive systems has been restricted mainly to the case where the cost criterion is bounded or grows at most linearly. One would like the nonlinear theory to subsume the LEQG case. We consider nonlinear risk–sensitive systems where the cost criterion may grow quadratically. This leads to difficulties which are surprisingly formidable. For instance, the HJB equation typically has multiple classical solutions.

1. Introduction

We consider nonlinear risk–sensitive control problems on an infinite time–horizon. In [FMc], [Mc1] it was demonstrated for certain nonlinear systems that in the risk–averse limit, the value function associated with infinite time–horizon risk–sensitive control problems converges to the value of the limiting H_∞ control problem. In that case, it was assumed that the cost function grew at most linearly. Of course, one would like the nonlinear theory to subsume the LEQG (linear–exponential–quadratic–gaussian) theory, so this was naturally a significant constraint. A number of other authors have also recently done work on nonlinear infinite time–horizon risk–sensitive control such as [FS] and [INT]. A full discussion of related work (such as for example [JBE], [CE]) is too long to include here, but see [FMc], [Mc2]. In [FS], it is assumed that the cost is bounded. In [INT], the HJB equation related to a specific class of risk–sensitive problems is examined.

When one allows the cost function to grow quadratically, some rather unpleasant difficulties appear. In particular, there may be multiple solutions of the associated HJB equation, and certain estimates become quite delicate. The following results represent a first examination of this territory. We will obtain some basic results, and demonstrate some machinery that appears useful in this context. Since this represents new extensions and the arguments are already quite technical, we will consider here only the case of a fixed–feedback control.

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Let the dynamics be given by

$$dX_t = g(X_t)\,dt + \sqrt{\frac{\epsilon}{\gamma^2}}\sigma(X_t)\,dB_t \qquad (1.1)$$
$$X_0 = x \in \Re^n$$

where $B_\cdot$ is an m–dimensional Brownian motion on the probabilty triple $(\Omega, \{F_t\}, P)$. We will consider a cost criterion of the form

$$\widehat{\lambda} \doteq \lim_{T\to\infty} \frac{\epsilon}{T} \log \mathrm{E} \exp \frac{1}{\epsilon} \int_0^T l(X_t)\,dt \qquad (1.2)$$

where $\epsilon > 0$ is our risk–sensitive parameter.

We will assume throughout that $g \in C^1$, $g(0) = 0$ and that there exists $\eta \in (0, \infty)$ such that

$$(x-y)^T(g(x)-g(y)) \le -\eta|x-y|^2 \qquad \forall\, x, y \in \Re^n \qquad (A1)$$

so that the system is internally stable. We will assume that σ is a globally Lipschitz, C^1, $m \times n$ matrix–valued function, and that there exists $M < \infty$ such that

$$|\sigma(x)| \le M \qquad \forall\, x \in \Re^n. \qquad (A2a)$$

We define $a(x) \doteq \sigma(x)\sigma^T(x)$. We assume that a is uniformly non–degenerate, that is there exists $\underline{a} > 0$ such that

$$z^T a(x) z \ge \underline{a}|z|^2 \qquad \forall\, x, z \in \Re^n. \qquad (A2b)$$

We will assume that $l \in C^1(\Re^n)$ and that there exists $\alpha < \infty$ such that

$$0 \le l(x) \le \alpha|x|^2 \qquad \forall\, x \in \Re^n. \qquad (A3)$$

We assume

$$\frac{2\alpha M^2}{\eta^2\gamma^2} < 1 \qquad (A4)$$

which is identical to the inequality required in [Mc3] for the H_∞ limit problem. (Note that the constant $2\gamma^2$ there is replaced by γ^2 here.)

The remaining assumptions are required only for the existence portion of Theorem 1.1 below. We assume that there exist $c, \beta < \infty$ such that

$$\sqrt{c^2+|x|^2}\,|\sigma_x(x)\sigma^T(x)| \le \beta \qquad \forall\, x \in \Re^n \quad (A5)$$
$$|\nabla l(x)| \le 2\alpha\sqrt{c^2+|x|^2} \qquad \forall\, x \in \Re^n. \qquad (A6)$$

Finally, Defining

$$\tilde{\eta} = \eta - \frac{\epsilon}{2\gamma^2} \max_x \left(tr\, a(x) + \frac{|a_x(x)|^2}{4\underline{a}} \right),$$

we assume that

$$\frac{8\alpha\beta}{\tilde{\eta}^2\gamma^2} < 1, \qquad \frac{4\alpha M^2}{\tilde{\eta}^2\gamma^2} < 1 + \sqrt{1 - \frac{8\alpha\beta}{\tilde{\eta}^2\gamma^2}} \qquad (A7)$$

Theorem 1.1: There exists $\lambda \in \Re$, $V \in C^2(\Re^n)$, $c_2 < \infty$, and $c_1 < \frac{\eta\gamma^2}{2M^2}$ such that $V(0) = 0$ (we always normalize by setting $V(0) = 0$), $V(x) \geq -b$ for some $b < \infty$ and

$$|\nabla V(x)| \leq 2c_1|x| + c_2 \qquad \forall\, x \in \Re^n, \qquad (1.3)$$

and such that (λ, V) form a solution to

$$\begin{aligned} \lambda = &\frac{\epsilon}{2\gamma^2} \sum_{i,j=1}^{n} a_{i,j}(x) V_{x_i x_j} + \frac{1}{2\gamma^2} \nabla V^T a(x) \nabla V \\ &+ g^T(x)\nabla V + l(x). \end{aligned} \qquad (1.4)$$

Further,

$$\lambda = \widehat{\lambda}.$$

We note that (1.4) may have multiple solutions. For example, consider the one–dimensional case where $g(x) = -x$, $\sigma(x) \equiv 1$, $l(x) = x^2$ so that (1.4) becomes

$$\lambda = \frac{\epsilon}{2\gamma^2} V_{xx} + \frac{1}{2\gamma^2} \nabla V^2 - x\nabla V + x^2. \qquad (1.5)$$

Suppose we look for a quadratic solution of the equation $0 = \frac{1}{2\gamma^2}\nabla V^2 - x\nabla V + x^2$. This has two solutions

$$V^1(x) = \frac{\gamma^2}{2}\left(1 + \sqrt{1 - \frac{2}{\gamma^2}}\right)x^2$$
$$V^2(x) = \frac{\gamma^2}{2}\left(1 - \sqrt{1 - \frac{2}{\gamma^2}}\right)x^2$$

as long as $\gamma > \sqrt{2}$. Then noting that the second derivative is constant, and letting

$$\lambda^1 = \frac{\epsilon}{2}[1 + \sqrt{1 - (2/\gamma^2)}]$$
$$\lambda^2 = \frac{\epsilon}{2}[1 - \sqrt{1 - (2/\gamma^2)}],$$

one sees that (λ^1, V^1) and (λ^2, V^2) are both classical solutions to (1.5). Notice however, that V^1 violates (1.3) while V^2 satisfies (1.3).

Now, index the above risk–sensitive quantities by ϵ to indicate their dependence on the risk–sensitive parameter.

Theorem 1.2: As $\epsilon \downarrow 0$, $\lambda^\epsilon \to 0$. Suppose that $V^\epsilon(x) \geq -b_\epsilon$ for all $x \in \Re^n$ where $b_\epsilon \to 0$ and $\epsilon \downarrow 0$. Then $V^\epsilon \to V^0$ uniformly on compact sets where V^0 is the unique viscosity solution to

$$0 = \frac{1}{2\gamma^2}\nabla V^T a(x) \nabla V + g^T(x)\nabla V + l(x)$$

in the class of solutions satisfying

$$0 \leq V^0(x) \leq c_1|x|^2 \qquad \forall\, x \in \Re^n.$$

Further, V^0 is the value of the H_∞ problem defined below.

Remark: The supposition that $b_\epsilon \to 0$ as $\epsilon \downarrow 0$ should follow from the earlier assumptions, but that has not yet been proved.

The limit H_∞ problem is as follows. The dynamics are given by

$$\frac{dX}{dt} = g(X) + \sigma(X)w, \qquad X(0) = x$$

where in this case, $X(\cdot)$ is deterministic and $w(\cdot)$ is in $L_2^{loc} = \{w : [0,\infty) \to \Re^m : \int_0^T |w(t)|^2\, dt < \infty \quad \forall T < \infty\}$. The payoff and value are given by

$$P(x, w, T) = \int_0^T l(X(t)) - \gamma^2|w(t)|^2\, dt$$
$$V^0(x) = \sup_{w\in\mathcal{W}} \sup_{T<\infty} P(x, w, T)$$

In Section 2, we will prove a verification theorem given existence of a solution to (1.4). The proof of existence of the desired solution to (1.4) and the equicontinuity estimate (i.e. (1.3)) is delayed to Section 4 since it is quite long and of a technical nature. In Section 3, we will show that the limit result of Theorem 1.2 holds using the previous results and a respresentation result for the limit PDE.

2. Representation

In this section, we will suppose that there exists a C^2 solution to (1.4) meeting conditions (1.3) (as well as $V(0) = 0$ and $V(x) \geq -b$) and show that $\lambda = \widehat{\lambda}$. This will require an application of Girsanov's Theorem. One difficulty is that the Novikov condition is not satisfied for arbitrarily long intervals. The first step will be to show that it is satisfied for sufficiently short intervals.

Lemma 2.1:

$$\exp\left[\int_0^t \sqrt{K} X_r \cdot dB_r - \frac{1}{2}\int_0^t K|X_r|^2\, dr\right]$$

is a martingale for any $K \in (0,\infty)$.

Outline of proof: Let $\overline{d} > 0$ be sufficiently small so that $\eta\gamma^2 - \overline{d}\epsilon M^2 > 0$. Let

$$\delta \doteq \min\{\frac{\overline{d}}{2}, \frac{2}{\epsilon}(\eta\gamma^2 - \overline{d}\epsilon M^2)\}.$$

First one works with the stopped process given by letting $\tau_k \doteq \{t : |X_t| \geq k\}$ and

$$X_t^k = \begin{cases} X_t & \text{if } t \leq \tau_k \\ X_{\tau_k} & \text{if } t > \tau_k. \end{cases}$$

Let $U_t^k = \exp[\overline{d}|X_t^k|^2]$. Applying Ito's rule and assumptions (A1) and (A2a), one finds

$$\begin{aligned} \mathrm{E}U_t^k &\leq e^{\overline{d}|x|^2} \\ &+ \frac{\epsilon}{\gamma^2}\mathrm{E}\left\{\int_0^{t\wedge\tau_k} U_r^k[-\delta\log(U_r^k) + \overline{d}M^2]\,dr\right\}, \end{aligned}$$

and the integrand is non–positive for $U_r^k \geq \exp[\overline{d}M^2/\delta]$. Consequently,

$$\mathrm{E}e^{\overline{d}|X_t^k|^2} = \mathrm{E}U_t^k \leq e^{\overline{d}|x|^2} + c_1 t \qquad \forall\, t$$

for some $c_1 < \infty$ independent of k. This implies, by Fatou's Lemma, that

$$\mathrm{E}e^{\overline{d}|X_t|^2} \leq e^{\overline{d}|x|^2} + c_1 t \qquad \forall\, t \tag{2.1}$$

This will allow us to assert that a certain term just below is a square–integrable martingale.

Let $U_t = \exp[d|X_t|^2]$ where $d \doteq (\overline{d} - \delta)/2 > 0$. Applying Ito's rule,

$$\begin{aligned} U_t = U_0 &+ \int_0^t 2aU_r X_r^T g(X_r)\,dr \\ &+ \frac{\epsilon}{2\gamma^2}\int_0^t U_r[4a^2 X_r^T \sigma\sigma^T X_r + 2a\sum \sigma_{i,j}^2]\,dr \\ &+ \sqrt{\frac{\epsilon}{\gamma^2}}2a\int_0^t U_r X_r^T \sigma(X_r)\,dB_r. \end{aligned}$$

Noting that $|U_r X_r|^2 \leq \exp[\overline{d}|X_r|^2]$ for large $|X_r|$, we see by (2.1) that the last term is a square–integrable martingale. Therefore, taking expectations and applying the assumptions yields

$$\mathrm{E}U_t \leq U_0 + \int_0^t \mathrm{E}\left\{U_r\left(-\hat{\delta}\log(U_r) + \frac{\epsilon d M^2}{\gamma^2}\right)\right\}\,dr$$

where $\hat{\delta} = 2\eta - (2aM^2\epsilon/\gamma^2) > 0$. Noting that the term within the expectation is concave, and applying Jensen's inequality yields

$$\mathrm{E}e^{d|X_t|^2} = \mathrm{E}U_t \leq \max\left\{e^{a|x|^2}, \exp\left[\frac{\epsilon d M^2}{\hat{\delta}\gamma^2}\right]\right\} \doteq C_x \tag{2.2}$$

for all t.

Now let $\tau > 0$ be a fixed (deterministic) time step. (We will choose a particular value later.) Let $t_N = N\tau$ for all N. Then, with $K \in (0,\infty)$ being any constant, we have

$$\begin{aligned} &\mathrm{E}\exp\left[\int_{t_N}^{t_N+1} K|X_t|^2\,dt\right] \\ &= \lim_{m\to\infty}\mathrm{E}\exp\left[K\sum_{i=0}^{m-1}|X_{t_N+i\tau/m}|^2\frac{\tau}{m}\right] \end{aligned}$$

which by Holder's inequality

$$\leq \lim_{m\to\infty}\left[\prod_{i=0}^{m-1}\mathrm{E}\exp\left(K|X_{t_N+i\tau/m}|^2\tau\right)\right]^{\frac{1}{m}}. \tag{2.3}$$

Now we choose $\tau = d/K$. Then, by (2.3) and (2.2)

$$\mathrm{E}\exp\left[\int_{t_N}^{t_N+1}|X_t|^2\,dt\right] \leq \lim_{m\to\infty}\left[\prod_{i=0}^{m-1}C_x\right]^{\frac{1}{m}} = C_x. \tag{2.4}$$

The result follows by the well–known relaxation of the Novikov condition (see for example [KS, Cor. 3.5.14]). □

Theorem 2.2: Suppose that there exist $\lambda \in \Re$, $V \in C^2(\Re^n)$ and $c_1 < \frac{\eta\gamma^2}{2M^2}$ such that $V(0) = 0$, $V(x) \geq -b$ for some $b < \infty$, and (λ, V) form a solution to (1.4) satisfying (1.3). Then $\lambda = \widehat{\lambda}$.

Outline of proof: We introduce a change of drift into (1.1) and define $\widehat{B}$ by

$$\begin{aligned} X_t &= x + \int_0^t g(X_r)\,dr + \sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t \sigma(X_r)\,dB_r \\ &\doteq x + \int_0^t [g(X_r) + \sigma(X_r)w_r]\,dr + \sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t \sigma(X_r)\,d\widehat{B}_r \end{aligned}$$

where we let

$$w_t \doteq \frac{1}{\gamma^2}\sigma^T(X_t)\nabla V(X_t). \tag{2.5}$$

By Lemma 2.1 and Girsanov's Theorem, there exists a probability measure, $\widehat{P}$, under which $\widehat{B}_\cdot$ is a Brownian motion given by

$$\begin{aligned} &P(d\omega) \\ &= \exp\left[\sqrt{\frac{\gamma^2}{\epsilon}}\int_0^t w_r^T\,dB_r - \frac{\gamma^2}{2\epsilon}\int_0^t |w_r|^2\,dr\right]\widehat{P}(d\omega). \end{aligned}$$

Letting $\widehat{\mathrm{E}}$ be expectation under this new measure, we have

$$\begin{aligned} \mathrm{E}\exp\left[\frac{1}{\epsilon}\int_0^T l(X_t)\,dt\right] &= \widehat{\mathrm{E}}\exp\left\{\frac{1}{\epsilon}\left[\int_0^T l(X_t)\right.\right. \\ &\left.\left. - \frac{\gamma^2}{2}|w_t|^2\,dt - \sqrt{\gamma^2\epsilon}\int_0^T w_t^T\,d\widehat{B}_t.\right]\right\} \end{aligned} \tag{2.6}$$

On the other hand, employing Ito's rule, (2.5) and (1.4), we have

$$V(X_T) - V(x) = \int_0^T \lambda - l(X_t) + \frac{\gamma^2}{2}|w_t|^2\,dt + \sqrt{\frac{\epsilon}{\gamma^2}}\int_0^T (\nabla V)^T \sigma(X_t)\,d\widehat{B}_t. \tag{2.7}$$

Combining (2.6) and (2.7) (and using (2.5)) yields

$$\mathrm{E}\exp\left[\frac{1}{\epsilon}\int_0^T l(X_t)\,dt\right] = e^{(\lambda T + V(x))/\epsilon}\widehat{\mathrm{E}}[e^{(-\frac{V(X_T)}{\epsilon})}]$$

which implies

$$\begin{aligned}&\lim_{T\to\infty}\frac{\epsilon}{T}\log \mathrm{E}\exp\left[\frac{1}{\epsilon}\int_0^T l(X_t)\,dt\right]\\ &\quad = \lambda + \lim_{T\to\infty}\frac{\epsilon}{T}\log\widehat{\mathrm{E}}\exp\left[-\frac{V(X_T)}{\epsilon}\right]\end{aligned} \tag{2.8}$$

(where we will demonstrate the existence of the limit below). We would like to demonstrate that this last term is zero. Since V is bounded from below by $-b$, we see that $\widehat{\mathrm{E}}\exp[-V(X_T)/\epsilon] \le \exp(b/\epsilon)$. We only need to show that it is bounded from below by a positive constant. Then we will be done.

Using Holder's inequality, (with $1/p + 1/q = 1$)

$$\widehat{\mathrm{E}}\exp[-V(X_T)/\epsilon] \ge \left[\widehat{\mathrm{E}}\exp\left(\frac{V(X_T)q}{\epsilon p}\right)\right]^{-p/q}. \tag{2.9}$$

By (1.3),

$$\widehat{\mathrm{E}}\exp\left[\frac{qV(X_T)}{\epsilon p}\right] \le \widehat{\mathrm{E}}\exp[\hat{c}_1|X_T|^2/\epsilon] \tag{2.10}$$

where we can make $\hat{c}_1$ arbitrarily small by letting $p \uparrow \infty$.

Let $U_t \doteq \exp[\hat{c}_1|X_t|^2/\epsilon]$. Applying Ito's rule to U_t with the new dynamics, and employing (2.5), (1.3) and the assumptions, one finds

$$\begin{aligned}U_t &\le U_0\\ &+ \frac{\hat{c}_1}{\epsilon}\int_0^t \Big\{U_r|X_r|^2\left[-2\eta + \frac{2\hat{c}_1M^2}{\gamma^2} + \frac{4c_1M^2}{\gamma^2}\right]\\ &+ U_r|X_r|\frac{2M^2c_2}{\gamma^2} + U_r\frac{\epsilon M^2}{\gamma^2}\Big\}\,dr\\ &+ \frac{2\hat{c}_1}{\epsilon}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t U_rX_r^T\sigma(X_r)\,d\widehat{B}_r.\end{aligned}$$

By a small modification of the argument in Lemma 2.1, this last term is a square–integrable martingale. Using this and the fact that $2|X_r| \le \tilde{\delta}^2|X_r|^2 + 1/\tilde{\delta}^2$ for any $\tilde{\delta} > 0$, this yields

$$\begin{aligned}\widehat{\mathrm{E}}U_t &= U_0 + \frac{\hat{c}_1}{\epsilon}\int_0^t \widehat{\mathrm{E}}\Big\{2U_r|X_r|^2\Big[-\eta + \frac{\hat{c}_1M^2}{\gamma^2}\\ &+ \frac{2c_1M^2}{\gamma^2} + \frac{\tilde{\delta}^2M^2c_2}{2\gamma^2}\Big] + U_r\left[\frac{\epsilon M^2}{\gamma^2} + \frac{M^2c_2}{\tilde{\delta}^2\gamma^2}\right]\Big\}\,dr.\end{aligned}$$

Since $c_1 < \eta\gamma^2/(2M^2)$, we can make the first term in brackets $-\overline{\delta} < 0$ by taking $\hat{c}_1, \tilde{\delta}$ sufficiently small, and consequently (letting $k \doteq \epsilon + (c_2/\tilde{\delta}^2)$)

$$\widehat{\mathrm{E}}U_t \le U_0 + \frac{\hat{c}_1}{\epsilon}\int_0^t \widehat{\mathrm{E}}\left\{U_r\left[-\frac{\overline{\delta}}{\hat{c}_1}\log(U_r) + \frac{M^2}{\gamma^2}k\right]\right\}\,dr$$

which by Jensen's inequality

$$\le U_0 + \frac{\hat{c}_1}{\epsilon}\int_0^t \widehat{\mathrm{E}}(U_r)\left[-\frac{\overline{\delta}}{\hat{c}_1}\log(\widehat{\mathrm{E}}U_r) + \frac{M^2}{\gamma^2}k\right]\,dr.$$

Therefore, there exists $\widehat{C}_x < \infty$ such that $\widehat{\mathrm{E}}[\hat{c}_1|X_t|^2/\epsilon] \le \widehat{C}_x$ for all t. Combining this with (2.9) and (2.10), we are done. □

3. Limit Results

In this section, we turn to the proof of Theorem 1.2. We superscript the solutions as λ^ϵ and V^ϵ to indicate their dependence on the parameter ϵ. Recall that we wish to prove that this solution converges to that of the H_∞ limit problem. We must prove the convergence of both λ^ϵ and V^ϵ. We first indicate a proof that $\lambda^\epsilon \to 0$ as $\epsilon \downarrow 0$. For the V^ϵ, we may employ the Ascoli-Arzela Theorem by Theorem 1.1. We then employ a uniqueness/representation result from [Mc3] to obtain the uniqueness of the limit and it's representation as the value function of the limit H_∞ problem.

Lemma 3.1: $\lambda^\epsilon \to 0$ as $\epsilon \downarrow 0$.

Outline of proof: Let

$$U_t = \exp\left[\frac{\alpha}{\epsilon}\int_0^t |X_r|^2\,dr\right]. \tag{3.1}$$

Then

$$\frac{dU_t}{dt} = \frac{\alpha}{\epsilon}|X_t|^2U_t. \tag{3.2}$$

Also, by (1.1) and the assumptions

$$\frac{\alpha}{\epsilon}|X_t|^2 \le \frac{\alpha}{\epsilon}|x|^2 - 2\eta\frac{\alpha}{\epsilon}\int_0^t |X_r|^2\,dr + \frac{2\alpha M}{\gamma^2}t + 2\frac{\alpha}{\epsilon}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t X_r^T\sigma(X_r)\,dB_r. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\frac{dU_t}{dt} \le \left[\frac{\alpha}{\epsilon}|x|^2 - 2\eta\log(U_t) + \frac{2\alpha M}{\gamma^2}t + 2\frac{\alpha}{\epsilon}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t X_r^T\sigma(X_r)\,dB_r\right]U_t. \tag{3.4}$$

Since the right hand side becomes negative if U_t becomes too large, we have (after taking expectations)

$$\mathrm{E}U_t \le \max\left\{U_0, \exp\left[\frac{1}{2\eta}\left(\frac{\alpha}{\epsilon}|x|^2 + \frac{2\alpha M}{\gamma^2}t\right)\right] \cdot \mathrm{E}\exp\left[\frac{\alpha}{\epsilon\eta}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t X_r^T\sigma(X_r)\,dB_r\right]\right\}. \tag{3.5}$$

We need only worry about the second term. Using a Holder inequality (with $1/p + 1/q = 1$), we find

$$\begin{aligned}
&\mathrm{E}\exp\left[\frac{\alpha}{\epsilon\eta}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t X_r^T\sigma(X_r)\,dB_r\right] \\
&\le \left\{\mathrm{E}\exp\left[\frac{\alpha p}{\epsilon\eta}\sqrt{\frac{\epsilon}{\gamma^2}}\int_0^t X_r^T\sigma\,dB_r - \frac{1}{2}\left(\frac{\alpha p}{\epsilon\eta}\right)^2\frac{\epsilon}{\gamma^2}\int_0^t|\sigma^TX_r|^2\,dr\right]\right\}^{1/p} \\
&\cdot\left\{\mathrm{E}\exp\left[\left(\frac{\alpha}{\epsilon\eta}\right)^2\frac{pq}{2}\frac{\epsilon}{\gamma^2}\int_0^t|\sigma^TX_r|^2\,dr\right]\right\}^{1/q}
\end{aligned}$$

and since as in Lemma 2.1, the term inside the first expectation on the right hand side is a martingale

$$\le \left\{\mathrm{E}\left[U_t^{\frac{\alpha pqM^2}{2\eta^2\gamma^2}}\right]\right\}^{1/q}$$

which by (A4) with $p = q = 2$

$$\le [\mathrm{E}U_t]^{\frac{\alpha M^2}{\eta^2\gamma^2}} = [\mathrm{E}U_t]^{\hat\delta} \tag{3.6}$$

where $\hat\delta \in (0,1)$.

Combining (3.5) and (3.6) yields

$$\mathrm{E}U_t \le \exp\left[\left(\frac{1}{1-\hat\delta}\right)\frac{1}{2\eta}\left(\frac{\alpha}{\epsilon}|x|^2 + \frac{2\alpha M}{\gamma^2}t\right)\right]. \tag{3.7}$$

Then by (1.2), (A3), (3.1) and (3.7), one finds that $\lambda^\epsilon \le \alpha M\epsilon/(2\eta\gamma^2(1-\hat\delta))$. On the other hand, by (1.2), (A3), $\lambda^\epsilon \ge 0$ which completes the proof. □

Theorem 1.2 now follows by [Mc3, Th. 2.5].

4. Existence Proof

The existence portion of the proof of Theorem 1.1 (which was delayed above) is outlined here.

Step 1 Consider the equation of the form

$$V_t = \hat H(x,\nabla V) + \tilde\epsilon\, tr a(x)V_{x_i,x_j}. \tag{4.1}$$

where we assume that $\hat H$ is C^1 and there exist $c_1 > 0$, $c_2 > 0$ and $c_3(x) \in L^2(R^n)$ such that

$$|\hat H(x,p) - \hat H(x,q)| \le c_1\,|p-q|, \qquad \hat H(x,0) \in L^2(R^n)$$

and

$$|\hat H_x(x,p)| \le c_3(x) + c_2\,|p|.$$

We show that equation (4.1) has a unique solution V in

$$H^1(0,\tau;H^1(R^n)) \cap L^2(0,\tau;H^3(R^n)) \cap C(0,\tau;H^2(R^n)).$$

Step 2 We define the Lipschitz solution to

$$V_t = H(x,\nabla V) + \tilde\epsilon\, tr\, a(x)V_{x_i,x_j} \tag{4.2}$$

where $H(x,p) = g(x)\cdot p + h(\sigma(x)p) + l(x)$. In particular, for $h(v) = \frac{1}{2\gamma^2}|v|^2$, if the growth condition

$$\eta^2 \ge \frac{4}{\gamma^2}\,|\nabla l|_\infty|\sigma_x\sigma^T|_\infty$$

holds, then (4.2) has a solution V satisfying

$$\sup_x\,|\nabla V(t)| \le \frac{2\,|\nabla l|_\infty}{\eta + \sqrt{\eta^2 - \frac{4}{\gamma^2}\,|\nabla l|_\infty|\sigma_x\sigma^t|_\infty}}.$$

The existence of the Lipschitz solution is shown as follows. First consider a family of solutions V_c defined by Step 1 for

$$V_t = \psi(x)\,H(x,\nabla V) + \tilde\epsilon\, tr\, a(x)V_{x_i,x_j}$$

where $\psi(x) = c^2/(c^2+|x|^2)$, and we assume h is Lipschitz with constant $\hat\alpha$. Letting $c \to \infty$, we show $\lim V_c$ defines the unique solution V in $L^2(0,\tau;H^2_{loc}(R^n)) \cap H^1(0,\tau;L^2_{loc}(R^n)) \cap C(0,\tau;H^1_{loc}(R^n))$ to (4.2) in the sense that

$$(V_t,\zeta\,\xi) = (H(x,\nabla V),\zeta\,\xi) + \tilde\epsilon\,(tr\, aV_{x_i,x_j},\zeta\,\xi) \tag{4.3}$$

for all $\zeta \in C_c^\infty(R^n)$ and $\xi \in L^2_{loc}(R^n)$ and

$$(U_t,\zeta\,\phi) = -(H(x,U),\nabla\cdot(\zeta\,\phi)) - \tilde\epsilon\,(a\,U_x,(\zeta\,\phi)_x) \tag{4.4}$$

for all $\zeta \in C_c^\infty(R^n)$ and $\phi \in H^1_{loc}(R^n)$ and V satisfies

$$\sup_x\,|\nabla V(t)| \le \int_0^t e^{(-\eta+\hat\alpha\,|\sigma_x|_\infty)(t-s)}\,|\nabla l|_\infty\,ds.$$

Next, we consider for $h \in C^1$ the cut-off function of $h(\sigma p)$ by

$$h^M(\sigma p) = \sigma h'(\sigma M\frac{p}{|p|})(p - M\frac{p}{|p|}) + h(\sigma M\frac{p}{|p|}) \quad \text{on } |p| > M$$

$$h^M(\sigma p) = h(\sigma p) \qquad \text{on } |p| \le M$$

Then, the equation

$$V_t = g(x)\cdot\nabla V + h^M(\sigma(x)\nabla V) + l(x) + \tilde{\epsilon}\, tra\, V_{x_i,x_j}$$

defines a sequence V^M and letting $M\to\infty$ we show that the limit solution is the Lipschitz continuous solution to (4.2) satisfying $\sup_x |\nabla V(t)| \le M$ for all t. $M>0$ is the smallest constant satisfying $|\nabla l|_\infty/(\eta - \hat{\alpha}(M)|\sigma_x|_\infty) \le M$ where

$$\hat{\alpha} = \hat{\alpha}(M) = \sup_{x\in R^n,\, |p|\le M} h'(\sigma(x)p).$$

Step 3 We prove the existence of a quadratically growing solution. For $l\in C^1$ satisfying (A6) We define a sequence $\{l_k\}$ of globally Lipschitz approximations of l by

$$l_k(x) = \begin{cases} l(x) & \text{if } |x|\le k \\ \nabla l(k\,\frac{x}{|x|})(x - k\,\frac{x}{|x|}) + l(k\,\frac{x}{|x|}) & \text{if } |x|\ge k. \end{cases}$$

Let $s: R^+\to R^+$ be a C^2 and convex function satisfying

$$s(t) = c_0,\quad 0\le t\le \frac{1}{2},\quad s(t) = t,\quad t\ge 1$$

and $s'(t)\sqrt{s(t)} \le \sqrt{t}$ for $t\ge 0$ where $0<c_0<1$ is a constant. Then we show that the sequence solution $\{V_k(\cdot)\}$ of (4.2) corresponding to l_k converges to

$$V(\cdot) \in L^2(0,\tau; H^2_{loc}(R^n)) \cap H^1(0,\tau; L^2_{loc}(R^n)) \cap C(0,\tau; H^1_{loc}(R^n))$$

and $V(\cdot)$ satisfies (4.3)–(4.4) and

$$\sup_x \frac{|\nabla V(t)|}{\sqrt{s(|x|^2)}} \le \frac{4\alpha}{\tilde{\eta} + \sqrt{\tilde{\eta}^2 - 8\alpha\beta}}. \qquad (4.5)$$

provided that

$$8\alpha\beta < \tilde{\eta}^2 \quad \text{and} \quad 4C\alpha M^2 < \tilde{\eta}\,(\tilde{\eta} + \sqrt{\tilde{\eta}^2 - 8\alpha\beta}) \quad (4.6)$$

hold, where we assumed $|h'(\sigma U)\sigma| \le CM^2|U|$ and

$$\sup_x \sqrt{s(|x|^2)}\, h'(\sigma(x)U)\sigma_x(x)| \le \beta\,|U|.$$

If $h(v) = \frac{1}{2\gamma^2}|v|^2$ then (4.5)–(4.6) coincide with the estimate in Theorem 1.1.

The following steps are essentially same as in [FMc] since the estimate (4.5) is independent of $t>0$, $\rho>0$ and $\epsilon>0$.

Step 4 For $\rho>0$ let $V_\rho(t)$ be the solution to the equation

$$V_t + \rho V = H(x,\nabla V) + \tilde{\epsilon}\, tr\, a(x) V_{x_i,x_j}$$

that corresponds to the discounted cost. Then, the estimate (4.5) holds independent of $t>0$, $\rho>0$. If we let $t\to\infty$, then $V_\rho(t,x)\to V_\rho(x)$ uniformly on any compact subset of R^n and V_ρ satisfies

$$\rho V = H(x,\nabla V) + \tilde{\epsilon}\, tr\, a(x) V_{x_i,x_j}$$

Step 5 If we define $\lambda = \lim_{\rho\to 0}\rho V_\rho(0)$, then a sequential limit $V_\rho(x) - V_\rho(0)$ yields a solution to (1.4). □

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