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A MAX–PLUS BASED ALGORITHM FOR AN HJB EQUATION OF NONLINEAR FILTERING*

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Abstract. The Hamilton–Jacobi–Bellman (HJB) equation associated with the Robust/H_∞ Filter (as well as the Mortensen Filter) is considered. These filters employ a model where the disturbances have finite power. The HJB equation for the filter information state is a first–order equation with a term which is quadratic in the gradient. Yet the solution operator is linear in the max–plus algebra. This property is exploited by the development of a numerical algorithm where the effect of the solution operator on a set of basis functions is computed off–line. The pre–computed solutions are stored as vectors of coefficients of the basis functions. These coefficients are then used directly in the real–time computations.

Key words. nonlinear filtering, robust, H_∞ , max–plus algebra, nonlinear HJB equations

AMS subject classifications. 3595, 49L25, 93B36, 93B99, 93C10, 93C90, 93E11

1. Introduction. This paper is concerned with an algorithm for construction of viscosity solutions of a certain HJB equation. The solution of this partial differential equation (PDE) is a necessary step in the application of certain filtering techniques (such as Robust Filtering) to nonlinear systems. This will be discussed further below. First, we note that the PDE is first–order but nonlinear – possessing a term which is quadratic in the gradient. However, in the max–plus algebra, the solution operator is linear. The algorithm takes advantage of this linearity by using a max–plus basis representation of the solution. Propagation of the solution is reduced to a max–plus matrix multiplication on the coefficients of the basis representation.

We consider the problem of estimating the state of some system evolving continuously in time. The state space is $\Re^n$. The problem of estimating the current state based on knowledge (or lack thereof) of the initial state and the measurements up to the current moment is the filtering problem. One approach to the problem is to model the system via stochastic differential equations. This leads to the stochastic nonlinear filtering problem (we assume that the extended linear Kalman Filter is not appropriate). The key step becomes the solution of a second–order stochastic partial differential–integral equation - the Kushner equation [19], or a linear, second–order stochastic partial differential equation (SPDE) - the Zakai equation [6]. Alternatively, this can be reformulated via pathwise filtering, leading to a deterministic PDE parameterized by the measurement process [6], [11], [28].

One particular approach to the solution of the Zakai SPDE is the splitting method (see, for instance, [21]). If the measurements occur at discrete times, this approach takes advantage of the linearity of the solution operator via a basis function representation of the solution of the SPDE. In the continuous–time measurements case, one

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takes a limit [21], [16].

An alternate approach is taken by Robust/H_∞ filters ([24], [9], [5]) and the Mortensen filter [29]. In this case, the noise in the system dynamics is modeled by an L^2 process which is unknown a priori, but to which we do not associate a probability measure. The measurement process is also affected by an a priori unknown disturbance. In contrast to the Kalman Filter approach, for nonlinear systems this problem necessitates solution of a *first–order* PDE. (Note that in the formulations considered here this is a a finite–dimensional PDE.) This first–order nature has certain implications. The foremost is that one has a "range–of–dependence" property: a change in initial conditions at one point does not instantaneously affect the solution everywhere in $\Re^n$; the solution propagates with finite speed. This has certain numerical advantages, the most obvious of which is the applicability of generalized characteristic methods [24], [27]. Secondly, the solution, even in non–degenerate cases, is likely to be nonsmooth leading to the use of viscosity solution definitions of solution. However, this first–order PDE is not linear; it possesses a quadratic term in the gradient.

In the max–plus algebra (which may more correctly be termed a commutative semifield), the "addition operation" is defined by $a \oplus b = \max\{a, b\}$, and the "multiplication operation" is defined by $a \otimes b = a + b$. Interestingly, the solution operator for the PDE associated with the Robust/H_∞ Filter is linear in the max–plus algebra. As noted above, the algorithm described herein takes advantage of this linearity and a max–plus basis representation of the solution. This approach is analogous to the splitting method described above for the Kalman Filter. In fact, that method provided, by analogy, an early influence on this work.

Both the case of discrete–time measurements and the case of continuous–time measurements are considered. In the discrete–time measurements case, the solution of the PDE between measurement times (2.14b) is of the form discussed just above, and the above approach is used, leading to the algorithm discussed in Section 6. In the case of continuous–time measurements, a proof of convergence of this algorithm as the measurement time–step goes to zero is presented in Section 7.

Section 2 reviews the Robust/H_∞ Filter (and Mortensen Filter), and provides motivation for the solution of the relevant HJB equations. In Section 3, the max–plus algebra is described, and the max–plus linearity of the solution operator is proved. The basis representation will be obtained in the class of continuous, semiconvex functions. Proofs of the semiconvexity of the solution of the PDE (under two sets of assumptions) are given in Section 4. A semiconvex duality–based representation of the solution is also included there. In Section 5, the semiconvex duality representation is used to obtain the basis representation, and certain properties of this representation are also presented. Sections 6 and 7 are as discussed above.

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2. Review of the Robust Filter. In this section, the Robust Filter (as well as the Mortensen Filter) are briefly reviewed. For a fuller discussion, see [24], [9], [10], [29].

Let $x(t)$ denote the state to be estimated at time $t \geq 0$, with $x(t) \in \Re^n$. The state dynamics are

$$\frac{dx}{dt} = f(x) + \sigma(x)w, \tag{2.1}$$

where $w \in L_2^{loc}([0, \infty); \Re^m)$ is the disturbance in the dynamics, and σ is an $n \times m$ matrix–valued function. First we will describe the case of continuous–time measurements. Let the observations, taking values in $\Re^k$, be given by

$$y(t) = h(x(t)) + \rho(x(t))v(t) \tag{2.2}$$

where $v \in L_2^{loc}([0,\infty);\Re^l)$ is the observation disturbance, ρ is a $k \times l$ matrix–valued function, and $k \leq l$. Let $\phi(x_0)$ be a measure of our uncertainty about the initial state $x(0) = x_0$.

We will assume that

$$\left.\begin{array}{l} f,\ \sigma,\ h \text{ and } \rho \text{ are } C^2 \\ f,\ \sigma \text{ and } h \text{ are globally Lipschitz in } x. \end{array}\right. \tag{$A1$}$$

The differentiability assumptions are required in certain calculus of variations arguments. We assume that there exists $M < \infty$ such that

$$|\sigma(x)| \leq M \qquad \forall\, x \in \Re^n \tag{$A2$}$$

and define $a \doteq \sigma\sigma^T$. We will also assume that

$$\text{Range}(\rho(x)) = \Re^k \qquad \forall\, x \in \Re^n \tag{$A3a$}$$

which guarantees that for any y, x there exists some v satisfying (2.2).

Finally, we define ρ^{-1} by

$$\rho^{-1}(x)b = \text{argmin}\{|v| : \rho(x)v = b\}. \tag{2.3}$$

Assume that ρ^{-1} is uniformly bounded, that is, there exists $C_\rho < \infty$ such that

$$\left.\begin{array}{l} |\rho^{-1}(x)b| \leq C_\rho|b| \qquad \forall\, x \in \Re^n,\ \ \forall\, b \in \Re^l, \\ \rho^{-1} \in C^2. \end{array}\right. \tag{$A3b$}$$

Note also that these assumptions imply that if we view the integral version of (2.1)

$$x_T \doteq x(T) = x_0 + \int_0^T f(x(t)) + \sigma(x(t))w(t)\, dt \tag{2.4}$$

as a mapping from x_0 to x_T then this mapping is one–to–one and onto for any $w \in \mathrm{L}_2$.

Let $\phi(x_0)$ be a measure of our uncertainty about the initial state x_0, and suppose there exists $D_1 < \infty$ such that

$$\left.\begin{array}{l} \phi(x) \leq D_1 \qquad \forall\, x \in \Re^n, \\ \phi \text{ is locally lipschitz} \end{array}\right. \tag{$A4$}$$

It should be noted here that the algorithm described in this paper will produce the solution of the HJB equation corresponding to the Robust/H_∞ and Mortensen Filters. The algorithm could also be used to solve similar HJB equations arising from different applications. If it is not obvious that the desired solution is the same as the solution for the Robust Filter, then one would want a result indicating uniqueness of the (viscosity) solution of the HJB equation in some class. This uniqueness is known under the additional assumption that ϕ satisfies a quadratic growth assumption (references are given below). Since this uniqueness is not required for the specific result here, we did not include a quadratic growth assumption on ϕ in the above.

Suppose we wish to estimate the state at time T. Consider a cost criterion of the form

$$J(T, x_T, w(\cdot)) = \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2 + |v(t)|^2\,dt \tag{2.5a}$$

$$= \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2 + |\rho^{-1}(x(t))[y(t) - h(x(t))]|^2\,dt \tag{2.5b}$$

where x_0 is given by (2.4) for any particular w. The information state is given by

$$P(T, x_T) = \sup_{w \in L_2} J(T, x_T, w). \tag{2.6}$$

We now briefly discuss the extraction of filter estimates from the information state. After this, we will return to a detailed discussion of aspects of (and representations for) the above information state. Further details may be found in the references included below. Readers who are interested mainly in the computational algorithm may choose to skip this discussion.

Mortensen [29] developed an estimator based on this information state in the 60's. Mortensen's estimate is $\widehat{x}_T \in \mathrm{argmax}_x P(T, x)$ (the existence of which follows easily under reasonable assumptions [24], [9]).

For a robust filter, one is interested in a state estimate for which one has a bound on the effect of the disturbances on our estimate error by the product of some constant and the disturbance energy (where one hopes that this constant is small). In particular, one would like an estimate, $\hat{e}_T$, such that for some $\gamma^2 < \infty$ [24],

$$|x(T) - \hat{e}_T|^2 \le \gamma^2 \left[-\phi(x_0) + \frac{1}{2}\int_0^T [|w_t|^2 + |v_t|^2]\,dt \right] \tag{2.7}$$

for all $x_0, w_\cdot, v_\cdot$ in the case of continuous–time measurements with an analogous inequality for discrete–time measurements. (Of course this would require an additional assumption that $\phi(x) \le -k|x - \overline{x}|^2$ for some $k > 0$ and $\overline{x} \in \Re^n$, see [24].) Under reasonable assumptions (see [24]), there exists $\gamma^* < \infty$ such that for all $\gamma \ge \gamma^*$, such a robust estimator is given by

$$\hat{e}_T \doteq \mathrm{argmin}_e \max_x \left[|x - e|^2 + \gamma^2 P(T, x) \right]. \tag{2.8}$$

It should perhaps be noted that there will generally be multiple estimates that yield disturbance attenuation (2.7), and in fact, under certain conditions, the Mortensen estimator is a robust estimator in this sense [9]. We also note that $\hat{e}_T$ (given by (2.8)) is the risk–averse limit of a risk–sensitive stochastic filter [10]. Some readers may note that the above information state differs from some which include an integral running cost term. Such an information state is particularly suited to the problem of H_∞ control under partial information as opposed the problem of robust state estimation considered here; see [1], [17].

Now we turn to the promised further discussion of the information state, P, itself. It will be convenient to note that:

LEMMA 2.1. *Let $T, R < \infty$, $|x_T| \le R$. There exists M_1 (depending on T, R) such that for $\epsilon \le 1$, ϵ–optimal w for problem (2.6) satisfy $\|w\|^2_{L_2[0,T]} \le M_1$ Further, there exists an optimal w^* and M_2 (depending on T, R) such that $|w^*(t)| \le M_2$ for all $t \in [0, T]$.*

Proof. The first assertion follows by the standard technique of comparison with $w^0 \equiv 0$. Let $x^0(\cdot)$ be the solution of (2.1) corresponding to w^0 and $x^0(T) = x_T$. By (A1), there exist $K, K_h < \infty$ such that $|f(x)| \le K(1 + |x|)$ and $|h(x)| \le K_h(1 + |x|)$. Then $\frac{d}{dt}|x^0|^2 \le K(3|x^0|^2 + 1)$, and by Gronwall's inequality, one obtains $|x^0(t)|^2 \le R_1^2 \doteq (R^2 + KT)(1 + 3KTe^{3KT})$ for all $t \in [0, T]$.

Let $\mu(\rho) = \min\{\phi(x) : |x| \le \rho\}$, and note that $\mu(\rho) > -\infty$ for all $\rho < \infty$ by (A4). Then

$$J(T, x_T, w^0(\cdot)) \ge \mu(R_1) - \frac{C_\rho^2}{2}[2K_h^2(1 + R_1^2) + \|y\|^2_{L_2[0,T]}] \doteq \overline{M}_1.$$

Consequently, for ϵ–optimal w^ϵ, $-\frac{1}{2}\int_0^T |w^\epsilon|^2\,dt \ge \overline{M}_1 - \epsilon$ which yields the first assertion. The second assertion then follows by the proof of [9, Lemma 2.1(a)]. □

The following theorem and remark are easily obtained in the case where $y(\cdot)$ is continuous, and we outline the proof in that case. They are not directly needed in the sequel, and so we do not prove them in the case where $y \in L_2 \setminus C$. (Of course, they could be needed in an adaption of this algorithm to the solution of PDE's not necessarily arising from this application.) Extensions of viscosity solutions to cases where the time–dependency of the Hamiltonian is only measurable are well–known but more technical (see, for instance, [4], [14], [22], [33]).

THEOREM 2.2. *P is a continuous viscosity solution of the HJB equation*

$$\begin{aligned} &0 = P_T + f^T(x)\nabla_x P - \frac{1}{2}\nabla_x P^T a(x)\nabla_x P + \frac{1}{2}|\rho^{-1}(x)(y(T) - h(x))|^2 \\ &P(0, x) = \phi(x). \end{aligned} \tag{2.9}$$

Proof. Fix $R < \infty$, and let $|x_T| \le R$. By Lemma 2.1, the optimal trajectory $x^*(\cdot)$ terminating at x_T satisfies $|x^*(t)| \le R_1$ for some $R_1 < \infty$ (depending on T, R) for any $x_t \in B_R \doteq \{x \in \Re^n : |x| \le R\}$. Let

$$\widetilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } |x| \le R_1 \\ \phi(R_1 x/|x|) & \text{if } |x| > R_1. \end{cases}$$

Let $\widetilde{P}$ be the value given by (2.6) with $\widetilde{\phi}$ replacing ϕ. Then, by standard results (for instance, [26]), $\widetilde{P}$ is a viscosity solution of (2.9) with $\widetilde{\phi}$ replacing ϕ. Since $\widetilde{P} = P$ on $[0, T] \times B_R$ and $\widetilde{\phi} = \phi$ on B_{R_1}, P is a viscosity solution of (2.9) on $[0, T] \times B_R$. Letting $R \to \infty$ one obtains the result. □

Remark 2.3. In the case where $\phi(x) \ge -D_2(1 + |x|^2)$ for some $D_2 < \infty$ and σ is constant, uniqueness among the class of continuous, quadratically growing viscosity solutions was obtained in [25]. Recent results of DaLio–McEneaney [7], Bardi–DaLio [2] and Ishii [15] allow the removal of the constant σ assumption.

Let us also develop the information state in the discrete–time measurement case.

In the discrete–time measurement case, the dynamics (2.1) are unchanged, but the measurement model is as follows. Suppose that at each measurement time, $t_j > 0$, we receive an observation which is modeled as

$$y_j = h(x(t_j)) + \rho(x(t_j))v_j. \tag{2.10}$$

Let the number of measurements in time $[0, T]$ be N. One may adapt the cost criterion (2.5) and information state (2.6) in the following manner. Let the cost criterion be

$$J(T, x_T, w(\cdot)) = \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2\,dt - \frac{\delta}{2}\sum_{j=1}^N |v_j|^2 \tag{2.11a}$$

$$= \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2\,dt - \frac{\delta}{2}\sum_{j=1}^N |\rho^{-1}(x(t_j))[y_j - h(x(t_j))]|^2 \tag{2.11b}$$

where δ is the time interval between measurements, and let the value again be given by

$$P(T, x_T) = \sup_{w\in L_2} J(T, x_T, w). \tag{2.12}$$

Due to the discrete nature of the measurements, it is helpful recall the form of the dynamic programming principle for such a system. Let $P(t_j^+, x)$ be the value at time t_j just after measurement j, and $P(t_j^-, x)$ be the value at time t_j just before the measurement. If T is not a measurement time, and $t \in (t_{k-1}, t_k)$, we have

$$\begin{aligned} P(T, x) = \sup_{w\in L_2([t,T];\Re^m)} \Big\{ &P(t, x(t)) - \frac{1}{2}\int_t^T |w(r)|^2\,dr \\ &- \frac{\delta}{2}\sum_{j=k}^N |\rho^{-1}(x(t_j))[y_j - h(x(t_j))]|^2 \Big\} \end{aligned} \tag{2.13a}$$

and when T occurs at the time of measurement j, we have

$$P(T^+, x) = P(T^-, x) - \frac{\delta}{2}|\rho^{-1}(x)[y_j - h(x)]|^2. \tag{2.13b}$$

The corresponding dynamic programming equations are

$$P(0, x) = \phi(x) \qquad t = 0 \tag{2.14a}$$

$$\begin{aligned} 0 &= P_t - \sup_{w\in\Re^m}\Big\{ -[f(x) + \sigma(x)w]^T\nabla_x P - \frac{1}{2}|w|^2 \Big\} && t \in (t_j, t_{j+1}) \\ &= P_t + f^T(x)\nabla_x P - \frac{1}{2}(\nabla_x P)^T a(x)\nabla_x P && t \in (t_j, t_{j+1}) \end{aligned} \tag{2.14b}$$

$$P(t_j^+, x) = P(t_j^-, x) - \frac{\delta}{2}|\rho^{-1}(x)[y_j - h(x)]|^2 \qquad t = t_j \text{ for some } j. \tag{2.14c}$$

As in the continuous–time measurements case, we have the following [24]. (Since y does not appear on the right hand side of (2.14b), there is no difficulty with discontinuous y in this case.)

THEOREM 2.4. *The information state, P, given by (2.13) is continuous, and is a viscosity solution of (2.14) between measurement times.*

In the case where ϕ satisfies a quadratic growth condition, uniqueness follows in a similar manner as that discussed in Remark 2.3.

As in the continuous–time measurement case, one obtains the robust estimator from (2.8), and an attenuation inequality similar to (2.7) follows (with summation of v_j replacing the integral of v_t).

Finally, see [1], [5] and [18] for related work.

3. Max–Plus Linearity. Recall the definition of max–plus addition, $\oplus$, and multiplication, $\otimes$, for elements of $\Re$ as

$$\begin{aligned} a \oplus b &\doteq \max\{a, b\} \\ a \otimes b &\doteq a + b. \end{aligned} \tag{3.1}$$

Define these operations for $-\infty$ in the obvious way. It is well–known that $(\Re \cup \{-\infty\}, \oplus, \otimes)$ is a commutative semifield which is referred to as the max–plus algebra. See [3], [20], [13] for a fuller discussion. (We remark that this can be extended to a field by appending $\Re^-$, loosely speaking a mirror image of $\Re$ on the other side of $-\infty$, to the set so that one has additive inverses. With the proper extension of $\oplus$, $\otimes$ on these new elements, one obtains a field $(\Re \cup \{-\infty\} \cup \Re^-, \oplus, \otimes)$. However, for the Robust/H_∞ and Mortensen Filter applications, we are interested only in real–valued quantities, and so nothing is gained by this extension. Consequently, we will not pursue this here.) The authors have recently learned that Theorem 3.1 was previously reported in 1987, [23].

As noted earlier, we are interested only in the discrete–time measurements case in this section; the continuous–time measurements case will be discussed in Section 7. Let $\mathcal{S}_T$ be the solution operator for HJB equation (2.14b). Specifically, given initial condition $P(t_0, \cdot)$, the solution to (2.14b) at time $t_0 + T$ is given by

$$P(t_0 + T, x) = \mathcal{S}_T[P(t_0, \cdot)](x). \tag{3.2}$$

From the dynamic programming principle (2.13a) (noting that there are no measurements in $(t_0, t_0 + T)$ in order for (2.14b) to be applicable), one has

$$\begin{aligned} \mathcal{S}_T[P(t_0, \cdot)](x) &= P(t_0 + T, x) \\ &= \sup_{w \in L_2([t_0, t_0+T]; \Re^m)} \left\{ P(t_0, x(t_0)) - \frac{1}{2} \int_{t_0}^{t_0+T} |w(r)|^2 \, dr \right\} \end{aligned} \tag{3.3}$$

where $x(t_0)$ is given by (2.1) with $x(t_0 + T) = x$.

We now prove that $\mathcal{S}_T$ is linear in the max–plus algebra. Let $c \in \Re$ and let $\phi, \psi : \Re^n \to \Re$ be locally Lipschitz and bounded from above. By (3.3), one easily sees that for any $x \in \Re^n$

$$\mathcal{S}_T[c \otimes \phi](x) = \mathcal{S}_T[c + \phi](x) = c + \mathcal{S}_T[\phi] = c \otimes \mathcal{S}_T[\phi](x). \tag{3.4}$$

Also, for any $x \in \Re^n$,

$$\begin{aligned} \mathcal{S}_T[\phi \oplus \psi](x) &= \sup_{w \in L_2([0,T]; \Re^m)} \left\{ \max[\phi(x(0)), \psi(x(0))] - \frac{1}{2} \int_0^T |w(r)|^2 \, dr \right\} \quad (3.5) \\ &\geq \sup_{w \in L_2([0,T]; \Re^m)} \left\{ \phi(x(0)) - \frac{1}{2} \int_0^T |w(r)|^2 \, dr \right\} \\ &= \mathcal{S}_T[\phi](x). \end{aligned}$$

By symmetry, one obtains an analogous inequality for ψ. Consequently,

$$\mathcal{S}_T[\phi \oplus \psi](x) \geq \max\{\mathcal{S}_T[\phi](x), \mathcal{S}_T[\psi](x)\} = \mathcal{S}_T[\phi](x) \oplus \mathcal{S}_T[\psi](x). \tag{3.6}$$

Now let w^* be optimal in (3.5). (The existence of the optimizer is discussed in Section 2.) Then

$$\mathcal{S}_T[\phi \oplus \psi](x) = \max[\phi(x(0)), \psi(x(0))] - \frac{1}{2}\int_0^T |w^*(r)|^2\,dr,$$

and, in the case where $\phi(x(0)) \geq \psi(x(0))$

$$\begin{aligned} &= \phi(x(0)) - \frac{1}{2}\int_0^T |w^*(r)|^2\,dr \\ &\leq \mathcal{S}_T[\phi](x). \end{aligned}$$

Considering also the case where $\phi(x(0)) < \psi(x(0))$ one obtains

$$\mathcal{S}_T[\phi \oplus \psi](x) \leq \mathcal{S}_T[\phi](x) \oplus \mathcal{S}_T[\psi](x).$$

Combining this with (3.6) yields

$$\mathcal{S}_T[\phi \oplus \psi](x) = \mathcal{S}_T[\phi](x) \oplus \mathcal{S}_T[\psi](x). \tag{3.7}$$

By (3.4) and (3.7), one has

THEOREM 3.1. *The solution operator for (2.14b) is linear in the max–plus algebra.*

4. Semi–Convexity. A function ψ is called *semiconvex* if for every $R < \infty$, there exists C_R such that

$$\widehat{\psi}(x) \doteq \psi(x) + \frac{C_R}{2}|x|^2 \tag{4.1}$$

is convex on the ball $B_R \doteq \{x \in \Re^n : |x| \leq R\}$. The infimum over such C_R will be known as the semiconvexity constant for ψ over B_R. Note that any real–valued semiconvex function is locally Lipschitz. In fact, if $\widehat{\psi}$ is convex on the ball B_R, then $\widehat{\psi}$ is Lipschitz on $B_{R/2}$ with a Lipschitz constant depending only on $\max_{|x|\leq R}\widehat{\psi}(x) - \min_{|x|\leq R}\widehat{\psi}(x)$ (see [8, p. 111]).

In this section, it will be shown that if either the initial information state is semiconvex, or if σ is nondegenerate, then the information state will be semiconvex for all $T > 0$. Next, the convex duality representation for convex functions is adapted to the semiconvex case. Then, since the information state is semiconvex, one sees that this provides a semiconvex duality representation for the information state. This representation will be used in Section 5 to obtain the basis representation.

Recall from Section 3 that the solution operator (in the absence of observations) is defined as

$$\begin{aligned} \mathcal{S}_T[\phi](x) &= \sup_{w\in L_2([0,T];\Re^m)} J^0(T, x; w(\cdot)), \\ J^0(T, x; w(\cdot)) &= \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2\,dt \end{aligned} \tag{4.2}$$

where $x_0 = x(0)$ given $x(T) = x$ and dynamics (2.1).

THEOREM 4.1. *If ϕ is semiconvex, then $\mathcal{S}_T[\phi]$ is semiconvex for each $T > 0$. In fact, there exists $\Gamma_R(T)$ such that for $0 \leq \tau \leq T$, $R > 0$, $\mathcal{S}_\tau[\phi](x) + \Gamma_R(T)|x|^2$ is convex on the ball B_R.*

Proof. Let

$$\begin{aligned}\widetilde{J}(\tau, x; w(\cdot)) &= J^0(\tau, x; w(\cdot)) + \Gamma|x|^2,\\ J^0(\tau, x, w(\cdot)) &= \phi(x_0) - \frac{1}{2}\int_0^\tau |w(t)|^2\,dt,\end{aligned}$$

where $\Gamma = \Gamma_R(T)$ is to be chosen suitably. Since the supremum of any family of convex functions is a convex function, it suffices to show that $\widetilde{J}(\tau, \cdot; w(\cdot))$ is convex on B_R for each $w(\cdot)$.

By smoothing via convolution with approximations to the identity, we may assume that ϕ is smooth. Then semiconvexity of ϕ is equivalent to $v^T\phi_{xx}(x)v \geq -C_R|v|^2$ for all $v \in \Re^n$, when $|x| \leq R$. To show convexity of $\widetilde{J}$ it suffices to show that

$$v^T J^0_{xx}(\tau, x; w(\cdot))v \geq -2\Gamma \tag{4.3}$$

whenever $|x| \leq R$, $|v| = 1$ and $||w||^2 \leq M_1(T, R)$ (see Lemma 2.1). let $\zeta^1(t), \zeta^2(t)$ denote the first and second derivatives of $x(t)$ in the direction v (considered as a function of $x = x(\tau)$). Then

$$v^T J^0_{xx}(\tau, x; w(\cdot))v = \phi_x(x_0)\cdot\zeta^2(0) + (\zeta^1(0))^T\phi_{xx}(x_0)\zeta^1(0).$$

By the "range–of–dependence" property (which follows directly from Lemma 2.1), $|x_0| \leq R_1$ for suitable R_1. If we can also obtain bounds on $|\zeta^1(0)|, |\zeta^2(0)|$ depending only on T, R, then we will have (4.3) for suitable $\Gamma = \Gamma_R(T)$ which will complete the proof. (In fact, one could explicitly compute $\Gamma_R(T)$ from these bounds if so desired.) Bounds on $\zeta^1(0), \zeta^2(0)$ are obtained as follows. Let

$$\begin{aligned}\dot{x}^h &= f(x^h) + \sigma(x^h)w\\ x^h(\tau) &= x + hv.\end{aligned}$$

Then with $\zeta^1(t) \doteq \frac{d}{dh}x^h(t)\big|_{h=0}$, one has

$$\begin{aligned}\dot{\zeta}^1 &= f_x(x^0)\zeta^1 + [\sigma_x(x^0)\zeta^1]w\\ \zeta^1(\tau) &= v.\end{aligned}$$

By Lemma 2.1, one has bounds on x^0, w, and then by (A1), this last equation yields a bound on $|\zeta^1(0)|$.

Letting $\zeta^{1,h}(t) = \frac{d}{dh}x^h(t)$ for h not necessarily zero, one has $\zeta^2(t) = \frac{d}{dh}\zeta^{1,h}(t)\big|_{h=0}$. Then

$$\begin{aligned}\dot{\zeta}^2 &= f_{xx}(x^0)\zeta^1\cdot\zeta^1 + f_x(x^0)\zeta^2 + [\sigma_{xx}(x^0)\zeta^1\cdot\zeta^1 + \sigma_x(x^0)\zeta^2]w\\ \zeta^2(\tau) &= 0.\end{aligned}$$

Again, employing bounds from Lemma 2.1 and (A1) with this equation leads to a bound for $|\zeta^2(0)|$. □

4.1. Fundamental Solutions. If we assume nondegeneracy of the disturbance term in (2.1), then $\mathcal{S}_T[\phi]$ is semiconvex for $T > 0$ without assuming semiconvexity of

ϕ. (The authors wish to thank P. Dupuis for pointing this out.) By nondegeneracy we mean that $m \geq n$ and

$$\sigma^{-1}(x) \text{ is bounded uniformly in } x, \text{ and also assume } \sigma^{-1} \in C^2 \tag{A6}$$

where we define the inverse by

$$\sigma^{-1}(x)b \doteq \operatorname{argmin}\{|a| : \sigma(x)a = b\}$$

for any $b \in \Re^n$. Note that assumption (A6) is only required if the initial information function, ϕ, is not semiconvex.

Let us introduce the following function, $V(T, x_0, x_T)$, which we call a *fundamental solution* of the PDE (2.14b). For $x_0, x_T \in \Re^n$, $T > 0$, let

$$V(T, x_0, x_T) = \sup_{w \in L_2([0,T];\Re^m)} \left\{ -\frac{1}{2} \int_0^T |w(t)|^2 \, dt \, : \, x(0) = x, \, x(T) = x_T \right\} \tag{4.4}$$

where $x(\cdot)$ satisfies (2.1). From (4.2),

$$\mathcal{S}_T[\phi](x) = \sup_{x_0 \in \Re^n} \{\phi(x_0) + V(T, x_0, x)\}. \tag{4.5}$$

We can rewrite V in terms of the following calculus of variations problem with fixed end conditions. Let

$$L(x, \dot{x}) = \frac{1}{2} |\sigma^{-1}(x)(\dot{x} - f(x))|^2,$$
$$I(T, x_0, x_T; x(\cdot)) = -\int_0^T L(x(t), \dot{x}(t)) \, dt.$$

Then

$$V(T, x_0, x_T) = \sup\{I(T, x_0, x_T; x(\cdot)) : x(0) = x_0, \, x(T) = x_T\}$$

with $x(\cdot)$ given by (2.1). Equivalence of (4.4) and this calculus of variations version follows easily by noting that for each path, $x(\cdot)$, there is a corresponding unique minimal–norm $w(\cdot)$ given in feedback form with this definition of σ^{-1}.

LEMMA 4.2. *Assume (A6). Then $V(T, x_0, \cdot)$ is semiconvex. In fact, given $0 < T_1 < T_2 < \infty$, R, R_0, there exists Δ (depending on T_1, T_2, R, R_0) such that $V(T, x_0, x) + \Delta |x|^2$ is convex on the ball $B_R \doteq \{x \in \Re^n : |x| \leq R\}$, for any $T \in [T_1, T_2]$ and any $|x_0| \leq R_0$.*

Proof. Given x_0, x, there exists $x^*(\cdot)$ such that $x^*(0) = x_0$, $x^*(T) = x$ and

$$V(T, x_0, x) = I(T, x_0, x; x^*(\cdot)).$$

Now suppose $0 < T_1 \leq T \leq T_2$, $|x_0| \leq R_0$ and $|x| \leq R$. Some "range–of–dependence" estimates will be obtained. For comparison with the optimal trajectory, let $\overline{x}(t) \doteq x_0 + (t/T)(x - x_0)$. Then there exists $C < \infty$ (depending on R_0, R, T_1) such that

$$\left|\dot{\overline{x}} - f(\overline{x}(t))\right| \leq C \qquad \forall\, t \in [0, T],$$

and then by (A6), there exists $M_i < \infty$ such that

$$-L(\overline{x},\dot{\overline{x}}) \geq -\frac{1}{2}M_i^2C^2.$$

Then by the optimality of x^*,

$$I(T,x_0,x;x^*(\cdot)) \geq -\frac{1}{2}M_i^2C^2T,$$

and using (A2), this yields

$$\int_0^T |\dot{x}^* - f(x^*(t))|^2\,dt \leq M^2M_i^2C^2T. \tag{4.6}$$

Then note that by (A1), there exists $K < \infty$ such that

$$\begin{aligned}\frac{d}{dt}|x^*|^2 &= 2(x^*)^T(f(x^*) + (\dot{x}^* - f(x^*)))\\ &\leq 3K|x^*|^2 + K + 2|x^*||\dot{x}^* - f(x^*)|\\ &\leq (3K+1)|x^*|^2 + K + |\dot{x}^* - f(x^*)|^2.\end{aligned}$$

Employing Gronwall's inequality and (4.6), one finds that there exist $R_1, \overline{M} < \infty$ (depending on R_0, R, T_1, T_2) such that $|x^*(t)| \leq R_1$ for all $t \in [0,T]$ and $\|\dot{x}^*(\cdot)\|_{L_2[0,T]} \leq \overline{M}$.

Next, consider any direction $v \in \Re^n$ ($|v| = 1$) and scalar $h \in [-1,1]$. Let

$$x^h(t) \doteq x^*(t) + \frac{t}{T}hv.$$

Then $x^h(0) = x_0$ and $x^h(T) = x + hv$. Moreover,

$$\frac{d^2}{dh^2}I(T,x_0,x+hv;x^h(\cdot)) = v^T\Lambda^h v,$$

where

$$\Lambda^h \doteq \frac{1}{T^2}\int_0^T \left[t^2L_{xx}^h + 2tL_{x\dot{x}}^h + L_{\dot{x},\dot{x}}^h\right]\,dt,$$

$L_{xx}^h = L_{xx}(x^h(t),\dot{x}^h(t))$, and the other derivatives defined similarly. Since $|x^h(t)|$ and $\|\dot{x}^h(\cdot)\|_{L_2[0,T]}$ are uniformly bounded,

$$\frac{d^2}{dh^2}I(T,x_0,x+hv;x^h(\cdot)) \geq -2\Delta$$

for some constant Δ depending on T_1, T_2, R, R_0. Then $I(T,x_0,x+hv;x^h(\cdot)) + \Delta h^2$ is a convex function of h for $|h| \leq 1$. Then

$$\begin{aligned}&V(T,x_0,x+hv) + V(T,x_0,x-hv) - 2V(T,x_0,x)\\ &\geq I(T,x_0,x+hv;x^h(\cdot)) + I(T,x_0,x-hv;x^{-h}(\cdot)) - 2I(T,x_0,x;x^*(\cdot))\\ &\geq -2\Delta h^2.\end{aligned}$$

This implies that $V(T,x_0,x) + \Delta|x|^2$ is convex on the ball $\{x \in \Re^n : |x| \leq R\}$. ∎

THEOREM 4.3. *Assume (A6). For any $\phi \le 0$, $T > 0$, $\mathcal{S}_T[\phi]$ is semiconvex.*

Proof. If $|x_T| \le R$, by the "range–of–dependence" property, it suffices to consider those $x(\cdot)$ such that $|x(t)| \le R_1$, where $x_0 = x(0)$. If $x = x_T$, then by (4.5)

$$\mathcal{S}_T[\phi](x) = \sup_{x_0 \in \Re^n} \{\phi(x_0) + V(T, x_0, x)\},$$
$$\mathcal{S}_T[\phi](x) + \Delta|x|^2 = \sup_{x_0 \in \Re^n} \{\phi(x_0) + V(T, x_0, x) + \Delta|x|^2\}$$

with Δ as in Lemma 4.2. Since the supremum of any family of convex functions is convex, $\mathcal{S}_T[\phi](x) + \Delta|x|^2$ is convex on the ball $\{x \in \Re^n : |x| \le R\}$. ☐

Theorem 4.1 and Theorem 4.3 imply (under different assumptions) that the information state (in the absence of measurement updates) will be semiconvex. To complete this discussion, one also needs to show that the measurement updates (2.14c) maintain semiconvexity.

LEMMA 4.4. *Suppose ψ is semiconvex. Then $\psi(x) - \frac{1}{2}|\rho^{-1}(x)[y - h(x)]|^2$ is semiconvex for any $y \in \Re^k$.*

Proof. Let $G(x) \doteq |\rho^{-1}(x)[y - h(x)]|^2$. Fix $R < \infty$. Then, since ψ is semiconvex, there exists $C_R < \infty$ such that $\psi(x) + \frac{C_R}{2}|x|^2$ is convex. But by (A1) and (A3b), there exists $C_{R,y} < \infty$ such that $|G_{xx}(x)| \le C_{R,y}$ for all $x \in B_R$. Consequently $\psi(x) - \frac{1}{2}|\rho^{-1}(x)[y - h(x)]|^2 + \frac{C_R + C_{R,y}}{2}|x|^2$ is convex. ☐

By Theorem 2.4, Lemma 4.4 and either Theorem 4.1 or Theorem 4.3, we know that, under the given assumptions, the information state, P, will be continuous and semiconvex. Consequently, we define

$$\mathcal{P} \doteq \{\psi : \Re^n \to \Re : \psi \text{ is continuous and semiconvex}\}.$$

The algorithm to follow will depend on a max–plus basis representation of any $\psi \in \mathcal{P}$. The starting point for this representation will be the convex dual.

Let $R > 0$ and (as above) $B_R \doteq \{x \in \Re^n : |x| \le R\}$. Let $\widehat{\psi} : B_R \to \Re$ be convex and Lipschitz over B_R with Lipschitz constant $L(R)$, and let $\widehat{\psi}(x) = +\infty$ for all $x \notin B_R$. The following result is a minor variation of a standard result in Convex Duality (see, for instance, [30], [31]). The slight variation here is that, in this case, one has existence of the maximizers in the statement of the Lemma, which is a result of the restrictive class of functions, $\widehat{\psi}$, being considered here.

LEMMA 4.5 (convex duality representation).

$$\widehat{\psi}(x) = \max_{p \in B_{L(R)}} [p^T x + \overline{a}_p] \qquad \forall\, x \in B_R$$

where

$$\overline{a}_p = -\max_{x \in B_R} [p^T x - \widehat{\psi}(x)],$$

and note that $\overline{a}_p$ may be $-\infty$.

Consider any $\psi \in \mathcal{P}$ over some ball B_R. Let $c > 0$ be a constant such that

$$\widehat{\psi}(x) \doteq \begin{cases} \psi(x) + \frac{c}{2}|x|^2 & \text{if } x \in B_R \\ +\infty & \text{otherwise} \end{cases} \quad \text{is convex.} \tag{4.7}$$

(The existence of c is guaranteed by $\psi \in \mathcal{P}$.) As noted above, by [8, p. 111], $\widehat{\psi}$ is Lipschitz over B_R with some constant $L(R)$. Further, if $|\psi(x)| \leq D_2(1 + |x|^2)$ for some D_2, then it can be shown using the explicit local Lipschitz bound in [8] that $L(R) \leq D_3(1 + R)$ for some D_3. The convex duality representation of $\widehat{\psi}$ in Lemma 4.5 leads easily to the following representation for ψ.

THEOREM 4.6 (semiconvex duality representation).

$$\psi(x) = \max_{\widetilde{x} \in B_{L(R)/c}} [-\frac{c}{2}|x - \widetilde{x}|^2 + a_{\widetilde{x}}] = \max_{\widetilde{x} \in \Re^n} [-\frac{c}{2}|x - \widetilde{x}|^2 + a_{\widetilde{x}}] \qquad \forall\, x \in B_R$$

where

$$a_{\widetilde{x}} = -\max_{x \in B_R} [-\frac{c}{2}|x - \widetilde{x}|^2 - \psi(x)].$$

Proof. For any $x \in B_R$

$$\psi(x) = \widehat{\psi}(x) - \frac{c}{2}|x|^2$$

which by Lemma 4.5

$$\begin{aligned} &= \max_{p \in B_{L(R)}} [p^T x - \frac{c}{2}|x|^2 + \overline{a}_p] \\ &= \max_{p \in B_{L(R)}} [-\frac{c}{2}|x - p/c|^2 + \frac{1}{2c}|p|^2 + \overline{a}_p] \end{aligned}$$

which by letting $\widetilde{x} = p/c$ becomes

$$\begin{aligned} &= \max_{\widetilde{x} \in B_{L(R)/c}} [-\frac{c}{2}|x - \widetilde{x}|^2 + \frac{c}{2}|\widetilde{x}|^2 + \overline{a}_{c\widetilde{x}}] \\ &= \max_{\widetilde{x} \in B_{L(R)/c}} [-\frac{c}{2}|x - \widetilde{x}|^2 + a_{\widetilde{x}}] \end{aligned}$$

where

$$\begin{aligned} a_{\widetilde{x}} &= \frac{c}{2}|\widetilde{x}|^2 + \overline{a}_{c\widetilde{x}} \\ &= -\max_{x \in B_R} [(c\widetilde{x})^T x - \widehat{\psi}(x) - \frac{c}{2}|\widetilde{x}|^2] \end{aligned}$$

which by (4.7)

$$\begin{aligned} &= -\max_{x \in B_R} [(c\widetilde{x})^T x - \psi(x) - \frac{c}{2}|x|^2 - \frac{c}{2}|\widetilde{x}|^2] \\ &= -\max_{x \in B_R} [-\frac{c}{2}|x - \widetilde{x}|^2 - \psi(x)]. \qquad \square \end{aligned}$$

5. Basis Representation. In this section we modify the representation of $\psi \in \mathcal{P}$ given by Theorem 4.6 to obtain a countable basis representation. Again, let $\psi \in \mathcal{P}$ and $R > 0$ with corresponding $\overline{c}$ such that (4.7) holds.

Suppose we have some countable dense subset $\{x_i\}_{i=1}^{\infty} \subseteq \Re^n$. Let $\widehat{c} \in (\overline{c}, \infty)$. Let

$$g_i(x) = -\frac{\widehat{c}}{2}|x - x_i|^2 \tag{5.1a}$$

and

$$a_i = -\max_{x \in B_R}[g_i(x) - \psi(x)]. \tag{5.1b}$$

Let $c \in (\widehat{c}, \infty)$, and

$$\overline{g}_{\widetilde{x}}(x) = -\frac{c}{2}|x - \widetilde{x}|^2.$$

By Theorem 4.6,

$$\psi(x) = \max_{\widetilde{x} \in B_{L(R)/c}} [\overline{g}_{\widetilde{x}}(x) + a_{\widetilde{x}}] \qquad \forall\, x \in B_R$$

where $a_{\widetilde{x}}$ is given in Theorem 4.6. It will next be shown that, in fact,

$$\psi(x) = \sup_i [a_i + g_i(x)] = \bigoplus_{i=1}^{\infty}[a_i \otimes g_i(x)] \qquad \forall\, x \in B_R$$

where $\bigoplus_{i=1}^{\infty}$ indicates max–plus summation. Some additional properties of this basis function representation will also be obtained. Lemma 5.1 will be useful in the two theorems to follow.

LEMMA 5.1. *Given $\widetilde{x} \in B_{L(R)/c}$ and $\epsilon > 0$, there exists i such that*

$$\psi(x) \geq g_i(x) + a_i \geq \overline{g}_{\widetilde{x}}(x) + a_{\widetilde{x}} - \epsilon \qquad \forall\, x \in B_R.$$

Further, for any $\delta > 0$ one may specify $|x_i - \widetilde{x}| \leq \frac{c - \widehat{c}}{c\widehat{c}}(cR + L(R)) + \delta$.

Proof. Let $\widetilde{x}$ and $\epsilon > 0$ be given. By Theorem 4.6, there exists $\overline{x} \in B_R$ such that

$$a_{\widetilde{x}} - \frac{c}{2}|\overline{x} - \widetilde{x}|^2 = \psi(\overline{x}). \tag{5.2}$$

Choose $\widehat{x} \in \Re^n$ such that

$$-\widehat{c}(\overline{x} - \widehat{x}) = -c(\overline{x} - \widetilde{x}) \tag{5.3}$$

(so that the gradient of $-\frac{\widehat{c}}{2}|x - \widehat{x}|^2$ matches that of $\overline{g}_{\widetilde{x}}$ at $\overline{x}$). Note that this implies $|\widehat{x} - \widetilde{x}| \leq \frac{c - \widehat{c}}{c\widehat{c}}(cR + L(R))$. By the density of $\{x_i\}_{i=1}^{\infty}$, given $\delta > 0$, there exists i such that

$$|x_i - \widehat{x}| < \delta. \tag{5.4}$$

Let $\widehat{a}_i$ be such that

$$\psi(\overline{x}) = \widehat{a}_i - \frac{\widehat{c}}{2}|\overline{x} - x_i|^2. \tag{5.5}$$

Define

$$\psi^0(x) = \begin{cases} \psi(x) + \frac{\overline{c}}{2}|x - \overline{x}|^2 & \text{if } x \in B_R \\ +\infty & \text{if } x \notin B_R \end{cases}$$

so that ψ^0 is convex. By the definition of $\overline{x}$,

$$-c(\overline{x} - \widetilde{x})^T \in \partial\psi^0(\overline{x})$$

and so by (5.3)

$$-\widehat{c}(\overline{x} - \widehat{x})^T \in \partial\psi^0(\overline{x})$$

which, by the definition of subgradient and that $\psi^0(\overline{x}) = \psi(\overline{x})$, implies

$$\psi(\overline{x}) - \widehat{c}(\overline{x} - \widehat{x})^T(x - \overline{x}) \le \psi^0(x) \qquad \forall\, x \in \Re^n. \tag{5.6}$$

It will next be shown that the a_i given by (5.1b) is not much smaller than $\widehat{a}_i$ given by (5.5). Define

$$F(x) \doteq \widehat{a}_i - \frac{\widehat{c}}{2}|x - x_i|^2 + \frac{\overline{c}}{2}|x - \overline{x}|^2 - [\psi(\overline{x}) - \widehat{c}(\overline{x} - \widehat{x})^T(x - \overline{x})]. \tag{5.7}$$

The maximum of F occurs at

$$x_0 = \overline{x} + \frac{\widehat{c}}{\widehat{c} - \overline{c}}(x_i - \widehat{x}),$$

and the maximum value is

$$F(x_0) = \frac{\widehat{c}^2}{2(\widehat{c} - \overline{c})}|x_i - \widehat{x}|^2$$

which by (5.4)

$$< \frac{\widehat{c}^2}{2(\widehat{c} - \overline{c})}\delta^2. \tag{5.8}$$

We choose $\delta > 0$ sufficiently small such that

$$\frac{\widehat{c}^2}{2(\widehat{c} - \overline{c})}\delta^2 < \frac{\epsilon}{2}. \tag{5.9}$$

Then, by (5.7), (5.8) and (5.9),

$$\widehat{a}_i - \frac{\widehat{c}}{2}|x - x_i|^2 + \frac{\overline{c}}{2}|x - \overline{x}|^2 < \psi(\overline{x}) - \widehat{c}(\overline{x} - \widehat{x})^T(x - \overline{x}) + \frac{\epsilon}{2} \qquad \forall\, x \in \Re^n,$$

and so by (5.6),

$$\widehat{a}_i - \frac{\widehat{c}}{2}|x - x_i|^2 + \frac{\overline{c}}{2}|x - \overline{x}|^2 < \psi^0(x) + \frac{\epsilon}{2} \qquad \forall\, x \in \Re^n,$$

which, by the definition of ψ^0, implies

$$\widehat{a}_i - \frac{\widehat{c}}{2}|x - x_i|^2 < \psi(x) + \frac{\epsilon}{2} \qquad \forall\, x \in B_R.$$

Consequently, with a_i given by (5.1b), one has

$$a_i > \widehat{a}_i - \frac{\epsilon}{2}. \tag{5.10}$$

Now it will be shown that $a_i - \frac{\widehat{c}}{2}|x-x_i|^2 = a_i + g_i(x)$ is not more that ϵ below $a_{\widetilde{x}} + \overline{g}_{\widetilde{x}}(x)$ (for sufficiently small δ). Define

$$H(x) = a_{\widetilde{x}} - \frac{c}{2}|x-\widetilde{x}|^2 - (a_i - \frac{\widehat{c}}{2}|x-x_i|^2). \tag{5.11}$$

By (5.2)

$$H(\overline{x}) = \psi(\overline{x}) - (a_i - \frac{\widehat{c}}{2}|\overline{x}-x_i|^2)$$

which by (5.10) and (5.5)

$$< \frac{\epsilon}{2}.$$

By (5.11)

$$\frac{dH}{dx}(\overline{x}) = -c(\overline{x}-\widetilde{x})^T + \widehat{c}(\overline{x}-x_i)^T$$

which by (5.3)

$$= -\widehat{c}(\overline{x}-\widehat{x})^T + \widehat{c}(\overline{x}-x_i)^T$$

so that

$$\left|\frac{dH}{dx}(\overline{x})\right| \leq \widehat{c}|\widehat{x}-x_i|.$$

Also,

$$\frac{d^2H}{dx^2}(x) = (\widehat{c}-c)I < 0$$

(where by the last inequality, we mean that $(\widehat{c}-c)I$ is negative definite). Consequently,

$$H(x) < \frac{\epsilon}{2} + \widehat{c}|\widehat{x}-x_i||x-\overline{x}| - \frac{c-\widehat{c}}{2}|x-\overline{x}|^2 \qquad \forall\, x \in \Re^n,$$

and taking the maximum of the right hand side yields

$$H(x) < \frac{\epsilon}{2} + \frac{\widehat{c}^2}{2(c-\widehat{c})}|\widehat{x}-x_i|^2 \qquad \forall\, x \in \Re^n$$

which by (5.4)

$$\leq \frac{\epsilon}{2} + \frac{\widehat{c}^2}{2(c-\widehat{c})}\delta^2 \qquad \forall\, x \in \Re^n,$$

and by possibly reducing the size of δ from that in (5.9), one can make this

$$< \epsilon. \tag{5.12}$$

By (5.11) and (5.12),

$$a_i - \frac{\widehat{c}}{2}|x-x_i|^2 \geq a_{\widetilde{x}} - \frac{c}{2}|x-\widetilde{x}|^2 - \epsilon \qquad \forall\, x \in \Re^n.$$

This is the desired right inequality, and the desired left inequality is a direct consequence of (5.1b). □

THEOREM 5.2.

$$\psi(x) = \sup_i [a_i + g_i(x)] = \bigoplus_{i=1}^{\infty} [a_i \otimes g_i(x)] \qquad \forall\, x \in B_R. \tag{5.13}$$

Further, for any $\delta > 0$ it is sufficient to consider only a set of $\{x_i\}$ which is a dense subset of $B_{\delta+L(R)/c}$.

Proof. Let $\overline{x} \in B_R$. By Theorem 4.6, there exists $\widetilde{x} \in \Re^n$ such that

$$\psi(\overline{x}) = \overline{g}_{\widetilde{x}}(\overline{x}) + a_{\widetilde{x}}$$

and by Lemma 5.1, given $\epsilon > 0$, there exists i such that

$$\leq g_i(\overline{x}) + a_i + \epsilon.$$

Since $\epsilon > 0$ was arbitrary,

$$\psi(\overline{x}) \leq \sup_i [g_i(\overline{x}) + a_i],$$

and the reverse follows by the definition of the a_i. Also, the last assertion follows from Theorem 4.6 and the density of the x_i. □

For the remainder of the section, let $\{x_i\}$ be dense over all of $\Re^n$ in order to reduce complication. One has the countable basis function representation (5.13) for any such countable dense set $\{x_i\}$ and particular choice of constant $\widehat{c} \in (\overline{c}, \infty)$ where $\overline{c}$ is any constant such that (4.7) holds. Some further guidance on appropriate choices of $\widehat{c}$ will now be obtained. Consider several possible choices of basis function sets indexed by j,

$$\mathcal{B}_j \doteq \left\{ g_{j,i}(x) = -\frac{\widehat{c}_j}{2}|x - x_i|^2 : \{x_i\}_{i=1}^{\infty} \right\}. \tag{5.14}$$

From Theorem 5.2, we know that

$$\psi(x) = \bigoplus_{i=1}^{\infty} [a_i \otimes g_{j,i}(x)] \tag{5.15}$$

for any j such that $\widehat{c}_j > \overline{c}$. Conceptually then, given some $\psi \in \mathcal{P}$ and $R > 0$, one could choose any set of basis functions, $\mathcal{B}_j \in \{\mathcal{B}_j\}$ such that

$$\widehat{c}_j > \overline{c}. \tag{5.15}$$

It will now be shown that it is in some sense optimal to choose the set $\mathcal{B}_j$ with the smallest (loosely speaking) $\widehat{c}_j$ which satisfies (5.15). This may be rigorously stated as follows.

THEOREM 5.3. *Suppose $R > 0$ and $\psi \in \mathcal{P}$ with semiconvexity constant $\overline{c}$ (so that (4.7) is satisfied). Let $\mathcal{B}_{j_1}, \mathcal{B}_{j_2} \in \{\mathcal{B}_j\}$ be such that $\widehat{c}_{j_2} > \widehat{c}_{j_1} > \overline{c}$. Then, for any $n < \infty$ and $\epsilon > 0$, there exists $\{k_i\}_{i=1}^{n}$ such that*

$$\psi(x) \geq \bigoplus_{i=1}^{n} a_{j_1,k_i} \otimes g_{j_1,k_i}(x) \geq \bigoplus_{i=1}^{n} a_{j_2,i} \otimes g_{j_2,i}(x) - \epsilon \qquad \forall\, x \in B_R$$

where $a_{j_1,i}, a_{j_2,i}$ are given by (5.1b) for all i.

Before proceeding with the proof, note that this implies that one can always do arbitrarily close to better by choosing the smaller choice of $\widehat{c}_j$, but that the particular order of x_i and the function ψ may affect this. Typically of course, one would expect better performance with the smaller choice of $\widehat{c}_j$ since the result is obviously not true if $\widehat{c}_{j_2} < \widehat{c}_{j_1}$.

Proof. Let $1 \le i \le n < \infty$ and $\epsilon > 0$. It is sufficient to show that there exists $k_i < \infty$ such that

$$\psi(x) \ge a_{j_1,k_i} \otimes g_{j_1,k_i}(x) \ge a_{j_2,i} \otimes g_{j_2,i}(x) - \epsilon \qquad \forall\, x \in B_R. \tag{5.16}$$

But this is simply Lemma 5.1. $\square$

Lastly we turn to motivation for the choice of coefficients a_i in the expansion. Specifically we motivate the choice given by (5.1b). Note that this theorem will be true independent of the particular choice of basis functions $\{g_i\}$ in (5.1a).

THEOREM 5.4. *Let $\{a_i\}_{i=1}^{\infty}$ be given by (5.1b) and let $\{\widetilde{a}_i\}_{i=1}^{\infty}$ be an alternative set of coefficients. If $\widetilde{a}_{i_0} > a_{i_0}$ for some i_0, then*

$$\max_{x \in B_R} \left| \bigoplus_{i=1}^{\infty} \widetilde{a}_i \otimes g_i(x) - \psi(x) \right| > 0. \tag{5.17}$$

Alternatively, if $\widetilde{a}_i \le a_i$ for all i, then

$$\max_{x \in B_R} \left| \bigoplus_{i=1}^{n} \widetilde{a}_i \otimes g_i(x) - \psi(x) \right| \ge \max_{x \in B_R} \left| \bigoplus_{i=1}^{n} a_i \otimes g_i(x) - \psi(x) \right| \qquad \forall\, n. \tag{5.18}$$

Remark 5.5. Before proceeding to the proof, let us note that this theorem implies that of the possible choices for $\{a_i\}_{i=1}^{\infty}$ such that (5.13) holds, the choice given by (5.1b) is optimal. However, this does not imply that for any fixed $n < \infty$,

$$\max_{x \in B_R} \left| \bigoplus_{i=1}^{n} a_i \otimes g_i(x) - \psi(x) \right| \le \max_{x \in B_R} \left| \bigoplus_{i=1}^{n} \widetilde{a}_i \otimes g_i(x) - \psi(x) \right|.$$

That is, there may be better choices of $\{a_i\}_{i=1}^{n}$ for finite approximations, but such a choice cannot be extended to yield ψ in the limit without changing the a_i's for $i \le n$.

Proof. For the first part, note that by (5.1a), there exists $x_0 \in B_R$ such that

$$\psi(x_0) - a_{i_0} \otimes g_{i_0}(x_0) = 0,$$

and consequently,

$$\psi(x_0) - \bigoplus_{i=1}^{\infty} \widetilde{a}_i \otimes g_i(x_0) \le \psi(x_0) - \widetilde{a}_{i_0} \otimes g_{i_0}(x_0)$$

which since $\widetilde{a}_{i_0} > a_{i_0}$

$$< \psi(x_0) - a_{i_0} \otimes g_{i_0}(x_0) = 0.$$

For the second part, note that for any $n \geq 1$ and any $x \in B_R$,

$$\psi(x) \geq \bigoplus_{i=1}^{n} a_i \otimes g_i(x) = \max_{1\leq i\leq n}[a_i + g_i(x)]$$
$$\geq \max_{1\leq i\leq n}[\widetilde{a}_i + g_i(x)] = \bigoplus_{i=1}^{n} \widetilde{a}_i \otimes g_i(x). \qquad \square$$

6. Algorithm. Before explicitly describing the computational algorithm, we discuss how one may combine basis representation (5.13) with the max–plus linearity of the solution operator (Theorem 3.1) to update the information state between measurement times. This is actually a general algorithm for solving HJB equations of the form (2.14b) with semiconvex initial conditions, and so is more general than the particular application considered in this paper.

Specific error estimates have not been included in this section. The goal of this paper is to outline a new approach to computations for the Mortensen and robust nonlinear filters. Since this is a new and unusual approach, and since the paper is already of substantial length, error estimates are delayed to a future paper where computational comparisons could also be examined.

Recall the discrete–time filter discussed in Section 2. Suppose $P(t_l^+, \cdot)$ is semiconvex, and that we approximate it over some ball, B_R, by some finite number of elements of the basis representation (5.13), that is

$$P(t_l^+, x) = \bigoplus_{i=1}^{\infty}[a_i \otimes g_{j_1,i}(x)] \simeq \bigoplus_{i=1}^{n}[a_i \otimes g_{j_1,i}(x)] \qquad \forall\, x \in B_R$$

where

$$g_{j_1,i} = -\frac{\widehat{c}_{j_1}}{2}|x - x_i|^2$$

with appropriate choice of $\widehat{c}_{j_1}$ and with a_i given by (5.1b). Then $P({t_{l+1}}^-, x)$ is given by $\mathcal{S}_\delta[P(t_l^+, \cdot)](x)$ where $\mathcal{S}_\delta$ is the solution operator for (2.14b) where $\delta = t_{l+1} - t_l$.

We know by Theorem 3.1 that

$$\begin{aligned} P({t_{l+1}}^-, x) &= \mathcal{S}_\delta[P(t_l^+, \cdot)](x) \\ &\simeq \mathcal{S}_\delta[\bigoplus_{i=1}^{n}[a_i \otimes g_{j_1,i}(\cdot)](x) \\ &= \bigoplus_{i=1}^{n}\{a_i \otimes \mathcal{S}_\delta[g_{j_1,i}](x)\}. \end{aligned} \tag{6.1}$$

It is assumed that one has pre–computed $\mathcal{S}_\delta[g_{j,i}]$ for all appropriate j, i where the solutions are represented in the form

$$\mathcal{S}_\delta[g_{j,i}](x) \simeq \bigoplus_{k=1}^{n} b_{k,i} \otimes g_{j_2,k}(x) \tag{6.2}$$

where $g_{j_2,k}(x) = -\frac{\widehat{c}_{j_2}}{2}|x - x_k|^2$, $\widehat{c}_{j_2}$ is greater than the largest semiconvexity constant over the set $\{\mathcal{S}_\delta g_{j_1,i}\}_{i=1}^n$, and $b_{k,i}$ is given by (5.1b), i.e. $b_{k,i} = -\max_{x\in B_R}[g_{j_2,k}(x) -$

$\mathcal{S}_\delta[g_i](x)]$. (Note that one might choose to vary the set of interest, B_R, from one time to another, but for simplicity of notation we do not include that here.) In other words, one has pre–computed and stored the matrices $[b_{k,i}]$ for all appropriate j_1, j_2. then by (6.1), (6.2)

$$\begin{aligned} P(t_{l+1}{}^-, x) &\simeq \bigoplus_{i=1}^n \{a_i \otimes \mathcal{S}_\delta[g_{j_1,i}](x)\} \\ &\simeq \bigoplus_{i=1}^n \{a_i \otimes [\bigoplus_{k=1}^n b_{k,i} \otimes g_{j_2,k}(x)]\} \\ &= \bigoplus_{i=1}^n \bigoplus_{k=1}^n [b_{k,i} \otimes a_i \otimes g_{j_2,k}(x)] \\ &= \bigoplus_{k=1}^n [(\bigoplus_{i=1}^n (b_{k,i} \otimes a_i)) \otimes g_{j_2,k}(x)]. \end{aligned} \tag{6.3}$$

Let B be the $n \times n$ matrix $[b_{k,i}]$, and let A be the vector $[a_i]$. Define max–plus matrix multiplication in the natural way, that is let $C = B \otimes A$ be given by $C = [c_k]$ where $c_k = \bigoplus_{i=1}^n [b_{k,i} \otimes a_i]$. Then,

$$P(t_{l+1}^-, x) = \bigoplus_{i=1}^n [c_i \otimes g_{j_2,i}(x)]$$

where

$$C = B \otimes A. \tag{6.4}$$

In other words, the update $P(t_{l+1}^-, x) = \mathcal{S}_\delta[P(t_l^+, \cdot)](x)$ is performed simply by the max–plus matrix multiplication (6.4).

Now we will outline the structure of the information state propagation numerical algorithm for the discrete–time measurement case. Assume that the measurements occur at fixed time intervals $t_l = l\delta$ for $l = 1, 2, \ldots$. Assume that $P(0^+, \cdot) = \phi(\cdot)$ is semiconvex. (If not, then we know from Theorem 4.3 that $P(\delta, \cdot)$ will be semiconvex for $\delta > 0$, under the nondegeneracy assumption discussed there.)

Let $l = 0$.

1) Compute an estimate of the semiconvexity constant, $\overline{c}$, for $P(t_l^+, \cdot)$ over B_R. (Since $P(t_l^+, \cdot)$ will be known only on some finite set, this is estimated by second–order differences.)

2) Choose a set of basis functions $\mathcal{B}_{j_1}$ with $\widehat{c}_{j_1} > \overline{c}$. (In actual operations, one may choose to use more than one set, $\{\mathcal{B}_{j_1}, \mathcal{B}_{j_2}, \ldots \mathcal{B}_{j_n}\}$, but we require $\max_{j_m} \widehat{c}_{j_m} > \overline{c}$.) Also, determine the particular subset $\{x_i\}_{i=1}^n \subset \{x_i\}_{i=1}^\infty$ to use.

3) Compute $A^l = [a_i^l]$ for $i = 1, 2 \ldots n$ from (5.1b). One now has the approximation

$$P(t_l^+, x) \simeq \bigoplus_{i=1}^n [a_i^l \otimes g_{j_1,i}(x)]$$

where $g_{j_1,i}(x) = -\frac{\widehat{c_{j_1}}}{2}|x-x_i|^2$, and so the information state $P(t_l^+,\cdot)$ is represented simply by the vector A^l.

4) Propagate the solution of (2.14b) forward in time to time t_{l+1}. From the above discussion, we see that this takes the form of the max–plus matrix multiplication

$$A^{l+1} = B \otimes A^l.$$

5) Compute $P({t_{l+1}}^-,\cdot)$ from

$$P({t_{l+1}}^-,x) = \bigoplus_{i=1}^{n}[a_i^{l+1} \otimes g_{j_2,i}(x)].$$

This step takes the representation of the information state in the form A^{l+1}, and converts it back to a function of x (actually evaluated only on some grid over B_R).

6) Given measurement y_{l+1}, perform the measurement update

$$P({t_{l+1}}^+,x) = P({t_{l+1}}^-,x) - \frac{1}{2}|\rho^{-1}(x)[y_{l+1} - h(x)]|^2$$

(for all points on the grid over B_R).

7) Increment l, and return to step 1.

It is interesting to note that in the absence of measurement updates, the solution could be propagated forward in time simply by a series of max–plus matrix multiplications. This would be the case if there were other applications where one was only interested in solution of an HJB equation of the form (2.14b) with semiconvex initial conditions or nondegenerate σ. However, in the robust filter application, the largest portion of the computational effort is likely to be expended in step 3 where one needs to compute the coefficients in the expansion. In many other types of expansions, the coefficients are computed by integration. Here, we see that the integration step is replaced by a maximization over the relevant portion of the state space, B_R.

7. Continuous–Time Measurements. Let us return to the case of continuous–time measurements satisfying (2.2), and let $P(T,x)$ be the information state given by (2.6). Let $P^\delta(T,x)$ be the discrete–time measurement information state with time–step δ. (To be specific, we take $P^\delta(t,\cdot)$ to be $P(t^+,\cdot)$ at the measurement times, although $P(t^-,\cdot)$ would work as well.) In this section, we will show that P is the limit of P^δ as $\delta \downarrow 0$ (Theorem 7.3). Section 6 described a numerical algorithm for finding $P^\delta(T,x)$ approximately, and hence for approximate solution of the HJB equation (2.9) for $P(T,x)$.

We use the simplest discretization of the problem, letting $t_l = l\delta$ for $l = 0,1,2...$ At each $l \geq 1$, the discrete approximation model receives a measurement y_l given by

$$y_l = \frac{1}{\delta}\int_{t_{l-1}}^{t_l} y(t)\,dt. \tag{7.1}$$

Given T, let N be the largest integer such that $t_N \le T$, and let

$$J^\delta(T,x,w) = \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2\,dt - \frac{\delta}{2}\sum_{l=1}^N |\rho^{-1}(x(t_l))[y_l - h(x(t_l))]|^2, \tag{7.2}$$

where $x_0 = x(0)$ given $x(T) = x$ and dynamics (2.1), and

$$P^\delta(T,x) = \sup_{w\in L_2} J^\delta(T,x,w). \tag{7.3}$$

Note that if $T = N\delta$, one has $P^\delta(T,x)$ being what was denoted as $P(T^+,x)$ in (2.13b) rather than $P(T^-,x)$.

It will first be shown that the semiconvexity constants for the approximations P^δ (which we recall affect the choice of basis functions in the algorithm of Section 6) do not blow up as $\delta \downarrow 0$.

THEOREM 7.1. *Assume that ϕ is semiconvex. Then for any T^0, R there exists a constant $\widehat{\Gamma}$ (depending on T^0, R) such that for any $T \in [0,T^0]$, $P^\delta(T,x) + \widehat{\Gamma}|x|^2$ is convex on B_R.*

Proof. Let

$$J^{\delta,0}(T,x,w) = \phi(x_0) - \frac{1}{2}\int_0^T |w(t)|^2\,dt$$

$$J^{\delta,1}(T,x,w) = -\frac{\delta}{2}\sum_{l=1}^N |\rho^{-1}(x(t_l))[y_l - h(x(t_l))]|^2$$

so that $J^\delta = J^{\delta,0} + J^{\delta,1}$. In the proof of Theorem 4.1, one finds that given T^0, R, there exists Γ such that

$$v^T J^{\delta,0}_{xx}(x,T,w)v \ge -2\Gamma \tag{7.4}$$

for all $|x| \le R$, $T \in [0,T^0]$, $|v| = 1$ and ϵ_0–optimal w with $\epsilon_0 \le 1$. In particular, $\Gamma = \Gamma_R(T^0)$ is independent of δ. All that remains is to prove a similar result for $J^{\delta,1}$. It will now be shown that there exists $\overline{\Gamma}$ (depending on T^0, R) such that

$$v^T J^{\delta,1}_{xx}(x,T,w)v \ge -2\overline{\Gamma}$$

for all $|x| \le R$, $T \in [0,T^0]$, $|v| = 1$ and ϵ_0–optimal w with $\epsilon_0 \le 1$.

As in the proof of Theorem 4.1, let $\zeta^1(t), \zeta^2(t)$ denote the first and second derivatives of $x(t)$ in the direction v (considered as a function of $x = x(T)$). Then

$$\begin{aligned} v^T J^{\delta,1}_{xx}(x,T,w)v = &-\frac{\delta}{2}\sum_{l=1}^N \Big\{ (|\rho^{-1}(x)[y_l - h(x)]|^2)_x \Big|_{x=x(t_l)} \zeta^2(t_l) \\ &+ \Big[(|\rho^{-1}(x)[y_l - h(x)]|^2)_{xx} \Big|_{x=x(t_l)} \zeta^1(t_l) \Big] \cdot \zeta^1(t_l) \Big\}. \end{aligned} \tag{7.5}$$

From Lemma 2.1, we know there exists $R_1 = R_1(T^0, R)$ such that $|x(t)| \le R_1$ for all $t \in [0,T]$, which leads, via (A1), to bounds on all x–dependent terms in (7.5). That is, there exists $M_1 = M_1(T^0, R)$ such that

$$v^T J^{\delta,1}_{xx}(x,T,w)v \ge -\frac{\delta}{2}\sum_{l=1}^N C_\rho^2 \left\{ (M_1 + |y_l|^2)\left(|\zeta^2(t_l)| + |\zeta^1(t_l)|^2\right) \right\}.$$

But as shown in the proof of Theorem 4.1, $|\zeta^2(t)|, |\zeta^1(t)|$ are bounded with bounds only dependent on T^0, R. Therefore, there exists $M_2 = M_2(T^0, R)$ such that

$$\begin{aligned} v^T J^{\delta,1}_{xx}(x,T,w)v &\geq -\frac{\delta}{2}\sum_{l=1}^{N}\{M_2(1+|y_l|^2)\} \\ &= -\frac{\delta}{2}\sum_{l=1}^{N}\left\{M_2\left(1+\left|\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y(t)\,dt\right|^2\right)\right\} \\ &\geq -\frac{\delta}{2}\sum_{l=1}^{N}\left\{M_2\left(1+\frac{1}{\delta}\int_{t_{l-1}}^{t_l} |y(t)|^2\,dt\right)\right\} \\ &\geq \frac{-M_2}{2}\left(T+\|y\|^2_{L_2[0,T]}\right) \end{aligned}$$

which we define to be

$$\doteq -2\overline{\Gamma}. \tag{7.6}$$

Combining (7.4) and (7.6), one has

$$v^T J^{\delta}_{xx}(x,T,w)v \geq -2(\Gamma+\overline{\Gamma})$$

for all $|x| \leq R$, ϵ_0–optimal w with $\epsilon_0 \leq 1$ where $\Gamma, \overline{\Gamma}$ are independent of δ. This implies the desired result. □

Remark 7.2. The P^δ satisfy a uniform (in δ) local Lipschitz condition in x. That is, given $R, T < \infty$, there exists $K = K(T,R)$ such that $|P^\delta(T,x) - P^\delta(T,\widetilde{x})| \leq K|x-\widetilde{x}|$ for all $x, \widetilde{x} \in B_R$. This follows directly from the uniform semiconvexity, see [8, p. 111].

Lastly, we obtain the convergence result.

THEOREM 7.3. *$P^\delta(T,x) \to P(T,x)$ as $\delta \downarrow 0$ for all $(T,x) \in [0,\infty) \times \Re^n$.*

Proof. Fix $T, \overline{R} < \infty$ and $x \in B_{\overline{R}}$. Let $\epsilon_0 \in [0,1]$. Let J be given by (2.5) and J^δ by (7.2). By Lemma 2.1 (and a similar result for the discrete case), we know ϵ_0–optimal w for either problem satisfy $\|w\|_{L_2[0,T]} \leq M_{\overline{R},T}$ for appropriate $M_{\overline{R},T}$.

Let $\|w\|_{L_2[0,T]} \leq M_{\overline{R},T}$. It is sufficient to show that

$$|J(T,x,w) - J^\delta(T,x,w)| \to 0$$

as $\delta \downarrow 0$ (with an estimate which is independent of w). We have

$$\begin{aligned} 2|J(T,x,w) - J^\delta(T,x,w)| = \bigg| &\int_0^T |\rho^{-1}(x(t))[y(t)-h(x(t))]|^2\,dt \\ &- \delta\sum_{l=1}^{N}|\rho^{-1}(x(t_l))[y_l - h(x(t_l))]|^2\bigg|. \end{aligned} \tag{7.7}$$

We now show that given $\epsilon > 0$, the right hand side of (7.7) can be made $< \epsilon$ for $\delta > 0$ sufficiently small. The proof will involve first an approximation of y by an L_∞ function, and then a mollification.

Let $R>0$ and

$$y^R(t) \doteq \begin{cases} y(t) & \text{if } |y(t)|\le R \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Xi_R = \{t\in[0,T] : |y(t)|>R\}$. Then

$$\int_0^T \Big[|\rho^{-1}(x(t))[y(t)-h(x(t))]|^2 \\ -|\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2\Big]\,dt = \int_{\Xi_R} |\rho^{-1}(x(t))[y(t)-h(x(t))]|^2\,dt$$

which by (A3), (A1) and Lemma 2.1

$$\le 2C_\rho^2 \int_{\Xi_R} (M_1+|y(t)|^2)\,dt$$

for proper choice of M_1, and then for R sufficiently large

$$<\frac{\epsilon}{4}. \tag{7.8}$$

Also,

$$\delta\sum_{l=1}^N \Bigg| \left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y(t)\,dt - h(x(t_l))\right]\right|^2 \\ - \left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(t)\,dt - h(x(t_l))\right]\right|^2\Bigg|$$

$$\le \delta C_\rho^2 \sum_{l=1}^N \left|\frac{1}{\delta}\int_{t_{l-1}}^{t_l} (y(t)+y^R(t))\,dt - 2h(x(t_l))\right| \left|\frac{1}{\delta}\int_{t_{l-1}}^{t_l} (y(t)-y^R(t))\,dt\right|$$

and using Lemma 2.1 and Cauchy–Schwarz, one finds that for proper choice of M_2,

$$\le \delta C_\rho^2 \sum_{l=1}^N \Bigg\{ \frac{1}{\sqrt{\delta}}\left[\int_{t_{l-1}}^{t_l} |y(t)+y^R(t)|^2\,dt\right]^{\frac12} \frac{1}{\sqrt{\delta}}\left[\int_{t_{l-1}}^{t_l} |y(t)-y^R(t)|^2\,dt\right]^{\frac12} \\ + \frac{M_2}{\delta}\int_{t_{l-1}}^{t_l} |y(t)-y^R(t)|\,dt\Bigg\}$$

which using $2ab\le ca^2+b^2/c$ for any $c>0$,

$$\le C_\rho^2\left\{\frac{c}{2}\int_0^T |y+y^R|^2\,dt + \frac{1}{2c}\int_0^T |y-y^R|^2\,dt + M_2\int_0^T |y-y^R|\,dt\right\}$$

which for c sufficiently small and then R sufficiently large

$$<\frac{\epsilon}{4}. \tag{7.9}$$

From (7.7),

$$
\begin{aligned}
&2|J(T,x,w)-J^{\delta}(T,x,w)| \\
&\le \left|\int_0^T |\rho^{-1}(x(t))[y(t)-h(x(t))]|^2 - |\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2,dt\right| \\
&\quad + \left|\int_0^T |\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2\,dt\right. \\
&\qquad\qquad \left. - \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left|\rho^{-1}(x(t))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t))\right]\right|^2 dt\right| \\
&\quad + \left|\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left\{\left|\rho^{-1}(x(t))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t))\right]\right|^2\right.\right. \\
&\qquad\qquad \left.\left. - \left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t_l))\right]\right|^2\right\} dt\right| \\
&\quad + \left|\sum_{l=1}^{N}\delta\left\{\left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t_l))\right]\right|^2\right.\right. \\
&\qquad\qquad \left.\left. - \left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y(r)\,dr - h(x(t_l))\right]\right|^2\right\} dt\right|
\end{aligned}
$$

which by (7.8), (7.9) and some rearrangement is

$$
\begin{aligned}
&< \left|\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left\{|\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2\right.\right. \\
&\qquad\qquad \left.\left. - \left|\rho^{-1}(x(t))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t))\right]\right|^2\right\} dt\right| \\
&\quad + \left|\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left\{\left|\rho^{-1}(x(t))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t))\right]\right|^2\right.\right. \\
&\qquad\qquad \left.\left. - \left|\rho^{-1}(x(t_l))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t_l))\right]\right|^2\right\} dt\right| \\
&\quad + \int_{t_N}^{T} |\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2\,dt + \frac{\epsilon}{2}
\end{aligned}
$$

which by the continuity of $x(\cdot)$ over $[0,T]$ for $\|w\| < M_{\overline{R},T}$

$$
\begin{aligned}
&< \left|\sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left\{|\rho^{-1}(x(t))[y^R(t)-h(x(t))]|^2\right.\right. \\
&\qquad\qquad \left.\left. - \left|\rho^{-1}(x(t))\left[\frac{1}{\delta}\int_{t_{l-1}}^{t_l} y^R(r)\,dr - h(x(t))\right]\right|^2\right\} dt\right| + M_3\delta + \frac{3\epsilon}{4}
\end{aligned}
$$

$$= \left| \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left\{ \left| \rho^{-1}(x(t)) \left[\frac{1}{\delta} \int_{t_{l-1}}^{t_l} y^R(t)\, dr - h(x(t)) \right] \right|^2 \right.\right.$$
$$\left.\left. - \left| \rho^{-1}(x(t)) \left[\frac{1}{\delta} \int_{t_{l-1}}^{t_l} y^R(r)\, dr - h(x(t)) \right] \right|^2 \right\} dt \right| + M_3\delta + \frac{3\epsilon}{4}$$

which for proper choice of $\xi(t) \in B_R$ for all t,

$$< 2C_\rho^2 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} |\xi(t) - h(x(t))| \left| \frac{1}{\delta} \int_{t_{l-1}}^{t_l} (y^R(t) - y^R(r))\, dr \right| dt + M_3\delta + \frac{3\epsilon}{4}$$

which for proper choice of $M_4 = M_4(R, T)$

$$< M_4 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left| \frac{1}{\delta} \int_{t_{l-1}}^{t_l} (y^R(t) - y^R(r))\, dr \right| dt + M_3\delta + \frac{3\epsilon}{4}. \tag{7.10}$$

Now let $y^{R,\rho}$ be a C^∞ mollification of y^R such that $||y^{R,\rho} - y^R||_{L_1[0,T]} \to 0$ as $\rho \downarrow 0$. Rearranging (7.10) yields

$$2|J(T,x,w) - J^\delta(T,x,w)|$$
$$< M_4 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left| \frac{1}{\delta} \int_{t_{l-1}}^{t_l} (y^{R,\rho}(t) - y^{R,\rho}(r))\, dr \right| dt$$
$$+ M_4 \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} \left| \left| \frac{1}{\delta} \int_{t_{l-1}}^{t_l} (y^R(t) - y^R(r))\, dr \right| - \left| \frac{1}{\delta} \int_{t_{l-1}}^{t_l} (y^{R,\rho}(t) - y^{R,\rho}(r))\, dr \right| \right| dt$$
$$+ M_3\delta + \frac{3\epsilon}{4},$$

and letting $\mu^\rho(\cdot)$ be the modulus of continuity of $y^{R,\rho}$, and using the inequality $\big||a-b| - |c-d|\big| \le |a-c| + |b-d|$ on the second sum, one finds that this is

$$\le M_4 T \mu^\rho(\delta) + M_4 \sum_{l=1}^{N} \left[\int_{t_{l-1}}^{t_l} |y^R(t) - y^{R,\rho}(t)|\, dt + \int_{t_{l-1}}^{t_l} |y^R(r) - y^{R,\rho}(r)|\, dr \right]$$
$$+ M_3\delta + \frac{3\epsilon}{4},$$
$$= M_4 T \mu^\rho(\delta) + 2M_4 ||y^R - y^{R,\rho}||_{L_1[0,T]} + M_3\delta + \frac{3\epsilon}{4},$$

which for ρ sufficiently small

$$< M_4 T \mu^\rho(\delta) + M_3\delta + \frac{7\epsilon}{8},$$

which for δ sufficiently small

$$< \epsilon. \qquad \square$$

Remark 7.4. In the case where $y(\cdot)$ is continuous, one can obtain an explicit estimate of the convergence rate. Specifically, using only the bound $||w||_{L_2[0,t]} \le M_{\overline{R},T}$, for ϵ_0–optimal disturbances for both continuous and discrete–time problems with $\delta \le 1$, one can obtain an estimate of the form $|P^\delta(T,x) - P(T,x)| \le \mathcal{M}[\sqrt{\delta} + \mu(\delta)]$ where $\mu(\cdot)$ is the modulus of continuity for y. We expect that with additional effort, one can obtain a uniform L_∞ bound on the optimal disturbances for the discrete–time problems (in analogy with the continuous–time bound of Lemma 2.1). This would lead to an estimate of the form $|P^\delta(T,x) - P(T,x)| \le \mathcal{M}[\delta + \mu(\delta)]$. However, the effort would likely require more journal space than would seem appropriate for this initial exploration.

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