THE PRINCIPLE OF LEAST ACTION AND FUNDAMENTAL SOLUTIONS OF MASS-SPRING AND N-BODY TWO-POINT BOUNDARY VALUE PROBLEMS *

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Abstract. Two-point boundary value problems for conservative systems are studied in the context of the least action principle. One obtains a fundamental solution, whereby two-point boundary value problems are converted to initial value problems via an idempotent convolution of the fundamental solution with a cost function related to the terminal data. The classical mass-spring problem is included as a simple example. The N-body problem under gravitation is also studied. There, the least action principle optimal control problem is converted to a differential game, where an opposing player maximizes over an indexed set of quadratics to yield the gravitational potential. Solutions are obtained as indexed sets of solutions of Riccati equations.

Key words. Least action, two-point boundary value problem, differential game, N-body problem, optimal control.

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1. Introduction.

We suppose a conservative system follows a trajectory which is a stationary point of the action functional, this being known as the principle of least (more correctly, stationary) action or as Hamilton's principle (c.f., [10, 11]). This allows the dynamical model to be posed in terms of various optimal control problems. Solution of these control problems allows one to convert two-point boundary value problems (TPBVPs) for the dynamical system into initial value problems (IVPs). For purposes of illustration, we will consider a simple mass-spring system, wherein solution of an associated Riccati equation generates the fundamental solution, and allows one to answer a variety of TPBVPs via a simple min-plus integral (equivalently, a supremum). We will also consider the N-body problem in orbital mechanics. There, the analysis becomes more technical. Nonetheless, one can construct machinery for guaranteed solution of various TPBVPs.

1.1. Least action, optimal control, and TPBVPs.

We begin with a somewhat formal discussion; specification of the exact assumptions will follow in the next section. Suppose the position component of the state at time, t, is denoted by $\xi(t) \in I\!\!R^n$, where also, we will use $x \in I\!\!R^n$ to denote generic positions. Let the potential energy at $x \in I\!\!R^n$ be denoted by $V(x)$. The kinetic energy at time t will be denoted by $T(\dot\xi(t)) \doteq \frac{1}{2}\dot\xi'(t)\mathcal{M}\dot\xi(t)$. If $\xi(t)$ is a point mass, $\mathcal{M}$ is simply mI, where m is the mass of the body; in a multi-body system, this is generalized in the obvious way. The action functional corresponding to $\{\xi(r) \,|\, r \in [0, t]\}$ is

$$\mathcal{F}(\xi(\cdot)) \doteq \int_0^t -V(\xi(r)) + T(\dot\xi(r))\,dr.$$

The original principle of least action stated that a system evolves so as to minimize the action functional. More recently, it has been understood that systems evolve so as to achieve a stationary point of the action functional (c.f., [11]).

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One can also interpret this in terms of the characteristic equations corresponding to the Hamiltonian of the system. Let the initial position be $\xi(0) = x \in \mathbb{R}^n$, and let the dynamics be $\dot{\xi}(r) = u(r)$ for all $r \in (0,t)$, where $u = u(\cdot) \in \mathcal{U}^{s,t}$, with $\mathcal{U}^{s,t} \doteq \mathscr{L}_2([s,t);\mathbb{R}^n)$. Also let

$$\mathcal{U}^\infty \doteq \{u : [0,\infty) \to \mathbb{R}^n \,|\, u_{[0,t)} \in \mathcal{U}^{0,t}\, \forall t \in [0,\infty)\}, \tag{1.1}$$

where $u_{[0,t)}$ denotes the restriction of the function to domain $[0,t)$. Define the control formulation payoff, $J^0 : [0,\infty) \times \mathbb{R}^n \times \mathcal{U}^\infty \to \mathbb{R} \cup \{-\infty,+\infty\}$, as

$$J^0(t,x,u) \doteq \int_0^t -V(\xi(r)) + T(u(r))\,dr = \int_0^t -V(\xi(r)) + \tfrac{1}{2}u'(r)\mathcal{M}u(r)\,dr, \tag{1.2}$$

where $\mathcal{M}$ is positive-definite symmetric, and the corresponding value function as

$$W^0(t,x) \doteq \inf_{u \in \mathcal{U}^\infty} J^0(t,x,u). \tag{1.3}$$

Clearly a solution of this problem yields an $\xi(\cdot)$ satisfying the least action principle, and so is the trajectory of the conservative system under potential energy field V, when the stationary action is the least.

Let $\mathcal{D} \doteq (0,t) \times \mathbb{R}^n$, $\bar{\mathcal{D}} \doteq [0,t] \times \mathbb{R}^n$, and $\hat{C}^1 \doteq C(\bar{\mathcal{D}}) \cap C^1(\mathcal{D})$. Under quite reasonable conditions on V, one can expect that $W^0 \in \hat{C}^1$, and that on $\mathcal{D}$, W^0 satisfies

$$\begin{aligned} 0 &= -\frac{\partial}{\partial r}W(r,x) + \inf_{v \in \mathbb{R}^n} \left\{ v \cdot \nabla_x W(r,x) + \tfrac{1}{2}v'\mathcal{M}v \right\} - V(x) && (1.4)\\ &= -\frac{\partial}{\partial r}W(r,x) - V(x) - \frac{1}{2}[\nabla_x W(r,x)]'\mathcal{M}^{-1}\nabla_x W(r,x) \\ &\doteq -\bar{H}\Big(r,x,\frac{\partial}{\partial r}W(r,x),\nabla_x W(r,x)\Big) \doteq -\frac{\partial}{\partial r}W(r,x) - H\Big(r,x,\nabla_x W(r,x)\Big). && (1.5) \end{aligned}$$

It is also well-established that under sufficiently strong conditions, first-order Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) such as (1.5) can be solved via the method of characteristics (c.f., [18]). The characteristic equations associated with (1.5) are

$$\frac{dr}{d\rho} = \bar{H}_q(r,\hat{\xi},q,\hat{p}) = 1, \qquad \frac{d\hat{\xi}}{d\rho} = \bar{H}_p(r,\hat{\xi},q,\hat{p}) = \mathcal{M}^{-1}\hat{p}(\rho) \tag{1.6}$$

$$\frac{dq}{d\rho} = -\bar{H}_r(r,\hat{\xi},q,\hat{p}) = 0, \qquad \frac{d\hat{p}}{d\rho} = -\bar{H}_x(r,\hat{\xi},q,\hat{p}) = -\nabla_x V(\hat{\xi}(\rho)). \tag{1.7}$$

These have associated initial and terminal conditions

$$\hat{\xi}(t) = x, \quad r(t) = 0, \quad \hat{p}(0) = 0, \quad q(0) = -V(\hat{\xi}(0)) - \frac{1}{2}(\hat{p}(0))'\mathcal{M}^{-1}\hat{p}(0) = -V(\hat{\xi}(0)), \tag{1.8}$$

where $\hat{p}(0) = 0$ follows from the lack of a terminal cost here. Because of (1.6), we may take $r = \rho$. Noting (1.7) and (1.8), we see that $q(r) = V(\hat{\xi}(0))$ for all r. Also, in order to return to forward time, we may take $s = t - r$, $\xi(s) = \hat{\xi}(t-s)$ and $p(s) = \hat{p}(t-s)$, in which case we have

$$\frac{d\xi}{ds} = -\mathcal{M}^{-1}p(s), \quad \frac{dp}{ds} = \nabla_x V(\xi(s)), \tag{1.9}$$

or,

$$\frac{d^2\xi}{ds^2} = -\mathcal{M}^{-1}\nabla_x V(\xi(s)), \tag{1.10}$$

which of course, is the classical Newton's second law formulation. Note that in the above development, the trajectory was not fully specified, as only the initial position, not the initial state (position and velocity), was given. Of course, (1.9) implies that the additive inverse of the co-state $p(r)$, is the momentum. (One might also note that the optimal velocity in (1.4) is attained at $v = -\mathcal{M}^{-1}\nabla_x W = -\mathcal{M}^{-1}p$.) Given both the initial position and initial velocity, forward integration of (1.9) is the classical IVP form for the system dynamics.

Suppose however, that one attaches a terminal cost to J^0 yielding, say

$$\bar{J}(t,x,u) = J^0(t,x,u) + \bar{\psi}(\xi(t)), \tag{1.11}$$

$$\overline{W}(t,x) = \inf_{u\in\mathcal{U}^\infty} \bar{J}(t,x,u), \tag{1.12}$$

where $\mathcal{U}^\infty$ is given by (1.1). The dynamic programing equation (DPE) and characteristic equations (1.9) remain unchanged. However, although the initial condition is still $\xi(0) = x$, the terminal condition is defined by $\bar{\psi}$. That is, we have a TPBVP where we control the terminal condition.

TPBVPs are common in classical optimal control theory, where the above characteristic equations appear in Calculus of Variations and Pontryagin Maximum Principle approaches (c.f., [20]). There, one is required to solve the relevant TPBVP to obtain the desired optimal control problem solution. Classical methods used a shooting approach, and more modern methods such as pseudo-spectral algorithms (c.f., [19]) have greatly advanced the state of the art.

Here we have a slightly different goal; we desire to solve TPBVPs that are constrained by conservative dynamics, i.e. those dynamics that conserve the (instantaneous) total energy defined as the sum of the potential and kinetic energies V and T. For the trajectory $\xi(\cdot)$ of (1.9), this total energy at time $s \in I\!\!R$ is given by

$$V(\xi(s)) + T(\dot{\xi}(s)) = H(s,\, \xi(s), \mathcal{M}\,\dot{\xi}(s))\,,$$

where H is the Hamiltonian of (1.5). Noting that this Hamiltonian is invariant with respect to its first argument, differentiation with respect to s along the trajectory $\xi(\cdot)$ yields via (1.10) that

$$\begin{aligned}\tfrac{d}{ds}H(s,\xi(s),\mathcal{M}\,\dot{\xi}(s)) &= \langle H_x(s,\xi(s),\mathcal{M}\,\dot{\xi}(s)),\dot{\xi}(s)\rangle + \langle H_p(s,\xi(s),\mathcal{M}\,\dot{\xi}(s)),\,\mathcal{M}\,\ddot{\xi}(s)\rangle \\ &= \langle \nabla_x V(\xi(s)),\,\dot{\xi}(s)\rangle + \langle \dot{\xi}(s),\,-\nabla_x V(\xi(s))\rangle = 0\,,\end{aligned}$$

for all $s \in [0,t]$. That is, the conservative dynamics of interest here are precisely those defined by (1.9), which in turn are defined by the characteristic equations (1.6) and (1.7) associated with the optimal control problem (1.3). With the addition of terminal cost $\bar{\psi}$ in this optimal control problem, the boundary conditions for (1.9), and hence the conservative dynamics of interest, consist of initial and terminal conditions

$$\xi(0) = x, \qquad p(t) = \nabla_x\bar{\psi}(\xi(t)). \tag{1.13}$$

If one takes, for example, $\bar{\psi}(x) = -x'\mathcal{M}\bar{v}$ for some given $\bar{v} \in I\!\!R^n$, then the terminal condition in (1.13) becomes $p(t) = -\mathcal{M}\bar{v}$. That is, one has boundary conditions

$$\xi(0) = x \quad \text{and} \quad \dot{\xi}(t) = \bar{v}. \tag{1.14}$$

Alternatively, if one takes $z \in I\!\!R^n$ and $\bar{\psi}(x) = \psi^\infty(x) \doteq \delta_0^-(x - z)$ where

$$\delta_0^-(y) \doteq \begin{cases} 0 & \text{if } y = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{1.15}$$

(i.e., the min-plus delta function, c.f., [17, 25]), then the solution of control problem (1.12) yields solution of the conservative system (1.9) with boundary conditions

$$\xi(0) = x \quad \text{and} \quad \xi(t) = z. \tag{1.16}$$

Clearly, other boundary conditions can be generated as well.

1.2. Fundamental solutions. The goal here will be the development of *fundamental solutions* for TPBVPs corresponding to conservative systems of the form (1.9). For a problem involving dynamical systems, we use the term *fundamental solution* to indicate an object which, once obtained for a specific time-horizon, yields the solution of that problem for other input data via an operation on the object and the input data, without re-propagation over time. For example, the operator $e^{At} \in \mathcal{L}(I\!R^n)$ defined with respect to $A \in I\!R^{n\times n}$ is a fundamental solution for the finite-dimensional linear initial value problem, $\dot{\xi}(s) = A\,\xi(s)$, $\xi(0) = x \in I\!R^n$, $s \in [0,t]$, as we have $\xi(t) = e^{At}\,x$ for any specific initial data $x \in I\!R^n$. As a substep in the analysis to follow, we will obtain a one-parameter semigroup of min-plus linear min-plus integral operators $\{\mathcal{G}^\oplus(t)\}_{t\in I\!R_{\ge 0}}$ that serves as a *min-plus primal space fundamental solution semigroup* [5, 31] for the optimal control problem (1.11), (1.12), or equivalently, the HJB PDE (1.5). As this optimal control problem is formulated to encapsulate the least action principle, the fundamental solution semigroup $\{\mathcal{G}^\oplus(t)\}_{t\in I\!R_{\ge 0}}$ can also be used as a fundamental solution semigroup for TPBVPs constrained by the conservative dynamics (1.9). In particular, given any terminal data of the form $\xi(t) = z$ or $\dot{\xi}(t) = \bar{v}$ for these dynamics, $\{\mathcal{G}^\oplus(t)\}_{t\in I\!R_{\ge 0}}$ can be used to evaluate the corresponding value function $\overline{W}(t,x)$ of (1.12), and the hence solve the corresponding TPBVP (with initial data $\xi(0) = x \in I\!R^n$). Also, in the case of the N-body problem, it will shown that the fundamental solution can be interpreted in terms of a convex set, $\widehat{\Sigma}(t) \subset I\!R^{2N^2+N+1}$, which can be used to generate the fundamental solution kernel $\overline{W}^\infty(t,\cdot,\cdot)$.

A specific element $\mathcal{G}^\oplus(t)$ of the aforementioned min-plus primal space fundamental solution semigroup $\{\mathcal{G}^\oplus(t)\}_{t\in I\!R^n_{\ge 0}}$ is an operator that propagates *any* terminal payoff through to the corresponding value function of the optimal control problem (1.11), (1.12) at horizon $t \in I\!R_{\ge 0}$ via a (min-plus linear) min-plus integration (or *convolution*, cf. [1, 12, 22, 24]). That is,

$$\overline{W}(t,\cdot) = \mathcal{G}^\oplus(t)\,\bar{\psi} \doteq \int_{I\!R^n}^\oplus \overline{W}^\infty(t,\cdot,z) \otimes \bar{\psi}(z)\,dz\,, \tag{1.17}$$

in which the min-plus integral is defined in general by $\int_{I\!R^n}^\oplus f(z)\,dz = \inf_{z\in I\!R^n} f(z)$ for any functional $f : I\!R^n \to I\!R \cup \{\infty\}$, and $\overline{W}^\infty(t,\cdot,\cdot) : I\!R^n \times I\!R^n \to I\!R \cup \{\infty\}$ denotes the associated *kernel*. Existence of the min-plus primal-space fundamental solution semigroup $\{\mathcal{G}^\oplus(t)\}_{t\in I\!R}$ is guaranteed by dynamic programming, which requires that

$$\overline{W}(t+\tau,\cdot) = \mathcal{S}(t+\tau)\,\bar{\psi} = \mathcal{S}(\tau)\,\mathcal{S}(t)\,\bar{\psi}\,, \tag{1.18}$$

$$\mathcal{S}(t)\,\bar{\psi} \doteq \int_{\mathcal{U}^\infty}^\oplus J^0(t,x,u) \otimes \bar{\psi}(\xi(t))\,du, \tag{1.19}$$

for all $t,\tau \in I\!R$ for which the value is finite, where $\mathcal{S}(t)$ is the *dynamic programming evolution operator*. Operator $\mathcal{S}(t)$ is itself a min-plus linear min-plus integral operator, by definition (1.19). The dynamic programming principle, as stipulated by the right-hand equality of (1.18), defines a one-parameter semigroup of these min-plus linear min-plus integral dynamic programming evolution operators $\{\mathcal{S}(t)\}_{t\in I\!R_{\geq 0}}$ for the optimal control problem (1.11), (1.12). (This is the so-called *Lax-Oleinik semigroup.*) Consequently, as (1.17) and (1.18) imply that $\mathcal{G}^{\oplus}(t)\,\bar{\psi} = \mathcal{S}(t)\,\bar{\psi}$ for any horizon $t \in I\!R_{\geq 0}$ and terminal payoff $\bar{\psi}$, it follows immediately that

$$\mathcal{G}^{\oplus}(t+\tau) = \mathcal{G}^{\oplus}(\tau)\,\mathcal{G}^{\oplus}(t) \tag{1.20}$$

for any $t, \tau \in I\!R_{\geq 0}$ for which the value function $\overline{W}(t+\tau,\cdot)$ is finite. That is, $\{\mathcal{G}^{\oplus}(t)\}_{t\in I\!R_{\geq 0}}$ defines a one-parameter semigroup of min-plus linear min-plus integral operators as per (1.17). Furthermore, an explicit representation for the kernel $\overline{W}^{\infty}(t,\cdot,\cdot)$ of $\mathcal{G}^{\oplus}(t)$ follows by inspection of (1.17), (1.18), (1.19). In particular, the identity $\bar{\psi}(x) = \int_{I\!R^n}^{\oplus} \delta_0^-(x-z)\otimes\bar{\psi}(z)\,dz$, which holds for all $x \in I\!R^n$ given the min-plus delta function δ_0^- of (1.15), the min-plus linearity of $\mathcal{S}(t)$ evident by definition (1.19), and the left-hand equality of (1.18), together imply that

$$\overline{W}(t,\cdot) = \mathcal{S}(t)\,\bar{\psi} = \mathcal{S}(t)\,\int_{I\!R^n}^{\oplus} \delta_0^-(\cdot - z)\otimes\bar{\psi}(z)\,dz = \int_{I\!R^n}^{\oplus} [\mathcal{S}(t)\,\delta_0^-(\cdot - z)]\otimes\bar{\psi}(z)\,dz\,,$$

so that by (1.17),

$$\overline{W}^{\infty}(t,x,z) = [\mathcal{S}(t)\,\delta_0^-(\cdot - z)](x)\,. \tag{1.21}$$

That is, the kernel $\overline{W}^{\infty}(t,x,z)$ of the min-plus primal-space fundamental solution $\mathcal{G}^{\oplus}(t)$ is itself the value of an optimal control problem defined with respect to initial and final states $x, z \in I\!R^n$ by (1.21).

REMARK 1.1. Using the notation of (1.17), (1.19), (1.20), it is important to note that the Lax-Oleinik semigroup $\{\mathcal{S}(t)\}_{t\in I\!R_{\geq 0}}$ does *not* define a min-plus primal-space fundamental solution semigroup, as its elements are defined as min-plus integral min-plus linear operators over $\mathcal{U}^{\infty}$ rather than $I\!R^n$. While this may seem to be a formal detail, it is crucial from the point of view of computation. In particular, in applying elements of $\{\mathcal{G}^{\oplus}(t)\}_{t\in I\!R_{\geq 0}}$, as opposed to $\{\mathcal{S}(t)\}_{t\in I\!R_{\geq 0}}$, minimization over a finite dimensional (rather than infinite dimensional) space is required. ◇

1.3. Application. With regard to the specific optimal control problem (1.11), (1.12), in the case where the potential energy T takes a linear-quadratic form, the kernel $\overline{W}^{\infty}(t,\cdot,\cdot)$ of the min-plus primal space fundamental solution $\mathcal{G}^{\oplus}(t)$ can be obtained through solution of an associated Riccati equation. Here, we will use only a simple mass-spring example to demonstrate the concept, although a combination of this approach with previously developed machinery for solution of certain infinite-dimensional problems [6, 7, 8] has yielded corresponding min-plus primal-space fundamental solutions for certain TPBVPs for infinite-dimensional systems [5].

We will also apply the approach to N-body problems under the gravitational potential. In that case, the potential does not take a linear-quadratic form. However, we will see that one may take a dynamic game approach to gravitation, where the potential is a linear-quadratic form in the position variable. This requires an additional max-plus integral, over the opponent controls, beyond that which is required in the purely linear-quadratic potential case.

In order to give a sense of the usefulness of the approach, two example problem classes are considered. The first is a simple mass-spring oscillator, which should be useful due to its simplicity, and this is discussed in Section 3. A deeper problem class, that of N-body problems, is considered in Section 4.

In the mass-spring case, one obtains an explicit solution for $\overline{W}^{\infty}$ in (3.10)–(3.11). Fix $t > 0$, $\sin(\omega t) \neq 0$ (for more details, see Section 3). Suppose one has the fundamental solution in form (3.10)–(3.11). Then, for a TPBVP generated by initial and terminals positions, say x and z in $I\!R$, the initial velocity solving the TPBVP is given by

$$v = \dot{\xi}(0) = \mathcal{M}^{-1}(Q_t^{\infty} x + R_t^{\infty} z), \tag{1.22}$$

where $\mathcal{M}$ is the mass. If instead, one is given a TPBVP generated by an initial position, x and terminal velocity, $\bar{v} \in I\!R$, the corresponding initial velocity is given by (1.22), but with $z = z^* = (\mathcal{M}\bar{v} - Q_t^{\infty} x)/R_t^{\infty}$.

The N-body problem class is more challenging, and as noted above, the fundamental solution, $\overline{W}^{\infty}$ is represented as the finite-dimensional convex set, $\widehat{\Sigma}(t)$. Given a set of N initial positions as vector $x \in I\!R^{3N}$ and a set of terminal positions as vector $z \in I\!R^{3N}$ (along with time-duration, $t \in (0, \bar{t})$, with $\bar{t}$ satisfying (4.56), and body masses and radii m_i, R_i, $i \in \{1, 2, \dots N\}$, where the radii are included for technical modeling reasons indicated below), the initial velocities solving the TPBVP are given by (4.106),(4.107). That is, once $\widehat{\Sigma}(t)$ is computed (more exactly, approximated), one may repeatedly use $\widehat{\Sigma}(t)$ for changing values of x and z. Specifically, in (4.100) for any x, z pair, $\overline{W}^{\infty}(t, x, z)$ is obtained as the supremum of a linear functional over the finite-dimensional convex set $\widehat{\Sigma}(t)$. The N-body case where initial position and terminal velocity are given is a bit more complex, and discussed further in Section 4.7.

Remark 1.2. It is worth noting that this is not the first instance in which idempotent methods have been used to address problems in this class. Notably, [26] takes a similar viewpoint on this topic, within a larger context. On another front, other authors have found it useful to introduce a game-theoretic interpretation as an aid in the study of dynamical systems, cf. [21]. ⋄

2. General Theory.

We now begin the rigorous development. As indicated above, we consider conservative systems, and take the least-action approach. (That is, in this paper, we concentrate on the case where the stationary action is least – see Lemma 4.17 and [10, 11, 13] as well.)

2.1. Optimal control problem.

We model the dynamics of position as

$$\dot{\xi}(r) = u(r), \qquad \xi(0) = x \in I\!R^n, \tag{2.1}$$

with $u \in \mathcal{U}^{\infty}$. Let the potential and kinetic energy functions be denoted by $V(x)$ and $T(y) = \frac{1}{2} y' \mathcal{M} y$, respectively. Recalling (1.2), we now have

$$J^0(t, x, u) = \int_0^t L(\xi(r), u(r))\, dr \doteq \int_0^t T(u(r)) - V(\xi(r))\, dr. \tag{2.2}$$

Throughout this section, we employ the following assumptions:

$\mathcal{M}$ is positive-definite and symmetric. $(A.\mathcal{M})$

There exists $D_V < \infty$ such that $V(x) \leq D_V$ for all $x \in I\!R^n$. $(A.V1)$

There exists $K_L, K_L^1 < \infty$ such that $|V(x) - V(z)| \leq K_L |x - z|$, and $|V(x)| \leq K_L^1 (1 + |x|)$. $(A.V2)$

(Of course, in $(A.V2)$, the existence of such a K_L^1 follows from the existence of K_L, but we find it useful to introduce both constants.)

REMARK 2.1. We note that $(A.V1), (A.V2)$ are violated in the case of the idealized harmonic oscillator given in Section 3. However, in that example one may nonetheless obtain a closed-form solution. In the N-body application class of Section 4, although $(A.V2)$ is violated if one assumes point-mass bodies, it is satisfied if one assumes the bodies have positive radius and bounded density. Such models are discussed further in Section 4. $\diamond$

For $c \in [0,\infty)$, let $\psi^c : I\!R^n \times I\!R^n \to [0,\infty)$ be given by

$$\psi^c(x,z) = \frac{c}{2}|x-z|^2. \tag{2.3}$$

Also let $\psi^\infty : I\!R^n \times I\!R^n \to [0,\infty]$ (where $[0,\infty] \doteq [0,\infty) \cup \{+\infty\}$) be given by

$$\psi^\infty(x,z) = \lim_{c\to\infty} \psi^c(x,z) = \delta_0^-(x-z), \tag{2.4}$$

where δ_0^- is given in (1.15). Define the finite time-horizon payoffs $\bar{J}^c : [0,\infty) \times I\!R^n \times \mathcal{U}^\infty \to I\!R \cup \{-\infty, +\infty\}$ by

$$\bar{J}^c(t,x,u,z) \doteq J^0(t,x,u) + \psi^c(\xi(t),z), \tag{2.5}$$

for $c \in [0,\infty]$, where we specifically note that $J^0(t,x,u) = \int_0^t L(\xi(s), u(s))\,ds$. Also, for $c \in [0,\infty]$, we let

$$\overline{W}^c(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} \bar{J}^c(t,x,u,z). \tag{2.6}$$

Value functions where one also notes dependence on terminal state components sometimes appear in the literature as "generating functions", specifically in reference to two-point boundary value problems (c.f., [14]). As in the introduction, for generic terminal cost, $\bar{\psi} \in \mathscr{L}_2(I\!R^n; I\!R)$, we continue to let

$$\bar{J}(t,x,u) \doteq J^0(t,x,u) + \bar{\psi}(\xi(t)), \quad \text{and} \quad \overline{W}(t,x) \doteq \inf_{u\in\mathcal{U}^\infty} \bar{J}(t,x,u). \tag{2.7}$$

We begin with general theory; results specific to application in mass-spring and N-body systems will follow in later sections.

LEMMA 2.2. *$\overline{W}^c(t,x,z) \ge -D_V t$ for all $x,z \in I\!R^n$ and all $t \ge 0$. Also, suppose there exists $D,R < \infty$ such that $V(y) \ge -D$ for all $y \in B_R(0)$. Then $\overline{W}^c(t,x,z) \le Dt + \frac{1}{2}\min\{c, \|\mathcal{M}\|/t\}|x-z|^2 \le Dt + \psi^c(x,z)$ for all $x,z \in B_R(0)$ and $t \ge 0$. More generally, for $\bar{\psi} : I\!R^n \to I\!R$, $\overline{W}^c(t,x) \in I\!R$ for all $t \in (0,\infty)$ and all $x \in I\!R^n$. Lastly, note that for $c \ge \hat{c}$, $\overline{W}^c(t,x,z) \ge \overline{W}^{\hat{c}}(t,x,z)$ for all $t \ge 0$ and $x,z \in I\!R^n$.*

Proof. To obtain the first assertion, note that for any $u \in \mathcal{U}^\infty$, $\bar{J}^c(t,x,u,z) \ge \int_0^t -V(\xi(r))\,dr \ge -D_V t$, where D_V is given in Assumption $(A.V1)$. For the second assertion, let $\bar{u}(s) = 0$ for all $s \in (0,t)$. Then, $\bar{J}^c(t,x,\bar{u},z) = \int_0^t -V(x)\,dr + \psi^c(x,z) \le Dt + \psi^c(x,z)$, which implies $\overline{W}^c(t,x,z) \le Dt + \psi^c(x,z)$. Alternatively, let $\tilde{u}(r) = \frac{1}{t}(z-x)$ for all $r \in (0,t)$. Then, the corresponding trajectory satisfies $\tilde{\xi}(r) \in B_R(0)$ for all $r \in [0,t]$, and we have $\bar{J}^c(t,x,\tilde{u},z) = \int_0^t \frac{1}{2t^2}(z-x)'\mathcal{M}(z-x) - V(\tilde{\xi}(r))\,dr + \psi^c(z,z) \le \frac{\|\mathcal{M}\|}{2t}|x-z|^2 + Dt$, which implies $\overline{W}^c(t,x,z) \le \frac{\|\mathcal{M}\|}{2t}|x-z|^2 + Dt$. For the third assertion, simply take $\bar{u} \equiv 0$. The final assertion is immediate by the definition. $\square$

One expects that $\overline{W}^c$ will be a viscosity solution on $(0,\infty) \times I\!R^n$ of

$$0 = -\partial_1 W(t,x,z) - H(t,x,\nabla_x W(t,x,z)) = -\bar{H}(t,x,\partial_1 W(t,x,z), \nabla_x W(t,x,z)) \tag{2.8}$$

$$W(0,x,z) = \psi^c(x,z) \quad x \in I\!R^n, \tag{2.9}$$

where we will find it handy to use the notation ∂_1 to denote the partial derivative with respect to the first variable throughout, and $H, \bar{H}$ are the Hamiltonians (1.5). In fact, we have the following:

THEOREM 2.3. *Given $c \in [0,\infty)$ and $z \in I\!R^n$, the value function $\overline{W}^c(\cdot,\cdot,z)$ of (2.7) is Lipschitz continuous on compact sets, and is the unique viscosity solution of (2.8),(2.9).*

Proof. This follows immediately from [3], where we specifically use Proposition 1.3 and Theorems 2.1, 2.2 and 3.2 there. □

2.2. A limit property. In order to characterize the fundamental solution to the optimal control problem (2.7), it is useful to first demonstrate that a specific limit property holds. In particular, it is demonstrated via a sequence of lemmas that $\lim_{c\to\infty} \overline{W}^c = \overline{W}^\infty$. Lemmas 2.5 and 2.6 provide bounds on near-optimal trajectories defined with respect to $\overline{W}^c$, leading to a sandwiching of $\overline{W}^c$ using $\overline{W}^\infty$. The required limit property is then stated via Theorem 2.7 and Corollary 2.8.

By the positive-definiteness of $\mathcal{M}$, there exists $m > 0$ such that

$$T(v) \ge \frac{m}{2}|v|^2, \quad \forall v \in I\!R^n. \tag{2.10}$$

Let $t > 0$. The "straight-line" control from x to z is given by $u^s_r = (1/t)[z - x]$ for all $r \in [0,t]$, and we let the corresponding trajectory be denoted by $\xi^s_\cdot$. The resulting cost is

$$\widetilde{W}^s(t,x,z) \doteq \bar{J}^c(t,x,u^s,z) \le K_L^1(1+|x|+|z|)t + \frac{\|\mathcal{M}\||z-x|^2}{2t},$$

which for an appropriate choice of $D_1 = D_1(t) < \infty$,

$$\le D_1(t)[1+|x|^2+|z|^2], \quad \forall x,z \in I\!R^n. \tag{2.11}$$

REMARK 2.4. We have $\overline{W}^\infty(t,x,z) \le \widetilde{W}^s(t,x,z) \le D_1(t)[1+|x|^2+|z|^2]$ for all $t \in (0,\infty)$ and all $x,z \in I\!R^n$. ⋄

LEMMA 2.5. *There exists $\hat{D} = \hat{D}(t) < \infty$ such that for any ϵ-optimal trajectory, ξ^ϵ (i.e., any trajectory ξ^ϵ corresponding to an ϵ-optimal input in the definition (2.6)) with $\epsilon \in (0,1]$, $|\xi^\epsilon(r)| \le \hat{D}[1+|x|+|z|]$ for all $0 \le r \le t < \infty$ and $x,z \in I\!R^n$.*

Proof. Let $t > 0$ and $x,z \in I\!R^n$. Let $u^\epsilon \in \mathcal{U}^\infty$ be ϵ-optimal in the definition (2.6) of $\overline{W}^c$ with $\epsilon \in (0,1]$, and let ξ^ϵ be the corresponding trajectory. Let

$$R \doteq \max\{|\xi^\epsilon(r)| \,|\, r \in [0,t]\}, \quad \tau \in \operatorname{argmax}\{|\xi^\epsilon(r)| \,|\, r \in [0,t]\}. \tag{2.12}$$

Note that by Hölder's inequality,

$$R = |\xi^\epsilon(\tau)| \le \sqrt{\tau}\|u^\epsilon\|_{L_2(0,\tau)} + |x| \le \sqrt{t}\|u^\epsilon\|_{L_2(0,t)} + |x|. \tag{2.13}$$

Now, using Assumption $(A.V2)$, (2.10) and (2.12),

$$\bar{J}^c(t,x,u^\epsilon,z) \ge \int_0^t -V(\xi^\epsilon(r)) + T(u^\epsilon(r))\,dr \ge -K_L^1(1+R)t + \frac{m}{2}\|\dot{\xi}^\epsilon\|^2_{L_2(0,t)},$$

which by (2.13),

$$\ge -K_L^1(1+R)t + \frac{m}{2t}(R-|x|)^2.$$

Consequently, considering the quadratic inequality in R given by

$$-K_L^1(1+R)t + \frac{m}{2t}(R-|x|)^2 - \left[K_L^1(1+|x|+|z|)t + \frac{\|\mathcal{M}\||z-x|^2}{2t} + 1\right] \ge 0,$$

and solving the quadratic equality by classical methods, we see that there exists $\hat{D} = \hat{D}(t) < \infty$ such that

$$\bar{J}^c(t,x,u^\epsilon,z) > K_L^1(1+|x|+|z|)t + \frac{\|\mathcal{M}\||z-x|^2}{2t} + 1 \geq \widetilde{W}^s(t,x,z)+1 \geq \overline{W}^c(t,x,z)+\epsilon$$

if $R > \hat{D}[1+|x|+|z|]$, which contradicts the ϵ-optimality of u^ϵ. Hence, $R \leq \hat{D}(1+|x|+|z|)$, completing the proof. □

LEMMA 2.6. *There exists $\tilde{D} = \tilde{D}(t) < \infty$ such that for ϵ-optimal controls, $u^{c,\epsilon} \in \mathcal{U}^\infty$, with $\epsilon \in (0,1]$, $|\xi^{c,\epsilon}(t) - z| \leq \frac{\tilde{D}[1+|x|+|z|]}{\sqrt{c}}$, for all $c,t > 0$ and $x,z \in I\!R^n$.*

Proof. Let $\epsilon \in (0,1]$, $c,t > 0$ and $x,z \in I\!R^n$. By Assmp. $(A.V2)$ and Lemma 2.5,

$$\bar{J}^c(t,x,u^{c,\epsilon},z) \geq -K_L^1(1+\hat{D}[1+|x|+|z|])t + \frac{c}{2}|\xi^{c,\epsilon}(t)-z|^2. \tag{2.14}$$

On the other hand,

$$\overline{W}^\infty(t,x,z) \geq \overline{W}^c(t,x,z) \geq \bar{J}^c(t,x,u^{c,\epsilon},z) - \epsilon \geq \bar{J}^c(t,x,u^{c,\epsilon},z) - 1. \tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\begin{aligned}\frac{c}{2}|\xi^{c,\epsilon}(t)-z|^2 &\leq \overline{W}^\infty(t,x,z) + K_L^1(1+\hat{D}[1+|x|+|z|])t \\ &\leq \widetilde{W}^s(t,x,z) + K_L^1(1+\hat{D}[1+|x|+|z|])t,\end{aligned}$$

which by (2.11),

$$\leq D_1(t)[1+|x|^2+|z|^2] + K_L^1(1+\hat{D}[1+|x|+|z|])t. \quad \square$$

THEOREM 2.7. *There exists $\check{D} = \check{D}(t) < \infty$ such that*

$$\overline{W}^\infty(t,x,z) - \frac{\check{D}}{\sqrt{c}}[1+|x|+|z|]^2 \leq \overline{W}^c(t,x,z) \leq \overline{W}^\infty(t,x,z),$$

for all $t \in (0,\infty)$, $x,z \in I\!R^n$ and $c \geq 1$.

Proof. Clearly, $\overline{W}^c(t,x,z) \leq \overline{W}^\infty(t,x,z)$ for all $t,c \in (0,\infty)$ and $x,z \in I\!R^n$. We concentrate on the other bound. Let $u^{c,\epsilon}$ be ϵ-optimal for $\overline{W}^c(t,x,z)$, with $\epsilon \in (0,1]$, and let $\xi^{c,\epsilon}$ denote the corresponding trajectory. Also for $r \in [0,t]$, let

$$\hat{u}^{c,\epsilon}(r) \doteq u^{c,\epsilon}(r) + (1/t)[z - \xi^{c,\epsilon}(t)], \quad \text{which yields} \quad \hat{\xi}^{c,\epsilon}(t) = z. \tag{2.16}$$

Further, using Lemma 2.6, this implies

$$|\hat{\xi}^{c,\epsilon}(r) - \xi^{c,\epsilon}(r)| = \frac{r}{t}|\xi^{c,\epsilon}(t) - z| \leq \frac{r\tilde{D}[1+|x|+|z|]}{t\sqrt{c}}, \quad \forall r \in [0,t]. \tag{2.17}$$

Next, note that

$$\begin{aligned}\bar{J}^c(t,x,u^{c,\epsilon},z) &= \int_0^t -V(\xi^{c,\epsilon}(r)) + T(u^{c,\epsilon}(r))\,dr + \psi^c(\xi^{c,\epsilon}(t),z) \\ &\leq \overline{W}^c(t,x,z) + \epsilon \leq \widetilde{W}^s(t,x,z) + 1,\end{aligned}$$

which implies

$$\frac{m}{2}\|u^{c,\epsilon}\|^2_{L_2(0,t)} \leq \int_0^t V(\xi^{c,\epsilon}(r))\,dr + \widetilde{W}^s(t,x,z) + 1,$$

which by Assumption $(A.V2)$, (2.11) and Lemma 2.5

$$\le K_L^1(1+\hat{D}(t)[1+|x|+|z|])t + D_1(t)[1+|x|^2+|z|^2]+1.$$

This implies there exists $D_2 = D_2(t) < \infty$ such that

$$\|u^{c,\epsilon}\|_{L_2(0,t)} \le D_2(t)[1+|x|+|z|]. \tag{2.18}$$

Now, recalling that $|a-b|^2 \le |a-b|(|a|+|b|)$ for all $a, b \in I\!R^n$, one has

$$\Big|\int_0^t T(u^{c,\epsilon}(r))\,dr - \int_0^t T(\hat{u}^{c,\epsilon}(r))\,dr\Big| \le \frac{\|\mathcal{M}\|}{2}\int_0^t |u^{c,\epsilon}(r)-\hat{u}^{c,\epsilon}(r)|(|u^{c,\epsilon}(r)|+|\hat{u}^{c,\epsilon}(r)|)\,dr$$

which by the definition of $\hat{u}^{c,\epsilon}$,

$$\le \frac{\|\mathcal{M}\|}{2t}\,|z-\xi^{c,\epsilon}(t)|\int_0^t (|u^{c,\epsilon}(r)|+|\hat{u}^{c,\epsilon}(r)|)\,dr$$
$$\le \frac{\|\mathcal{M}\|}{2t}\,|z-\xi^{c,\epsilon}(t)|\int_0^t \left[\frac{|z-\xi^{c,\epsilon}(t)|}{t}+2|u^{c,\epsilon}(r)|\right]dr$$
$$\le \frac{\|\mathcal{M}\|}{2t}\,|z-\xi^{c,\epsilon}(t)|\big[|z-\xi^{c,\epsilon}(t)|+2\sqrt{t}\|u^{c,\epsilon}\|_{L_2(0,t)}\big]$$

(where the last bound follows by Hölder's inequality), which by Lemma 2.6 and (2.18),

$$\le \frac{D_3(t)[1+|x|+|z|]^2}{\sqrt{c}}, \tag{2.19}$$

for all $x, z \in I\!R^n$ and all $c \in [1,\infty)$ for proper choice of $D_3(t) < \infty$. Also, by Assumption $(A.V2)$,

$$\left|\int_0^t -V(\xi^{c,\epsilon}(r))\,dr - \int_0^t -V(\hat{\xi}^{c,\epsilon}(r))\,dr\right| \le K_L\int_0^t \big|\xi^{c,\epsilon}(r)-\hat{\xi}^{c,\epsilon}(r)\big|\,dr,$$

which by (2.17),

$$\le \frac{K_L\tilde{D}[1+|x|+|z|]t}{2\sqrt{c}} \tag{2.20}$$

By (2.16), (2.19), (2.20) (and noting that $\psi^c \ge 0$),

$$\bar{J}^c(t,x,u^{c,\epsilon},z) - \bar{J}^c(t,x,\hat{u}^{c,\epsilon},z) \ge -\frac{D_3(t)[1+|x|+|z|]^2}{\sqrt{c}} - \frac{K_L\tilde{D}[1+|x|+|z|]t}{2\sqrt{c}}$$
$$\ge -\frac{\breve{D}(t)[1+|x|+|z|]^2}{\sqrt{c}},$$

for an appropriate choice of $\breve{D}(t) < \infty$. This implies

$$\bar{J}^c(t,x,u^{c,\epsilon},z) \ge \overline{W}^\infty(t,x,z) - \frac{\breve{D}(t)[1+|x|+|z|]^2}{\sqrt{c}},$$

and since this is true for all $\epsilon \in (0,1]$, $\overline{W}^c(t,x,z) \ge \overline{W}^\infty(t,x,z) - \frac{\breve{D}(t)[1+|x|+|z|]^2}{\sqrt{c}}$, which completes the proof. □

Of course, Theorem 2.7 immediately implies:

Corollary 2.8. *The value functions $\overline{W}^c$ and $\overline{W}^\infty$ of (2.6) satisfy the limit property* $\lim_{c\to\infty}\overline{W}^c(t,x,z) = \overline{W}^\infty(t,x,z)$ *for all $t \in (0,\infty)$, $x, z \in I\!R^n$.*

REMARK 2.9. Note that

$$\begin{aligned}\overline{W}^c(t,x,z) &= \inf_{u\in\mathcal{U}^\infty} \Big\{ \int_0^t L(\xi(s),u(s))\,ds + \tfrac{c}{2}|\xi(t)-z|^2 \Big\} \\ &= \inf \Big\{ \int_0^t L(\xi(s),u(s))\,ds + \psi^\infty(\xi(t),y) + \tfrac{c}{2}|y-z|^2 \,\Big|\, u\in\mathcal{U}^\infty,\ y\in I\!R^n \Big\} \\ &= \inf_{y\in I\!R^n} \big\{ \overline{W}^\infty(t,x,y) + \tfrac{c}{2}|y-z|^2 \big\},\end{aligned}$$

which implies that $\overline{W}^c(t,x,\cdot)$ is a Moreau envelope of $\overline{W}^\infty(t,x,\cdot)$. Consequently, an alternative means for obtaining Corollary 2.8 is through verification of necessary conditions for convergence of the Moreau envelope to $\overline{W}^\infty(t,x,\cdot)$; see, for example, [28], Theorem 1.25. ◇

2.3. Fundamental solution. A reachability problem of interest is defined via the value function $\widetilde{W} : I\!R_{\geq 0} \times I\!R^n \times I\!R^n \mapsto I\!R$, where

$$\widetilde{W}(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} \Big\{ \int_0^t L(\xi(s),u(s))\,ds \,\Big|\, \xi(t)=z \Big\}, \tag{2.21}$$

where ξ satisfies (2.1) with $\xi(0)=x$. Using $\widetilde{W}$ of (2.21), it is convenient to define the function $\widehat{W} : I\!R_{\geq 0} \times I\!R^n \mapsto I\!R$ by

$$\widehat{W}(t,x) \doteq \inf_{z\in I\!R^n} \Big\{ \widetilde{W}(t,x,z) + \bar\psi(z) \Big\}. \tag{2.22}$$

PROPOSITION 2.10. *$\overline{W}(t,x) = \widehat{W}(t,x)$ for all $t\in I\!R_{\geq 0}$ and $x\in I\!R^n$.*
Proof. Fix $t\in I\!R_{\geq 0}$, $x\in I\!R^n$. By (2.21) and (2.22),

$$\begin{aligned}\widehat{W}(t,x) &= \inf_{z\in I\!R^n} \Big\{ \inf_{u\in\mathcal{U}^\infty} \Big[\int_0^t L(\xi(s),u(s))\,ds \,\Big|\, \xi(t)=z \Big] + \bar\psi(z) \Big\} \\ &= \inf \Big\{ \int_0^t L(\xi(s),u(s))\,ds + \bar\psi(\xi(t)) \,\Big|\, u\in\mathcal{U}^\infty,\ z\in I\!R^n,\ z=\xi(t) \Big\} \\ &= \inf \Big\{ \int_0^t L(\xi(s),u(s))\,ds + \bar\psi(\xi(t)) \,\Big|\, u\in\mathcal{U}^\infty \Big\} = \overline{W}(t,x). \qquad \square\end{aligned}$$

From Theorem 2.10, we see that $\widetilde{W}$ is a fundamental solution for optimal control problem (2.1),(2.7). In particular, given a terminal cost function $\bar\psi(\cdot)$, we have

$$\overline{W}(t,x) = \inf_{z\in I\!R^n} \big[\widetilde{W}(t,x,z) + \bar\psi(z) \big] = \int_{I\!R^n}^{\oplus} \widetilde{W}(t,x,z) \otimes \bar\psi(z)\,dz,$$

where in the last expression we use min-plus algebra notation (cf. [1, 22, 24]). We next demonstrate the equivalence of $\widetilde{W}$ and $\overline{W}^\infty$, thereby showing that $\overline{W}^\infty$ is a fundamental solution for (2.1),(2.7).

PROPOSITION 2.11. *$\overline{W}^\infty(t,x,z) = \widetilde{W}(t,x,z)$ for all $t\in I\!R_{>0}$ and $x,z\in I\!R^n$, and $\overline{W}(t,x) = \widehat{W}(t,x) = \inf_{z\in I\!R^n}\{\overline{W}^\infty(t,x,z) + \bar\psi(z)\}$ for all $t\in I\!R_{>0}$ and $x\in I\!R^n$.*

Proof. Fix $t\in I\!R_{>0}$ and $x,z\in I\!R^n$. Considering control $\tilde u(t) = \frac{1}{t}(z-x)$ as in the proof of Lemma 2.2, we immediately see

$$\overline{W}^\infty(t,x,z) < \infty. \tag{2.23}$$

By (2.4)–(2.7) (in the case $c = \infty$) and (2.23), we see

$$\begin{aligned}\overline{W}^{\infty}(t,x,z) &= \inf\left\{ \bar{J}^{\infty}(t,x,u,z) \,\middle|\, u \in \mathcal{U}^{\infty},\, \xi(t) = z \right\} \\ &= \inf\left\{ \bar{J}^{0}(t,x,u) \,\middle|\, u \in \mathcal{U}^{\infty},\, \xi(t) = z \right\} = \widetilde{W}(t,x,z),\end{aligned}$$

which yields the first assertion. The second assertion then follows from Proposition 2.10. □

3. Application: a simple mass-spring system.

3.1. Model. We consider the standard example: A mass $\mathcal{M} \in (0,\infty)$ is fixed to a vertical wall via an elastic spring with spring constant $K \in (0,\infty)$, with the mass free to move horizontally. Friction is neglected. Newton's second law implies that the position ξ satisfies the ordinary differential equation (ODE)

$$0 = \ddot{\xi}(t) + \omega^2\, \xi(t) \tag{3.1}$$

where $\omega \doteq \sqrt{K/\mathcal{M}}$ is the frequency of oscillation. The potential and kinetic energy associated with the spring and mass respectively are given by

$$V(x) \doteq \tfrac{K}{2}\, x^2\,, \quad T(\dot{\xi}) \doteq \tfrac{\mathcal{M}}{2}\, (\dot{\xi})^2\,. \tag{3.2}$$

In this case, our Hamiltonian becomes

$$H(x,p) = \tfrac{K}{2} x^2 - \inf_{v \in I\!R} \left\{ vp + \tfrac{\mathcal{M}}{2} p^2 \right\} = \tfrac{K}{2} x^2 + \tfrac{1}{2\mathcal{M}} p^2. \tag{3.3}$$

As the potential energy for this idealized spring is quadratic (with potential energy possibly going to $+\infty$), Assumptions $(A.V1)$ and $(A.V2)$ are violated, and we cannot employ Lemma 2.2 or Theorem 2.3. However, we will have an explicit solution of the HJB PDE, and consequently, we will use the following instead.

THEOREM 3.1. *Let $c \in (0,\infty)$, $z \in I\!R^n$, $0 < t < \widehat{T} < \infty$. Suppose $W \in C([0,\widehat{T}) \times I\!R^n \times I\!R^n; I\!R) \cap C^1((0,\widehat{T}) \times I\!R^n \times I\!R^n; I\!R)$ satisfies (2.8),(2.9). Then, $W(t,x,z) \le \bar{J}^c(t,x,u,z)$ for all $x \in I\!R^n$, $u \in \mathcal{U}^{\infty}$. Furthermore, $W(t,x,z) = \bar{J}^c(t,x,u^*,z)$ for the input $u^*(s) \doteq -\mathcal{M}^{-1}\nabla_x W(t-s,\xi^*(s),z)$, $s \in [0,t]$, where ξ^* is the solution of dynamics (2.1), driven by u^*. Consequently $W(t,x,z) = \overline{W}^c(t,x,z)$.*

Proof. With $z \in I\!R^n$ fixed, let W denote a solution of (2.8), (2.9) as per the theorem statement. Fix any $t \in [0,\widehat{T})$ and any $\bar{u} \in \mathcal{U}^{\infty}$. Define $\pi(v) \doteq p \cdot v + \frac{1}{2} v'\mathcal{M}v$, $p \in I\!R^n$, and note that by completion of squares that $\pi(v) \ge -\frac{1}{2}\, p'\mathcal{M}^{-1}p$. Select $v = \bar{u}(s)$ and $p = \nabla_x W(t-s,\bar{\xi}(s),z)$ at each $s \in [0,t]$, where $\bar{\xi}$ denotes the trajectory satisfying (2.1) corresponding to input $\bar{u}$. Then,

$$\begin{aligned}&\nabla_x W(t-s,\bar{\xi}(s),z) \cdot \bar{u}(s) + \tfrac{1}{2}\, \bar{u}(s)'\mathcal{M}\bar{u}(s) \\ &\ge -\tfrac{1}{2}[\nabla_x W(t-s,\bar{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W(t-s,\bar{\xi}(s),z)\,,\end{aligned}$$

so that (2.8) and (1.5) imply that for all $s \in [0,t]$,

$$\begin{aligned}0 &= -\partial_1 W(t-s,\bar{\xi}(s),z) - V(\bar{\xi}(s)) - \tfrac{1}{2}\, [\nabla_x W(t-s,\bar{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W(t-s,\bar{\xi}(s),z) \\ &\le -\partial_1 W(t-s,\bar{\xi}(s),z) + \nabla_x W(t-s,\bar{\xi}(s),z) \cdot \bar{u}(s) + \tfrac{1}{2}\, \bar{u}(s)'\mathcal{M}\bar{u}(s) - V(\bar{\xi}(s)) \\ &= \tfrac{\partial}{\partial s} W(t-s,\bar{\xi}(s),z) + \tfrac{1}{2}\, \bar{u}(s)'\mathcal{M}\bar{u}(s) - V(\bar{\xi}(s))\,.\end{aligned}$$

Integrating with respect to s over $[0,t]$ then yields (via the fundamental theorem of calculus and (2.9)) that

$$W(t,x,z) \le \int_0^t L(\bar{\xi}(s),\, \bar{u}(s))\, ds + \psi^c(\bar{\xi}(t),z) = \bar{J}^c(t,x,\bar{u},z)\,, \tag{3.4}$$

proving the first assertion. To prove the second assertion, fix $\bar{u} \doteq u^*$, where u^* is as indicated in the theorem statement. Repeating the above argument yields equality in (3.4), so that $W(t,x,z) = \bar{J}^c(t,x,u^*,z) = \overline{W}^c(t,x,z)$ as required. □

3.2. Fundamental solution of the mass-spring system. Analogues of Theorems 2.7 and 2.10 and Proposition 2.11 provide a path for solution of the optimal control problem with value function $\overline{W}$ of (2.7) associated with the principle of least action. In particular, Theorem 2.10 provides a characterization of $\overline{W}$ in terms of $\widetilde{W}$ of (2.21), which is in turn equivalent to $\overline{W}^\infty$ of (2.6) by Proposition 2.11. However, $\overline{W}^\infty$ may be obtained as the limit case of $\overline{W}^c$ of (2.6) as $c \to \infty$ by Theorem 2.7, when sufficiently smooth, where $\overline{W}^c$ may be obtained by solving (2.8),(2.9). To this end, let $\widehat{T} \doteq \pi/\omega$, and define the time-indexed quadratic function $\breve{W}^c : [0,\widehat{T}) \times \mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\breve{W}^c(t,x,z) = \tfrac{1}{2}\, P_t\, x^2 + Q_t\, x\, z + \tfrac{1}{2}\, R_t\, z^2\,, \tag{3.5}$$

where $P_t,\, Q_t,\, R_t \in \mathbb{R}$ satisfy the IVPs on $[0,\widehat{T})$ given by

$$\dot{P}_t = -K - \tfrac{1}{\mathcal{M}} P_t^2, \qquad \dot{Q}_t = -\tfrac{1}{\mathcal{M}}\, P_t\, Q_t, \qquad \dot{R}_t = -\tfrac{1}{\mathcal{M}}\, Q_t^2\,, \tag{3.6}$$

$$P_0 = c, \qquad Q_0 = c, \qquad R_0 = c. \tag{3.7}$$

THEOREM 3.2. *The value function $\overline{W}^c$ of (2.6) and the explicit function $\breve{W}^c$ of (3.5) are equivalent. That is, $\overline{W}^c(t,x,z) = \breve{W}^c(t,x,z)$ for all $t \in [0,\widehat{T})$, $x, z \in \mathbb{R}$.*

Proof. By inspection of (3.5), note that

$$\partial_1 \breve{W}^c(t,x,z) = \tfrac{1}{2}\, \dot{P}_t\, x^2 + \dot{Q}_t\, x\, z + \tfrac{1}{2}\, \dot{R}_t\, z^2\,, \tag{3.8}$$

$$\nabla_x \breve{W}^c(t,x,z) = P_t\, x + Q_t\, z\,. \tag{3.9}$$

By inspection of (3.6), (3.7), (3.8) and (3.9), observe that for all $t \in (0,\widehat{T})$ and $x, z \in \mathbb{R}$,

$$\begin{aligned} 0 &= -\left[\tfrac{1}{2}\, \dot{P}_t\, x^2 + \dot{Q}_t\, x\, z + \tfrac{1}{2}\, \dot{R}_t\, z^2 + (\tfrac{K}{2})\, x^2 + (\tfrac{1}{2\mathcal{M}})\, (P_t\, x + Q_t\, z)^2\right] \\ &= -\partial_1 \breve{W}(t,x,z) - H(x,\, \breve{W}(t,x,z))\,, \end{aligned}$$

where H is the Hamiltonian (3.3). That is, (2.8) holds for $\breve{W}$. Also observe that $\breve{W}^c(0,x,z) = \frac{c}{2}\, x^2 - c\, x\, z + \frac{c}{2}\, z^2 = \psi^c(x,z)$, where ψ^c is as per (2.3). That is, (2.9) also holds for $\breve{W}$. Hence, Theorem 3.1 yields the desired result. □

Theorem 3.2 and the unbounded-potential analogue of Corollary 2.8 may be used to explore the limit case of $\overline{W}^c$ of (2.6) as $c \to \infty$. This limiting case can be approached explicitly by solving (3.6), (3.7) for arbitrary fixed $c \in \mathbb{R}_{>0}$ followed by taking the aforementioned limit. Applying Theorem 2.7 then yields $\overline{W}^\infty$ of (2.6), and hence $\widetilde{W}$ of (2.21) by Proposition 2.11. By inspection of (3.6), first note that it is convenient to compute the inverse of P_t to facilitate computation of the limiting case. To this end, define $\alpha\, \pi_t \doteq P_t^{-1}$, or $P_t\, \pi_t = \frac{1}{\alpha}$, where $\alpha \in \mathbb{R}_{>0}$ is fixed. Differentiation yields $\dot{P}_t\, \pi_t + P_t\, \dot{\pi}_t = 0$, or

$$\dot{\pi}_t = -\alpha\, \pi_t\, \dot{P}_t\, \pi_t = -\alpha\, \pi_t \left(-K - \tfrac{1}{\mathcal{M}}\, P_t^2\right) \pi_t = \alpha\, K\, \pi_t^2 + \tfrac{1}{\alpha\, \mathcal{M}} = \alpha\, K \left(\pi_t^2 + \tfrac{1}{\alpha^2\, K\, \mathcal{M}}\right)\,.$$

For convenience, select $\alpha \doteq \frac{1}{\sqrt{K\,\mathcal{M}}}$, so that $\left(\frac{1}{1+\pi_t^2}\right) \dot{\pi}_t = \omega$. Let $t \in (0,\widehat{T})$. Integration over the interval $[0,t]$ yields $\tan^{-1} \pi_t \big|_0^t = \omega\, t$, or $\pi_t = \tan\left(\tan^{-1} \pi_0 + \omega\, t\right) =$

$\tan\left(\omega\, t + \tan^{-1}(\frac{1}{\alpha\, c})\right)$. As $c \to \infty$, $\pi_t \to \pi_t^\infty$, where $\pi_t^\infty \doteq \tan(\omega\, t)$. Equivalently,

$$P_t \to P_t^\infty \doteq \frac{1}{\alpha\, \pi_t^\infty} = \frac{1}{\alpha\, \tan(\omega\, t)}$$

as $c \to \infty$. Similarly, one obtains

$$Q_t = \frac{-c\, \sin \tan^{-1}(\frac{1}{\alpha\, c})}{\sin\left(\omega\, t + \tan^{-1}(\frac{1}{\alpha\, c})\right)} \to Q_t^\infty \doteq \frac{-1}{\alpha\, \sin \omega\, t} \qquad \text{as } c \to \infty,$$

and

$$R_t = \tfrac{1}{\alpha^2\, c} \left(\frac{1}{1 + (\frac{1}{\alpha\, c})^2} \right) + \tfrac{1}{\alpha} \left(\frac{1}{1 + (\frac{1}{\alpha\, c})^2} \right) \cot\left(\omega\, t + \tan^{-1}(\tfrac{1}{\alpha\, c})\right) \to R_t^\infty \doteq \frac{1}{\alpha\, \tan(\omega\, t)}$$

as $c \to \infty$. Hence, in the case of the mass-spring system, Proposition 2.11 and Theorem 3.2 and the unbounded-potential analogue of Corollary 2.8 imply that for $t \in (0, \pi/\omega)$,

$$\widetilde{W}(t,x,z) = \overline{W}^\infty(t,x,z) = \tfrac{1}{2}\, P_t^\infty\, x^2 + Q_t^\infty\, x\, z + \tfrac{1}{2}\, R_t^\infty\, z^2, \tag{3.10}$$

$$\text{where} \quad P_t^\infty = (\tfrac{1}{\alpha}) \cot(\omega\, t)\,, \quad Q_t^\infty = -(\tfrac{1}{\alpha})\text{cosec}(\omega\, t)\,, \quad R_t^\infty = (\tfrac{1}{\alpha}) \cot(\omega\, t)\,. \tag{3.11}$$

3.3. Usage in a two-point boundary value problem. As an application of Theorem 2.10, consider the case where the terminal velocity $\bar{v}$ is known. As the state of (2.1) corresponds to the position of the mass, the additive inverse of the co-state defined via the value function $\overline{W}$ of (2.7) corresponds to the momentum of the mass. As the final co-state is $\nabla_x \bar{\psi}(x(t))$, knowledge of the final momentum $\mathcal{M}\, \bar{v}$ implies that $\nabla_x \bar{\psi}(x(t)) = -\mathcal{M}\, \bar{v}$, which in turn implies a terminal cost of

$$\bar{\psi}(z) = -\mathcal{M}\, \bar{v}\, z\,. \tag{3.12}$$

Let $t \in (0, \pi/\omega)$. Applying Theorem 2.10, and using (2.22), the terminal position $z^*(t,x) \in I\!\!R$ corresponding to initial position $x \in I\!\!R$ and terminal velocity $\bar{v} = \dot{x}(t)$ is

$$\begin{aligned} z^*(t,x,\bar{v}) &\doteq \operatorname*{argmin}_{z \in I\!\!R} \left\{ \widetilde{W}(t,x,z) - \mathcal{M}\, \bar{v}\, z \right\} \qquad (3.13) \\ &= \operatorname*{argmin}_{z \in I\!\!R} \left\{ \tfrac{1}{2}\, P_t^\infty\, x^2 + Q_t^\infty\, x\, z + \tfrac{1}{2}\, R_t^\infty\, z^2 - \mathcal{M}\, \bar{v}\, z \right\}\,. \end{aligned}$$

Hence, by inspection, $0 = Q_t^\infty\, x + R_t^\infty\, z^*(t,x,\bar{v}) - \mathcal{M}\, \bar{v}$, so that

$$z^*(t,x,\bar{v}) = \frac{\mathcal{M}\, \bar{v} - Q_t^\infty\, x}{R_t^\infty} = (\tfrac{\bar{v}}{\omega})\, \tan(\omega\, t) + \sec(\omega\, t)\, x\,. \tag{3.14}$$

In order to check (3.14), the dynamics of the mass-spring system may be integrated explicitly. In particular, using general solution $\xi(t) = A \cos(\omega\, t) + B \sin(\omega\, t)$, and solving for A, B from $\xi(0) = x$ and $\dot{\xi}(t) = \bar{v}$, one may check the above solution.

4. The N-body problem. Here, we address the solution of TPBVPs with N bodies acting under gravitational acceleration. That is, we obtain a means for conversion of TPBVPs into initial value problems. The key to application of the above approach to this class of problems lies in a variation of convex duality, leading to an interpretation of the least action principle as a zero-sum, differential game. Due to the particular, simple form of this game, one may invert the order of the minimization and maximization operations, after which the inner game problem is reduced to solution of differential Riccati equations. This leads to a representation of $\overline{W}^\infty$ as a maximization of a linear functional over a finite-dimensional, convex set.

4.1. A representation for the gravitational potential energy. We begin with a representation of the gravitational potential energy of the N-body problem as a pointwise maximum of quadratic forms.

LEMMA 4.1. *For $\rho \in (0,\infty)$, one has*

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2} \max_{\alpha\in(0,\infty)} \alpha\left[1 - \frac{(\alpha\rho)^2}{2}\right] = \left(\frac{3}{2}\right)^{3/2} \max_{\alpha\in[0,\sqrt{2/3}\rho^{-1}]} \alpha\left[1 - \frac{(\alpha\rho)^2}{2}\right].$$

Proof. Suppose $f : (0,\infty) \to I\!R$ is given by $f(\hat\rho) = \hat\rho^{-1/2}$. By standard methods of convex duality (c.f., [27, 28, 29]), one has the convex duality pair

$$f(\hat\rho) = \sup_{\hat\beta<0}\left[\hat\beta\hat\rho + a(\hat\beta)\right] \quad \forall \hat\rho \in (0,\infty),$$
$$a(\hat\beta) = -\sup_{\hat\rho>0}\left[\hat\beta\hat\rho - f(\hat\rho)\right] \quad \forall \hat\beta \in (-\infty,0).$$

Further, $a(\hat\beta) = \frac{-3}{2}(2\hat\beta)^{1/3}$ for all $\hat\beta \in (-\infty,0)$. Next, letting $\beta \doteq -\hat\beta$, this yields

$$\hat\rho^{-1/2} = \sup_{\beta>0}\left[\frac{3}{2}(2\beta)^{1/3} - \beta\hat\rho\right], \quad \forall \hat\rho > 0.$$

Letting $\alpha = \sqrt{\frac{2}{3}}(2\beta)^{1/3}$ for $\beta > 0$, one finds

$$\hat\rho^{-1/2} = \sup_{\alpha\geq 0}\left[\left(\frac{3}{2}\right)^{3/2}\alpha - \left(\frac{3}{2}\right)^{3/2}\frac{\alpha^3\hat\rho}{2}\right], \quad \forall \hat\rho > 0.$$

Finally, letting $\hat\rho = \rho^2$ for $\rho > 0$, one sees that this becomes

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2} \sup_{\alpha\geq 0} \alpha\left[1 - \frac{(\alpha\rho)^2}{2}\right], \quad \forall \rho > 0.$$

Lastly, note that the supremum is always attained, and does so at $\sqrt{\frac{2}{3}\frac{1}{\rho}}$. □

From Lemma 4.1, one immediately obtains the following.

LEMMA 4.2. *Given any $\bar\delta \in (0,\infty)$ and any $\rho \in [\bar\delta,\infty)$, one has*

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2} \max_{\alpha\in[0,\sqrt{2/3}\bar\delta^{-1}]} \alpha\left[1 - \frac{(\alpha\rho)^2}{2}\right],$$

while for $\rho \in (0,\bar\delta)$, one has

$$\max_{\tilde\rho\geq\bar\delta}\frac{1}{\tilde\rho} \leq \left(\frac{3}{2}\right)^{3/2} \max_{\alpha\in[0,\sqrt{2/3}\bar\delta^{-1}]} \alpha\left[1 - \frac{(\alpha\rho)^2}{2}\right] < \frac{1}{\rho}.$$

Recall that the gravitational potential energy due to two point masses of mass m_1 and m_2, separated by distance $\rho > 0$, is given by

$$\mathcal{G}^{m_1,m_2}(\rho) = \frac{-Gm_1m_2}{\rho},$$

where G is the universal gravitational constant. Of course, this is also valid for spherically symmetric bodies when the distance is greater than the sum of the radii of the bodies. Using Lemma 4.1, we see that this may be represented as

$$-\mathcal{G}^{m_1,m_2}(\rho) = \widehat{G}\, m_1 \max_{\alpha_{1,2}\geq 0}(\alpha_{1,2}m_2)\left[1-\frac{(\alpha_{1,2}\rho)^2}{2}\right],$$

where the universal gravitational constant is replaced by $\widehat{G} \doteq \left(\frac{3}{2}\right)^{3/2} G$. In the case of N bodies at locations $x^i \in I\!R^3$ for $i \in \mathcal{N} \doteq]1, N[$ (where for integers $i < j$, we let $]i, j[$ denote $\{i, i+1, i+2, \dots j\}$ throughout), the additive inverse of the potential is given by

$$-\widetilde{V}(x) = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i \max_{\alpha_{i,j}\geq 0}(\alpha_{i,j}m_j)\left[1-\frac{(\alpha_{i,j}|x^i-x^j|)^2}{2}\right] = \sum_{(i,j)\in\mathcal{I}^\Delta}\frac{Gm_im_j}{|x^i-x^j|}, \quad (4.1)$$

where $\mathcal{I}^\Delta \doteq \{(i,j) \in]1,N[^2 \,|\, j > i\}$ and $x = \{x^1, x^2, \dots x^N\} \in I\!R^n \doteq (I\!R^3)^N$. In view of Lemma 4.2, we fix some $\bar{\delta} > 0$, and use instead,

$$-V(x) = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i \max_{\alpha_{i,j}\in[0,\sqrt{2/3}\bar{\delta}^{-1}]}(\alpha_{i,j}m_j)\left[1-\frac{(\alpha_{i,j}|x^i-x^j|)^2}{2}\right]. \quad (4.2)$$

Throughout, we will largely suppress the dependence of V on the body masses.

Remark 4.3. The classical N-body problem formulation employs gravitational potential energy models of the form (4.1). That is, it assumes the gravitational potential model Gm/r_c for a central body of mass m, where r_c denotes the distance from the body center. This model has beautiful simplicity, but is only exact in the case of a point-mass, i.e., a body of zero radius and infinite density. Further, and importantly, this model possesses an asymptote in the potential at $r = 0$, which is both technically problematic and purely an artifact of the model. If one instead assumes a spherical, positive-radius body of uniform density, the gravitational potential is obtained as

$$-V^p(x) = \begin{cases} \frac{Gm}{r_c} & \text{if } r_c \in [R_c, \infty) \\ \widehat{G}m\breve{\alpha}\left[1-\frac{\breve{\alpha}^2 r_c^2}{2}\right] & \text{if } r_c \in [0, R_c], \end{cases} \quad (4.3)$$

where $\breve{\alpha} \doteq [\sqrt{3/2}R_c]^{-1}$ and R_c denotes the body radius. (We note that this is also correct for $r \in [R_c, \infty)$ when the density is only spherically symmetric.) Note that such potentials are not only more realistic for macroscopic bodies, but also eliminate the vertical asymptote of the classical problem-definition model. The next result demonstrates that for the realistic case where the bodies have positive radii, one may choose $\bar{\delta}$ such that V and $\widetilde{V}$ yield identical solutions. In particular, if $\bar{\delta}$ is less than the minimal sum of body radii, then both models are identical for any solution in which the bodies do not collide. It is perhaps worth remarking that in the case of one body of uniform density and a second body which is essentially a point-mass, then model (4.2), being similar to (4.3), would be correct when the point-mass was within the radius of the larger body. (This is not an entirely academic point, as gravitation within diffuse bodies is relevant, for example, in the study of the motion of stars within galaxies.) Allowing $\bar{\delta}$ to vary as a function of the pairs of bodies, one can go a bit further along that line [15]. ⋄

Lemma 4.4. *Suppose* $|x^i - x^j| \geq \bar{\delta}$ *for all* $(i,j) \in \mathcal{I}^\Delta$. *Then* $-V(x) = -\widetilde{V}(x)$. *Otherwise,* $-V(x) \leq -\widetilde{V}(x)$.

The above lemma is immediate from Lemma 4.2, and we do not include a proof.

4.2. The differential-game model. The above representation for the gravitational potential, (4.2), will be used to create a zero-sum game representation of the N-body least-action problem, where an additional player will maximize over time-indexed functions taking values in $\mathcal{A}$ (defined in (4.4)).

$$\mathcal{A} \doteq \left\{ \alpha = \{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}^\Delta} \,\middle|\, \alpha_{i,j} \in [0, \sqrt{2/3}\bar{\delta}^{-1}]\, \forall (i,j) \in \mathcal{I}^\Delta \right\}, \tag{4.4}$$

and note that $\mathcal{A} \subset I\!R^{I^\Delta}$ where $I^\Delta \doteq \#\mathcal{I}^\Delta$. Then (4.2) may be written as

$$-V(x) = \max_{\alpha\in\mathcal{A}}\{-\hat{V}(x,\alpha)\}, \quad -\hat{V}(x,\alpha) \doteq \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i(\alpha_{i,j} m_j) \left[1 - \frac{(\alpha_{i,j}|x^i - x^j|)^2}{2}\right]. \tag{4.5}$$

Let $\xi(\cdot)$ be a trajectory of the N-body system satisfying (2.1). The running cost will again be

$$L(\xi(r), \dot{\xi}(r)) = T(\dot{\xi}(r)) - V(\xi(r)), \tag{4.6}$$

where now V is given by (4.5). Also, let

$$\mathcal{M} \doteq \mathrm{diag}(m_1, m_1, m_1, m_2, m_2, m_2, \ldots m_N) = \mathrm{diag}(m_1, m_2, \cdots, m_N) \otimes I_3 \tag{4.7}$$

(where $\otimes$ denotes the Kronecker product, cf., [16]), $\bar{m} \doteq \min_{i\in\mathcal{N}} m_i > 0$, and $\bar{M} \doteq \max_{i\in\mathcal{N}} m_i$. Note that we may write

$$T(y) = \tfrac{1}{2} y' \mathcal{M} y, \quad \forall y \in I\!R^n. \tag{4.8}$$

We also continue to take ψ^c as given in Section 2.1 (i.e., by (2.3) and (2.4)) for $c \in [0, \infty]$. With these specific definitions, the least-action payoff, $\bar{J}^c$ given by (2.5), becomes

$$\bar{J}^c(t,x,u,z) = \int_0^t T(u(r)) - V(\xi(r))\, dr + \psi^c(\xi(t), z) \tag{4.9}$$

$$= \int_0^t T(u(r)) + \max_{\alpha\in\mathcal{A}}\{-\hat{V}(\xi(r),\alpha)\}\, dr + \psi^c(\xi(t), z). \tag{4.10}$$

As in (2.6), we let the value be given by

$$\overline{W}^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty} \bar{J}^c(t,x,u,z). \tag{4.11}$$

Let $\tilde{\bar{J}}^c : [0,\infty) \times I\!R^n \times \mathcal{U}^\infty \times I\!R^n \to I\!R$ and $\widetilde{\overline{W}}^c : [0,\infty) \times I\!R^n \times I\!R^n \to I\!R$ be given by

$$\tilde{\bar{J}}^c(t,x,u,z) = \int_0^t T(u(r)) - \widetilde{V}(\xi(r))\, dr + \psi^c(\xi(t), z), \tag{4.12}$$

$$\widetilde{\overline{W}}^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty} \tilde{\bar{J}}^c(t,x,u,z). \tag{4.13}$$

Fix $\bar{\delta}_0 \geq \bar{\delta}$, and let

$$D^{\bar{\delta}_0} \doteq \left\{ x \in I\!R^n \,\middle|\, |x^i - x^j| > \bar{\delta}_0\, \forall (i,j) \in \mathcal{I}^\Delta \right\}. \tag{4.14}$$

Fix $t > 0$ and $x, z \in D^{\bar{\delta}_0}$. We assume:

$$\begin{aligned}&\exists\,\bar c=\bar c(t,x,z)<\infty,\ \bar\epsilon=\bar\epsilon(t,x,z)>0 \text{ such that } \forall\ \epsilon\text{-optimal } u^\epsilon\in\mathcal{U}^\infty \text{ in}\\ &(4.11) \text{ with } \epsilon\in(0,\bar\epsilon], \text{ and with } \xi^\epsilon \text{ denoting the corresponding trajectory,}\\ &\text{we have } |(\xi^\epsilon)^i(r)-(\xi^\epsilon)^j(r)|\ge\bar\delta\ \forall\, r\in[0,t],\ \forall\,(i,j)\in\mathcal{I}^\Delta.\end{aligned} \tag{$A.N1$}$$

We remark that Assumption $(A.N1)$ is used only in Theorem 4.5 and Corollary 4.6, which demonstrate that the two problem models just above are equivalent for problems where collision is not possible for ϵ-optimal approximate solutions.

THEOREM 4.5. *Let $t\in(0,\infty)$ and $x,z\in D^{\bar\delta_0}$. Let $c\ge\bar c(t,x,z)$. Suppose $u^*\in\mathcal{U}^\infty$ minimizes $\bar J^c(t,x,\cdot,z)$. Then u^* also minimizes $\tilde{\bar J}^c(t,x,\cdot,z)$.*

Proof. Fix $t\in[0,\infty)$ and $x,z\in I\!R^n$. Let $u^*\in\mathcal{U}^\infty$ minimize $\bar J^c(t,x,\cdot,z)$. Let $\tilde u\in\mathcal{U}^\infty$. By (4.10), (4.12), Lemma 4.4, and then by the choice of u^*,

$$\tilde{\bar J}^c(t,x,\tilde u,z)\ge\bar J^c(t,x,\tilde u,z)\ge\bar J^c(t,x,u^*,z),$$

which by Assumption $(A.N1)$ and Lemma 4.4,

$$=\tilde{\bar J}^c(t,x,u^*,z).\ \square$$

COROLLARY 4.6. *Let $t\in[0,\infty)$ and $x,z\in D^{\bar\delta_0}$. Then, $\overline{W}^c(t,x,z)=\widetilde{\overline{W}}^c(t,x,z)$ for all $c\ge\bar c(t,x,z)$.*

Henceforth, we work only with $V,\bar J^c,\overline{W}^c$, rather than $\widetilde V,\tilde{\bar J}^c,\widetilde{\overline{W}}^c$. Let

$$\mathcal{A}^\infty\doteq\big\{\alpha:[0,\infty)\to\mathcal{A}\,|\,\exists K<\infty,\{\tau_k\}_{k\in]0,K[} \text{ such that } \tau_0=0,\ \tau_K=t, \text{ and}$$
$$\tau_{(k-1)}<\tau_k \text{ and } \alpha_{[\tau_{k-1},\tau_k)}\in C([\tau_{k-1},\tau_k);\mathcal{A})\,\forall k\in]1,K[\,\big\}, \tag{4.15}$$
$$\bar{\mathcal{A}}^\infty\doteq\mathcal{L}_\infty([0,\infty);\mathcal{A}), \tag{4.16}$$

and we note that, of course, $C([0,\infty);\mathcal{A})\subset\mathcal{A}^\infty\subseteq\bar{\mathcal{A}}^\infty$. Also, we replace the time-independent potential energy function, $V(\cdot)$, with

$$-V^\alpha(r,x)\doteq-\hat V(x,\alpha(r))=\sum_{(i,j)\in\mathcal{I}^\Delta}\widehat G\,m_i(\alpha_{i,j}(r)m_j)\left[1-\frac{(\alpha_{i,j}(r)|x^i-x^j|)^2}{2}\right]. \tag{4.17}$$

Let $J^c:[0,\infty)\times I\!R^n\times\mathcal{U}^\infty\times\bar{\mathcal{A}}^\infty\times I\!R^n\to I\!R$ be given by

$$J^c(t,x,u,\alpha,z)\doteq\int_0^t T(u(r))-V^\alpha(r,\xi(r))\,dr+\psi^c(\xi(t),z). \tag{4.18}$$

THEOREM 4.7. *Let $t\ge0$ and $x,z\in I\!R^n$. Then,*

$$\bar J^c(t,x,u,z)=\max_{\alpha\in\mathcal{A}^\infty}J^c(t,x,u,\alpha,z)=\max_{\alpha\in\bar{\mathcal{A}}^\infty}J^c(t,x,u,\alpha,z),\ \ \forall u\in\mathcal{U}^\infty, \tag{4.19}$$
$$\text{and }\ \overline{W}^c(t,x,z)=\inf_{u\in\mathcal{U}^\infty}\max_{\alpha(\cdot)\in\mathcal{A}^\infty}J^c(t,x,u,\alpha,z)=\inf_{u\in\mathcal{U}^\infty}\max_{\alpha(\cdot)\in\bar{\mathcal{A}}^\infty}J^c(t,x,u,\alpha,z). \tag{4.20}$$

Proof. Fix $t\ge0$ and $x,z\in I\!R^n$. Let $u\in\mathcal{U}^\infty$, and recall from (4.5) and (4.10) that

$$\bar J^c(t,x,u,z)=\int_0^t T(u(r))+\max_{\alpha\in\mathcal{A}}\sum_{(i,j)\in\mathcal{I}^\Delta}\widehat G\,m_i(\alpha_{i,j}m_j)\left[1-\frac{(\alpha_{i,j}|\xi^i(r)-\xi^j(r)|)^2}{2}\right]dr$$
$$+\psi^c(\xi(t),z). \tag{4.21}$$

By (4.15), (4.16), (4.18) and (4.21), any $\alpha(r)$ is suboptimal in the maximization in (4.21) for any $r\in[0,t]$ and any $\alpha\in\bar{\mathcal{A}}^\infty\supseteq\mathcal{A}^\infty$, and in particular,

$$\bar{J}^c(t,x,u,z)\geq\max_{\alpha(\cdot)\in\bar{\mathcal{A}}^\infty}J^c(t,x,u,\alpha,z)\geq\max_{\alpha(\cdot)\in\mathcal{A}^\infty}J^c(t,x,u,\alpha,z), \tag{4.22}$$

and we do not include the obvious details.

Let $\bar{\alpha}^*:I\!R^n\to\mathcal{A}$ be given by $\bar{\alpha}^*(x)=\{\bar{\alpha}^*_{i,j}(x^i,x^j)\}_{(i,j)\in\mathcal{I}^\Delta}$, where

$$\begin{aligned}\bar{\alpha}^*_{i,j}(x^i,x^j)&\doteq\operatorname*{argmax}_{\alpha\in[0,\sqrt{2/3}\bar{\delta}^{-1}]}\alpha\left[1-\frac{(\alpha|x^i-x^j|)^2}{2}\right],\quad\forall(i,j)\in\mathcal{I}^\Delta,\ \forall x\in I\!R^n\\&=\operatorname*{argmax}_{\alpha\in[0,\sqrt{2/3}\bar{\delta}^{-1}]}\widehat{G}\,m_i(\alpha m_j)\left[1-\frac{(\alpha|x^i-x^j|)^2}{2}\right],\ \forall(i,j)\in\mathcal{I}^\Delta,\ x\in I\!R^n.\end{aligned} \tag{4.23}$$

Let ξ denote the state trajectory corresponding to u and $\xi_0=x$. Let

$$\alpha^*(r)=\alpha^*(r;u(\cdot))=\{\alpha^*_{i,j}(r)\,|\,(i,j)\in\mathcal{I}^\Delta\}\in\mathcal{A}^\infty, \tag{4.24}$$

where the $(i,j)^{th}$ element of α^* is given by

$$\alpha^*_{i,j}(r)=\bar{\alpha}^*_{i,j}(\xi^i(r),\xi^j(r)),\quad\forall r\in[0,t). \tag{4.25}$$

Note that $\alpha^*\in\mathcal{A}^\infty$. Also note that by (4.23) and (4.25),

$$\alpha^*_{i,j}(r)=\operatorname*{argmax}_{\alpha\in[0,\sqrt{2/3}\bar{\delta}^{-1}]}\widehat{G}\,m_i(\alpha m_j)\left[1-\frac{(\alpha|\xi^i(r)-\xi^j(r)|)^2}{2}\right]\ \forall(i,j)\in\mathcal{I}^\Delta,\ r\in[0,t). \tag{4.26}$$

Then, by (4.2), (4.17) and (4.26),

$$-V^{\alpha^*}(r,\xi(r))=-V(\xi(r))\quad\forall r\in[0,t). \tag{4.27}$$

By (4.9), (4.18), and (4.27),

$$\bar{J}^c(t,x,u,z)=J^c(t,x,u,\alpha^*,z)\leq\max_{\alpha\in\mathcal{A}^\infty}J^c(t,x,u,\alpha,z). \tag{4.28}$$

By (4.22) and (4.28), we have (4.19). That, in turn, immediately implies (4.20). □

We specifically note that the problem of finding the fundamental solution of the TPBVP for the N-body problem has been converted to a differential game. In a heuristic sense, one may think of the problem now as not only a search over possible world lines of the bodies, but as also including a search over negotiated potentials between the bodies. Again heuristically, one may think of the potentials, not as fields existing throughout space but as the opposing player in a game interpretation. The first player minimizes the action at each moment, with immediate effect on the kinetic term and integrated effect on the other terms, while the second player maximizes the potential term at each moment. The analytical advantage obtained through the use of this viewpoint is that one may express the potential energy as a quadratic form.

Remark 4.8. We note that (4.20) is a non-standard form for dynamic games. The inf / sup is neither in terms of non-anticipative strategies (c.f., [2, 9]), nor in terms of state feedback controls. This is due to the very simple form of the maximizing player, which is only a representation for the running cost. ◇

Remark 4.9. Note that with V given by (4.2) and $\mathcal{M}$ given by (4.7), V and $\mathcal{M}$ satisfy conditions $(A.\mathcal{M})$, $(A.V1)$ and $(A.V2)$ of Section 2.1. ◇

4.3. Semiconvexity and the HJB PDE. We next proceed to obtain the expected relationship to the associated HJB PDE.

LEMMA 4.10. *$\overline{W}^c(t,x,z) \in [0, \bar{D}t + \psi^c(x,z)]$ for all $t \geq 0$ and all $x, z \in I\!R^n$, where $\bar{D} = (G/\bar{\delta}) \sum_{(i,j)\in\mathcal{I}^\Delta} m_i m_j$.*

Proof. The result follows by Remark 4.9 and Lemma 2.2. □

LEMMA 4.11. *Let $\epsilon \in (0,1]$. Given ϵ-optimal u^ϵ in the definition, (4.11), of $\overline{W}^c(t,x,z)$, we have $\|u^\epsilon\|^2_{L_2(0,t)} \leq \frac{2}{\bar{m}}(\bar{D}t + \psi^c(x,z) + 1)$.*

Proof. Let $\epsilon \in (0,1]$, and let u^ϵ be as per the lemma statement. Let the corresponding trajectory be denoted by ξ^ϵ. Then, using Lemma 4.10,

$$\int_0^t T(u^\epsilon(r)) - V(\xi^\epsilon(r))\,dr + \psi^c(\xi^\epsilon(t), z) \leq \overline{W}^c(t,x,z) + 1 \leq \bar{D}t + \psi^c(x,z) + 1.$$

Hence, noting the non-positivity of the potential, one has

$$\int_0^t T(u^\epsilon(r))\,dr \leq \bar{D}t + \psi^c(x,z) + 1 + \int_0^t V(\xi^\epsilon(r))\,dr \leq \bar{D}t + \psi^c(x,z) + 1.$$

That is, $\frac{1}{2}\int_0^t (u^\epsilon)'(r)\mathcal{M}u^\epsilon(r)\,dr \leq \bar{D}t + \psi^c(x,z) + 1$. This immediately implies that $\|u^\epsilon\|^2_{L_2(0,t)} \leq (2/\bar{m})[\bar{D}t + \psi^c(x,z) + 1]$. □

LEMMA 4.12. *For any $t_0 > 0$, $\overline{W}^c(t,x,z)$ is semiconcave in x, uniformly in $(t,x,z,c) \in [t_0,\infty) \times I\!R^n \times I\!R^n \times [0,\infty)$.*

Proof. Let $t_0 > 0$, $t \in [t_0,\infty)$, $x, z \in I\!R^n$, $c \in [0,\infty)$ and $\epsilon \in (0,1]$. Let $\gamma \in I\!R^n$, $|\gamma| < \bar{\delta}/4$ where $\bar{\delta}, \epsilon$ are as in Assumption $(A.N1)$. Let u be an ϵ-optimal input in the definition, (4.11), or $\overline{W}^c(t,x,z)$. We will obtain an upper bound on second-order difference, $[\overline{W}^c(t,x+\gamma,z) + \overline{W}^c(t,x-\gamma,z) - 2\overline{W}^c(t,x,z)]/|\gamma|^2$, where this implies the asserted semiconcavity (c.f., [4, 24]). Let

$$u^+(r) \doteq \begin{cases} u(r) - \frac{1}{t_0}\gamma & \text{if } r \in [0,t_0] \\ u(r) & \text{if } r \in (t_0,t), \end{cases} \quad \text{and} \quad u^-(r) \doteq \begin{cases} u(r) + \frac{1}{t_0}\gamma & \text{if } r \in [0,t_0] \\ u(r) & \text{if } r \in (t_0,t). \end{cases} \tag{4.29}$$

By the ϵ-optimality of u with respect to $\overline{W}^c(t,x,z)$ and the suboptimality of $u^\pm$ with respect to $\overline{W}^c(t,x\pm\gamma,z)$,

$$\begin{aligned} &\overline{W}^c(t,x+\gamma,z) + \overline{W}^c(t,x-\gamma,z) - 2\overline{W}^c(t,x,z) \\ &\quad < \bar{J}^c(t,x+\gamma,u^+,z) + \bar{J}^c(t,x-\gamma,u^-,z) - 2\bar{J}^c(t,x,u,z) + 2\epsilon. \end{aligned} \tag{4.30}$$

Let ξ, ξ^+ and ξ^- be the trajectories resulting from these controls with $\xi(0) = x$, $\xi^+(0) = x + \gamma$ and $\xi^-(0) = x - \gamma$, and note that

$$|\xi^+(r) - \xi(r)| = |\xi^-(r) - \xi(r)| \leq |\gamma|, \quad \forall r \in [0,t_0), \tag{4.31}$$

$$\xi(r) = \xi^+(r) = \xi^-(r), \quad \forall r \in [t_0,t]. \tag{4.32}$$

We see that (4.9) and (4.30) imply

$$\begin{aligned} &\overline{W}^c(t,x+\gamma,z) + \overline{W}^c(t,x-\gamma,z) - 2\overline{W}^c(t,x,z) \\ &< \int_0^t T(u^+(r)) + T(u^-(r)) - 2T(u(r))\,dr + \int_0^t 2V(\xi(r)) - V(\xi^+(r)) - V(\xi^-(r))\,dr \\ &\quad + \psi^c(\xi^+(t),z) + \psi^c(\xi^+(t),z) - 2\psi^c(\xi(t),z) + 2\epsilon, \end{aligned}$$

which by (4.29) and (4.32),

$$= \int_0^{t_0} T(u^+(r)) + T(u^-(r)) - 2T(u(r))\,dr + \int_0^{t_0} 2V(\xi(r)) - V(\xi^+(r)) - V(\xi^-(r))\,dr + 2\epsilon. \tag{4.33}$$

We examine each of the second-order differences separately. A simple calculation (and using notation (4.7)) verifies that $T(u^+(r)) + T(u^-(r)) - 2T(u(r)) = \frac{1}{t_0^2}\gamma'\mathcal{M}\gamma \leq \frac{\bar{M}}{t_0^2}|\gamma|^2$. Integrating, this yields

$$\int_0^{t_0} T(u^+(r)) + T(u^-(r)) - 2T(u(r))\,dr \leq \frac{\bar{M}}{t_0}|\gamma|^2. \tag{4.34}$$

By the choice of controls, Assumption $(A.N1)$, and the fact that $|\gamma| < \bar{\delta}/4$, for all $(i,j) \in \mathcal{I}^\Delta$,

$$\begin{aligned} \left|(\xi^+)^i(r) - (\xi^+)^j(r)\right| &\geq \left|\xi^i(r) - \xi^j(r)\right| - \Big[\left|(\xi^+)^i(r) - \xi^i(r)\right| + \left|(\xi^+)^j(r) - \xi^j(r)\right|\Big] \\ &\geq \bar{\delta}/2, \quad \forall r \in [0,t], \end{aligned} \tag{4.35}$$

and similarly for ξ^-. One may also show that there exists $\bar{K}_2 < \infty$ such that $|V_{xx}(y)| \leq \bar{K}_2 \quad \forall y \in I\!R^n$ such that $|y^i - y^j| \geq \bar{\delta}/2$ for all $(i,j) \in \mathcal{I}^\Delta$. Then, using (4.31), (4.35) and a similar argument to that for $T(\cdot)$, one finds that there exists $K_2 < \infty$ such that

$$\int_0^{t_0} 2V(\xi(r)) - [V(\xi^+(r)) + V(\xi^-(r))]\,dr \leq K_2|\gamma|^2. \tag{4.36}$$

Employing (4.34) and (4.36) in (4.33), one has

$$\overline{W}^c(t, x+\gamma, z) + \overline{W}^c(t, x-\gamma, z) - 2\overline{W}^c(t,x,z) \leq \left[\frac{\bar{M}}{t_0} + K_2\right]|\gamma|^2 + 2\epsilon.$$

As this is true for all sufficiently small $\epsilon > 0$, we obtain the desired result. □

The HJB PDE associated with our problem here is

$$\begin{aligned} 0 &= -\partial_1 W(t,x,z) - H(x, \nabla_x W(t,x,z)) \\ &\doteq -\partial_1 W(t,x,z) + \inf_{v \in I\!R^n} \sup_{\alpha \in \mathcal{A}} \left\{ \tfrac{1}{2} v'\mathcal{M}v - \hat{V}(x,\alpha) + v'\nabla_x W(t,x,z) \right\}. \end{aligned} \tag{4.37}$$

Note that the right-hand side of (4.37) is separated (and in fact, the Isaacs condition is satisfied). Consequently, we may write (4.37) as

$$0 = -\partial_1 W(t,x,z) + \min_{v \in I\!R^n} \left\{ \tfrac{1}{2} v'\mathcal{M}v + v'\nabla_x W(t,x,z) \right\} + \sup_{\alpha \in \mathcal{A}} \{-\hat{V}(x,\alpha)\} \tag{4.38}$$

$$= -\partial_1 W(t,x,z) - \tfrac{1}{2}\big(\nabla_x W(t,x,z)\big)'\mathcal{M}^{-1}\nabla_x W(t,x,z) + \sup_{\alpha \in \mathcal{A}} \{-\hat{V}(x,\alpha)\}, \tag{4.39}$$

which by (4.5),

$$= -\partial_1 W(t,x,z) - \tfrac{1}{2}\big(\nabla_x W(t,x,z)\big)'\mathcal{M}^{-1}\nabla_x W(t,x,z) - V(x). \tag{4.40}$$

The initial conditions, indexed by $z \in I\!R^n$, corresponding to value function $\overline{W}^c$ are

$$W(0,x,z) = \psi^c(x,z), \quad \forall x \in I\!R^n. \tag{4.41}$$

For $t > 0$, let

$$\mathcal{D}_t \doteq C([0,t] \times I\!R^n) \cap C^1((0,t) \times I\!R^n). \tag{4.42}$$

THEOREM 4.13. *Let $c \in [0, \infty)$ and $z \in I\!R^n$. Value function $\overline{W}^c(\cdot,\cdot,z)$ is Lipschitz continuous on compact sets, and is the unique viscosity solution of HJB PDE (4.37) (equivalently, (4.38)–(4.40)) and initial condition (4.41). Let $t > 0$, and suppose further that $W(\cdot,\cdot,z) \in \mathcal{D}_t$ and satisfies (4.37) (equivalently, (4.38)–(4.40)) and initial condition (4.41). Let $x \in I\!R^n$, and let u^* be given by $u^*(s) = \tilde{u}(s, \tilde{\xi}(s))$ where $\tilde{\xi}(s)$ is generated by (2.1) with feedback $\tilde{u}(s,x) \doteq -\mathcal{M}^{-1}\nabla_x W(t-s,x,z)$ and initial condition $\tilde{\xi}(0) = x$. Then, $W(t,x,z) = \bar{J}^c(t,x,u^*,z) = \overline{W}^c(t,x,z)$.*

Proof. By Remark 4.9, conditions $(A.\mathcal{M})$, $(A.V1)$ and $(A.V2)$ of Section 2.1 are satisfied. Consequently, the first assertion follows directly from Theorem 2.3. (We remark that the local Lipschitz assertion also follows from Lemma 4.12.)

We turn to the second assertion. Fix $t > 0$. Let $c, z, W(\cdot,\cdot,z), u^*, \tilde{u}, \tilde{\xi}$ be as indicated. Let $s \in (0,t)$. Then,

$$\begin{aligned} &\nabla_x W(t-s, \tilde{\xi}(s), z) \cdot u^*(s) + \tfrac{1}{2}\, u^*(s)'\mathcal{M}u^*(s) \\ &= -\tfrac{1}{2}[\nabla_x W(t-s,\tilde{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W(t-s,\tilde{\xi}(s),z)\,. \end{aligned} \tag{4.43}$$

From (4.39) and then (4.43),

$$\begin{aligned} 0 &= -\partial_1 W(t-s,\tilde{\xi}(s),z) + \sup_{\alpha\in\mathcal{A}}\{-\hat{V}(\tilde{\xi}(s),\alpha)\} \\ &\quad - \tfrac{1}{2}\,[\nabla_x W(t-s,\tilde{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W(t-s,\tilde{\xi}(s),z) \\ &= -\partial_1 W(t-s,\tilde{\xi}(s),z) + \sup_{\alpha\in\mathcal{A}}\{-\hat{V}(\tilde{\xi}(s),\alpha)\} \\ &\quad + \nabla_x W(t-s,\tilde{\xi}(s),z)\cdot u^*(s) + \tfrac{1}{2}\,u^*(s)'\mathcal{M}u^*(s) \\ &= \tfrac{\partial}{\partial s} W(t-s,\tilde{\xi}(s),z) + \tfrac{1}{2}\,u^*(s)'\mathcal{M}u^*(s) + \sup_{\alpha\in\mathcal{A}}\big\{ -\hat{V}(\tilde{\xi}(s),\alpha)\big\}\,. \end{aligned}$$

Note that $u^* \in \mathcal{U}^\infty$ by definition of $\mathcal{D}_t$. Integrating with respect to s over $[0,t]$ (noting that the integrand is $\mathcal{L}_1$ by $u^* \in \mathcal{U}^\infty$, the form of $-\hat{V}$, (4.8), and Assumption $(A.N1)$) yields

$$\begin{aligned} 0 &= W(0,\tilde{\xi}(t),z) - W(t,x,z) + \int_0^t T(u^*(s)) + \sup_{\alpha\in\mathcal{A}}\{-\hat{V}(\tilde{\xi}(s),\alpha)\}\,ds \\ &= W(0,\tilde{\xi}(t),z) - W(t,x,z) + \int_0^t T(u^*(s)) - V(\tilde{\xi}(s))\,ds\,, \end{aligned} \tag{4.44}$$

which, by applying (4.41), yields

$$W(t,x,z) = \int_0^t T(u^*(s)) - V(\tilde{\xi}(s))\,ds + \psi^c(\tilde{\xi}(t),z) = \bar{J}^c(t,x,u^*,z).$$

To obtain the last equality, note that by assumption and the definition of a viscosity solution, W is also a viscosity solution of (4.37),(4.41). Therefore, by the uniqueness obtained in the first assertion, $W(t,x,z) = \overline{W}^c(t,x,z)$. □

We now proceed to consider the game where the order of infimum and supremum are reversed. Due to the very simple form of this particular game, with the α controller acting only on the running cost and that being in a separated form, an unusual equivalence can be obtained. Recalling (4.18), let

$$\underline{W}^c(t,x,z) \doteq \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \tag{4.45}$$

By the usual reordering inequality, (4.20) immediately implies that

$$\underline{W}^c(t,x,z) \le \overline{W}^c(t,x,z) \quad \forall (t,x,z)\in[0,\infty)\times I\!R^n\times I\!R^n. \tag{4.46}$$

It will be helpful to introduce more notation. For $c\in[0,\infty]$ and $\alpha\in\bar{\mathcal{A}}^\infty$, we let

$$\mathcal{W}^{\alpha,c}(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \tag{4.47}$$

The corresponding Hamiltonian is

$$H^\alpha(r,x,p) \doteq V^\alpha(r,x) + \frac{1}{2} p'\mathcal{M}^{-1}p. \tag{4.48}$$

Of course, one immediately sees that

$$\underline{W}^c(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z) \quad \forall (t,x,z)\in[0,\infty)\times I\!R^n\times I\!R^n. \tag{4.49}$$

In a similar fashion to verification Theorem 4.13, we have the following.

THEOREM 4.14. *Let $c\in(0,\infty)$, $z\in I\!R^n$ and $\alpha\in\bar{\mathcal{A}}^\infty$. In particular, suppose that α is piecewise continuous, with possible discontinuities only at $0<\tau_1<\tau_2<\dots\tau_{K-1}<t$ with $K<\infty$. Let $\tau_0=0$, $\tau_K=t$ and $\mathcal{O}^t \doteq \bigcup_{k\in]0,K-1[}(\tau_k,\tau_{k+1})$. Suppose $W^\alpha(\cdot,\cdot,z)\in C(I\!R_{\ge 0}\times I\!R^n;I\!R)\cap C^1(\mathcal{O}^t\times I\!R^n;I\!R)$ satisfies*

$$0 = -\partial_1 W^\alpha(r,x,z) - H^\alpha(t-r,x,\nabla_x W^\alpha(r,x,z)), \qquad (r,x)\in\mathcal{O}^t\times I\!R^n, \tag{4.50}$$
$$W^\alpha(0,x,z) = \psi^c(x,z), \qquad x\in I\!R^n. \tag{4.51}$$

Then, $W^\alpha(t,x,z)\le J^c(t,x,u,\alpha,z)$ for all $x\in I\!R^n$, $u\in\mathcal{U}^\infty$. Further, $W^\alpha(t,x,z) = J^c(t,x,u^*,\alpha,z)$ where $u^*(s)=\tilde{u}(s,\tilde{\xi}(s))$ with $\tilde{\xi}(s)$ given by (2.1) with $\tilde{u}(s,x)\doteq -\mathcal{M}^{-1}\nabla_x W(t-s,x,z)$ and $\tilde{\xi}(0)=x$. Consequently $W^\alpha(t,x,z)=\mathcal{W}^{\alpha,c}(t,x,z)$.

Proof. Fix $t>0$, $c\in(0,\infty)$, $z\in I\!R^n$ and $\alpha\in\bar{\mathcal{A}}^\infty$. Let W^α be as asserted, and let $\bar{u}\in\mathcal{U}^\infty$. We use induction on k. Let $k\in]0,K-1[$, and suppose $W^\alpha(t-\tau_{k+1},x,z)\le J^c(t-\tau_{k+1},x,\bar{u},\alpha,z)$ for all $x\in I\!R^n$, which is certainly true for $k+1=K$. Define $\pi(v)\doteq p\cdot v+\frac{1}{2}v'\mathcal{M}v$, $p\in I\!R^n$, and note that by completion of squares that $\pi(v)\ge -\frac{1}{2}p'\mathcal{M}^{-1}p$. Select $v=\bar{u}(s)$ and $p=\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)$ at each $s\in(\tau_k,\tau_{k+1})$, where $\bar{\xi}$ denotes the trajectory satisfying (2.1) corresponding to input $\bar{u}$. Then,

$$\begin{aligned}&\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)\cdot\bar{u}(s)+\tfrac{1}{2}\,\bar{u}(s)'\mathcal{M}\bar{u}(s)\\ &\ge -\tfrac{1}{2}[\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)\,,\end{aligned}$$

so that (4.48) and (4.50) imply that for all $s\in(\tau_k,\tau_{k+1})$,

$$\begin{aligned}0 &= -\partial_1 W^\alpha(t-s,\bar{\xi}(s),z) - V^\alpha(s,\bar{\xi}(s))\\ &\quad -\tfrac{1}{2}\,[\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)]'\mathcal{M}^{-1}\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)\\ &\le -\partial_1 W^\alpha(t-s,\bar{\xi}(s),z)+\nabla_x W^\alpha(t-s,\bar{\xi}(s),z)'\bar{u}(s)+\tfrac{1}{2}\,\bar{u}(s)'\mathcal{M}\bar{u}(s)-V^\alpha(s,\bar{\xi}(s))\\ &= \tfrac{d}{ds}W^\alpha(t-s,\bar{\xi}(s),z)+\tfrac{1}{2}\,\bar{u}(s)'\mathcal{M}\bar{u}(s)-V^\alpha(s,\bar{\xi}(s))\,.\end{aligned}$$

Integrating with respect to s over (τ_k, τ_{k+1}) then yields that

$$0 \le W^\alpha(t-\tau_{k+1}, \bar\xi(\tau_{k+1}), z) - W^\alpha(t-\tau_k, \bar\xi(\tau_k), z) + \int_{\tau_k}^{\tau_{k+1}} T(\bar u(s)) - V^\alpha(s, \bar\xi(s))\, ds,$$

or equivalently,

$$W^\alpha(t-\tau_k, \bar\xi(\tau_k), z) \le \int_{\tau_k}^{\tau_{k+1}} T(\bar u(s)) - V^\alpha(s, \bar\xi(s))\, ds + W^\alpha(t-\tau_{k+1}, \bar\xi(\tau_{k+1}), z),$$

which by supposition,

$$\le \int_{\tau_k}^{\tau_{k+1}} T(\bar u(s)) - V^\alpha(s, \bar\xi(s))\, ds + J^c(t-\tau_{k+1}, \bar\xi(\tau_{k+1}), \bar u, \alpha, z) = J^c(t-\tau_k, \bar\xi(\tau_k), \bar u, \alpha, z). \tag{4.52}$$

By induction, we have the first assertion. To prove the second assertion, fix $\bar u \doteq u^*$, where u^* is as indicated in the theorem statement. Repeating the above argument yields equality in (4.52), so that $W^\alpha(t,x,z) = J^c(t,x,u^*,z) = \mathcal{W}^{\alpha,c}(t,x,z)$ as required. □

4.4. Interchange of the order of minimization and maximization. Due to the particular, simple form of the game, the order of the minimization and maximization operations may be reversed. This will be key to the numerical approach to follow.

LEMMA 4.15. *Let $t \in (0,\infty)$ and $x, z \in I\!R^n$. Let $u^\dagger \in \mathcal{U}^\infty$ be a critical point of $\bar J^c(t,x,\cdot,z)$ of (4.9), and let the corresponding state trajectory be denoted by $\xi^\dagger$. Let $\alpha^*(r) \doteq \bar\alpha^*(\xi^\dagger(r))$ for all $r \in [0,t)$ where $\bar\alpha^*$ is given by (4.23). Then, $u^\dagger$ is a critical point of $J^c(t,x,\cdot,\alpha^*,z)$, where J^c is given in (4.18).*

Proof. Let $\nu \in \mathcal{U}^\infty$ and $\delta > 0$. We examine differences in the direction ν from $u^\dagger$. In particular, by inspection of (4.17), (4.18) and (4.8),

$$\begin{aligned}
&J^c(t,x,u^\dagger+\delta\nu,\alpha^*,z) - J^c(t,x,u^\dagger,\alpha^*,z)\\
&= \int_0^t \delta[u^\dagger(r)]'\mathcal{M}\nu(r) - \delta\big[\nabla_x \hat V(\xi^\dagger(r),\alpha^*(r))\big]' \int_0^r \nu(\rho)\,d\rho\,dr\\
&\quad + \delta\big(\nabla_x\psi^c(\xi^\dagger(t),z)\big)' \int_0^t \nu(r)\,dr + \mathcal{O}(\delta^2)\\
&= \int_0^t \delta[u^\dagger(r)]'\mathcal{M}\nu(r) - \delta\big[\nabla_x \hat V(\xi^\dagger(r),\bar\alpha^*(\xi^\dagger(r)))\big]' \int_0^r \nu(\rho)\,d\rho\,dr\\
&\quad + \delta\big(\nabla_x\psi^c(\xi^\dagger(t),z)\big)' \int_0^t \nu(r)\,dr + \mathcal{O}(\delta^2).
\end{aligned} \tag{4.53}$$

Now recall from (4.5) that $-V(x) = \max_{\alpha\in\mathcal{A}}\big[-\hat V(x,\alpha)\big]$, where the maximum is uniquely attained at $\bar\alpha^*(x)$. Consequently, $-\nabla_x V(x) = -\nabla_x \hat V(x,\bar\alpha^*(x))$, and therefore with $\alpha^*(r) \doteq \bar\alpha^*(\xi^\dagger(r))$, we see that (4.53) becomes

$$\begin{aligned}
&J^c(t,x,u^\dagger+\delta\nu,\alpha^*,z) - J^c(t,x,u^\dagger,\alpha^*,z)\\
&= \int_0^t \delta[u^\dagger(r)]'\mathcal{M}\nu(r) - \delta\big[\nabla_x V(\xi^\dagger(r))\big]' \int_0^r \nu(\rho)\,d\rho\,dr\\
&\quad + \delta\big(\nabla_x\psi^c(\xi^\dagger(t),z)\big)' \int_0^t \nu(r)\,dr + \mathcal{O}(\delta^2)
\end{aligned}$$

$$= \bar{J}^c(t, x, u^\dagger + \delta\nu, z) - \bar{J}^c(t, x, u^\dagger, z) + \mathcal{O}(\delta^2),$$

and since $u^\dagger$ is a critical point of $\bar{J}^c(t, x, \cdot, z)$,

$$= \mathcal{O}(\delta^2). \tag{4.54}$$

That is, $u^\dagger$ is a critical point of $J^c(t, x, \cdot, \alpha^*, z)$. □

Now let u^* be an optimal control for our original problem (with potential energy function, $V(\cdot)$), that is, we let

$$u^* \in \operatorname*{argmin}_{u \in \mathcal{U}^\infty} \bar{J}^c(t, x, \cdot, z) \tag{4.55}$$

where $\bar{J}^c$ is as per (2.5). As u^* is a minimizer of $\bar{J}^c(t, x, \cdot, z)$, Lemma 4.15 immediately yields the following.

LEMMA 4.16. *Let $t \in (0, \infty)$ and $x, z \in I\!R^n$. Then, u^* given by (4.55) is a critical point of $J^c(t, x, \cdot, \alpha^*, z)$.*

LEMMA 4.17. *Let*

$$\bar{t} = \bar{t}(\bar{\delta}) \doteq \sqrt{\frac{\bar{\delta}^3}{G \max_{i \ni]1,n[}(\sum_{j>i} m_j)}}. \tag{4.56}$$

Let $x, z \in I\!R^n$ and $t \in (0, \bar{t})$. Then $J^c(t, x, \cdot, \alpha^*, z)$ is strictly convex, and further, u^* given by (4.55) is the minimizer of $J^c(t, x, \cdot, \alpha^*, z)$.

Proof. Let $\delta \in (0, \infty)$ and $\tilde{u} \in \mathcal{U}^\infty$, and let u^* be as per (4.55). Let $\hat{\xi}(\cdot)$ denote the trajectory of (2.1) corresponding to initial state $x \in I\!R^n$ and input $\hat{u} \doteq u^* + \delta\, \tilde{u} \in \mathcal{U}^\infty$. Note that

$$\begin{aligned} \xi^*(r) &= x + \int_0^r u^*(s)\, ds \\ \hat{\xi}(r) &= x + \int_0^r u^*(s) + \delta\, \tilde{u}(s)\, ds = \xi^*(r) + \delta\, \tilde{\xi}(r)\,, \quad \tilde{\xi}(r) \doteq \int_0^r \tilde{u}(s)\, ds\,. \end{aligned} \tag{4.57}$$

In order to demonstrate convexity of $J^c(t, x, \alpha^*, \cdot, z)$, it is convenient to represent $-V^\alpha(r, \cdot)$ of (4.17) as an explicit quadratic function of the vector $x \in I\!R^n$ of all initial states. To this end, let $E^i \in I\!R^{1\times N}$, $i \in]1, N[$, denote the i^{th} elementary basis vector in $I\!R^N$, and define $E^{i,j} \in I\!R^{N\times N}$ by

$$E^{i,j} \doteq (E^i - E^j)'\, (E^i - E^j)\,. \tag{4.58}$$

Similarly, define the matrix $\mathcal{E}^i \in I\!R^{3\times n}$ by

$$\mathcal{E}^i = E^i \otimes I_3\,, \quad \mathcal{E}^{i,j} \doteq (\mathcal{E}^i - \mathcal{E}^j)'(\mathcal{E}^i - \mathcal{E}^j) = E^{i,j} \otimes I_3\,, \tag{4.59}$$

in which $\otimes$ denotes the Kronecker product. Using (4.59), define the quadratic function $\Psi^{i,j} : I\!R^n \to I\!R$ by

$$\Psi^{i,j}(x) \doteq \tfrac{1}{2}\, x' \mathcal{E}^{i,j}\, x = \tfrac{1}{2}\, |(\mathcal{E}^i - \mathcal{E}^j)\, x|^2 = \tfrac{1}{2}\, |x^i - x^j|^2\,. \tag{4.60}$$

Employing (4.60) in the definition (4.17) of $-V^\alpha(r, \cdot)$ yields that

$$-V^\alpha(r, x) = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, (\alpha_{i,j}(r)\, m_j) \left[1 - (\alpha_{i,j}(r))^2\, \Psi^{i,j}(x)\right]\,. \tag{4.61}$$

It is evident by inspection of (4.8), (2.3), and (4.61) that the functions T, $\psi^c(\cdot, z)$, and $V^\alpha(r, \cdot)$ are quadratic. In general, a quadratic function $\psi : I\!R^n \to I\!R$ satisfies

$$\psi(x + h) = \psi(x) + \nabla_x \psi(x) \cdot h + \tfrac{1}{2}\, (\nabla_{xx}\psi(x)\, h) \cdot h \tag{4.62}$$

for all $x, h \in I\!R^n$. Here, $\nabla_x \psi : I\!R^n \to I\!R^n$ and $\nabla_{xx}\psi : I\!R^n \to I\!R^{n\times n}$ denote respectively the derivative and Hessian of ψ. The first inner-product term on the right-hand side is the directional derivative of ψ at $x \in I\!R^n$ in direction $h \in I\!R^n$. In the special case where $\psi(x) \doteq \frac{1}{2} x' \mathcal{P} x$ is a quadratic function with $\mathcal{P} \in I\!R^{n\times n}$, $\nabla_x \psi(x) = \frac{1}{2}(\mathcal{P} + \mathcal{P}') x$ and $\nabla_{xx}\psi(x) = \frac{1}{2}(\mathcal{P} + \mathcal{P}')$. So, applying (4.62) to (4.8), (2.3), (4.60),

$$T(u^*(r) + \delta\, \tilde{u}(r)) = T(u^*(r)) + \delta\, (\mathcal{M}\, u^*(r)) \cdot \tilde{u}(r) + \tfrac{\delta^2}{2}\, (\mathcal{M}\, \tilde{u}(r)) \cdot \tilde{u}(r)\,, \tag{4.63}$$

$$\Psi^{i,j}(\xi^*(r) + \delta\, \tilde{\xi}(r)) = \Psi^{i,j}(\xi^*(r)) + \delta\, (\mathcal{E}^{i,j}\, \xi^*(r)) \cdot \tilde{\xi}(r) + \tfrac{\delta^2}{2}\, (\mathcal{E}^{i,j}\, \tilde{\xi}(r)) \cdot \tilde{\xi}(r)\,, \tag{4.64}$$

$$\psi^c(\xi^*(t) + \delta\, \tilde{\xi}(t), z) = \psi^c(\xi^*(t), z) + \delta\, (c\, (\xi^*(t) - z)) \cdot \tilde{\xi}(t) + \tfrac{c\delta^2}{2}\, |\tilde{\xi}(t)|^2\,. \tag{4.65}$$

In particular, (4.61) and (4.64) imply that

$$\begin{aligned}
& - V^\alpha(r, \xi^*(r) + \delta\, \tilde{\xi}(r)) = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, (\alpha_{i,j}(r)\, m_j) \left[1 - (\alpha_{i,j}(r))^2\, \Psi^{i,j}(\xi^*(r) + \delta\, \tilde{\xi}(r))\right] \\
& = \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, (\alpha_{i,j}(r)\, m_j) \Big[1 - (\alpha_{i,j}(r))^2 \big(\Psi^{i,j}(\xi^*(r)) + \delta\, (\mathcal{E}^{i,j}\, \xi^*(r)) \cdot \tilde{\xi}(r) \\
& \qquad\qquad + \tfrac{\delta^2}{2}\, (\mathcal{E}^{i,j}\, \tilde{\xi}(r)) \cdot \tilde{\xi}(r)\big)\Big] \\
& = -V^\alpha(r, \xi^*(r)) - \delta \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(r))^3\, (\mathcal{E}^{i,j}\, \xi^*(r)) \cdot \tilde{\xi}(r) \\
& \qquad - \tfrac{\delta^2}{2} \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(r))^3\, (\mathcal{E}^{i,j}\, \tilde{\xi}(r)) \cdot \tilde{\xi}(r)\,.
\end{aligned} \tag{4.66}$$

Hence, combining (4.18), (4.63), (4.65) and (4.66),

$$\begin{aligned}
& J^c(t, x, \alpha^*, u^* + \delta\, \tilde{u}, z) - J^c(t, x, \alpha^*, u^*, z) \\
& = \int_0^t \delta (\mathcal{M}\, u^*(r))' \tilde{u}(r) + \tfrac{\delta^2}{2}\, (\mathcal{M}\, \tilde{u}(r))' \tilde{u}(r) - \delta \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(r))^3\, (\mathcal{E}^{i,j}\, \xi^*(r))' \tilde{\xi}(r) \\
& \qquad - \tfrac{\delta^2}{2} \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(r))^3\, (\mathcal{E}^{i,j}\, \tilde{\xi}(r))' \tilde{\xi}(r)\, dr \\
& \quad + \delta\, (c\, (\xi^*(t) - z)) \cdot \tilde{\xi}(t) + \tfrac{c\delta^2}{2}\, |\tilde{\xi}(t)|^2\,.
\end{aligned} \tag{4.67}$$

The corresponding expression for $J^c(t, x, \alpha^*, u^* - \delta\, \tilde{u}, z) - J^c(t, x, \alpha^*, u^*, z)$ follows by substituting δ with $-\delta$ in (4.67). Adding the result of this substitution to (4.67) yields the second difference of $J^c(t, x, \alpha^*, \cdot, z)$, namely,

$$\begin{aligned}
& J^c(t, x, \alpha^*, u^* + \delta\, \tilde{u}, z) + J^c(t, x, \alpha^*, u^* - \delta\, \tilde{u}, z) - 2\, J^c(t, x, \alpha^*, u^*, z) \\
& = \int_0^t \delta^2\, (\mathcal{M}\, \tilde{u}(r))' \tilde{u}(r) - \delta^2 \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i\, m_j\, \left(\alpha^*_{i,j}(r)\right)^3 (\mathcal{E}^{i,j}\, \tilde{\xi}(r))' \tilde{\xi}(r)\, dr + c\delta^2\, |\tilde{\xi}(t)|^2\,.
\end{aligned} \tag{4.68}$$

It remains to bound this second difference from below. To this end, write $\tilde{u}(r) = \left[\begin{array}{ccc} \tilde{u}^1(r)' & \cdots & \tilde{u}^N(r)' \end{array}\right]'$ and $\tilde{\xi}(r) = \left[\begin{array}{ccc} \tilde{\xi}^1(r)' & \cdots & \tilde{\xi}^N(r)' \end{array}\right]'$, in which $\tilde{u}^i(r), \tilde{\xi}^i(r) \in I\!R^3$ for each $i \in]1, n[$ and $r \in [0, t]$. So, recalling the definition of $\mathcal{M}$,

$$\int_0^t (\mathcal{M}\, \tilde{u}(r))' \tilde{u}(r) = \sum_{i=1}^N m_i\, \|\tilde{u}^i\|^2_{\mathscr{L}_2[0,t]}\,. \tag{4.69}$$

Similarly, recalling the definition of $\mathcal{E}^{i,j}$ and the fact that $\alpha^*_{i,j}(r) \in \left[0, \frac{1}{\delta}\,(\frac{2}{3})^{\frac{1}{2}}\right)$ by Assumption $(A.N1)$,

$$\big(\alpha^*_{i,j}(r)\big)(\mathcal{E}^{i,j}\,\tilde\xi(r))'\tilde\xi(r) \le \tfrac{1}{\delta^3}\,(\tfrac{2}{3})^{\frac{3}{2}}|\tilde\xi^i(r) - \tilde\xi^j(r)|^2, \quad \forall r \in [0,t]. \tag{4.70}$$

So, in order to bound the summation term in (4.68), note that by (4.57), Hölder's inequality, and a reordering of the summations involved,

$$\begin{aligned}
&\int_0^t \sum_{(i,j)\in\mathcal{I}^\Delta} m_i\, m_j\, |\tilde\xi^i(r) - \tilde\xi^j(r)|^2\, dr \le 2 \sum_{(i,j)\in\mathcal{I}^\Delta} m_i\, m_j \int_0^t \left(|\tilde\xi^i(r)|^2 + |\tilde\xi^j(r)|^2\right) dr \\
&= 2 \sum_{(i,j)\in\mathcal{I}^\Delta} m_i\, m_j \int_0^t \left(\left|\int_0^r \tilde u^i(s)\, ds\right|^2 + \left|\int_0^r \tilde u^j(s)\, ds\right|^2\right) dr \\
&\le 2 \sum_{(i,j)\in\mathcal{I}^\Delta} m_i\, m_j \int_0^t \left(\left(\int_0^r |\tilde u^i(s)|\, ds\right)^2 + \left(\int_0^r |\tilde u^j(s)|\, ds\right)^2\right) dr \\
&\le 2 \sum_{(i,j)\in\mathcal{I}^\Delta} m_i\, m_j \left(\int_0^t r\, dr\right)\left(\|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} + \|\tilde u^j\|^2_{\mathscr{L}_2[0,t]}\right) \\
&= \tfrac{t^2}{2} \sum_{i=1}^N m_i \sum_{j=1,\, j\ne i}^N m_j \left(\|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} + \|\tilde u^j\|^2_{\mathscr{L}_2[0,t]}\right) \\
&= \tfrac{t^2}{2} \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \sum_{j=1}^N m_j + \tfrac{t^2}{2} \sum_{i=1}^N m_i \sum_{j=1}^N m_j\, \|\tilde u^j\|^2_{\mathscr{L}_2[0,t]} - t^2 \sum_{i=1}^N m_i^2\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \\
&= t^2 \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \sum_{j=1}^N m_j - t^2 \sum_{i=1}^N m_i^2\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} = t^2 \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \sum_{j=1,\, j\ne i}^N m_j\,.
\end{aligned} \tag{4.71}$$

Combining (4.69)–(4.71) in (4.68) (and noting there that $c \in [0,\infty)$),

$$\begin{aligned}
&J^c(t,x,\alpha^*,u^* + \delta\,\tilde u, z) + J^c(t,x,\alpha^*,u^* - \delta\,\tilde u, z) - 2\,J^c(t,x,\alpha^*,u^*,z) \\
&\ge \delta^2 \left(\int_0^t (\mathcal{M}\,\tilde u(r))\cdot\tilde u(r) - \int_0^t \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat G\, m_i\, m_j \left(\alpha^*_{i,j}(r)\right)^3 (\mathcal{E}^{i,j}\,\tilde\xi(r))\cdot\tilde\xi(r)\, dr\right) \\
&\ge \delta^2 \left(\sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} - \tfrac{\widehat G}{\delta^3}\,(\tfrac{2}{3})^{\frac{3}{2}}\, t^2 \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \sum_{j=1,\, j\ne i}^N m_j\right) \\
&= \delta^2 \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} \left(1 - (\tfrac{G}{\delta^3})\, t^2 \sum_{j=1,\, j\ne i}^N m_j\right) \\
&\ge \delta^2 \left(1 - (\tfrac{G}{\delta^3})\, t^2 \max_{i\in]1,N[} \sum_{j=1,\, j\ne i}^N m_j\right) \sum_{i=1}^N m_i\, \|\tilde u^i\|^2_{\mathscr{L}_2[0,t]} > 0
\end{aligned} \tag{4.72}$$

if $\delta \in (0,\infty)$, $\|\tilde u\|_{\mathscr{L}_2[0,t]} > 0$, and $t \in (0,\bar t)$. That is, $J^c(t,x,\alpha^*,\cdot,z)$ is strictly convex if $t \in (0,\bar t)$, as required. □

THEOREM 4.18. *Suppose* $t \in [0,\bar{t})$ *where* $\bar{t}$ *is as per (4.56). Then one has* $\underline{W}^c(t,x,z) = \overline{W}^c(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z)$ *for all* $x,z \in I\!\!R^n$.

Proof. Let $x,z \in I\!\!R^n$ and $t \in [0,\bar{t})$. By the choice of u^*, viz., (4.55) and (4.28), we have

$$\overline{W}^c(t,x,z) = \bar{J}^c(t,x,u^*,z) = J^c(t,x,u^*,\alpha^*,z),$$

which by Lemma 4.17,

$$= \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha^*,z) \le \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z) = \underline{W}^c(t,x,z).$$

On the other hand, by (4.46), $\underline{W}^c(t,x,z) \le \overline{W}^c(t,x,z)$, and consequently we have the first equality. The second follows immediately from (4.49). □

4.5. The limit property. Recall that the fundamental solution of interest is obtained in the $c \to \infty$ limit. Consequently, we note that we have:

THEOREM 4.19. $\overline{W}^\infty(t,x,z) = \lim_{c\to\infty} \overline{W}^c(t,x,z) = \sup_{c\in[0,\infty)} \overline{W}^c(t,x,z)$, *where the convergence is uniform on compact subsets of* $[0,\bar{t}) \times \mathbb{R}^N \times \mathbb{R}^N$, *with* $\bar{t}$ *as per (4.56).*

Proof. This follows directly from Remark 4.9, Theorem 2.7 and the monotonicity of $\overline{W}^c(t,x,z)$ with respect to c. □

THEOREM 4.20. $\overline{W}^\infty(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,\infty}(t,x,z)$ *for all* $t \in [0,\bar{t})$ *and* $x,z \in I\!\!R^n$, *where* $\bar{t}$ *is as per (4.56).*

Proof. Fix $t \in [0,\bar{t})$ and $x,z \in I\!\!R^n$. By Theorems 4.18 and 4.19,

$$\overline{W}^\infty(t,x,z) = \sup_{c\in[0,\infty)} \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \sup_{c\in[0,\infty)} \mathcal{W}^{\alpha,c}(t,x,z). \tag{4.73}$$

Now, $J^c(t,x,u,\alpha,z)$ is monotonically increasing in c for all $(t,x,u,\alpha,z) \in [0,\bar{t}) \times I\!\!R^n \times \mathcal{U}^\infty \times \bar{\mathcal{A}}^\infty \times I\!\!R^n$. From this one easily sees that $\mathcal{W}^{\alpha,c}(t,x,z)$ is monotonically increasing in c. (Take ϵ-optimal controls, u.) Therefore,

$$\mathcal{W}^{\alpha,\infty}(t,x,z) = \lim_{c\to\infty} \mathcal{W}^{\alpha,c}(t,x,z) = \sup_{c\in[0,\infty)} \mathcal{W}^{\alpha,c}(t,x,z), \tag{4.74}$$

where we note that the finiteness of the supremum follows easily by using constant-velocity trajectories from x to z, and we do not include the details. Combining (4.73) and (4.74) yields the result. □

4.6. Fundamental solution as a set of Riccati solutions. We will find that the fundamental solution may be given in terms of a set of solutions of Riccati equations. We look for a solution, $\breve{\mathcal{W}}^{\alpha,c}$, of the form

$$\breve{\mathcal{W}}^{\alpha,c}(r,x,z) = \tfrac{1}{2}\left[x'\breve{P}_r^c\,x + 2\,x'\,\breve{Q}_r^c\,z + z'\,\breve{R}_r^c\,z + \breve{\gamma}_r^c\right], \quad r \in [0,t], \tag{4.75}$$

where $\breve{P}_r^c$, $\breve{Q}_r^c$, $\breve{R}_r^c$, $\breve{\gamma}_r^c \in I\!\!R^{n\times n}$ depend implicitly on the choice of $\alpha \in \bar{\mathcal{A}}^\infty$ and satisfy

$$\breve{P}_r^c = P_r^c \otimes I_3\,, \quad \breve{Q}_r^c = Q_r^c \otimes I_3\,, \quad \breve{R}_r^c = R_r^c \otimes I_3\,, \quad \breve{\gamma}_r^c = \gamma_r^c\,. \tag{4.76}$$

Here, $P_r^c \otimes I_3$ denotes the Kronecker product of P_r^c with the identity matrix on $I\!\!R^3$, with P_r^c, Q_r^c, $R_r^c \in I\!\!R^{N\times N}$ and $\gamma_r^c \in I\!\!R$ satisfying the respective initial value problems

$$\dot{P}_r^c = -P_r^c\,\mathcal{M}_*^{-1}\,P_r^c + \nu_r\,, \qquad P_0^c = c\,I_N\,, \tag{4.77}$$

$$\dot{Q}_r^c = -P_r^c\,\mathcal{M}_*^{-1}\,Q_r^c\,, \qquad Q_0^c = -c\,I_N\,, \tag{4.78}$$

$$\dot{R}^c_r = -(Q^c_r)'\,\mathcal{M}_*^{-1}\,Q^c_r\,, \qquad R^c_0 = c\,I_N\,, \tag{4.79}$$

$$\dot{\gamma}^c_r = +2\sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,\alpha_{i,j}(t-r)\,, \qquad \gamma^c_0 = 0\,, \tag{4.80}$$

in which P^c_r and R^c_r are self-adjoint, $\mathcal{M}_* \doteq \mathrm{diag}(\{m_i\}_{i=1}^N)$, $\mathcal{M} \equiv \mathcal{M}_* \otimes I_3$, I_N denotes the identity matrix on $I\!R^N$,

$$\nu_r \doteq -\sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,(\alpha_{i,j}(t-r))^3\,E^{i,j}\,, \tag{4.81}$$

and $E^{i,j} \in I\!R^{N\times N}$ is as per (4.58).

Theorem 4.21. *The value function $\mathcal{W}^{\alpha,c}$ of (4.47) and the explicit function $\breve{\mathcal{W}}^{\alpha,c}$ of (4.75) are equivalent. That is, $\mathcal{W}^{\alpha,c}(r,x,z) = \breve{\mathcal{W}}^{\alpha,c}(r,x,z)$ for all $r\in[0,t]$, $x,z\in I\!R^n$, with $t\in[0,\bar{t})$.*

Proof. It is sufficient to show that $\breve{\mathcal{W}}^{\alpha,c}$ satisfies the conditions of Theorem 4.14. To this end, note by inspection of (4.75) that

$$\tfrac{\partial}{\partial r}\breve{\mathcal{W}}^{\alpha,c}(r,x,z) = \tfrac12\,[x'\,\dot{\breve{P}}^c_r\,x + 2\,x'\,\dot{\breve{Q}}^c_r\,z + z'\,\dot{\breve{R}}^c_r\,z + \dot{\breve{\gamma}}^c_r]\,, \tag{4.82}$$

$$\nabla_x\breve{\mathcal{W}}^{\alpha,c}(r,x,z) = \breve{P}^c_r\,x + \breve{Q}^c_r\,z\,. \tag{4.83}$$

Recalling the form (4.61) of $-V^\alpha(r,x)$, in which the quadratic function $\Psi^{i,j}$ of (4.60) is defined via matrix $\mathcal{E}^{i,j}\in I\!R^{n\times n}$ of (4.59), the Hamiltonian H of (4.48) is given by

$$\begin{aligned}
&-H(t-r,x,\nabla_x\breve{\mathcal{W}}^{\alpha,c}(r,x,z))\\
&= -V^\alpha(t-r,x) - \tfrac12\,(\nabla_x\breve{\mathcal{W}}^{\alpha,c}(r,x,z))'\,\mathcal{M}^{-1}\,\nabla_x\breve{\mathcal{W}}^{\alpha,c}(r,x,z)\\
&= \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,(\alpha_{i,j}(t-r)\,m_j)\left[1 - \tfrac12\,(\alpha_{i,j}(t-r))^2\,x'\,\mathcal{E}^{i,j}\,x\right]\\
&\quad - \tfrac12\,[x'\,\breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{P}^c_r\,x + 2\,x'\breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{Q}^c_r\,z + z'\,(\breve{Q}^c_r)'\,\mathcal{M}^{-1}\,\breve{Q}^c_r\,z]\,.
\end{aligned} \tag{4.84}$$

Hence, substituting (4.82) and (4.84) in the right-hand side of the DPE (4.50) yields

$$-\tfrac{\partial}{\partial r}\breve{\mathcal{W}}^{\alpha,c}(r,x,z) - H(t-r,x,\nabla_x\breve{\mathcal{W}}^{\alpha,c}(r,x,z)) = \tfrac12\,[x'\,\breve{X}^c_r\,x + 2\,x'\,\breve{Y}^c_r\,z + z'\,\breve{Z}^c_r\,z + \breve{\zeta}^c_r] \tag{4.85}$$

in which

$$\breve{X}^c_r \doteq -\dot{\breve{P}}^c_r - \breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{P}^c_r - \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,(\alpha_{i,j}(t-r))^3\,\mathcal{E}^{i,j}\,, \tag{4.86}$$

$$\breve{Y}^c_r \doteq -\dot{\breve{Q}}^c_r - \breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{Q}^c_r\,, \quad \breve{Z}^c_r \doteq -\dot{\breve{R}}^c_r - (\breve{Q}^c_r)'\,\mathcal{M}^{-1}\,\breve{Q}^c_r\,, \tag{4.87}$$

$$\breve{\zeta}^c_r \doteq -\dot{\breve{\gamma}}^c_r + 2\sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,\alpha_{i,j}(t-r)\,. \tag{4.88}$$

Standard properties of Kronecker products (c.f., [16]) applied to (4.76) and the various terms in (4.86), (4.87) imply that

$$\dot{\breve{P}}^c_r = \dot{P}^c_r\otimes I_3\,, \quad \dot{\breve{Q}}^c_r = \dot{Q}^c_r\otimes I_3\,, \quad \dot{\breve{R}}^c_r = \dot{R}^c_r\otimes I_3\,, \tag{4.89}$$

$$\begin{aligned}
&\breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{P}^c_r = (P^c_r\otimes I_3)\,(\mathcal{M}_*^{-1}\otimes I_3)\,(P^c_r\otimes I_3) = (P^c_r\,\mathcal{M}_*^{-1}\,P^c_r)\otimes I_3\,,\\
&\breve{P}^c_r\,\mathcal{M}^{-1}\,\breve{Q}^c_r = (P^c_r\,\mathcal{M}_*^{-1}\,Q^c_r)\otimes I_3\,, \quad (\breve{Q}^c_r)'\,\mathcal{M}^{-1}\,\breve{Q}^c_r = (Q^c_r\,\mathcal{M}_*^{-1}\,Q^c_r)\otimes I_3\,,
\end{aligned} \tag{4.90}$$

$$-\sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,(\alpha_{i,j}(t-r))^3\,\mathcal{E}^{i,j} = \left(-\sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\,m_i\,m_j\,(\alpha_{i,j}(t-r))^3\,E^{i,j}\right)\otimes I_3$$

$$= \nu_r \otimes I_3 \doteq \breve{\nu}_r\,, \tag{4.91}$$

in which $E^{i,j} \in I\!R^{N\times N}$ and ν_r are as per (4.58) and (4.81) respectively. In addition, given any $A,\, B \in I\!R^{N\times N}$, note that $A\otimes I_3 = B\otimes I_3$ implies that $A = B$. Consequently, substitution of (4.89), (4.90), (4.91) in (4.86), (4.87) yields that there exists $X_r^c,\, Y_r^c,\, Z_r^c \in I\!R^{N\times N}$ such that

$$\breve{X}_r^c = X_r^c \otimes I_3\,, \quad \breve{Y}_r^c = Y_r^c\otimes I_3\,, \quad \breve{Z}_r^c = Z_r^c \otimes I_3\,, \tag{4.92}$$

where
$$X_r^c \doteq -\dot{P}_r^c - P_r^c \mathcal{M}_*^{-1} P_r^c + \nu_r\,, \quad Y_r^c \doteq -\dot{Q}_r^c - P_r^c\mathcal{M}_*^{-1} Q_r^c\,, \tag{4.93}$$

$$Z_r^c \doteq -\dot{R}_r^c - (Q_r^c)'\mathcal{M}_*^{-1} Q_r^c\,, \tag{4.94}$$

in which ν_r is as per (4.81). However, definitions (4.77), (4.78), (4.79) of $P_r^c,\, Q_r^c,\, R_r^c$ imply via (4.93), (4.94) that $0 = X_r^c = Y_r^c = Z_r^c$. Hence, (4.92) immediately implies that $0 = \breve{X}_r^c = \breve{Y}_r^c = \breve{Z}_r^c$. Similarly, definition (4.80) of γ_r^c, (4.76), and (4.88) imply that $0 = \breve{\zeta}_r^c$. Hence, DPE (4.85) holds as required. □

Remark 4.22. Recalling that $\alpha_{i,j}$ is defined for all $(i,j)\in\mathcal{I}^\Delta$, it is useful to define $\alpha_{i,j} \doteq \alpha_{j,i}$ for each $j\in]1, i-1[$. Using this definition (and re-indexing using $k,\, l \in]1, N[,\ k\neq l$), the square matrix ν_r of (4.77),(4.81) is equivalently given by

$$\nu_r \doteq -\tfrac{1}{2}\sum_{k=1}^N \sum_{l=1,\, l\neq k}^N \widehat{G}\, m_k\, m_l\, (\alpha_{k,l}(t-r))^3\, E^{k,l}\,. \tag{4.95}$$

Contributions to the $(i,j)^{th}$ entry of ν_r from the sum in (4.95) are limited to four cases, namely where (i) $k = i = j$, (ii) $l = i = j$, (iii) $k = i,\, l = j$, (iv) $k = j,\, l = i$. Rewriting (4.95) as a sum of these four cases (with terms appearing in order of (i) to (iv)),

$$\nu_r^{i,j} = -\tfrac{1}{2}\,\widehat{G}\, m_i \sum_{l=1, l\neq i}^N m_l\, (\alpha_{i,l}(t-r))^3\, 1_{i=j} - \tfrac{1}{2}\,\widehat{G} \sum_{k=1,\, k\neq i}^N m_k\, m_i\, (\alpha_{k,i}(t-r))^3\, 1_{i=j}$$
$$+ \tfrac{1}{2}\,\widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(t-r))^3\, 1_{i\neq j} + \tfrac{1}{2}\,\widehat{G}\, m_j\, m_i\, (\alpha_{j,i}(t-r))^3\, 1_{i\neq j}\,,$$

in which $1_b \doteq 1$ if b holds (and $1_b \doteq 0$ otherwise). Hence,

$$\nu_r^{i,j} = \begin{cases} -\widehat{G}\, m_i \sum_{k=1,\, k\neq i}^N m_k\, (\alpha_{i,k}(t-r))^3\,, & i = j\,, \\ +\widehat{G}\, m_i\, m_j\, (\alpha_{i,j}(t-r))^3\,, & i\neq j\,. \end{cases} \tag{4.96}$$

Similarly, note that the initial value problem (4.80) may be rewritten as

$$\dot{\gamma}_r^c = \sum_{i=1}^N \sum_{j=1,\, j\neq i}^N \widehat{G}\, m_i\, m_j\, \alpha_{i,j}(t-r)\,, \qquad \gamma_0^c = 0\,. \tag{4.97}$$

◇

Now, note that by Theorem 4.20,

$$\overline{W}^\infty(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,\infty}(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \lim_{c\to\infty} \mathcal{W}^{\alpha,c}(t,x,z)$$

which by (4.75),

$$= \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \lim_{c\to\infty} \tfrac{1}{2}\left[x'\breve{P}_t^c\, x + 2x'\breve{Q}_t^c z + z'\breve{R}_t^c\, z + \gamma_t^c\right],$$

$$= \sup_{\alpha \in \bar{\mathcal{A}}^\infty} \tfrac{1}{2} \left[x' \breve{P}_t^\infty x + 2x' \breve{Q}_t^\infty z + z' \breve{R}_t^\infty z + \gamma_t^\infty \right]. \tag{4.98}$$

Let $\Sigma(t) = \Sigma(t; m_1, m_2, \ldots m_N)$ be given by $\Sigma(t) = \left\{ \left(\breve{P}_t^\infty, \breve{Q}_t^\infty, \breve{R}_t^\infty, \gamma_t^\infty \right) \,\middle|\, \alpha \in \bar{\mathcal{A}}^\infty \right\}$. We see that

$$\overline{W}^\infty(t, x, z) = \sup_{(P,Q,R,\gamma) \in \Sigma(t)} \tfrac{1}{2} \left[x'Px + 2x'Qz + z'Rz + \gamma \right], \tag{4.99}$$

which by the linearity in (P, Q, R, γ) of the expression inside the supremum,

$$= \sup_{(P,Q,R,\gamma) \in \widehat{\Sigma}(t)} \tfrac{1}{2} \left[x'Px + 2x'Qz + z'Rz + \gamma \right], \tag{4.100}$$

where $\widehat{\Sigma}(t) \doteq \langle \Sigma(t) \rangle$ (i.e., the convex hull of $\Sigma(t)$). Consequently, we will see that the set $\widehat{\Sigma}(t) = \widehat{\Sigma}(t; m_1, \ldots m_N)$ will represent the general solution of the N-body TPBVP. With the symmetry of P, R, one can see that this set lies in $I\!R^{2N^2+N+1}$, a finite-dimensional space.

Of course, one may be concerned about computation of the suprema in (4.98) and (4.100). Specifically, one would like to know whether the object inside the supremum in (4.98) is concave. In this regard, it is helpful to define

$$\mathcal{P}_t^c = \mathcal{P}_t^c(\alpha) \doteq \begin{pmatrix} \breve{P}_t^c & \breve{Q}_t^c \\ (\breve{Q}_t^c)' & \breve{R}_t^c \end{pmatrix}. \tag{4.101}$$

We will say that a matrix-valued function, say $\mathcal{P} : \bar{\mathcal{A}}^\infty \to \mathcal{L}(I\!R^{2n}; I\!R^{2n})$ is concave if its domain, $\bar{\mathcal{A}}^\infty$, is convex and $\mathcal{P}(\alpha^0 + \delta\hat{\alpha}) - 2\mathcal{P}(\alpha^0) + \mathcal{P}(\alpha^0 - \delta\hat{\alpha}) \preceq 0$ for all $\alpha^0 \in \bar{\mathcal{A}}^\infty$, $\hat{\alpha} \in \mathcal{L}_\infty([0, \infty); I\!R^{I^\Delta})$ and $\delta \in I\!R$ such that $\alpha^0 + \delta\hat{\alpha} \in \bar{\mathcal{A}}^\infty$, where we find it useful to include δ in this definition. Here, we use the standard partial order given by $\mathcal{P} \preceq \hat{\mathcal{P}}$ if and only if $\hat{\mathcal{P}} - \mathcal{P}$ is non-negative definite.

LEMMA 4.23. *$\mathcal{P}_t^c$ is a concave function of $\alpha \in \bar{\mathcal{A}}^\infty$.*

Proof. Let $\alpha^0 \in \bar{\mathcal{A}}^\infty$, $\hat{\alpha} \in \mathcal{L}_\infty([0, \infty); I\!R^{I^\Delta})$ be such that $\alpha^0 + \hat{\alpha}, \alpha^0 - \hat{\alpha} \in \bar{\mathcal{A}}^\infty$ and $\delta \in [-1, 1]$. Let $\alpha = \alpha^0 + \delta\hat{\alpha}$ Let $\nu_t^{i,j}$ be given by (4.96) with this α, where we may then view $\nu_t^{i,j}$ as a function of δ. It is easy to see that

$$\frac{d^2 \nu_r^{i,j}}{d\delta^2} = \begin{cases} -\sum_{k \neq i} 6\widehat{G} m_i m_k \alpha_{i,j}(t-r)(\hat{\alpha}_{i,j}(t-r))^2 & \text{if } i = j, \\ 6\widehat{G} m_i m_k \alpha_{i,j}(t-r)(\hat{\alpha}_{i,j}(t-r))^2 & \text{if } i \neq j, \end{cases} \tag{4.102}$$

where we note that this implies

$$\frac{d^2 \nu_t^{i,j}}{d\delta^2} \geq 0, \quad \forall t \geq 0, \ \ i \neq j. \tag{4.103}$$

Now, for any $y \in I\!R^n$, using (4.102) and (4.103), we see that

$$y' \frac{d^2 \nu_t}{d\delta^2} y = \sum_{i \in \mathcal{N}} \frac{d^2 \nu_t^{i,i}}{d\delta^2} y_i^2 + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \frac{d^2 \nu_t^{i,j}}{d\delta^2} y_i y_j = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \frac{d^2 \nu_t^{i,j}}{d\delta^2} (-y_i^2 + y_i y_j)$$
$$\leq \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \frac{d^2 \nu_t^{i,j}}{d\delta^2} (-y_i^2 + \frac{y_i^2}{2} + \frac{y_j^2}{2}) = \tfrac{1}{2} \sum_{(i,j) \in \mathcal{I}^\Delta} \Big[\frac{d^2 \nu_t^{i,j}}{d\delta^2} (y_j^2 - y_i^2) + \frac{d^2 \nu_t^{j,i}}{d\delta^2} (y_i^2 - y_j^2) \Big]$$
$$= 0,$$

where the last inequality follows by noting that $\frac{d^2\nu_t^{i,j}}{d\delta^2} = \frac{d^2\nu_t^{j,i}}{d\delta^2}$. Consequently, $\frac{d^2\nu_t}{d\delta^2}$ is non-positive definite for all $t \geq 0$. (It may be helpful to note that taking $\bar{y}_i = 1$ for all $i \in \mathcal{N}$, $\bar{y}' \frac{d^2\nu_t}{d\delta^2} \bar{y} = 0$, and so $\frac{d^2\nu_t}{d\delta^2}$ is never strictly negative definite.) Now let

$$\mathcal{I}_x \doteq \begin{pmatrix} \mathcal{M}^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad \hat{\mathbf{N}}_t \doteq \begin{pmatrix} \frac{d\nu_t}{d\delta} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix}, \quad \text{and} \quad \mathbf{N}_t \doteq \begin{pmatrix} \frac{d^2\nu_t}{d\delta^2} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix},$$

where $0_{n\times n}$ denotes the $n \times n$ matrix with all entries zero. By the non-positive definiteness of $\frac{d^2\nu_t}{d\delta^2}$, $\mathbf{N}_r$ is non-positive definite for all $r \geq 0$. By (4.77)–(4.79) and (4.101), and recalling from (4.91) that $\breve{\nu}_r \doteq \nu_r \otimes I_3$,

$$\dot{\mathcal{P}}_t^c = -\mathcal{P}_t^c \mathcal{I}_x \mathcal{P}_t^c + \breve{\nu}_t. \tag{4.104}$$

Let $\Pi_t^c \doteq \frac{d\mathcal{P}_t^c}{d\delta}$ and $\sigma_t^c \doteq \frac{d^2\mathcal{P}_t^c}{d\delta^2}$. Differentiating (4.104) with respect to δ, we find

$$\begin{aligned} &\dot{\Pi}_t^c = -(\Pi_t^c)'\mathcal{I}_x\mathcal{P}_t^c - \mathcal{P}_t^c\mathcal{I}_x\Pi_t^c + \hat{\mathbf{N}}_t, \\ &\dot{\sigma}_t^c = -(\sigma_t^c)'\mathcal{I}_x\mathcal{P}_t^c - \mathcal{P}_t^c\mathcal{I}_x\sigma_t^c - 2(\Pi_t^c)'\mathcal{I}_x\Pi_t^c + \mathbf{N}_t \doteq -(\sigma_t^c)'\mathcal{I}_x\mathcal{P}_t^c - \mathcal{P}_t^c\mathcal{I}_x\sigma_t^c + \Omega_t^c. \end{aligned}$$

For $0 \leq r \leq t < \infty$, define $\mathcal{S}_{r,t}^c \doteq \exp\left\{-\int_r^t \mathcal{P}_s^c \, ds \, \mathcal{I}_x\right\}$. We find that

$$\sigma_t^c = \int_0^t \mathcal{S}_{r,t}^c \Omega_r^c \big(\mathcal{S}_{r,t}^c\big)' \, dr. \tag{4.105}$$

Note that by the non-positive definiteness of $\mathbf{N}_r$, Ω_r^c is non-positive definite for all $r \geq 0$. This implies that $\mathcal{S}_{r,t}^c \Omega_r^c \big(\mathcal{S}_{r,t}^c\big)'$ is non-positive definite for all $0 \leq r \leq t < \infty$, and consequently, by (4.105), σ_t^c is non-positive definite for all $t \geq 0$. Finally, note

$$\begin{aligned} &\mathcal{P}_t^c(\alpha^0 + \delta\hat{\alpha}) - 2\mathcal{P}_t^c(\alpha^0) + \mathcal{P}_t^c(\alpha^0 - \delta\hat{\alpha}) = \int_0^\delta \Pi_t^c(\alpha^0 + r\hat{\alpha})\, dr - \int_0^\delta \Pi_t^c(\alpha^0 + (r-\delta)\hat{\alpha})\, dr \\ &= \int_0^\delta \left[\int_0^r \sigma_t^c(\alpha^0 + s\hat{\alpha})\, ds + \Pi_t^c(\alpha^0)\right] dr - \int_0^\delta \left[\Pi_t^c(\alpha^0) - \int_{r-\delta}^0 \sigma_t^c(\alpha^0 + s\hat{\alpha})\, ds\right] dr \\ &= \int_0^\delta \int_{r-\delta}^r \sigma_t^c(\alpha^0 + s\hat{\alpha})\, ds\, dr \preceq 0 \quad \forall t \geq 0, \end{aligned}$$

where the last ordering follows from the non-positive definiteness of σ_t^c. ☐

THEOREM 4.24. *For all $t \in [0, \bar{t})$, $c > 0$ and $x, z \in I\!\!R^n$, both $\mathcal{W}^{\alpha,c}(t,x,z)$ and $\mathcal{W}^{\alpha,\infty}(t,x,z)$ are concave in α.*

Proof. First, as noted above, $\bar{\mathcal{A}}^\infty$ is convex. Next, note that $\mathcal{W}^{\alpha,c}(t,x,z)$ is linear in $\mathcal{P}_t^c$ and γ_t. Also note that γ_t is linear in α. Now, recall

$$\mathcal{W}^{\alpha,c}(t,x,z) = \tfrac{1}{2}\begin{pmatrix} x \\ z \end{pmatrix}' \mathcal{P}_t^c(\alpha) \begin{pmatrix} x \\ z \end{pmatrix} + \gamma_t^c(\alpha).$$

Then, by Lemma 4.23, $\mathcal{W}^{\alpha,c}(t,x,z)$ is concave in α.

Next, let $\alpha^0 \in \bar{\mathcal{A}}^\infty$, $\hat{\alpha} \in L_\infty((0,\infty); I\!\!R^{I^\Delta})$ be such that $\alpha^0 + \hat{\alpha}, \alpha^0 - \hat{\alpha} \in \bar{\mathcal{A}}^\infty$ and $\delta \in [-1,1]$. Then,

$$\begin{aligned} &\mathcal{W}^{\alpha^0+\delta\hat{\alpha},\infty}(t,x,z) - 2\mathcal{W}^{\alpha^0,\infty}(t,x,z) + \mathcal{W}^{\alpha^0-\delta\hat{\alpha},\infty}(t,x,z) \\ &\leq \limsup_{c\to\infty}\left[\mathcal{W}^{\alpha^0+\delta\hat{\alpha},c}(t,x,z) - 2\mathcal{W}^{\alpha^0,c}(t,x,z) + \mathcal{W}^{\alpha^0-\delta\hat{\alpha},c}(t,x,z)\right] \leq 0, \end{aligned}$$

where the last inequality follows from the concavity of $\mathcal{W}^{\alpha,c}(t,x,z)$. ☐

4.7. Usage in a two-point boundary value problem. The fundamental solution may be used to solve two-point boundary value problems in the same manner as indicated in Section 3.3. However, the details of the significant complication induced by the required optimization over α merit discussion. Recall that $\overline{W}^{\infty}(t,x,z)$ may be used to solve a variety of TPBVPs where the initial position vector, $x \in I\!R^{3N}$, is specified, and some combination of terminal position and velocity data are also specified. The two obvious cases are where terminal position vector $z \in I\!R^{3N}$ is specified, and where terminal velocity vector $\bar{v} \in I\!R^{3N}$ is specified.

We first consider the case where x and z are specified. The corresponding initial velocity, $v_0 = \dot{\xi}(0)$, is obtained from $v_0 = -\mathcal{M}^{-1}\nabla_x \overline{W}^{\infty}(t,x,z)$. Using (4.100), one see that this is

$$v_0 = -\mathcal{M}^{-1}[P^*x + Q^*z], \tag{4.106}$$

where

$$(P^*,Q^*,R^*,\gamma^*) \in \operatorname{argmax}\left\{\tfrac{1}{2}\Big[x'Px + 2x'Qz + z'R'z + \gamma\Big] \,\Big|\, (P,Q,R,\gamma) \in \widehat{\Sigma}(t)\right\}, \tag{4.107}$$

where the linearity of the argument in (P,Q,R,γ), the convexity and finite-dimensionality of $\widehat{\Sigma}(t)$, and the finiteness of the maximum (guaranteed by the finiteness of $\overline{W}^{\infty}$ and (4.100)) guarantee the non-emptyness of the argmax. Regarding implementation, we note that $\widehat{\Sigma}(t)$ may be computed offline, and stored. If this is done in a brute force manner, say computing quadruples $\big(\breve{P}^{\infty}_t, \breve{Q}^{\infty}_t, \breve{R}^{\infty}_t, \gamma^{\infty}_t\big)$ for a large set of piecewise constant $\alpha \in \bar{\mathcal{A}}^{\infty}$, one should store only those elements which are not obviously in the interior of the convex hull of other such points. Obviously, methods for efficient computation, storage and use of an approximation of $\widehat{\Sigma}(t)$ would be a substantial block of research, and is not considered here.

In the case where initial point vector, x, and terminal velocity vector, $\bar{v}$, are specified, additional effort is required. As in Section 3.3, one must obtain z^* such that the TPBVP for x, z^* has $\dot{\xi}^*(t) = \bar{v}$ (where $\xi^*(\cdot)$ is the solution trajectory from x to z^*). As is (3.13), and using representation (4.100), one sees that the problem becomes

$$z^* = \operatorname*{argmin}_{z \in I\!R^n} \max_{(P,Q,R,\gamma)\in\widehat{\Sigma}(t)} \left\{\tfrac{1}{2}\big[x'Px + 2x'Qz + z'Rz + \gamma\big] - \bar{v}'\mathcal{M}z\right\}.$$

One should note that the minimization and maximization are both over finite-dimensional spaces. Note also that for sufficiently small $t > 0$ (depending on α), one can expect R^{∞}_t to be positive definite, and consequently, one has a convex/concave argument. Although beyond the scope here, if one obtains conditions on the problem data such that conditions for a unique saddle point are satisfied (cf., [30]), then

$$\begin{aligned}
&\min_{z\in I\!R^n} \max_{(P,Q,R,\gamma)\in\widehat{\Sigma}(t)} \left\{\tfrac{1}{2}\big[x'Px + 2x'Qz + z'Rz + \gamma\big] - \bar{v}'\mathcal{M}z\right\} \\
&= \max_{(P,Q,R,\gamma)\in\widehat{\Sigma}(t)} \left\{\tfrac{1}{2}\Big[x'Px + \gamma - (\mathcal{M}\bar{v} - Q'x)'R^{-1}(\mathcal{M}\bar{v} - Q'x)\Big]\right\},
\end{aligned}$$

and the desired initial velocity is $v_0 = -\mathcal{M}^{-1}[P^*x + Q^*z^*]$ where $z^* = (R^*)^{-1}(\mathcal{M}\bar{v} - (Q^*)'x)$ and

$$(P^*,Q^*,R^*,\gamma^*) \in \operatorname*{argmax}_{(P,Q,R,\gamma)\in\widehat{\Sigma}(t)} \left\{x'Px + \gamma - (\mathcal{M}\bar{v} - Q'x)'R^{-1}(\mathcal{M}\bar{v} - Q'x)\right\}.$$

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