Max–Plus Eigenvector Representations for Nonlinear H_∞ Value Functions

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Abstract

The H_∞ problem for nonlinear systems is considered. The corresponding dynamic programming equation is a fully nonlinear, first–order, partial differential equation. Interestingly, if one switches from the normal definition of addition and multiplication to the max–plus algebra (which is no more complex), the solution operator becomes a linear operator. The solution can be expanded using a max–plus basis. The coefficients in this expansion satisfy a max–plus eigenvector equation for a matrix associated with this solution operator – thus transforming the nonlinear problem into a linear one. In fact there is a parameterized family of matrices for which this holds. Expressions and approximations for the coefficients in these matrices are given.

1 Introduction

We consider the H_∞ problem for a nonlinear system. Nonlinear H_∞ control and its relation to the associated dynamic programming equation (DPE) has been studied extensively (see [1], [6], [12], [2] among many notable others). For many nonlinear control problems, the value function is the unique (viscosity) solution of the DPE, and if this is sufficiently smooth, then one can obtain the optimal feedback control from it. For continuous systems, this DPE takes the form of a partial differential equation (PDE). In the case of nonlinear H_∞ , there are additional technical difficulties. First of all there are several value–like quantities of interest such as the available storage and the required storage [12]. Soravia [11] showed that for a wide class of nonlinear systems, the smallest nonnegative continuous viscosity solution of the PDE is the available storage. This is the solution we seek. Under additional assumptions, [8], [9], one can show that there is a unique viscosity solution satisfying a certain growth condition, and that this solution is the available storage. We will refer to the available storage as the value freely throughout.

Unfortunately, the H_∞ DPE is nonlinear – possessing a term which is quadratic in the gradient. In the Robust/H_∞ Filter problem, a new numerical method using the max–plus (to be defined below) linearity of the solution operator for a similar DPE was recently obtained [5]. This linearity was also noticed earlier in [7]. In this paper, we show that the solution of the H_∞ PDE is given by a max–plus eigenvector corresponding to a max–plus eigenvalue of 0 (the max–plus multiplicative identity) for a particular matrix. This matrix is associated with the solution operator of the PDE. We remind the reader that the max–plus algebra is a commutative semi–field on $\mathbf{R} \cup \{-\infty\}$ given by

$$a \oplus b = \max\{a, b\}, \qquad a \otimes b = a + b \tag{1}$$

We consider mainly the design case where one fixes a control and a value of the disturbance attenuation parameter, γ, and desires to compute the solution to the PDE. (If the PDE has a nonnegative solution which is zero at the origin, then one has an H_∞ controller.) We will also consider a class of active control cases where one replaces the max–plus algebra above by a min–plus algebra.

In Section 2 it will be shown that the value function for a certain finite–time horizon problem converges to the value of the H_∞ problem, and that this is the unique viscosity solution of the PDE in a certain class. In Section 3, this solution will be shown to satisfy a certain steady–state dynamic programming principle (DPP). The right hand side of this DPP is the solution operator of the DPE, and it will be shown that this solution operator is linear in the max–plus algebra. In Section 4, we show that the solution lies in the space of semi–convex functions, and we will indicate a max–plus basis for this space. In Section 5, we expand this steady–state solution in the max–plus basis, and show that the coefficients can be found through the solution of a max–plus eigenvector equation for a 0 eigenvalue

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of $a = B \otimes a$ where a is the vector of coefficients, and B is a matrix associated with the solution operator of the DPE. We discuss a simple approximation to B in Section 6. In Section 7, we consider a class of "active" control problems where the DPE has a concave Hamiltonian, and consequently, one can use a similar technique to that above but with the min–plus algebra. This approach is being used to numerically solve some nonlinear H_∞ problems.

2 Background on Existence and Uniqueness

The results in this section are not the focus of the paper, but are simply some particular previous results that can be used as a precursor to the new material to follow in later sections. We consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{2}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T]\ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are probably not necessary but are sufficient for the results to follow. We will assume that all the functions f, σ and l (given below) are smooth, although that is not required for the results. We will assume that there exist $K, c \in (0,\infty)$ such that

$$\begin{array}{ll} |f(x) - f(y)| \le K|x - y| & \forall\, x, y \in \mathbf{R}^n \\ (x-y)^T(f(x) - f(y)) \le -c|x-y|^2 & \forall\, x, y \in \mathbf{R}^n \\ f(0) = 0 & \end{array} \tag{A1}$$

Note that this automatically implies the closed–loop stability criterion of H_∞ control. We assume that

$$\begin{array}{ll} |\sigma(x) - \sigma(y)| \le K|x - y| & \forall\, x, y \in \mathbf{R}^n \\ |\sigma(x)| \le M & \forall\, x \in \mathbf{R}^n \end{array} \tag{A2}$$

for some $M < \infty$. Let $l(x)$ be the running cost (to which the L_2–norm disturbance penalty will be added). We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{array}{ll} l_{xx}(x) \le \beta & \forall\, x \in \mathbf{R}^n \\ 0 \le l(x) \le \alpha|x|^2 & \forall\, x \in \mathbf{R}^n. \end{array} \tag{A3}$$

(There is a reason for allowing β to be greater than 2α, which one might notice below.)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function (again, also referred to as the value function in the sequel), $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T<\infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt. \tag{3}$$

The corresponding DPE is

$$\begin{aligned} 0 &= \sup\nolimits_{w \in \mathbf{R}^m} \left\{ [f(x) + \sigma(x)w]^T \nabla W + l(x) - \tfrac{\gamma^2}{2}|w|^2 \right\} \\ &= \tfrac{1}{2\gamma^2} \nabla W^T \sigma(x)\sigma^T(x) \nabla W + f^T(x)\nabla W + l(x). \end{aligned} \tag{4}$$

It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \tag{5}$$

We note that there are examples where (5) fails and the available storage is ∞. One has the following result ([8],[9]). The proof can be found in the references.

Theorem 2.1 *There exists a unique continuous viscosity solution of (4) in the class*

$$0 \le W(x) \le c\frac{(\gamma-\delta)^2}{2M^2}|x|^2 \tag{6}$$

for some $\delta > 0$.

We also have the following. Consider the finite time horizon control problem with dynamics (2) and payoff and value given by

$$\begin{array}{l} J(T,x,w) = \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt \\ V(T,x) = \sup_{w \in \mathcal{W}} J(T,x,w). \end{array} \tag{7}$$

Theorem 2.2 *The unique continuous viscosity solution of the previous theorem is given by*

$$W(x) = \lim_{T \to \infty} V(T,x). \tag{8}$$

This proof can also be found in the references ([8], [9]).

3 Solution operator and max–plus linearity

We would like to show that W is a fixed point of the solution operator. To show this we need to use the DPP for the finite time horizon problem. Specifically, we note

Lemma 3.1 *$V(x,T) = \sup_{w \in \mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + V(X(\tau), T-\tau)\}$ (where X satisfies (2)) for any $0 \le \tau \le T < \infty$ and any $x \in \mathbf{R}^n$.*

Proof: Let

$$\begin{aligned} R(x,T) \;=\; & \sup_{w \in \mathcal{W}} \{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + \\ & V(X(\tau), T-\tau)\}. \end{aligned}$$

Then

$$\begin{aligned}R(x,T) &= \sup_{w_1\in\mathcal{W}}\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w_1(t)|^2\,dt + \\ &\sup_{w_2\in\mathcal{W}}(\int_\tau^T l(X(t))-\frac{\gamma^2}{2}|w_2(t)|^2\,dt)\} \\ &= \sup_{w\in\mathcal{W}}\{\int_0^T l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt\} \\ &= V(x,T). \square\end{aligned}$$

Define the solution operator $S_\tau[W(\cdot)](x) = \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\}$ where X satisfies (2) for any $x\in\mathbf{R}^n$.

Theorem 3.2 *For any $\tau\in[0,\infty)$, W given by (8) satisfies $S_\tau[W]=W$, and further, it is the unique solution in the class (6).*

Proof: We prove only the first assertion here; the proof of uniqueness is similar to that in [8], [9]. By Theorem 2.2 and then by Lemma 3.1, one has

$$\begin{aligned}W(x) &= \lim_{T\to\infty} V(x,T) \\ &= \lim_{T\to\infty}\sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt \\ &\quad +V(X(\tau),T-\tau)\}\end{aligned}$$

Let $T_2 > T_1$. Choose $w(t)$ on $[0,T_2]$ such that $w(t)=\begin{cases}\tilde{w}(t) & : \quad t\in[0,T_1] \\ 0 & : \quad t\in(T_1,T_2]\end{cases}$ and $\tilde{w}(t)$ is ε-optimal on $[0,T_1]$. Then

$$V(x,T_1)\le J(x,w,T_1)+\varepsilon \le J(x,w,T_2)+\varepsilon \le V(x,T_2)+\varepsilon.$$

Since our choice in ε was arbitrary, $V(x,\cdot)$ is monotonically increasing and therefore

$$\lim_{T\to\infty} V(x,T) = \sup_T V(x,T).$$

Continuing from above we find,

$$\begin{aligned}W(x) &= \sup_T\sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt + \\ &\qquad V(X(\tau),T-\tau)\} \\ &= \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt + \\ &\qquad W(X(\tau))\} \\ &= S_\tau[W(\cdot)](x). \square\end{aligned}$$

Note that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve. This will be discussed further below.

Now we will demonstrate that S_τ is linear in the max–plus algebra. The proof that max–plus multiplication passes through the operator is trivial, so we will not show it here.

For any $x\in\mathbf{R}^n$ and continuous ϕ,ψ,

$$\begin{aligned}S_\tau[\phi\oplus\psi](x) &= \sup_{w\in\mathcal{W}}\{\max[\phi(X(\tau)),\psi(X(\tau))]+ \\ &\quad \textstyle\int_0^\tau l(X(r))-\frac{\gamma^2}{2}|w(r)|^2\,dr\} \\ &\ge \sup_{w\in\mathcal{W}}\{\phi(X(\tau))+\textstyle\int_0^\tau l(X(r))- \\ &\quad \tfrac{\gamma^2}{2}|w(r)|^2\,dr\} = S_\tau[\phi](x)\end{aligned} \tag{9}$$

By symmetry, one obtains an analogous inequality for ψ. Consequently,

$$\begin{aligned}S_\tau[\phi\oplus\psi](x) &\ge \max\{S_\tau[\phi](x),S_\tau[\psi](x)\} \\ &= S_\tau[\phi](x)\oplus S_\tau[\psi](x).\end{aligned} \tag{10}$$

Now let w^ε be optimal in the first equation of (9) (with corresponding X^ε). Then

$$\begin{aligned}S_\tau[\phi\oplus\psi](x)\le\ & \max[\phi(X(\tau)),\psi(X(\tau))]+ \\ & \textstyle\int_0^\tau l(X^\varepsilon(r))-\frac{\gamma^2}{2}|w^\varepsilon(r)|^2\,dr+\varepsilon\end{aligned} \tag{11}$$

and so, in the case where $\phi(X(\tau))\ge\psi(X(\tau))$

$$\begin{aligned}S_\tau[\phi\oplus\psi](x) &\le \phi(X(\tau))+\textstyle\int_0^\tau l(X^\varepsilon(r)) \\ &\quad -\tfrac{\gamma^2}{2}|w^*(r)|^2\,dr+\varepsilon\le S_\tau[\phi](x)+\varepsilon\end{aligned} \tag{12}$$

Considering also the case where $\phi(X(\tau))<\psi(X(\tau))$ and taking $\varepsilon>0$ arbitrarily small, one obtains

$$S_\tau[\phi\oplus\psi](x)\le S_\tau[\phi](x)\oplus S_\tau[\psi](x).$$

Combining this with (10) yields

$$S_\tau[\phi\oplus\psi](x)= S_\tau[\phi](x)\oplus S_\tau[\psi](x). \tag{13}$$

One now has

Theorem 3.3 *The solution operator, S_τ, is linear in the max–plus algebra.*

4 Max–plus basis for semi–convex functions

In this section, we first outline the proof that the value, W, lies in the space of semi–convex functions, and then we describe the max–plus basis for this space.

Define the space of semi–convex functions, $\mathcal{S}$, as the set of $\psi:\mathbf{R}^n\to\mathbf{R}$ such that for any $R<\infty$ there exists $C_R < \infty$ such that $\psi(x) + \frac{C_R}{2}|x|^2$ is convex over $B_R = \{x \in \mathbf{R}^n : |x| \leq R\}$. We refer to such a C_R as a semi–convexity constant for ψ. Note that any $\psi \in \mathcal{S}$ is automatically locally Lipschitz [4].

The proof that the value, W, lies in $\mathcal{S}$ is rather technical. We include only a brief sketch here. Let $\varepsilon > 0$, $R < \infty$ and $|x| \leq R - 1$. Let $w^{\varepsilon,T}$ be ε–optimal for problem (7), and let $X^{\varepsilon,T}$ be the corresponding state process. Then there exists M_R *independent of* T *and* $x \in B_R$ such that

$$\begin{array}{l} \|w^{\varepsilon,T}\|_{L_2[0,T]} \leq M_R \\ \|X^{\varepsilon,T}\|_{L_2[0,T]} \leq M_R. \end{array} \tag{14}$$

Subscript the state process satisfying (2) by the initial value, that is denote the state process starting at $X(0) = z$ by $X_z(\cdot)$. Let $\delta < 1$. Then by looking at the second–order difference in V and using the assumptions and (14), one may obtain a semi–convexity constant for $V(T, \cdot)$ over B_R independent of T. Combining this with Theorem 2.2 yields the semi–convexity of W. More details of the proof for a similar result appear in [5].

Now we turn to the max–plus basis over $\mathcal{S}$. In [5], the following semi–convex duality was derived. It is derived from convex duality [10] in an easy manner.

Theorem 4.1 *(Fleming and McEneaney) Let* $\phi \in \mathcal{S}$. *Let* $c > C_R$ *where* $C_R > 0$ *is a semi–convexity constant for* ϕ, *and let* $L(R)$ *be the Lipschitz constant for* ϕ *over* B_R. *Then*

$$\phi(x) = \max_{\widetilde{x} \in B_{L(R)/c}} [-\frac{c}{2}|x - \widetilde{x}|^2 + a_{\widetilde{x}}] \quad \forall x \in B_R$$

where

$$a_{\widetilde{x}} = -\max_{x \in B_R} [-\frac{c}{2}|x - \widetilde{x}|^2 - \phi(x)].$$

Let $\phi \in \mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over B_R, and let $c > C_R$. Define

$$\psi_i(x) \doteq \frac{-c}{2}|x - x_i|^2$$

for each i. Then, using Theorem 4.1, one finds (see [5])

$$\begin{array}{rl} \phi(x) & = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \quad \forall x \in B_R \\ \text{where} & \\ a_i & \doteq -\max_{x \in B_R}[\psi_i(x) - \phi(x)]. \end{array} \tag{15}$$

This is a max–plus basis expansion.

5 The eigenvector equation

In order to reduce complexity, we suppose throughout the next two sections that W has a max–plus basis expansion with a finite number of terms. Eventually, of course, one must consider the error introduced by truncating the expansion at a finite number of terms.

Let $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i$, $a^T = (a_1 \; a_2 \cdots a_n)$, and $B_{j,i} = -\max_{x \in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$. Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation. By (17) and the semi–convexity preserving property of S_τ, note that

$$S_\tau[\psi_i](x) = \bigoplus_{i=1}^n B_{j,i} \otimes \psi_j(x) \tag{16}$$

where again we blithely truncated the expansion.

Theorem 5.1

1. *Suppose* W *is a solution to* $S_\tau[W] = W$, *and that any expansion* $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$ *requires* $a_i > -\infty \; \forall i$. *Then* $a = Ba$.
2. *Conversely, suppose* $a = Ba$ *and that* $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$ *then* $S_\tau[W] = W$.

Proof: We begin with the first assertion. We have $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$. By Theorem 3.3, (16), and the distributive property, we get

$$\begin{aligned} S_\tau[W(\cdot)](x) &= \bigoplus_{i=1}^n a_i \otimes S_\tau \psi_i(x) \\ &= \bigoplus_{j=1}^n \bigoplus_{i=1}^n ((a_i \otimes B_{j,i}) \otimes \psi_j(x)). \end{aligned}$$

Combining the above with $S_\tau[W(\cdot)](x) = W(x)$,

$$\bigoplus_{j=1}^n a_j \otimes \psi_j(x) = \bigoplus_{j=1}^n \bigoplus_{i=1}^n ((a_i \otimes B_{j,i}) \otimes \psi_j(x)).$$

Since it is necessary that $a_j > -\infty$, we know there exists a neighborhood $B_\delta(y_j)$ in which $a_j + \psi_j(x)$ is larger than all other $a_i + \psi_i(x)$. Thus, the only basis vector that we can vertically shift so that it is equivalent to $a_j + \psi_j(x)$ on $B_{\overline{\delta}}(y_j) \subset B_\delta(y_j)$ is $\psi_j(x)$ itself. Hence,

$$a_j = \max_i (a_i + B_{j,i}),$$

and therefore $a = Ba$.

For the second part of the theorem, start with the fact that if $a = Ba$ then

$$\bigoplus_{j=1}^n a_j \otimes \psi_j(x) \quad = \quad \bigoplus_{j=1}^n \bigoplus_{i=1}^n ((a_i \otimes B_{j,i}) \otimes \psi_j(x))$$

$$= \bigoplus_{i=1}^{n}(a_i \otimes \bigoplus_{j=1}^{n}(B_{j,i} \otimes \psi_j(x)))$$
$$= \bigoplus_{i=1}^{n} a_i \otimes S_\tau[\psi_i](x)$$

Now since $W(x) = \bigoplus_{j=1}^{n} a_j \otimes \psi_j(x)$, we get that

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes S_\tau[\psi_i](x) = S_\tau[\bigoplus_{i=1}^{n} a_i \otimes \psi_i](x).$$

Hence $W(x)$ is a solution to $S_\tau[W(\cdot)](x) = W(x)$. □

6 An approximation for B

Clearly the level of difficulty of the computations for the construction of B (or, more exactly, a B_τ in the parameterized set) will be a major concern with this approach. Recall that (for any specific τ)

$$B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \tfrac{\gamma^2}{2}|w(t)|^2\,dt + \psi_i(X(\tau))\}). \tag{17}$$

where we suppress the dependence of B on τ to reduce notation. Thus, one can obtain the entries of B by the solution of a *finite time horizon* control problem. Interestingly, we have the freedom to choose the value of τ. Clearly, choosing a small τ has the advantage that this control problem approximates a simple maximization problem with no time component at all. (This is given below.) The maximum of this limit problem may even be analytically solvable, thus removing a significant stumbling block. This section is still a work in progress. We will obtain a bound on the difference between the limit and B_τ for small τ. Although we can show this goes to zero as τ goes to zero, there appears to be a countervailing sensitivity to truncation errors in using a finite number of basis vectors, thereby preventing any "free lunch".

Choose w which is ε-optimal in (17). In [8], [9], it is demonstrated that there exists $D < \infty$ such that

$$\|w\|^2_{L_2(0,\tau)} \le D(1+|x|^2) \tag{18}$$

where D is independent of τ. Fix $R<\infty$. Then,

$$\begin{aligned}&\left|\int_0^\tau l(X(t)) - \tfrac{\gamma^2}{2}|w(t)|^2\,dt + \psi_i(X(\tau)) - \int_0^\tau l(x)\right.\\ &\left.-\tfrac{\gamma^2}{2}|w(t)|^2 + \nabla\psi_i(x)\cdot[f(x)+\sigma(x)w]\,dt - \psi_i(x)\right|\\ &= \left|\int_0^\tau l(X(t)) - l(x)\,dt + \int_0^\tau \nabla\psi_i(X(t))\cdot[f(X)\right.\\ &\left.+\sigma(X)w(t)] - \nabla\psi_i(x)\cdot[f(x)+\sigma(x)w(t)]\,dt\right|\\ &\le K_1\tau^{\frac{3}{2}} + K_2\tau\end{aligned} \tag{19}$$

where K_1, K_2 are independent of $\varepsilon > 0$ and $x \in B_R$, since by (18) and the assumptions

$$|l(X(t)) - l(x)| \le c_1\tau^{\frac{1}{2}},$$
$$|\nabla\psi_i(X(t))f(X) - \nabla\psi_i(x)f(x)| \le c_2\tau^{\frac{1}{2}},$$
$$|\nabla\psi_i(X(t))\sigma(X) - \nabla\psi_i(x)\sigma(x))| \le c_3\tau^{\frac{1}{2}},$$

and also using Cauchy-Schwartz,

$$\int_0^\tau (\nabla\psi_i(X(t))\sigma(X) - \nabla\psi_i(x)\sigma(x))w(t)\,dt \le c_4\tau.$$

Let

$$\begin{aligned}w_o &= argmax_{w\in\mathbf{R}^m}\{\tau[l(x) - \frac{\gamma^2}{2}|w|^2]\\ &\quad +\psi_i(x) + \tau\nabla\psi_i\cdot[f(x)+\sigma(x)w]\}.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^\tau l(x) - \tfrac{\gamma^2}{2}|w(t)|^2 + \nabla\psi_i\cdot[f(x)+\sigma(x)w]\,dt + \psi_i(x)\\ &\le \int_0^\tau l(x) - \tfrac{\gamma^2}{2}|w_o|^2\,dt + \psi_i(x) +\\ &\quad \tau\nabla\psi_i\cdot[f(x)+\sigma(x)w_o].\end{aligned} \tag{20}$$

Again let w be ε-optimal, then using the above,

$$\begin{aligned}B_{j,i} &\le -\max_x[\psi_j(x) - (l(x)\tau - \frac{\gamma^2}{2}|w_o|^2\tau\\ &+ \psi_i(x) + \tau\nabla\psi_i\cdot[f(x)+\sigma(x)w_o])]\\ &+ K_1\tau^{\frac{3}{2}} + K_2\tau\end{aligned}$$

since this inequality is true for all ε.

Now let $w_o(t) = w_o$

$$\begin{aligned}B_{j,i} &\ge -\max_x(\psi_j(x) - (l(x)\tau - \frac{\gamma^2}{2}|w_o|^2\tau\\ &+ \psi_i(x) + \tau\nabla\psi_i\cdot[f(x)+\sigma(x)w_o])) - K_3\tau^2.\end{aligned}$$

So, we define (still suppressing the τ dependence)

$$\begin{aligned}\widetilde{B}_{i,j} &\doteq -\max_x(\psi_j(x) - (l(x)\tau - \frac{\gamma^2}{2}|w_o|^2\tau + \psi_i(x)\\ &+ \tau\nabla\psi_i\cdot[f(x)+\sigma(x)w_o])),\end{aligned}$$

and we see that we have

$$|B_{j,i} - \widetilde{B}_{j,i}| \le \overline{K}_2\tau + \overline{K}_3\tau^2.$$

We will use $\widetilde{B}$ to compute vectors $\tilde{a}$ such that $\tilde{a} = \widetilde{B}\tilde{a}$. Define the norm $\|a\| = \sup_i |a_i|$. We have not yet determined the size of $\|a - \tilde{a}\|$ (where $a = Ba$) given the above bound on $|B_{j,i} - \widetilde{B}_{j,i}|$. However, note that since $W(x) = \bigoplus_{i=1}^{n} a_i \otimes \psi_i$, we will be able to bound $\|\widetilde{W}(x) - W(x)\|_{L_\infty}$ once we bound $\|a - \tilde{a}\|$.

7 Active control and the min–plus algebra

The analysis in the above sections concentrated on the testing of a control which is already included in dynamics $f(x)$ in order to determine the value function.

Here we consider the case where one chooses a value of the disturbance attenuation parameter, and attempts to find the value function for this "active" control problem. In this case, the problem takes the form of a game. We specifically consider the case where the dynamics take the form

$$\dot{X} = A(x) + D(X)u + \sigma(X)w, \qquad X(0) = x \tag{21}$$

where we have now added the control, $u(\cdot)$, taking values in $\mathbf{R}^l$, and D is an $n \times l$ matrix–valued multiplier on the control. In this section we will not include all of the details, but merely outline the manner in which one can use ideas analogous to those above. Consider a payoff and value which now take the form

$$J(T,x,u,w) = \int_0^T [l(x) + \tfrac{\eta^2}{2}|u|^2 - \tfrac{\gamma^2}{2}|w|^2]\,dt$$

$$W(x) = \inf_{\lambda\in\Lambda} \sup_{T<\infty} \sup_{w\in\mathcal{W}} J(T,x,\lambda[w],w) \tag{22}$$

where Λ is the set of non–anticipative strategies for the controller. More specifically, we define a non–anticipative strategy for the controller according to the Elliott–Kalton definition [3] as a non–anticipative mapping $\lambda : \mathcal{W} \to \mathcal{U}$ where $\mathcal{U} = \{u : [0,\infty) \to \mathbf{R}^l : u \in L_2[0,T]\ \forall T < \infty\}$.

The corresponding DPE is

$$0 == A(x)\cdot\nabla W + l(x) - (\nabla W)^T \left[\frac{DD^T}{2\eta^2} - \frac{\sigma\sigma^T}{2\gamma^2}\right]\nabla W \tag{23}$$

We will assume that the controller dominates the disturbance in that

$$\frac{QQ^T}{2} \doteq \frac{DD^T}{2\eta^2} - \frac{\sigma\sigma^T}{2\gamma^2}$$

is bounded and uniformly elliptic. Then DPE (23) is related to the problem

$$\begin{array}{l} \dot{X} = A(X) + Q(X)v, \qquad X(0) = 0 \\ W(x) = \inf_{u\in\mathcal{U}} \int_0^\infty l(X) - \frac{1}{2}|u|^2\,dt. \end{array} \tag{24}$$

when the long–run average cost per unit time is zero.

Then, as in the previous sections we find that W satisfies the steady–state DPP

$$\begin{aligned} W(x) &= S_\tau[W](x) \\ &= \inf_{u\in\mathcal{U}} \left\{ \int_0^\tau l(X) + \tfrac{1}{2}|u|^2\,dt + W(X(\tau)) \right\} \end{aligned} \tag{25}$$

for any $\tau > 0$. In this case, S_τ is linear in the min–plus algebra, where this is the commutative semi–field defined by the operations

$$a \oplus b = \min\{a,b\}, \qquad a \otimes b = a + b \tag{26}$$

over $\mathbf{R} \cup \{+\infty\}$.

Again, by analogy, W is now semi–concave. There is an analogous semi–concave basis, and we may write

$$W(x) = \bigoplus_{i=1}^{\infty} a_i \otimes \phi_i(x) \simeq \bigoplus_{i=1}^{n} a_i \otimes \phi_i(x)$$

where $\phi_i(x) = \frac{c}{2}|x - x_i|^2$ for a dense set $\{x_i\}$ and proper choice of c. Then we again obtain a relation similar to Theorem 5.1 relating the solution of (25) to the min–plus eigenvector equation, $a = B \otimes a$, where B is defined analogously.

References

[1] T. Basar and P. Bernhard, **$\mathbf{H}_\infty$–Optimal Control and Related Minimax Design Problems**, Birkhäuser (1991).

[2] J.A. Ball and J.W. Helton, "H_∞ control for nonlinear plants: connections with differential games", Proc. 28th IEEE Conf. Dec. Control, Tampa FL (1989), 956–962.

[3] R. J. Elliott and N. J. Kalton, "The existence of value in differential games", Memoirs of the Amer. Math. Society, **126** (1972).

[4] W.H. Fleming, **Functions of Several Variables**, Springer–Verlag (1977).

[5] W.H. Fleming and W.M. McEneaney, "A max–plus based algorithm for an HJB equation of nonlinear filtering", Submitted to SIAM J. Control and Optim.

[6] M.R. James, "A partial differential inequality for dissipative nonlinear systems", Systems and Control Letters, **21** (1993) 315–320.

[7] V.P. Maslov, "On a new principle of superposition for optimization problems", Russian Math. Surveys, **42** (1987) 43–54.

[8] W. M. McEneaney, "Elimination of troublesome disturbances with application to representation results for H_∞ control DPEs", Proc. Seventh International Symposium on Dynamic Games and Applications, Shonan Village (1996), 662–671.

[9] W.M. McEneaney, "A Uniqueness result for the Isaacs equation corresponding to nonlinear H_∞ control", Math. Controls, Signals and Systems (to appear).

[10] R.T. Rockafellar and R.J. Wets, **Variational Analysis**, Springer–Verlag (1997).

[11] P. Soravia, "H_∞ control of nonlinear systems: differential games and viscosity solutions", SIAM J. Control and Optim., **34** (1996), 1071–1097.

[12] A.J. van der Schaft, "L_2–Gain analysis of nonlinear systems and nonlinear state feedback H_∞ control", IEEE Trans. Auto. Control, **37** (1992) 770–784.