The Principle of Stationary Action and Numerical Methods for N-Body Problems*

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Abstract— **Two-point boundary-value problems for conservative systems are studied in the context of the stationary action principle. In particular, we consider the case where the initial boundary condition is the system position, and the terminal boundary condition may be a combination of position and velocity data. The emphasis is on the N-body problem under gravitation. When the duration is sufficiently short, one may use a differential game formulation to obtain a fundamental solution, where for specific initial position and terminal data, one obtains the particular solution via a min-plus convolution of a function related to the terminal data and another function associated with the fundamental solution. That latter function is obtained by minimization of a parameterized linear functional over a convex set. This convex set is the fundamental solution. For longer duration problems, one takes a stationary point rather than a minimum.**

I. INTRODUCTION

A conservative system follows a trajectory which is a stationary point of the action functional, this being known as the Principle of Stationary Action, derivable in a general context from fundamental principles (c.f., [3], [4]). For two-point boundary-value problems (TPBVPs) of sufficiently short time-duration, the stationary point is a minimum. In simple mass-spring problems, as well as the wave equation, the potential takes a quadratic form, and the fundamental solution is a quadratic form obtained by solution of associated Ricatti equations [1], [5], [6]. For specific initial position and terminal data comprised of a combination of position and velocity data, one obtains the solution of that TPBVP by min-plus convolution of the fundamental solution with a function defined by the terminal data.

Here, we are concerned mainly with the N-body problem under gravitation. One obtains a bound such that for time-durations below that bound, the stationary point is a minimum. Minimization of the action takes the form of an optimal control problem. However, in the N-body problem, the potential has the rather unpleasant $1/r$ form. In a minor generalization of classical convex duality, the additive inverse of the gravitational potential has a representation as a maximum of quadratic forms where the dual variable appears in a cubic form. One may reformulate the control problem as a game where the opposing player maximizes over functions mapping time into the space of the dual variable. More specifically, the original, minimizing, controller minimizes over controls which are the velocities of the N bodies, and the maximizing player controls these parameterized quadratics which define the potential. Interestingly, the upper and lower values of this game over open loop controls are equal. Inverting the order of infimum and supremum, the inner, minimizing, control problem is of quadratic form. The solution is represented in terms of solutions of Riccati equations. These solutions may be parameterized by maximizing-player controls. The collection of such solutions forms a set in a space related to the number of bodies. This set forms the fundamental solution for the TPBVPs. All TPBVPs of the given form, with those particular body masses and time-duration, are obtained by minimization of a parameterized linear functional over the convex hull of this set, followed by min-plus convolution against a function defined by the terminal data. One has guaranteed convergence.

For longer-duration problems, one divides the time-interval into subintervals meeting the aforementioned condition. Then, one may proceed similarly, but with the added computation of a stationary point over a set of intermediary positions of the N bodies.

As this is a direction which may not be familiar, we first review the means by which one obtains fundamental solutions for TPBVPs as min-plus convolutions via an optimal control formulation. We then, indicate how one may effectively handle the N-body case with aid of differential games. The details are available in the references. That theory is for limited time-duration problems. Lastly, we very briefly indicate the extension to indefinite-duration problems.

II. REVIEW OF FUNDAMENTAL SOLUTION FORM

Before proceeding to the N-body case, we review the nature of this min-plus fundamental solution form and its usage in a general context. For more detail, one may see [5], [6]. Suppose the position state variable at time, r, is denoted by $\xi(r) \in I\!R^n$. Let the potential energy at position $x \in I\!R^n$ be denoted by $V(x)$. The kinetic energy will be denoted by $T(\dot\xi(r)) \doteq \frac{1}{2}\dot\xi'(r)\mathcal{M}\dot\xi(r)$, where $\xi(r)$ refers to the position of a point mass, $\mathcal{M}$ is simply mI, where m is the mass of the body and $I \in I\!R^{n\times n}$ is the identity; in a multi-body system, this is generalized in the obvious way. The action functional evaluated on path $\{\xi(r)\,|\, r\in[0,t]\}$ is given by $\mathcal{F}(\xi) \doteq \int_0^t T(\dot\xi(r)) - V(\xi(r))\,dr$. We reformulate this in a more convenient form. Let the initial position be $\xi(0) = x \in I\!R^n$, and let the dynamics be $\dot\xi(r) = u(r)$ for all $r \in (0,t)$, where $u \in \mathcal{U}^\infty \doteq L_2^{loc}((0,\infty); I\!R^n)$. Define the payoff, $J^0 : [0,\infty)\times I\!R^n \times \mathcal{U}^\infty \to I\!R \cup \{-\infty,+\infty\}$, as

$$J^0(t,x,u) \doteq \int_0^t \tfrac{1}{2}u'(r)\mathcal{M}u(r) - V(\xi(r))\,dr, \quad (1)$$

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where $\mathcal{M}$ is positive-definite, symmetric. Suppose for the purposes of this discussion that t is sufficiently small such that the stationary point yields a minimum [5], [6]. Then, one considers the corresponding value function given as

$$W^0(t,x) \doteq \inf_{u\in\mathcal{U}^\infty} J^0(t,x,u). \tag{2}$$

Clearly a solution of this problem yields $\xi(\cdot)$ satisfying the stationary action principle, and so is the trajectory of the conservative system under potential energy field V.

In order to see how one may use the optimal control formulation to obtain fundamental solutions for TPBVPs, it is helpful to consider the associated Hamiltonian-Jacobi partial differential equation (HJ PDE), and the corresponding characteristic equations. Let $\mathcal{D} \doteq (0,t)\times I\!R^n$ and $\hat{C}^1 \doteq C(\bar{\mathcal{D}})\cap C^1(\mathcal{D})$. Under reasonable conditions on V, one can expect that $W^0 \in \hat{C}^1$, and that on $\mathcal{D}$,

$$0 = -\frac{\partial}{\partial t}W(r,x) - V(x) - \frac{1}{2}[\nabla_x W(r,x)]'\mathcal{M}^{-1}\nabla_x W(r,x)$$

where r denotes the time-to-go. The state and gradient characteristic equations corresponding to this HJ PDE are

$$\frac{d\hat{\xi}}{d\rho} = \mathcal{M}^{-1}p(\rho), \qquad \frac{d\hat{p}}{d\rho} = -\nabla_x V(\hat{\xi}(\rho)).$$

These have associated initial and terminal conditions $\hat{\xi}(t) = x$ and $p(0) = 0$. In order to convert to forward time, one may take $\xi(s) = \hat{\xi}(t-s)$ and $p(s) = \hat{p}(t-s)$, which yield

$$\frac{d\xi}{ds} = -\mathcal{M}^{-1}p(s), \quad \frac{dp}{ds} = \nabla_x V(\xi(s)), \tag{3}$$

or $\frac{d^2\xi}{ds^2} = -\mathcal{M}^{-1}\nabla_x V(\xi(s))$, which of course, is the classical Newton's second law formulation. Also, (3) implies that the additive inverse of the co-state $p(r)$ is the momentum. One might also note that the optimal velocity in the HJB PDE is attained at $v = -\mathcal{M}^{-1}\nabla_x W = -\mathcal{M}^{-1}p$.

Suppose that one attaches a terminal cost to J^0 yielding,

$$\overline{J}(t,x,u) = J^0(t,x,u) + \bar{\psi}(\xi(t)), \tag{4}$$

$$\overline{W}(t,x) = \inf_{u\in\mathcal{U}^\infty} \overline{J}(t,x,u). \tag{5}$$

The HJ PDE and characteristic equations (3) remain unchanged. However, although the initial condition is still $\xi(0) = x$, the terminal condition is defined by $\bar{\psi}$. That is, we have a TPBVP where the terminal condition corresponds to the choice of $\bar{\psi}$.

With the addition of $\bar{\psi}$, the boundary conditions for (3) consist of initial condition $\xi(0) = x$ and terminal condition $p(t) = \nabla_x \bar{\psi}(\xi(t))$. If one takes, for example,

$$\bar{\psi}(x) = -\bar{v}\cdot x \tag{6}$$

for some given $\bar{v} \in I\!R^n$, then the terminal condition becomes $p(t) = \bar{v}$. That is, one has boundary conditions

$$\xi(0) = x \quad \text{and} \quad \dot{\xi}(t) = \bar{v}. \tag{7}$$

Alternatively, if one takes $z \in I\!R^n$ and terminal cost $\psi^\infty(x) = \psi^\infty(x,z) \doteq \delta_0^-(x-z)$ where

$$\delta_0^-(y) \doteq \begin{cases} 0 & \text{if } \; \text{y=0} \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

(i.e., the min-plus "delta function"), then the solution of control problem (5) yields the solution of the TPBVP with

$$\xi(0) = x \quad \text{and} \quad \xi(t) = z. \tag{9}$$

Other boundary conditions can be generated as well.

The goal here will be the development of *fundamental solutions* for TPBVPs corresponding to conservative systems. These fundamental solutions will generate particular solutions for boundary conditions such as $\dot{\xi}(t) = \bar{v}$ via a min-plus convolution over $I\!R^n$.

For $c \in [0,\infty)$, let $\psi^c : I\!R^n \times I\!R^n \to [0,\infty)$ be given by

$$\psi^c(x,z) = \frac{c}{2}|x-z|^2, \tag{10}$$

where we note $\psi^\infty(x,z)$ is given just above (8).

Define the finite time-horizon payoff

$$J^c(t,x,u,z) \doteq \int_0^t L(\xi(s),\, u(s))\,ds + \psi^c(\xi(t),z),$$

for $c \in [0,\infty] \doteq [0,\infty)\cup\{+\infty\}$. Also, for $c \in [0,\infty]$, let

$$W^c(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,z). \tag{11}$$

Now, from (4), (5) and (11),

$$\begin{aligned}
\overline{W}(t,x) &= \inf_{u\in\mathcal{U}^\infty} \left\{J^0(t,x,u) + \bar{\psi}(\xi(t))\right\} \\
&= \inf_{u\in\mathcal{U}^\infty} \left\{ J^0(t,x,u) + \inf_{z\in I\!R^n}\left[\psi^\infty(\xi(t),z) + \bar{\psi}(\xi(t))\right]\right\} \\
&= \inf_{z\in I\!R^n}\left[\inf_{u\in\mathcal{U}^\infty}\left\{J^0(t,x,u) + \psi^\infty(\xi(t),z)\right\} + \bar{\psi}(\xi(t))\right] \\
&= \inf_{z\in I\!R^n}\left[W^\infty(t,x,z) + \bar{\psi}(z)\right], \qquad (12)
\end{aligned}$$

or equivalently, in min-plus semi-field form (with $a\oplus b \doteq \min\{a,b\}$, $a\otimes b \doteq a+b$),

$$= \int_{I\!R^n}^{\oplus} W^\infty(t,x,z)\otimes\bar{\psi}(z)\,dz. \tag{13}$$

By (12) or (13), we see that $W^\infty(t,x,z)$ may be regarded as the fundamental solution of the TPBVP. That is, one obtains $\overline{W}$ by min-plus covolution of $\bar{\psi}$ with W^∞, where we may choose $\bar{\psi}$ to yield certain classes of terminal data. In the case of a quadratic potential function (such as in the mass-spring [6], [5] and wave equation [1] examples), W^∞ is a quadratic function in (x,z) determined by the solution of a Ricatti equation. In the case of the linear $\bar{\psi}$ of (6) and the tautological case of $\bar{\psi}(\cdot) = \psi^\infty(\cdot,z)$, this min-plus convolution becomes trivial. Perhaps it should also be mentioned that the initial velocity that yields the solution of the TPBVP is given by $-\mathcal{M}^{-1}\nabla_x W^\infty(t,x,z)$, where we remind the reader that t indicates time-to-go.

III. The N-body problem

Now that we have indicated how one employs the fundamental solution in a general context, we move to the N-body problem, which requires a game formulation. We address the solution of TPBVPs with N bodies acting under gravitational acceleration. In particular, we obtain a means for conversion of TPBVPs to initial value problems. The key to application of our approach to this class of problems lies in a variation of convex duality, leading to an interpretation of the least action principle as a differential game. The following is easily obtained through methods of convex duality (c.f., [7]).

Lemma 1: For $\rho > 0$, one has

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2} \max_{\alpha \in (0,\infty)} \alpha \left[1 - \frac{(\alpha\rho)^2}{2}\right],$$

and the supremum is attained at $\alpha = \sqrt{2/3}\,\rho^{-1}$.

Recall that the gravitational potential energy due to two point-mass bodies of mass m_1 and m_2, separated by distance $\rho > 0$, is given by $\mathcal{G}^{m_1,m_2}(\rho) = \frac{-Gm_1m_2}{\rho}$, where G is the universal gravitational constant. Of course, this is also valid for spherically symmetric bodies when the distance is greater than the sum of the radii of the bodies, and we do not concern ourselves with this distinction further. Using Lemma 1, we see that this may be represented as

$$-\mathcal{G}^{m_1,m_2}(\rho) = \widehat{G}\, m_1 \max_{\alpha_{1,2}\geq 0} (\alpha_{1,2} m_2) \left[1 - \frac{(\alpha_{1,2}\rho)^2}{2}\right],$$

where the universal gravitational constant is replaced by $\widehat{G} \doteq \left(\frac{3}{2}\right)^{3/2} G$. In the case of N bodies at locations x^i for $i \in \mathcal{N} \doteq]1,N[$ (where for integers $i < j$, we let $]i,j[$ denote $\{i, i+1, i+2, \dots j\}$ throughout), the additive inverse of the potential is given by

$$-V(x) = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i \max_{\alpha_{i,j}\geq 0} (\alpha_{i,j} m_j) \left[1 - \frac{(\alpha_{i,j}|x^i - x^j|)^2}{2}\right] \tag{14}$$

where $\mathcal{I}^\Delta \doteq \{(i,j) \in]1,N[\,|\, j > i\}$ and $x = \{x^1, x^2, \dots x^N\} \in I\!R^n \doteq I\!R^{3N}$. Let $\mathcal{A} \doteq \{\alpha = \{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}^\Delta} \,|\, \alpha_{i,j} \in [0,\infty) \,\forall (i,j) \in \mathcal{I}^\Delta\}$, and note that $\mathcal{A} \subset I\!R^{I^\Delta}$ where $I^\Delta \doteq \#\mathcal{I}^\Delta$. Then (14) may be written as

$$-V(x) = \max_{\alpha\in\mathcal{A}} \{-\hat{V}(x,\alpha)\},$$
$$-\hat{V}(x,\alpha) \doteq \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i (\alpha_{i,j} m_j) \left[1 - \frac{(\alpha_{i,j}|x^i - x^j|)^2}{2}\right].$$

Let $\xi(\cdot) = ((\xi^1(\cdot))', (\xi^2(\cdot))', \dots, (\xi^N(\cdot))')'$ denote a trajectory of the N-body system satisfying $\dot{\xi}(r) = u(r) = ((u^1(r))', \dots (u^N(r))')'$. The running cost will be

$$T(\dot{\xi}(r)) - V(\xi(r)) = \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{m_i |u^i(r)|^2}{2} - V(\xi(r)), \tag{15}$$

where V is given by (15). Note that for $x, z \in I\!R^n$ and $c \in [0,\infty]$, we continue to take ψ^c as given by (10). With these definitions, the least-action payoff, $\bar{J}^c$, becomes

$$\bar{J}^c(t,x,u,z) = \int_0^t T(u(r)) + \max_{\alpha\in\mathcal{A}} \{-\hat{V}(\xi(r),\alpha)\}\,dr + \psi^c(\xi(t),z). \tag{16}$$

The value is given by (11) with payoff J^c replaced by (16).

We assume spatial separation of near-optimal trajectories, that is:

$$\begin{aligned} &\exists\, \bar{\delta}, \bar{\epsilon} > 0 \text{ such that } \forall\ \epsilon\text{-optimal } u^\epsilon \in \mathcal{U}^\infty \text{ with } \epsilon \in (0,\bar{\epsilon}], \text{ and letting } \xi^\epsilon \text{ denote the corresponding trajectory, we have } |(\xi^\epsilon)^i(r) - (\xi^\epsilon)^j(r)| > \bar{\delta} \\ &\forall\, r \in [0,t],\ \forall\,(i,j) \in \mathcal{I}^\Delta. \end{aligned} \tag{A.N}$$

Let $\bar{\mathcal{A}}^\infty \doteq C([0,\infty);\mathcal{A})$. Also, for $\alpha \in \bar{\mathcal{A}}^\infty$, replace the time-independent potential energy function with

$$\begin{aligned} -V^\alpha(r,x) &\doteq -\hat{V}(x,\alpha(r)) \\ &= \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i (\alpha_{i,j}(r) m_j) \left[1 - \frac{(\alpha_{i,j}(r)|x^i - x^j|)^2}{2}\right]. \end{aligned} \tag{17}$$

Theorem 2: For all $t \geq 0$ and all $x, z \in I\!R^n$,

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty} \max_{\alpha(\cdot)\in\bar{\mathcal{A}}^\infty} J^c(t,x,u,\alpha,z), \tag{18}$$

where

$$J^c(t,x,u,\alpha,z) \doteq \int_0^t T(u(r)) - V^\alpha(r,\xi(r))\,dr + \psi^c(\xi(t),z).$$

By inspection of (11) and Theorem 2, the problem of finding the fundamental solution of the TPBVP for the N-body problem has been converted to a differential game. The first player minimizes the action at each moment, with immediate effect on the kinetic term and integrated effect on the other terms, while the second player maximizes the potential term at each moment. With this viewpoint, one may express the potential energy as a quadratic form.

We note that (18) is a non-standard form for dynamic games, as it is not expressed in terms of non-anticipative strategies (c.f., [2]), nor in terms of state feedback controls. We note:

Lemma 3: For any $t_0 > 0$, $W^c(t,x,z)$ is semiconvex in x, uniformly in $(t,x,z,c) \in [t_0,\infty) \times I\!R^n \times I\!R^n \times [0,\infty]$.

With minor manipulation, one finds that the HJ PDE associated with our problem is

$$\begin{aligned} 0 = &-\tfrac{\partial}{\partial t} W(t,x,z) + \sup_{\alpha\in\mathcal{A}} \{-\hat{V}(x,\alpha)\} \\ &-\tfrac{1}{2}\big(\nabla_x W(t,x,z)\big)' \mathcal{M}^{-1} \nabla_x W(t,x,z), \end{aligned} \tag{19}$$

where $\mathcal{M} \doteq \mathrm{diag}(\{m_1, m_2, \dots m_n\})$. Let

$$D^{\bar{\delta}} \doteq \{x \in I\!R^n \,\big|\, |x^i - x^j| > \bar{\delta}\, \forall (i,j) \in \mathcal{I}^\Delta\},$$
$$\mathcal{D}_t^{\bar{\delta}} \doteq C([0,t] \times \bar{D}^{\bar{\delta}}) \cap C((0,t) \times D^{\bar{\delta}}).$$

Theorem 4: Let $c \in (0,\infty)$, $t > 0$ and $z \in D^{\bar{\delta}}$. Suppose $W \in \mathcal{D}_t^{\bar{\delta}}$ satisfies (19) on $(0,t) \times D^{\bar{\delta}}$, and initial condition

$$W(0,x,z) = \psi^c(x,z), \qquad x \in D^{\bar{\delta}}. \tag{20}$$

Then, $W(t,x,z) = W^c(t,x,z)$ for all $x \in D^{\bar{\delta}}$. Further, with the controller $u^*(s)$ given by $u^*(s) = \tilde{u}(s,\tilde{\xi}(s))$ where $\tilde{\xi}(s)$ is generated by feedback $\tilde{u}(s,x) \doteq \nabla_x W(t-s,x,z)$, one has $W(t,x,z) = \bar{J}^c(t,x,u^*,z)$.

We now consider the game where the order of infimum and supremum are reversed. Due to the simple form of this particular game, an unusual equivalence can be obtained. Let

$$\underline{W}^c(t,x,z) \doteq \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \tag{21}$$

By the usual reordering inequality, one immediately has

$$\underline{W}^c(t,x,z) \le W^c(t,x,z) \quad \forall (t,x,z) \in [0,\infty)\times I\!R^n \times I\!R^n.$$

For any $\alpha \in \bar{\mathcal{A}}^\infty$, we let

$$\mathcal{W}^{\alpha,c}(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \tag{22}$$

Then, $\underline{W}^c(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z)$ for all $(t,x,z)\in[0,\infty)\times I\!R^n \times I\!R^n$.

Now let u^* be the optimal controller for our original problem (with potential energy function, $V(\cdot)$), that is $u^*(s) = \tilde{u}(s,\tilde{\xi}(s))$ where $\tilde{\xi}(s)$ is generated by feedback $\tilde{u}(s,x) \doteq -\mathcal{M}^{-1}\nabla_x W^c(t-s,x,z)$. Let $\xi^*(s)$ be the resulting trajectory, where of course, $\xi^*\equiv\tilde{\xi}$. For $s\in[0,t]$, let $\alpha^*(s)$ track the optimal value of α for trajectory ξ^*. (trivially obtainable from the last assertion of Lemma 1).

Lemma 5: Let $t\in(0,\infty)$ and $x,z\in D^{\bar{\delta}}$. Then, u^* is a critical point of $J^c(t,x,\cdot,\alpha^*,z)$.

Lemma 6: Let $\bar{t} = \bar{t}(\bar{\delta}) \doteq \sqrt{\frac{\sqrt{3}\bar{\delta}^3}{\sqrt{2}\widehat{G}\max_{i\ni]1,n[}(\sum_{j>i} m_j)}}$. Let $x,z\in D^{\bar{\delta}}$. If $t\in(0,\bar{t})$, then $J^c(t,x,\cdot,\alpha^*,z)$ is strictly convex, and further, u^* is the minimizer of $J^c(t,x,\cdot,\alpha^*,z)$.

Theorem 7: Let $t\in[0,\bar{t})$ and and $x,z\in D^{\bar{\delta}}$. Then

$$\underline{W}^c(t,x,z) = W^c(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z),$$

and further, $W^\infty(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,\infty}(t,x,z)$.

A. Fundamental Solution as Set of Riccati Solutions

We will find that the fundamental solution of the N-body problem may be given in terms of a set of solutions of Riccati equations. In particular, we look for a solution of the form

$$\mathcal{W}^{\alpha,c}(t,x,z) = \tfrac{1}{2}\left[x'P_t^c\,x + 2x'Q_t^c z + \tfrac{1}{2}\,z'R_t^c\,z + \gamma_t^c\right], \tag{23}$$

where $P_\cdot^c, Q_\cdot^c, R_\cdot^c, \gamma_\cdot^c$ implicity depend on the choice of $\alpha\in\bar{\mathcal{A}}^\infty$. In particular, we suppose that P_t^c has the form $P_t^c = \bar{P}_t^c \otimes_K I_3$, where $\otimes_K$ denotes the Kronecker product, and I_3 denotes the 3×3 identity matrix, with analogous forms for Q_t^c and R_t^c.

Also let $\bar{\nu}_t$ denote the $N\times N$ matrix of terms given by

$$\bar{\nu}_t^{i,j} = \begin{cases} -\sum_{k\neq i}\widehat{G}m_i m_k(\alpha_{i,j}(r))^3 & \text{if } i=j, \\ \widehat{G}m_i m_j(\alpha_{i,j}(r))^3 & \text{if } i\neq j. \end{cases} \tag{24}$$

Then, we have the Riccati system

$$\dot{\bar{P}}_t^c = -\bar{P}_t^c\mathcal{M}^{-1}\bar{P}_t^c + \bar{\nu}_t, \quad \dot{\bar{Q}}_t^c = -\bar{P}_t^c\mathcal{M}^{-1}\bar{Q}_t^c \tag{25}$$

$$\dot{\bar{R}}_t^c = -\left[\bar{Q}_t^c\right]'\mathcal{M}^{-1}\bar{Q}_t^c, \quad \dot{\gamma}_t = 2\sum_{i\neq j}\widehat{G}m_i m_j \alpha_{i,j}(r), \tag{26}$$

with initial conditions $\bar{P}_0^c = \bar{R}_0^c = cI$, $\bar{Q}_0^c = -cI$, $\gamma_0 = 0$.

Now, note that by Theorem 7 and (23),

$$\begin{aligned} W^\infty(t,x,z) &= \sup_{\alpha\in\bar{\mathcal{A}}^\infty}\lim_{c\to\infty}\mathcal{W}^{\alpha,c}(t,x,z) \\ &= \sup_{\alpha\in\bar{\mathcal{A}}^\infty}\lim_{c\to\infty}\tfrac{1}{2}\left[x'P_t^c\,x + 2x'Q_t^c z + \tfrac{1}{2}\,z'R_t^c\,z + \gamma_t^c\right], \\ &= \sup_{\alpha\in\bar{\mathcal{A}}^\infty}\tfrac{1}{2}\left[x'P_t^\infty\,x + 2x'Q_t^\infty z + \tfrac{1}{2}\,z'R_t^\infty\,z + \gamma_t^\infty\right]. \end{aligned}$$

It is important to note that $\mathcal{W}^{\alpha,c}(t,x,z)$ and $\mathcal{W}^{\alpha,\infty}(t,x,z)$ are concave in α. Letting

$$\mathcal{G}(t) = \mathcal{G}(t;\{m_j\}_{j=1}^N) \doteq \left\{\left(P_t^\infty, Q_t^\infty, R_t^\infty, \gamma_t^\infty\right) \,\middle|\, \alpha\in\bar{\mathcal{A}}^\infty\right\},$$

$$\begin{aligned} W^\infty(t,x,z) &= \sup_{(P,Q,R,\gamma)\in\mathcal{G}(t)} \tfrac{1}{2}\left[x'Px + 2x'Qz + \tfrac{1}{2}\,z'Rz + \gamma\right] \\ &= \sup_{(P,Q,R,\gamma)\in\hat{\mathcal{G}}(t)} \tfrac{1}{2}\left[x'Px + 2x'Qz + \tfrac{1}{2}\,z'Rz + \gamma\right], \end{aligned}$$

where $\hat{\mathcal{G}}(t) \doteq \langle\mathcal{G}(t)\rangle$ where $\langle\cdot\rangle$ denotes convex hull (and we remind the reader that the velocity of the solution of the TPBVP is given by $u^*(s) = -\mathcal{M}^{-1}\nabla_x W^\infty(t-s,x,z)$ for all $s\in[0,t]$). Consequently, one sees that the set $\hat{\mathcal{G}}(t) = \hat{\mathcal{G}}(t;m_1,m_2,\dots m_N)$ represents the general solution of the N-body TPBVP for a given set of masses and time-duration.

IV. Longer-duration problems

We very briefly indicate the extension to arbitrary-duration problems. The above solution is only guaranteed to work for problems where $t<\bar{t}$ (see Lemma 6). For longer-durations, the $\inf_{u\in\mathcal{U}^\infty}$ is replaced by $\mathrm{stat}_{u\in\mathcal{U}^\infty}$. Here, we note that one defines $\operatorname{argstat}_{y\in\mathcal{G}_y}\mathcal{F}(y) \doteq \{y\in\mathcal{G}_\mathcal{Y} \,|\, \mathcal{F}_y(y)=0\}$ where $\mathcal{F}_y$ denotes the Fréchet differential, and $\mathrm{stat}_{y\in\mathcal{G}_y}\mathcal{F}(y) \doteq \{\mathcal{F}(y)\,|\,y\in\operatorname{argstat}_{y\in\mathcal{G}_y}\mathcal{F}(y)\}$. One seeks $W^\infty(t,x;z) = \mathrm{stat}_{u\in\mathcal{U}}\sup_{\alpha\in\mathcal{A}} J^\infty(t,x,u,\alpha;z)$. For $t>\bar{t}$, one may introduce intermediary times $\tau_k \doteq k\tau$, $k\in]1,K[$, where $\tau\in(0,\bar{t})$ and $t_K = t$, and intermediary points, $\zeta^k\in I\!R^n$ for $k\in]1,K-1[$. One obtains

$$W^\infty(t,x;z) = \operatorname*{stat}_{\hat{\zeta}\in R^{(K-1)n}}\sup_{\hat{\alpha}\in\hat{\mathcal{A}}}\widehat{W}^\infty(t,x,\hat{\zeta},\hat{\alpha};z),$$

where $\hat{\mathcal{A}}$ is composed of an outer product of $\bar{\mathcal{A}}^\infty$, $\hat{\zeta}\doteq\{\zeta^k\}_{k=1}^{K-1}$, and $\widehat{W}^\infty$ is an appropriately defined sum of minimal actions for trajectories connecting the endpoints ζ^k (with $\zeta^0\doteq x$ and $\zeta^K\doteq z$). Then, $\widehat{W}^\infty$ is represented as a set of solutions of Riccati equations, $\left\{\left(P_\tau^{k,\infty}(\alpha^k), Q_\tau^{k,\infty}(\alpha^k), R_\tau^{k,\infty}(\alpha^k)\right)\right\}_{k\in]1,K-1[}$, and

$$W^\infty(t,x;z) = \sup_{\hat{\alpha}\in\hat{\mathcal{A}}}\operatorname*{stat}_{\hat{\zeta}\in R^{(K-1)n}}\widehat{W}^\infty(t,x,\hat{\zeta},\hat{\alpha};z),$$

which yields an alternate numerical approach, where the stationarity over the $\hat{\zeta}$ is obtained from a linear system.

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