A max-plus affine power method for approximation of a class of mixed L_∞ / L_2 value functions.

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Abstract

This paper is concerned with a number of issues associated with the approximation via max-plus methods of value functions arising in a class of mixed L_2 / L_∞ optimisation. The class of problems is defined to be suitably general so as to admit future application to the computation of value functions associated with L_2-gain analysis, L_∞ bounded (LIB) dissipation and to the analysis of systems with the ISS (input to state stability) properties. Common to each these problems is the applicability of dynamic programming, which naturally leads to the formulation of max-plus methods using the resulting semigroups (and sub-semigroups). This paper provides the details of this formulation. In particular, we develop an affine power method that yields the correct solution of the dynamic programming principle (DPP) and hence the underlying optimisation problem, despite the inherent non-uniqueness of solutions of such DPPs.
Keywords: max-plus, L_2-gain dissipation, L_∞ bounded dissipation, practical stability, dynamic programming.

1 Introduction

Dynamic programming has proved to be an invaluable tool in the analysis and design of control systems. Of particular relevance is the application of dynamic programming in optimal control and in the verification of various performance related properties, including L_2-gain analysis and nonlinear H^∞-control [9, 23, 24], practical L_2-gain analysis [4, 5], L_∞-bounded dissipation (LIB) [12] and many other optimisation based control tools (including receding horizon control, etc). Recently, it was demonstrated in [10] that dynamic programming can also be used to quantify the notion of minimal gains in a number of L_∞ properties including input-to-state stability (ISS) [22]. Furthermore, it has also been demonstrated in [11] that dynamic programming can be applied in the synthesis of controllers for yielding closed loop ISS.

Given the breadth of dynamic programming applications, it is important to develop, where possible, approximation techniques for the solution of the associated dynamic programming principles (DPPs). A class of particularly promising approaches in this regard are the so-called max-plus methods (see for example [6, 8]).

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A second motivation comes from the development of the theory of max-plus additive functionals of solutions of max-plus stochastic differential equations. Max-plus stochastic processes are developed following the definition of max-plus probability measures. See [7] for more details on this theory.

In this paper, we apply max-plus methods to a particular class of optimisation problems. This class of optimisation problems is chosen to be sufficiently general so as to include the recent applications of dynamic programming cited above. Furthermore, in developing this approximation method, we demonstrate a number of interesting properties related to uniqueness of the attendant DPPs.

2 Problem Formulation

Throughout this paper, we consider nonlinear continuous time systems of the form

$$\dot{\xi}(t) \;=\; f(\xi(t), w(t)), \tag{1}$$

initialized at $\xi(0) = x_\circ$, where $\xi(t) \in \mathbf{R}^n$ and $w(t) \in \mathbf{R}^p$ are respectively the state and input at time t.

Let $l, L : \mathbf{R}^n \to \mathbf{R}$, $\gamma : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ be continuous functions, with $\gamma(0) = 0$ and $\gamma(s)$ strictly increasing in s. Then, system (1) satisfies the following mixed L_∞ / L_2-gain property iff there exists a locally bounded nonnegative function $\beta : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ such that

$$L(\xi(T)) + \int_0^T l(\xi(s)) - \gamma(|w(s)|)\; ds \;\leq\; \beta(x_\circ) \tag{2}$$

for all $x_\circ \in \mathbf{R}^n$, $w \in \mathcal{W}[0,T)$ and $T \geq 0$. Here, the input space $\mathcal{W}[0,T)$ is assumed apriori depending on the choice of costs l and L. With $L(\cdot) \equiv 0$, typically we choose $\mathcal{W}[0,T) = \mathcal{W}_\infty[0,T)$, where $\mathcal{W}_\infty[0,T) := \left\{ w : \mathbf{R} \to \mathbf{R}^p |\, \text{measurable}, \, \|w\|_\infty < \infty, \, \int_0^T \gamma(|w(s)|)\; ds < \infty \right\}$ and for $M \in (0,\infty)$, $\mathcal{W}_M[0,T) := \left\{ w : \mathbf{R} \to \mathbf{R}^p |\, \text{measurable}, \, \|w\|_\infty \leq M \;\; \int_0^T \gamma(|w(s)|)\; ds < \infty \right\}$. Otherwise, we choose $\mathcal{W}[0,T) = \mathcal{W}_M[0,T)$ for some fixed $M < \infty$. For simplicity of notation, we define

$$I_{[t_\circ,t]}(x,w) \;\doteq\; \int_{t_\circ}^t l(\xi(s)) - \gamma(|w(s)|)\, ds, \tag{3}$$

$$Q_{[t_\circ,t]}(x,w) \;\doteq\; L(\xi(t)) + I_{[t_\circ,t]}(x,w) = L(\xi(t)) + \int_{t_\circ}^t l(\xi(s)) - \gamma(|w(s)|)\, ds, \tag{4}$$

where $\xi(\cdot)$ satisfies (1) with $\xi(t_\circ) = x$. Of interest in this paper is the formulation and approximation of value functions for the following optimisation problems:

Problem 1: [stopping time problem]

$$W(x) \;=\; \sup_{T\geq 0} \sup_{w\in\mathcal{W}[0,T)} \left\{ Q_{[0,T]}(x,w) \right\} \tag{5}$$

Problem 2: [infinite horizon problem]

$$V(x) \;=\; \limsup_{T\to\infty} \sup_{w\in\mathcal{W}[0,T)} \left\{ Q_{[0,T]}(x,w) \right\} \tag{6}$$

In order to develop max-plus approximations for these value functions, it is essential to first write down a dynamic programming principle (DPP) for each of these problems. These DPPs then define respectively a sub-semigroup and a semigroup which prove to be linear in the max-plus sense, thereby admitting the application of so called max-plus power methods for approximation of the two value functions. Lastly, we note that the infinite horizon problem value function V will be of interest not only in its own right, but also in that it will pertain to a certain nonuniqueness difficulty one encounters in solving for W.

3 Dynamic Programming

3.1 Dynamic Programming Principles (DPPs)

As noted above, we will be using max-plus methods to analyze and develop tools for the computation of W and V, and these max-plus techniques follow from the dynamic programming principle (DPP). Typically, one uses the DPP to obtain the dynamic programming equation (DPE) – which takes the form of either a partial differential equation (PDE) or variational inequality (VI) for the problems considered here. Once one has the PDE or VI, one then typically applies some numerical technique such as finite elements to obtain the solution of the PDE or VI. In the case here, the DPPs themselves lead directly to the max-plus formulations and associated numerical methods without reference to the infinitesmal PDE and VI formulations. We now develop the DPPs for W and V. It is not difficult to show that the value $W(x)$ satisfies the following DPP.

Theorem 3.1 *W given by (5) satisfies*

$$W(x) = \max\Big\{ \sup_{t\in[0,\tau)} \sup_{w\in\mathcal{W}[0,t]} Q_{[0,t]}(x,w), \sup_{w\in\mathcal{W}[0,\tau)} I_{[0,\tau]}(x,w) + W(\xi(\tau)) \Big\} \qquad \forall\, x \in \mathbf{R}^n. \tag{7}$$

Proof: Fix $\tau > 0$. The supremum over $T \geq 0$ in (5) is equivalent to maximum of the suprema over $[0,\tau)$ and $[\tau,\infty)$. That is,

$$W(x) = \max\left(\sup_{T\in[0,\tau)} \sup_{w\in\mathcal{W}[0,T)} Q_T(x,w), \underbrace{\sup_{T\in[\tau,\infty)} \sup_{w\in\mathcal{W}[0,T)} \Big\{ L(\xi(T)) + I_{[0,T)}(x,w) : \xi(0) = x \Big\}}_{=:\ell_\tau(x)} \right)$$

Considering the second term $\ell_\tau(x)$ only,

$$\begin{aligned}
\ell_\tau(x) &= \sup_{T\in[\tau,\infty)} \sup_{w\in\mathcal{W}[0,T)} \Big\{ I_{[0,\tau]}(x,w) + L(\xi(T)) + I_{[\tau,T]}(x,w)\ ds : \xi(0) = x \Big\} \\
&= \sup_{T\in[\tau,\infty)} \sup_{w_1\in\mathcal{W}[0,\tau)} \sup_{w_2\in\mathcal{W}[\tau,T)} \Big\{ I_{[0,\tau]}(x,w_1) + L(\xi_2(T)) + I_{[\tau,T]}(\xi_2(\tau),w_2) : \xi_1(0) = x, \xi_2(\tau) = \xi_1(\tau) \Big\} \\
&= \sup_{w_1\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w_1) + \sup_{T\in[\tau,\infty)} \sup_{w_2\in\mathcal{W}[\tau,T)} \Big\{ L(\xi(T)) + I_{[\tau,T]}(\xi(\tau),w_2) \Big\} : \xi(0) = x \right\} \\
&= \sup_{w_1\in\mathcal{W}[0,\tau)} \Big\{ I_{[0,\tau]}(x,w_1) + W(\xi(\tau)) : \xi(0) = x \Big\},
\end{aligned}$$

where $\xi_1(\cdot)$, $\xi_2(\cdot)$ satisfy (1) and correspond to inputs $w_1(\cdot)$ and $w_2(\cdot)$ respectively. This completes the proof. ∎

In order to prove a dynamic programming principle for (6), it is useful to firstly consider the following auxiliary finite horizon optimization problem:

$$\tilde{W}(x,T) \;=\; \sup_{w\in\mathcal{W}[0,T)} \left\{ Q_{[0,T]}(x,w) \right\} \tag{8}$$

It follows immediately from (5), (6) and (8) that

$$W(x) \;=\; \sup_{T\geq 0} \tilde{W}(x,T), \tag{9}$$

$$V(x) \;=\; \limsup_{T\to\infty} \tilde{W}(x,T). \tag{10}$$

Identity (10) will be used to prove the DPP for V. First we need the following lemma, the proof of which is standard.

Lemma 3.2 *$\tilde{W}$ given by (8) satisfies*

$$\tilde{W}(x,T) \;=\; \sup_{w\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + \tilde{W}(\xi(\tau), T-\tau) : \xi(0)=x \right\} \qquad \forall\, x\in\mathbf{R}^n,\ \tau\in[0,T). \tag{11}$$

Proof: Fix $\tau\in[0,T)$.

$$\begin{aligned}
\tilde{W}(x,T) \;&=\; \sup_{w_1\in\mathcal{W}[0,\tau)} \sup_{w_2\in\mathcal{W}[\tau,T)} \Big\{ I_{[0,\tau]}(x,w_1) + L(\xi_2(T)) + I_{[\tau,T]}(\xi_2(\tau),w_2) \\
&\qquad\qquad : \xi_1(0)=x, \xi_2(\tau)=\xi_1(\tau) \Big\} \\
&=\; \sup_{w_1\in\mathcal{W}[0,\tau)} \Big\{ I_{[0,\tau]}(x,w_1) + \sup_{w_2\in\mathcal{W}[\tau,T)} \Big\{ L(\xi_2(T)) + I_{[\tau,T]}(\xi_2(\tau),w_2) \\
&\qquad\qquad : \xi_2(\tau)=\xi_1(\tau) \Big\} : \xi_1(0)=x \Big\} \\
&=\; \sup_{w_1\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w_1) + \tilde{W}(\xi_1(\tau),T-\tau) : \xi_1(0)=x \right\},
\end{aligned} \tag{12}$$

which completes the proof. ∎

Note that DPP (7) may also be proved using (11) by first rewriting (11) so that it holds for all $\tau\in[0,\infty)$.

Proof: [Theorem 3.1 using Lemma 3.2] Let $a\wedge b \doteq \min(a,b)$. Then, (11) implies that

$$\tilde{W}(x,T) \;=\; \sup_{w\in\mathcal{W}[0,\tau\wedge T)} \left\{ I_{[0,\tau\wedge T]}(x,w) + \tilde{W}(\xi(\tau\wedge T), T-(\tau\wedge T) \right\} \tag{13}$$

for all $\tau\in[0,\infty)$. Taking the supremum over $T\geq 0$ (of which τ is now independent) and applying (9),

$$\begin{aligned}
W(x) \;&=\; \sup_{T\geq 0} \sup_{w\in\mathcal{W}[0,\tau\wedge T)} \left\{ I_{[0,\tau\wedge T]}(x,w) + \tilde{W}(\xi(\tau\wedge T), T-(\tau\wedge T)) : \xi(0)=x \right\} \\
&=\; \max\Bigg(\sup_{T\in[0,\tau)} \sup_{w\in\mathcal{W}[0,T)} \left\{ I_{[0,T]}(x,w) + \tilde{W}(\xi(T), T-T)) : \xi(0)=x \right\}, \\
&\qquad \sup_{T\in[\tau,\infty)} \sup_{w\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + \tilde{W}(\xi(\tau), T-\tau)) : \xi(0)=x \right\} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \max\Bigg(\sup_{T\in[0,\tau)} \sup_{w\in\mathcal{W}[0,T)} \left\{ I_{[0,T]}(x,w) + L(\xi(T)) : \xi(0) = x \right\}, \\
&\qquad \sup_{w\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + \sup_{T\in[\tau,\infty)} \tilde{W}(\xi(\tau), T-\tau)) : \xi(0) = x \right\} \Bigg) \\
&= \max\left(\sup_{T\in[0,\tau)} \sup_{w\in\mathcal{W}[0,T)} Q_{[0,T]}(x,w),\ \sup_{w\in\mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + W(\xi(\tau)) \right\} \right)
\end{aligned}
$$

as obtained in (7). ∎

Lemma 3.2 can also be applied to prove a dynamic programming principle for the infinite horizon value function V, given by (6). First we need the following Lemma.

Lemma 3.3 *Fix $\rho \in [0,\infty)$, $\bar{\delta} \in [0,\infty)$, and $\tau \in [0,\infty)$. Given any $T \in [\tau,\infty)$, let $\xi^{\delta}_{x,\tau,T}(\tau) \in \mathbf{R}^n$ denote the solution of (1) at time τ when initialized with $\xi^{\delta}_{x,\tau,T}(0) = x \in B_\rho$ and driven by input $w^{\delta}_{x,\tau,T} \in \mathcal{W}[0,\tau]$, where $w^{\delta}_{x,\tau,T} \in \mathcal{W}[0,\tau]$ is δ-optimal in the DPP (11) for $\tilde{W}(x,T)$ (for the given τ), and $\delta \le \bar{\delta}$.*

Then, there exists $R_\rho \in [\rho,\infty)$ such that $\xi^{\delta}_{x,\tau,T}(\tau) \in B_{R_\rho}$ for all $x \in B_\rho$, $\delta \le \bar{\delta}$ and $T \ge \tau$.

Proof: Where $\mathcal{W}[0,\tau) = \mathcal{W}_M[0,\tau)$ for some $M < \infty$, the proof is immediate. In the case where $\mathcal{W}[0,\tau) = L_2[0,\tau)$, a similar proof can be found in [5]. ∎

Definition 3.4 *The limsup in (10) is attained uniformly on compact sets when the following condition holds: Given $\mathcal{X} \subset \mathbf{R}^n$ compact, $\delta \in [\frac{\bar{\delta}}{2}, \bar{\delta}]$, $\bar{\delta} \in [0,\infty)$ there exists $T_{\bar{\delta},\mathcal{X}} < \infty$ such that*

$$
T \ \ge\ T_{\bar{\delta},\mathcal{X}} \quad \Rightarrow \quad \sup_{\zeta\in\mathcal{X}} \left| \sup_{\sigma\ge T} W(\zeta,\sigma) - V(\zeta) \right| \ \le\ \delta. \tag{14}
$$

Note that $\sup_{\sigma\ge T} W(\zeta,\sigma) \ge V(\zeta)$ for all $\zeta \in \mathbf{R}^n$. Hence, the right hand side of (14) may be rewritten as

$$
0 \ \le\ \sup_{\zeta\in\mathcal{X}} \left\{ \sup_{\sigma\ge T} W(\zeta,\sigma) - V(\zeta) \right\} \ \le\ \delta.
$$

Lemma 3.5 *The limsup in (10) attained uniformly on compact sets implies that the following properties hold, given any $R \in [0,\infty)$, $\tau \in [0,\infty)$ $\bar{\delta} \in [0,\infty)$, and $\delta \in [\frac{\bar{\delta}}{2}, \bar{\delta}]$:*

(i). There exists a $\bar{T}^{\bar{\delta}}_{R,\tau} \in [\tau,\infty)$ such that

$$
V(\zeta) \ \ge\ \tilde{W}(\zeta, T-\tau) - \delta \qquad \forall\, \zeta \in B_R,\ T \ge \bar{T}^{\bar{\delta}}_{R,\tau}. \tag{15}
$$

(ii). There exists $\bar{i}(\zeta) < \infty$ such that

$$
V(\zeta) \ \le\ W(\zeta, T_i(\zeta)) + \delta \qquad \forall\ i \ge \bar{i}(\zeta), \tag{16}
$$

where $\{T_i(\zeta)\} \to \infty$ is a sequence such that $\lim_{i\to\infty} W(\zeta, T_i(\zeta)) = \limsup_{T\to\infty} W(\zeta, T)$.

Proof: Fix R, τ, $\bar{\delta}$, δ as per the statement of the Lemma. Fix $\mathcal{X} := B_R \subset \mathbf{R}^n$ (which is obviously compact) and any $\zeta \in \mathcal{X}$. Assertion (ii) is obvious by the definition of limit supremum, and so we prove only Assertion (i).

Assertion (i): Applying Definition 3.4, fix $T_{R,\tau}^{\bar{\delta}} = T_{\delta,\mathcal{X}} + \tau$. Then, (14) states that

$$\zeta \in \mathcal{X},\ T - \tau \geq T_{\bar{\delta},\mathcal{X}} \quad \Rightarrow \quad \sup_{\sigma \geq T-\tau} W(\zeta,\sigma) - V(\zeta) \ \leq \ \delta$$

which yields the implication (by selecting suboptimal $\sigma = T - \tau$)

$$\zeta \in \mathcal{X},\ T \geq T_{R,\tau}^{\bar{\delta}} \quad \Rightarrow \quad V(\zeta) \geq W(\zeta, T-\tau) - \delta$$

as required. ∎

Lemma 3.6 *Suppose that the limsup in (6) (equivalently, (10)) is achieved uniformly on compact sets. Then, V given by (6) satisfies*

$$V(x) \ = \sup_{w \in \mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + V(\xi(\tau)) : \xi(0) = x \right\} \qquad \forall\, x \in \mathbf{R}^n,\ \tau \in [0,\infty). \tag{17}$$

Proof: Fix $x \in \mathbf{R}^n$, $\tau \in [0,\infty)$. Applying (10) and (13),

$$\begin{aligned}
V(x) &= \limsup_{T\to\infty} \sup_{w \in \mathcal{W}[0,\tau\wedge T)} \left\{ I_{[0,\tau\wedge T]}(x,w) + \tilde{W}(\xi(\tau\wedge T), T - (\tau\wedge T)) : \xi(0) = x \right\} \\
&\geq \sup_{w \in \mathcal{W}[0,\tau)} \limsup_{T\to\infty} \left\{ I_{[0,\tau]}(x,w) + \tilde{W}(\xi(\tau), T-\tau) : \xi(0) = x \right\} \\
&= \sup_{w \in \mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + \limsup_{T\to\infty} \left\{ \tilde{W}(\xi(\tau), T-\tau) \right\} : \xi(0) = x \right\} \\
&= \sup_{w \in \mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + V(\xi(\tau)) : \xi(0) = x \right\}
\end{aligned} \tag{18}$$

which proves the inequality in one direction. We prove that (18) in fact holds with equality by contradiction.

Fix $\rho \in [0,\infty)$, $x \in B_\rho$, $\tau \in [0,\infty)$, $\bar{\delta} \in [0,\infty)$. Assume (18) holds with strict inequality. That is, there exists a $\delta \in (0,\bar{\delta}]$ (fixed) such that

$$V(x) \ \geq \sup_{w \in \mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + V(\xi(\tau)) : \xi(0) = x \right\} + 4\delta \tag{19}$$

Given ρ, τ and $\bar{\delta}$, Lemma 3.3 fixes $R := R_\rho < \infty$ such that $\xi_{x,\tau,T}^{\delta} \in B_R$ for all $x \in B_\rho$, $\delta \leq \bar{\delta}$ and $T \geq \tau$. The uniform limit assumption on V (via Lemma 3.5) then fixes $\bar{T} := \bar{T}_{R,\tau}^{\bar{\delta}} < \infty$ such that

$$V(\xi_{x,\tau,T}^{\delta}(\tau)) \ \geq \ \tilde{W}(\xi_{x,\tau,T}^{\delta}(\tau), T-\tau) - \delta \qquad \forall\, T \geq \bar{T}. \tag{20}$$

Given $x \in B_\rho$, Lemma 3.5 fixes $\bar{i}(x) < \infty$ such that

$$V(x) \ \leq \ \tilde{W}(x, T_i(x)) + \delta \qquad \forall\, i \geq \bar{i}(x). \tag{21}$$

Fix any $\bar{i} \geq \bar{i}(x)$ such that $T_{\bar{i}}(x) \geq \bar{T}$. Then, inequalities (19), (20) and (21) imply that

$$\begin{aligned}
\tilde{W}(x, T_{\bar{i}}(x)) &\geq \sup_{w \in \mathcal{W}[0,\tau)} \left\{ I_{[0,\tau]}(x,w) + V(\xi(\tau)) : \xi(0) = x \right\} + 3\delta \\
&\geq I_{[0,\tau]}(x, w_{x,\tau,T_{\bar{i}}(x)}^{\delta}) + V(\xi_{x,\tau,T_{\bar{i}}(x)}^{\delta}(\tau)) + 3\delta \\
&\geq I_{[0,\tau]}(x, w_{x,\tau,T_{\bar{i}}(x)}^{\delta}) + \tilde{W}(\xi_{x,\tau,T_{\bar{i}}(x)}^{\delta}(\tau), T_{\bar{i}}(x) - \tau) + 2\delta
\end{aligned}$$

where $w^{\delta}_{x,\tau,T_{\bar{i}}(x)} \in \mathcal{W}[0,\tau)$ is δ-optimal in the DPP for $\tilde{W}(x, T_{\bar{i}}(x))$. Hence,

$$\tilde{W}(x, T_{\bar{i}}(x)) \;\; \geq \sup_{w \in \mathcal{W}[0,\tau)} \Big\{ I_{[0,\tau]}(x,w) + \tilde{W}(\xi(\tau), T_{\bar{i}}(x) - \tau) : \xi(0) = x \Big\} + \delta$$

which is a contradiction by Lemma 3.2. Hence, the assumption is incorrect. That is, the inequality (18) holds with equality, proving (17). ■

3.2 Dynamic Programming Equations

By considering the DPPs (7) and (17) in the limit as $\tau \downarrow 0$, it is possible to show that the W and V satisfy respectively a variational inequality (VI) and a partial differetial equation (PDE). Since we will focus on max-plus mathods rather than PDE/VI based methods, the proof of the following theorem is omitted.

Theorem 3.7 *Define the Hamiltonian* $H(x,p) := l(x) + \sup_{w \in \mathcal{W}} \{p \cdot f(x,w) - \gamma(|w|)\}$. *Then, for all* $x \in \mathbf{R}^n$,

(i). W given by (5) satisfies the VI

$$\max\left(L(x) - W(x), H(x, \nabla_x W(x))\right) \; = \; 0, \quad \textit{and} \tag{22}$$

(ii). V given by (6) satisfies the PDE

$$H(x, \nabla_x V(x)) \; = \; 0 \tag{23}$$

4 Max-Plus Representations and Numerical Methods

Define the time-indexed operator (actually a sub-semigroup)

$$S_\tau[\phi] = \max\Big\{ \sup_{t \in [0,\tau)} \sup_{w \in \mathcal{W}[0,\tau)} Q_{[0,t]}(x,w), \sup_{w \in \mathcal{W}[0,\tau)} [I_\tau(x,w) + \phi(\xi(\tau))] \Big\} \tag{24}$$

and the semigroup

$$\mathcal{L}_\tau[\phi] = \sup_{w \in \mathcal{W}[0,\tau)} \Big[I_{[0,\tau]}(x,w) + \phi(\xi(\tau)) \Big] \tag{25}$$

where the domains are implicit. Then DPP (7) can be rewritten as

$$W(x) = S_\tau[W](x) = \max\Big\{ \sup_{t \in [0,\tau)} \sup_{w \in \mathcal{W}[0,\tau)} Q_{[0,t]}(x,w), \, \mathcal{L}_\tau[W](x) \Big\} \qquad \forall\, x \in \mathbf{R}^n, \tag{26}$$

and the DPP for V, (17) can be rewritten as

$$V(x) = \mathcal{L}_\tau[V](x) \qquad \forall\, x \in \mathbf{R}^n. \tag{27}$$

Recall that the max-plus algebra is defined over $\mathbf{R}_\oplus \doteq \mathbf{R} \cup \{-\infty\}$ where the max-plus addition and multiplication operations, $\oplus$ and $\otimes$, are defined as $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, respectively. One can easily show that $\mathcal{L}_\tau$ is a max-plus linear operator, that is, $\mathcal{L}_\tau[a\otimes\phi\oplus b\otimes\psi] = a\otimes\mathcal{L}_\tau[\phi]\oplus b\otimes\mathcal{L}_\tau[\psi]$ where ϕ,ψ lie in the domain of $\mathcal{L}_\tau$. Define $c_\tau : \mathbf{R}^n \to \mathbf{R}$ by

$$c_\tau(x) = \sup_{t\in[0,\tau)} \sup_{w\in\mathcal{W}[0,\tau)} Q_{[0,t]}(x,w) \qquad \forall\, x\in\mathbf{R}^n \tag{28}$$

where we implicitly assume that L, l, f are such that $c_\tau(x)$ is finite for all x (cf. [15]). Then, for all ϕ in the domain of $\mathcal{L}_\tau$,

$$S_\tau[\phi](x) = \max\{c_\tau(x), \mathcal{L}_\tau[\phi](x)\} \tag{29}$$
$$= \{c_\tau \oplus \mathcal{L}_\tau[\phi]\}(x) \qquad \forall\, x\in\mathbf{R}^n, \tag{30}$$

and consequently, S_τ is a max-plus affine operator. Note that our DPP for W, (7), now takes the form

$$W = c_\tau \oplus \mathcal{L}_\tau[W]. \tag{31}$$

Similarly, one easily has (see above) that our DPP for V, (17), takes the max-plus eigenvector form

$$0\otimes V = \mathcal{L}_\tau[V]. \tag{32}$$

We include the max-plus multiplication by the identity in this last equation to emphasize the eigenvector nature of the problem.

We need to define a space to which the value functions belong. We will use the space of semiconvex functions over $\mathbf{R}^n$, $S_c = S_c(\mathbf{R}^n)$. Note that $\phi\in S_c$ if given any $R<\infty$, there exists $c_R\in(0,\infty)$ such that $\phi(x)+\frac{c}{2}|x|^2$ is convex over $B_R(0)$. Since we will be mainly interested in solving for these value functions over some compact set, we also define $S_c^{R,c_R}(\mathbf{R}^n)$ to be the set of semiconvex functions for which $\phi(x)+\frac{c_R}{2}|x|^2$ is convex over $B_R(0)$, that is the set of those semiconvex functions which are semiconvex for a specific semiconvex constant c_R over over specific ball $B_R(0)$. The following result is typical under reasonable assumptions on the dynamics and cost. The proof is technical for infinite time-horizon problems such as those considered here, and so it is not included. See [15], [6] for similar results.

Theorem 4.1 *The value functions V and W are semiconvex, and consequently, given $R<\infty$, there exists $c_R<\infty$ such that $W, V\in S_c^{R,c_R}$.*

Semiconvex duality (a variant of convex duality) implies that for any symmetric positive definite matrix C such that $C - c_R I > 0$, one has

$$W(x) = \max_z\{a(z) + \widehat{\psi}(x,z)\} \tag{33}$$

with a given by

$$a(z) = \min_x\{W(x) - \widehat{\psi}(x,z)\} \tag{34}$$

for all $x\in B_R(0)$ where $\widehat{\psi}(x,z) \doteq -\frac{1}{2}(x-z)^T C(x-z)$; see [15], [6], [20], [19] for details. By restricting the z to some countable set $\{z_i\}_{i=1}^\infty$ (say the rationals) over a particular ball, this semiconvex duality becomes

$$W(x) = \sup_{i\in\{1,2,\ldots\}}\{a_i + \widehat{\psi}(x,z_i)\}) = \sup_{i\in\{1,2,\ldots\}}\{a(z_i) + \widehat{\psi}(x,z_i)\} \tag{35}$$

with the a_i given by

$$a_i = \min_x \{W(x) - \widehat{\psi}(x, z_i)\} \tag{36}$$

for all $x \in B_R(0)$; again see the references for details. Now we will suppose that the value function, $W(x)$ has a max-plus expansion with a specific, *finite* number of max-plus basis functions. This is not generally true of course, and one needs to perform an error analysis that indicates that the errors introduced by truncation of the basis expansion go to zero as the number of functions in the expansion goes to infinity. We delay the very long proof of such results so as to get to the core concepts. A proof for the H^∞case is given in [14], with portions appearing in [16] [17]. Thus, we assume for the purposes of this extended abstract that

$$W(x) = \max_{i \in \{1,2,\ldots,N\}} \{a_i + \widehat{\psi}(x, z_i)\} = \Big\{\bigoplus_{i=1}^{N} [a_i \otimes \psi_i]\Big\}(x) \tag{37}$$

with the a_i given as above. Similarly, suppose that for each $i \in \{1, 2, \ldots, N\}$ one has a finite max-plus expansion of $\mathcal{L}_\tau[\psi_i]$ which we denote by

$$\mathcal{L}_\tau[\psi_i] = \bigoplus_{j=1}^{N} [B_{j,i} \otimes \psi_j] \tag{38}$$

where $B_{j,i} = \min_x\{\mathcal{L}_\tau[\psi_i](x) - \psi_j(x)\}$, and also that

$$c_\tau = \bigoplus_{j=1}^{N} [c_j \otimes \psi_j] \tag{39}$$

where $c_j = \min_x\{c_\tau(x) - \psi_j(x)\}$. Then, by (31) and (37)

$$\bigoplus_{j=1}^{N} a_j \otimes \psi_j = c_\tau \oplus \mathcal{L}_\tau\Big[\bigoplus_{i=1}^{N} a_i \otimes \psi_i\Big]$$

which by (38) and (39)

$$\begin{aligned} &= \Big[\bigoplus_{j=1}^{N} c_j \otimes \psi_j\Big] \oplus \Big\{\bigoplus_{i=1}^{N} \otimes a_i \otimes \Big[\bigoplus_{j=1}^{N} B_{j,i} \otimes \psi_j\Big]\Big\} \\ &= \Big[\bigoplus_{j=1}^{N} c_j \otimes \psi_j\Big] \oplus \Big\{\bigoplus_{j=1}^{N} \Big[\bigoplus_{i=1}^{N} a_i \otimes B_{j,i}\Big] \otimes \psi_j\Big\} \\ &= \bigoplus_{j=1}^{N} \Big[c_j \oplus \bigoplus_{i=1}^{N} B_{j,i} \otimes a_i\Big] \otimes \psi_j \end{aligned} \tag{40}$$

Under an asummption that each basis function is active (i.e. required in the expansion), (40) implies that the vector of coefficients a_i, denoted simply by a satisfies the affine equation

$$a = c \oplus [B \otimes a] \tag{41}$$

where c is the vector of coefficients c_i and B is the $N \times N$ matrix of the $B_{j,i}$. In summary, under mild assumptions (and blindly truncating the max-plus basis expansions) one finds the following. (The reader should keep in mind that we are truncating the expansion, and the more exact statement would be $W \simeq \bigoplus_{i=1}^{N} a_i \otimes \psi_i$ with the error going to zero as $N \to \infty$ – see [14], [16], [17].)

Theorem 4.2 *The solution of DPP (7) is given by* $W = \bigoplus_{i=1}^N a_i \otimes \psi_i$ *where the vector of coefficients satisfies max-plus affine equation (41).*

Similarly, suppose for now that V has the finite expansion (but see [14], [16], [17] for discussion of the associated errors for this case for the slightly simpler case of $L \equiv 0$)

$$V(x) = \Big\{ \bigoplus_{i=1}^{N} [e_i \otimes \psi_i] \Big\}(x) \tag{42}$$

with the e_i given by

$$e_i = \min_x \{ V(x) - \widehat{\psi}(x, z_i) \}. \tag{43}$$

Then one has the following similar result.

Theorem 4.3 *The solution of DPP (17) is given by* $V = \bigoplus_{i=1}^N e_i \otimes \psi_i$ *where the vector of coefficients satisfies max-plus eigenvector equation*

$$0 \otimes e = e = B \otimes e. \tag{44}$$

4.1 Nonuniqueness for the Max-Plus Affine Equation

There are serious nonuniqueness issues for both the DPPs and the DPEs for both W and V. It will be simpler to quantify this lack of uniqueness with the technology below. Note that this nonuniqueness also appears in the above PDE and VI forms. Some (although not all) of these nonuniqueness issues also appear in the max-plus algebraic forms of these equations, (41) and (44).

In the case of V, the max-plus equation (44) is simply an eigenvector problem for eigenvalue zero. The following property can be shown to hold for some problem forms. In particular, it is shown to hold for the L_2-gain/H^∞ problem form under reasonable conditions on the dynamics and cost [15], [20], [19]. (Note that this form is equivalent to the case where $L = 0$ along with certain conditions on the integral cost form.)

B-Dissipation Property: Let $x_1 = 0$. $B_{1,1} = 0$, *and there exists* $N_i < \infty$, $\varepsilon > 0$ *such that for all* $\{k_i\}_{i=1}^{N_i}$ *such that* $k_1 = k_{N_i}$ *and not* $k_i = 1$ *for all* i, *one has* $\sum_{i=1}^{N_i - 1} B_{k_i, k_{i+1}} < -\varepsilon$.

We also suppose that $B_{j,i} \neq -\infty$ for all j, i, and this holds under reasonable conditions on the dynamics and choice of C in the basis functions. In particular, this has also been shown to hold in the H^∞case under reasonable assumptions [15]. The condition $B_{j,i} \neq -\infty$ for all j, i is sufficient (although not necessary) to guarantee that B has exactly one max-plus eigenvalue [3]. Further, under the additional condition of the B-Dissipation Property, there is a unique eigenvector (modulo max-plus multiplication by a scalar of course) corresponding to this eigenvalue [15], [20], [19]. (These uniqueness properties are obviously different from the properties one expects for eigenvalues and eigenvectors in the standard algebraic field.) Note that this is in contrast to the corresponding DPP and DPE. Further, this unique eigenvector is the yields the correct solution, i.e. the value function, for the original control problem.

Now, consider our max-plus affine problem (41). Suppose this problem has solution a^0. Also suppose that the eigenvector problem, (44), has solution e^0. Let $a^1 \doteq a^0 \oplus e^0$. Then

$$a^1 = a^0 \oplus e^0$$

$$= [c \oplus (B \otimes a^0)] \oplus (B \otimes e^0)$$
$$= c \oplus [(B \otimes a^0)] \oplus (B \otimes e^0)]$$
$$= c \oplus [B \otimes (a^0 \oplus e^0)] = c \oplus (B \otimes a^1).$$

Therefore, one has the following.

Theorem 4.4 *Solutions of (41) are at most unique modulo max-plus addition by a max-plus eigenvector corresponding to max-plus eigenvalue zero.*

This also yields a way to view nonuniqueness in the originating DPP and VI. More specifially, if W is a solution of the DPP or VI, and if V is a solution of the corresponding DPP or PDE for the problem with $\limsup$, then the pointwise maximum of W and V (i.e. max-plus sum of W and V) yields another solution of the DPP or VI for W.

4.2 The Affine Power Method

Given this lack of uniqueness in the DPP and variational inequality for W, and the corresponding lack of uniqueness in the max-plus affine equation (41), one would question how one would know that the solution that one computed to any of these characterizations was the correct solution (the value function). Interestingly, there is a method for solution of the max-plus affine equation (41) that yields this correct solution. The underlying reason that it yields the correct solution is that it corresponds to forward propagation of the original control problem. One particularly nice property of the solution method is that it converges exactly in a finite number of steps. This exact convergence rather than standard convergence (i.e given $\varepsilon > 0$, there exists $N_\varepsilon < \infty$ such that the solution is within ε of the limit after N_ε steps) is typical of problems in the max-plus algebra. Roughly speaking, it is due to the fact that $a \oplus b = a$ for any $b \leq a$, which is in contrast to the standard field where addition by anything other than the additive identity yields a sum different from the numbers being added.

Let $\mathbf{R}_\oplus \doteq \mathbf{R} \cup \{-\infty\}$. Let $F : \mathbf{R}_\oplus{}^N \to \mathbf{R}_\oplus{}^N$ be defined by

$$F[e] \doteq c \oplus (B \otimes e). \tag{45}$$

The max-plus affine power method will simply be repeated applicaton of F until one has convergence. The following lemma will be useful. It follows from the B-dissipation property. The proof can be found in [15], [20], [19], and so it is not included here.

Lemma 4.5 *Given any $a \in \mathbf{R}_\oplus{}^N$, and B (with elements in $\mathbf{R}_\oplus$), saitisfying the B-dissipation property, there exists $\overline{K} < \infty$ (dependent on a) such that $B^k \otimes a = B^{\overline{K}} \otimes a$ for all $k \geq \overline{K}$ where we note that the superscripts k and $\overline{K}$ on B indicate max-plus exponentiation (i.e the max-plus multiplication of B by itself k or $\overline{K}$ times).*

We will denote the initial vector for the afine power method as a^0. The following is a direct result of Lemma 4.5.

Lemma 4.6 *Given any $a^0, c \in \mathbf{R}_\oplus{}^N$, and B saitisfying the B-dissipation property, there exists $\widehat{K} < \infty$ such that*

$$B^k \otimes a^0 = B^{\widehat{K}} \otimes a^0, \text{ and } B^k \otimes c = B^{\widehat{K}} \otimes c$$

for all $k \geq \widehat{K}$.

Now note that for any $k \geq \widehat{K}$, one has

$$F^{k+1}[a^0] = \left[\bigoplus_{i=0}^{k} \left(B^i \otimes c\right)\right] \oplus \left(B^{k+1} \otimes a^0\right)$$
$$= \left[\bigoplus_{i=0}^{\widehat{K}} \left(B^i \otimes c\right)\right] \oplus \left(B^{\widehat{K}+1} \otimes a^0\right) = F^{\widehat{K}+1}[a^0].$$

Let

$$a^* \doteq \lim_{k\to\infty} F^k[a^0]. \tag{46}$$

Then

$$a^* = F^{\widehat{K}+1}[a^0]. \tag{47}$$

Further,

$$F[a^*] = F^{\widehat{K}+2}[a^0] = F^{\widehat{K}+1}[a^0] = a^*.$$

Consequently, one has

Theorem 4.7 *For any initial a^0, a^* given by (46) is a solution of (41).*

Not only is the limit a solution of (41) and achieved in a finite number of steps, it is also the correct solution of (41) in that it is the solution corresponding to the value function. Let the k^{th} iterate be $a^k = F^k[a^0]$. Also, define the corresponding k^{th} approximation of the solution to be $W^k(x) \doteq \bigoplus_{i=1}^N a_i^k \otimes \psi_i(x)$. Note that from the above, one has $W^k = W^{\widehat{K}+1}$ for all $k \geq \widehat{K}+1$. Let $W^*(x) \doteq \bigoplus_{i=1}^N a_i^* \otimes \psi_i(x) = \bigoplus_{i=1}^N a_i^{\widehat{K}+1} \otimes \psi_i(x)$. We will assume that a^0 is chosen such that

$$W^0(x) \leq W(x) \qquad \forall\, x \in \mathbf{R}^n \tag{IC}$$

where W is the value function. Note that this is an assumption on the a_i^0 coefficients. It is typically the case that one knows $W(x) \geq 0$ for all x. Consequently, if the basis functions are of the form $\psi(x) = -\frac{1}{2}(x-x_i)^T C(x-x_i)$ with C positive definite, then one only needs to take the $a_i^0 \leq 0$ in order to satisfy (IC).

Theorem 4.8 *W^* is the correct solution of the DPP (i.e. it is the value function of the original control problem).*

Proof: The result will follow by showing that repeated application of the F operator corresponds to forward propagation of the value function of a finite time-horizon problem. We only sketch the main points of the proof. First note that

$$W^1(x) = \bigoplus_{i=1}^N \Big\{ \Big[c_i \oplus (B \otimes a^0)_i\Big] \otimes \psi_i(x) \Big\}$$
$$= \left[\bigoplus_{i=1}^N c_i \otimes \psi_i(x)\right] \oplus \left[\bigoplus_{i=1}^N (B \otimes a^0)_i \otimes \psi_i(x)\right]$$

which by the definitions of c and B

$$= \max\Big\{c_\tau(x), \mathcal{L}_\tau[W^0](x)\Big\}$$
$$= \max\Big\{\sup_{t\in[0,\tau)} \sup_{w\in\mathcal{W}[0,\tau)} Q_{[0,t]}(x,w),\ \sup_{w\in\mathcal{W}[0,\tau)} [I_{[0,\tau]}(x,w) + W^0(\xi(\tau))]\Big\}. \tag{48}$$

Similarly,

$$W^2(x) = \max\Big\{ \sup_{t\in[0,\tau)} \sup_{w\in\mathcal{W}[0,\tau)} Q_{[0,t]}(x,w), \sup_{w\in\mathcal{W}[0,\tau)} [I_{[0,\tau(x,w)]} + W^1(\xi(\tau))] \Big\}$$

and using (48), this becomes

$$= \max\Big\{ \sup_{t\in[0,2\tau)} \sup_{w\in\mathcal{W}[0,2\tau)} Q_{[0,t]}(x,w), \sup_{w\in\mathcal{W}[0,2\tau)} [I_{[0,2\tau(x,w)]} + W^0(\xi(2\tau))] \Big\}. \tag{49}$$

By induction, one finds

$$W^k(x) = \max\Big\{ \sup_{t\in[0,k\tau)} \sup_{w\in\mathcal{W}[0,k\tau)} Q_{[0,t]}(x,w), \sup_{w\in\mathcal{W}[0,k\tau)} [I_{[0,k\tau(x,w)]} + W^0(\xi(k\tau))] \Big\}. \tag{50}$$

The next step is to note that given $\varepsilon > 0$, there exists $K_\varepsilon < \infty$ such that (by the definition of W)

$$W(x) \le \sup_{t\in[0,k\tau)} \sup_{w\in\mathcal{W}[0,k\tau)} Q_{[0,t]}(x,w) + \varepsilon$$

for any $k \ge K_\varepsilon$. Consequently, using (50), one has

$$W(x) \le W^k(x) + \varepsilon \tag{51}$$

for any $k \ge K_\varepsilon$. On the other hand, by the DPP of Theorem 3.1 one has

$$W(x) = \max\Big\{ \sup_{t\in[0,k\tau)} \sup_{w\in\mathcal{W}[0,k\tau)} Q_{[0,t]}(x,w), \sup_{w\in\mathcal{W}[0,k\tau)} [I_{[0,k\tau]}(x,w) + W(\xi(k\tau))] \Big\}$$

which by the condition (IC),

$$\begin{aligned} &\ge \max\Big\{ \sup_{t\in[0,k\tau)} \sup_{w\in\mathcal{W}[0,k\tau)} Q_{[0,t(x,w)]}, \sup_{w\in\mathcal{W}[0,k\tau)} [I_{[0,k\tau(x,w)]} + W^0(\xi(k\tau))] \Big\} \\ &= W^k(x). \end{aligned} \tag{52}$$

Combining (51) and (52) leads to the result. ∎

We have now obtained max-plus based techniques for the solution of the control problems addressed here. These techniques belong to an entirely new class of methods which are not related to finite elements or any other previously known class of methods. Along the way, we have developed a new way to represent the solutions of these control problems, such as by the max-plus affine formulation (41). This max-plus affine formulation leads to a new way to represent the nonuniqueness question in terms of max-plus linear algebra. Interestingly, the max-plus affine power method yields the correct solution of (41) in spite of the nonuniqueness.

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