The Principle of Least Action and Solution of Two-Point Boundary Value Problems on a Limited Time Horizon*

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Abstract

Two-point boundary problems for conservative systems are studied in the context of the least action principle. The emphasis is on the N-body problem under gravitation. There, the least action principle optimal control problem is converted to a differential game, where an opposing player maximizes over an indexed set of quadratics to yield the gravitational potential. For problems where the time-duration is below a specified bound, fundamental solutions are obtained as indexed sets of solutions of Riccati equations.

1 Introduction

We suppose a conservative system follows a trajectory minimizing the action functional, this being known as the principle of least action or as Hamilton's principle (c.f., [8, 9]). This allows the dynamical model to be posed in terms of various optimal control problems. Solution of the control problems allows one to convert two-point boundary-value problems (TPBVPs) for the dynamical system into initial value problems. In a simple mass-spring system, wherein solution of an associated Riccati equation generates the fundamental solution, this allows one to answer a variety of TPBVPs via a simple max-plus integral (equivalently, a supremum). We will concern ourselves mainly with the N-body problem in orbital mechanics. In this case, the analysis becomes more technical. Nonetheless, one can construct machinery for guaranteed solution of the various TPBVPs.

Suppose the position component of the state at time, t, is denoted by $\xi(t) \in I\!R^n$, where also, we will use $x \in I\!R^n$ to denote generic positions. Let the potential energy at $x \in I\!R^n$ be denoted by $V(x)$. The kinetic energy at time, t, will be denoted by $T(\dot\xi(t)) \doteq \frac{1}{2}\dot\xi'(t)\mathcal{M}\dot\xi(t)$, where if $\xi(t)$ refers to the position of a point mass, $\mathcal{M}$ is simply mI, where m is the mass of the body; in a multi-body system, this is generalized in the obvious way. The action functional corresponding to $\{\xi(r)\,|\,r \in [0,t]\}$ is given by

$$\mathcal{F}(\xi(\cdot)) \doteq \int_0^t -V(\xi(r)) + T(\dot\xi(r))\,dr.$$

The principle of least action states that a system evolves so as to minimize the action functional.

One can also interpret this in terms of the characteristic equations corresponding to the Hamiltonian of the system. Let the initial position be $\xi(0) = x \in I\!R^n$, and let the dynamics be $\dot\xi(r) = u(r)$ for all $r \in (0,t)$, where $u = u(\cdot) \in \mathcal{U}^{s,t} \doteq \mathcal{L}_2([s,t); I\!R^n)$. Also let $\mathcal{U}^\infty \doteq \{u : (0,\infty) \to I\!R^n \,|\, u_{(0,t)} \in \mathcal{U}^{0,t}\, \forall t \in (0,\infty)\}$, where $u_{(0,t)}$ denotes the restriction of the function to domain $(0,t)$. Define the control formulation payoff, $J^0 : [0,\infty) \times I\!R^n \times \mathcal{U}^\infty \to I\!R \cup \{-\infty,+\infty\}$, as

$$J^0(t,x,u) \doteq \int_0^t -V(\xi(r)) + T(u(r))\,dr = \int_0^t -V(\xi(r)) + \tfrac{1}{2}u'(r)\mathcal{M}u(r)\,dr, \tag{1.1}$$

where $\mathcal{M}$ is positive-definite symmetric, and the corresponding value function as

$$W^0(t,x) \doteq \inf_{u\in\mathcal{U}^\infty} J^0(t,x,u). \tag{1.2}$$

Clearly a solution of this problem yields an $\xi(\cdot)$ satisfying the least action principle, and so is the trajectory of the conservative system under potential energy field V.

Let $\mathcal{D} \doteq (0,t) \times I\!R^n$ (with $\bar{\mathcal{D}} \doteq [0,t] \times I\!R^n$) and $\hat C^1 \doteq C(\bar{\mathcal{D}}) \cap C^1(\mathcal{D})$. Under quite reasonable conditions on V, one can expect that $W^0 \in \hat C^1$, and that on $\mathcal{D}$, W^0 satisfies

$$\begin{aligned} 0 &= -\frac{\partial}{\partial t}W(r,x) - V(x) \\ &\quad -\frac{1}{2}[\nabla_x W(r,x)]'\mathcal{M}^{-1}\nabla_x W(r,x) \\ &\doteq -\bar H\Big(r,x,\frac{\partial}{\partial t}W(r,x), \nabla_x W(r,x)\Big) \\ &\doteq -\frac{\partial}{\partial t}W(r,x) - H\Big(r,x,\nabla_x W(r,x)\Big). \end{aligned} \tag{1.3}$$

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It is also well-established that under sufficiently strong conditions, first-order HJB PDEs such as (1.3) can be solved via the method of characteristics. The characteristic equations associated with (1.3) are

$$\frac{dr}{d\rho} = \bar{H}_q(r, \hat{\xi}, q, p) = 1 \tag{1.4}$$

$$\frac{d\hat{\xi}}{d\rho} = \bar{H}_p(r, \hat{\xi}, q, p) = \mathcal{M}^{-1} p(\rho) \tag{1.5}$$

$$\frac{dq}{d\rho} = -\bar{H}_r(r, \hat{\xi}, q, p) = 0 \tag{1.6}$$

$$\frac{d\hat{p}}{d\rho} = -\bar{H}_x(r, \hat{\xi}, q, p) = -\nabla_x V(\hat{\xi}(\rho)). \tag{1.7}$$

These have associated initial and terminal conditions

$$\begin{aligned} &\hat{\xi}(t) = x, \quad r(t) = 0, \quad \hat{p}(0) = 0 \\ &q(0) = -V(\hat{\xi}(0)) - \frac{1}{2}(\hat{p}(0))' \mathcal{M}^{-1} \hat{p}(0) = -V(\hat{\xi}(0)), \end{aligned} \tag{1.8}$$

where $\hat{p}(0) = 0$ follows from the lack of a terminal cost here. Because of (1.4), we may take $r = \rho$. Noting (1.6) and (1.8), we see that $q(r) = V(\hat{\xi}(0))$ for all r. Also, in order to return to forward time, we may take $s = t - r$, $\xi(s) = \hat{\xi}(t - s)$ and $p(s) = \hat{p}(t - s)$, in which case we have

$$\frac{d\xi}{ds} = -\mathcal{M}^{-1} p(s), \quad \frac{dp}{ds} = \nabla_x V(\xi(s)), \tag{1.9}$$

or,

$$\frac{d^2\xi}{ds^2} = -\mathcal{M}^{-1} \nabla_x V(\xi(s)), \tag{1.10}$$

which of course, is the classical Newton's second law formulation. Note that in the above development, the trajectory was not fully specified, as only the initial position, not the initial state (position and velocity), was given. Of course, (1.9) implies that the additive inverse of the co-state $p(r)$, is the momentum. (One might also note that the optimal velocity in the HJB PDE is attained at $v = -\mathcal{M}^{-1} \nabla_x W = -\mathcal{M}^{-1} p$.) Given both the initial position and initial velocity, forward integration of (1.9) is the classical initial value problem (IVP) form for the system dynamics.

Suppose however, that one attaches a terminal cost to J^0 yielding, say

$$\overline{J}(t, x, u) = J^0(t, x, u) + \bar{\psi}(\xi(t)), \tag{1.11}$$

$$\overline{W}(t, u) = \inf_{u \in \mathcal{U}^\infty} \overline{J}(t, x, u). \tag{1.12}$$

The dynamic programing equation (DPE) and characteristic equations (1.9) remain unchanged. However, although the initial condition is still $\xi(0) = x$, the terminal condition is defined by $\bar{\psi}$. That is, we have a TPBVP where we control the terminal condition.

TPBVPs are common in classical optimal control theory, where the above characteristic equations appear in Calculus of Variations and Pontryagin Maximum Principle approaches. There, one is required to solve the relevant TPBVP to obtain the desired optimal control problem solution. Classical methods used a shooting approach, and more modern methods such as pseudo-spectral algorithms (c.f., [10]) have greatly advanced the state of the art.

Here we have a slightly different goal; we desire to solve TPBVPs arising from dynamical systems governed by conservative dynamics. With the addition of terminal cost, $\bar{\psi}$, the boundary conditions for (1.9) consist of initial condition

$$\xi(0) = x, \tag{1.13}$$

and terminal condition

$$p(t) = \nabla_x \bar{\psi}(\xi(t)). \tag{1.14}$$

If one takes, for example, $\bar{\psi}(x) = -\bar{v} \cdot x$ for some given $\bar{v} \in I\!R^n$, then terminal condition (1.14) becomes $p(t) = \bar{v}$. That is, one has boundary conditions

$$\xi(0) = x \quad \text{and} \quad \dot{\xi}(t) = \bar{v}. \tag{1.15}$$

Alternatively, if one takes $z \in I\!R^n$ and $\bar{\psi}(x) = \psi^\infty(x) \doteq \delta_0^-(x - z)$ where

$$\delta_0^-(y) \doteq \begin{cases} 0 & \text{if y=0} \\ +\infty & \text{otherwise} \end{cases} \tag{1.16}$$

(i.e., the min-plus "delta function"), then the solution of control problem (1.12) yields solution of the conservative system with boundary conditions

$$\xi(0) = x \quad \text{and} \quad \xi(t) = z. \tag{1.17}$$

Clearly, other boundary conditions can be generated as well.

The goal here will be the development of *fundamental solutions* for TPBVPs corresponding to conservative systems. These fundamental solutions will generate particular solutions for boundary conditions such as $\dot{\xi}(t) = \bar{v}$ via a max-plus integration over $I\!R^n$.

In the case where the potential energy takes a linear-quadratic form, the fundamental solution may be obtained through solution of an associated Riccati equation. However, here we will apply the approach to N-body problems under the gravitational potential. In this case, the potential does not take a linear-quadratic form. However, we will see that one may take a dynamic game approach to gravitation, where the potential is a linear-quadratic form in the position variable. This requires an additional max-plus integral, over the opponent controls, beyond that which is required in the purely linear-quadratic potential case.

2 Background and Standard Theory

Because of space limitations, the bulk of the proofs of the results below are not included.

As indicated in the introduction, we consider conservative systems, and taking the least-action approach, we model the dynamics of position as

$$\dot{\xi}(r) = u(r), \qquad \xi(0) = x \in I\!R^n, \tag{2.18}$$

with $u \in \mathcal{U}^\infty$. With potential and kinetic energy functions $V(x)$ and $T(y) = \frac{1}{2}y'\mathcal{M}y$, the running cost is

$$L(\xi(r), \dot{\xi}(r)) = T(\dot{\xi}(r)) - V(\xi(r)).$$

Note that by (1.1),(1.11),(1.12),

$$\overline{W}(0,x) = \bar{\psi}(x), \tag{2.19}$$

and that $\overline{W}(t,x) = \mathcal{S}_t[\bar{\psi}](x)$ for $t > 0$. For $c \in [0,\infty)$, let $\psi^c : I\!R^n \times I\!R^n \to [0,\infty)$ be given by

$$\psi^c(x,z) = \frac{c}{2}|x-z|^2. \tag{2.20}$$

Also let $\psi^\infty : I\!R^n \times I\!R^n \to [0,\infty]$ (where we find it notationally convenient to let $[0,\infty] \doteq [0,\infty) \cup \{+\infty\}$) be given by

$$\psi^\infty(x,z) = \delta_0^-(x-z), \tag{2.21}$$

where δ_0^- is given in (1.16).

Define the finite time-horizon payoffs $J^c : [0,\infty) \times I\!R^n \times \mathcal{U}^\infty \times I\!R \to I\!R \cup \{-\infty, +\infty\}$ by

$$J^c(t,x,u,z) \doteq \int_0^t L(\xi(s),\, u(s))\, ds + \psi^c(\xi(t), z),$$

for $c \in [0,\infty]$, where we specifically note $J^0(t,x,u) = \int_0^t L(\xi(s),\, u(s))\, ds$. Also, for $c \in [0,\infty]$, we let

$$W^c(t,x,z) \doteq \inf_{u \in \mathcal{U}^\infty} J^c(t,x,u,z). \tag{2.22}$$

We establish existence in the special case where the potential energy function, V, is bounded. In the interest of space, and given the extensive existing literature, we do not prove this with potential energy functions corresponding to mass-spring systems and N-body problems.

THEOREM 2.1. *Let $c \in (0,\infty)$ and $z \in I\!R^n$. Suppose $W \in C(I\!R_{\geq 0} \times I\!R^n \times I\!R^n; I\!R) \cap C^1(I\!R_{>0} \times I\!R^n \times I\!R^n; I\!R)$ satisfies*

$$0 = -\tfrac{\partial}{\partial t} W(t,x,z) - H(r, x, \nabla_x W(t,x,z)), \tag{2.23}$$

$$W(0,x,z) = \psi^c(x,z) \tag{2.24}$$

for all $t \in I\!R_{\geq 0}$ and $x \in I\!R^n$, where $H/\bar{H}$ are the Hamiltonians (1.3). Then, $W(t,x,z) \leq J^c(t,x,u,z)$ for all $x \in I\!R^n$, $u \in \mathcal{U}^\infty$. Furthermore, $W(t,x,z) = J^c(t,x,u^*,z)$ for the input $u^*(s) \doteq -\mathcal{M}^{-1}\nabla_x W(t-s, \xi^*(s), z)$, $s \in [0,t]$, where ξ^* is the solution of dynamics (2.18), driven by u^*. Consequently $W(t,x,z) = W^c(t,x,z)$.

A reachability problem of interest is defined via the value function $\widetilde{W} : I\!R_{\geq 0} \times I\!R^n \times I\!R^n \mapsto I\!R$, where

$$\widetilde{W}(t,x,z) \doteq \inf_{u \in \mathcal{U}^\infty} \left\{ \int_0^t L(\xi(s), u(s))\, ds \right\}. \tag{2.25}$$

where (2.18) holds with $\xi(0) = x$, $\xi(t) = z$.

Using $\widetilde{W}$ of (2.25), it is convenient to define the function $\widehat{W} : I\!R_{\geq 0} \times I\!R^n \mapsto I\!R$ by

$$\widehat{W}(t,x) \doteq \inf_{z \in I\!R^n} \left\{ \widetilde{W}(t,x,z) + \bar{\psi}(z) \right\}. \tag{2.26}$$

THEOREM 2.2. *The value function $\overline{W}$ of (1.12) and the function $\widehat{W}$ of (2.26) are equivalent. That is,*

$$\overline{W}(t,x) = \widehat{W}(t,x) \tag{2.27}$$

for all $t \in I\!R_{\geq 0}$ and $x \in I\!R^n$.

In view of Theorem 2.2, $\widetilde{W}$ may be regarded as fundamental to the solution of the optimal control problem (1.12). In particular, characterization of $\widetilde{W}$ of (2.25) admits the evaluation of $\overline{W}$ of (1.12) for *any* terminal cost $\bar{\psi}$ via (2.26). To this end, establishing a relationship between W^∞ of (2.22) and $\widetilde{W}$ of (2.25) is vital.

THEOREM 2.3. *The value functions W^∞ of (2.22) and $\widetilde{W}$ of (2.25) are equivalent. That is,*

$$W^\infty(t,x,z) = \widetilde{W}(t,x,z) \tag{2.28}$$

for all $t \in I\!R_{\geq 0}$ and $x, z \in I\!R^n$.

3 The N-body problem

Here, we address the solution of TPBVPs with N bodies acting under gravitational acceleration. In particular, we obtain a means for conversion of TPBVPs to initial value problems. The key to application of our approach to this class of problems lies in a variation of convex duality, leading to an interpretation of the least action principle as a differential game.

The following is easily obtained through methods of convex duality (c.f., [14, 15, 16]), and we do not include a proof.

LEMMA 3.1. *Suppose $f(\hat{\rho}) = \hat{\rho}^{-1/2}$ for all $\hat{\rho} \in (0,\infty)$. Then,*

$$f(\hat{\rho}) = \sup_{\hat{\beta} < 0} \left[\hat{\beta}\hat{\rho} + a(\hat{\beta}) \right] \quad \forall \hat{\rho} > 0,$$

where

$$a(\hat\beta) = -\sup_{\hat\rho>0}\left[\hat\beta\hat\rho - f(\hat\rho)\right] \quad \forall\hat\beta<0,$$

and, in particular, $a(\hat\beta) = \frac{-3}{2}(2\hat\beta)^{1/3}$ *for all* $\hat\beta \in (-\infty,0)$.

Minor manipulation of this duality result yields:

LEMMA 3.2. *For* $\rho>0$, *one has*

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2}\max_{\alpha\in(0,\infty)}\alpha\left[1-\frac{(\alpha\rho)^2}{2}\right].$$

Proof. Letting $\beta \doteq -\hat\beta$, Lemma 3.1 implies

$$\hat\rho^{-1/2} = \sup_{\beta>0}\left[\frac{3}{2}(2\beta)^{1/3} - \beta\hat\rho\right], \quad \forall\hat\rho>0.$$

Next, letting $\alpha = \sqrt{\frac{2}{3}}(2\beta)^{1/3}$ for $\beta>0$, we find

$$\hat\rho^{-1/2} = \sup_{\alpha>0}\left[\left(\frac{3}{2}\right)^{3/2}\alpha - \left(\frac{3}{2}\right)^{3/2}\frac{\alpha^3\hat\rho}{2}\right], \quad \forall\hat\rho>0.$$

Finally, letting $\hat\rho=\rho^2$ for $\rho>0$, we see that the above becomes

$$\frac{1}{\rho} = \left(\frac{3}{2}\right)^{3/2}\sup_{\alpha>0}\alpha\left[1-\frac{(\alpha\rho)^2}{2}\right], \quad \forall\rho>0.$$

Lastly, note that the supremum is always attained, and does so at $\sqrt{\frac{2}{3}}\frac{1}{\rho}$. □

Recall that the gravitational potential energy due to two point-mass bodies of mass m_1 and m_2, separated by distance, $\rho>0$, is given by

$$\mathcal{G}^{m_1,m_2}(\rho) = \frac{-Gm_1m_2}{\rho},$$

where G is the universal gravitational constant. Of course, this is also valid for spherically symmetric bodies when the distance is greater than the sum of the radii of the bodies, and we do not concern ourselves with this distinction further. Using Lemma 3.2, we see that this may be represented as

$$-\mathcal{G}^{m_1,m_2}(\rho) = \widehat{G}\,m_1\max_{\alpha_{1,2}\ge0}(\alpha_{1,2}m_2)\left[1-\frac{(\alpha_{1,2}\rho)^2}{2}\right],$$

where the universal gravitational constant is replaced by $\widehat{G} \doteq \left(\frac{3}{2}\right)^{3/2}G$. In the case of N bodies at locations x^i for $i\in\mathcal{N}\doteq]1,N[$ (where for integers $i<j$, we let $]i,j[$ denote $\{i,i+1,i+2,\dots j\}$ throughout), the additive inverse of the potential is given by

$$-V(x) = \sum_{(i,j)\in\mathcal{I}^\Delta}\widehat{G}\,m_i\max_{\alpha_{i,j}\ge0}(\alpha_{i,j}m_j)\left[1-\frac{(\alpha_{i,j}|x^i-x^j|)^2}{2}\right] \tag{3.29}$$

where $\mathcal{I}^\Delta \doteq \{(i,j)\in]1,N[\,|\,j>i\}$ and $x=\{x^1,x^2,\dots x^N\}\in\mathcal{X}\doteq(I\!R^3)^N$. Throughout, we will largely suppress the dependence of V on the body masses. It is worth noting that the term in brackets in (3.29) is negative for $\alpha_{i,j} > \frac{\sqrt{2}}{|x^i-x^j|}$. Let $\mathcal{A}\doteq\{\,\alpha=\{\alpha_{i,j}\}_{(i,j)\in\mathcal{I}^\Delta}\,|\,\alpha_{i,j}\in[0,\infty)\,\forall(i,j)\in\mathcal{I}^\Delta\,\}$, and note that $\mathcal{A}\subset I\!R^{I^\Delta}$ where $I^\Delta\doteq\#\mathcal{I}^\Delta$. Then (3.29) may be written as

$$-V(x) = \max_{\alpha\in\mathcal{A}}\{-\hat V(x,\alpha)\} \tag{3.30}$$

where

$$-\hat V(x,\alpha) \doteq \sum_{(i,j)\in\mathcal{I}^\Delta}\widehat{G}\,m_i(\alpha_{i,j}m_j)\left[1-\frac{(\alpha_{i,j}|x^i-x^j|)^2}{2}\right]. \tag{3.31}$$

With $n\doteq3N$, let $\xi(\cdot)$ be a trajectory of the N-body system satisfying (2.18). The running cost will be

$$L(\xi(r),\dot\xi(r)) = T(\dot\xi(r)) - V(\xi(r)), \tag{3.32}$$

where V is given by (3.30) and

$$T(y) \doteq \sum_{(i,j)\in\mathcal{I}^\Delta}\frac{m_i|y^i|^2}{2}, \tag{3.33}$$

for $y\in\mathcal{X}$. Note also that for $x,z\in\mathcal{X}$, one continues to have

$$\psi^c(x,z) = \frac{c}{2}|x-z|^2 = \frac{c}{2}\sum_{i=1}^N|x^i-z^i|^2. \tag{3.34}$$

With these specific definitions, the least-action payoff, $\bar J^c$, becomes

$$\begin{aligned}\bar J^c(t,x,u,z) &= \int_0^t T(u(r)) + \sum_{(i,j)\in\mathcal{I}^\Delta}\left[\frac{Gm_im_j}{|\xi^i(r)-\xi^j(r)|}\right]dr\\ &\quad+\psi^c(\xi(t),z)\\ &= \int_0^t T(u(r)) + \max_{\alpha\in\mathcal{A}}\{-\hat V(\xi(r),\alpha)\}\,dr\\ &\quad+\psi^c(\xi(t),z).\end{aligned} \tag{3.35}$$

As in (2.22), we let the value be given by

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty}\bar J^c(t,x,u,z). \tag{3.36}$$

We assume:

$\exists\,\bar\delta,\bar\epsilon>0$ such that $\forall$ ϵ-optimal $u^\epsilon\in\mathcal{U}^\infty$ with $\epsilon\in(0,\bar\epsilon]$, and letting ξ^ϵ denote the corresponding trajectory, we have $|(\xi^\epsilon)^i(r)-(\xi^\epsilon)^j(r)|\ge\bar\delta\ \forall r\in[0,t],\ \forall(i,j)\in\mathcal{I}^\Delta$.

(A.N1)

Let

$$\mathcal{A}^\infty \doteq \{\ \alpha : [0,\infty) \to \mathcal{A} \,|\, \exists K < \infty,\ \{\tau_k\}_{k\in]0,K[}$$
$$\text{such that } \tau_0 = 0,\ \tau_K = t,\ \tau_{(k-1)} < \tau_k \,\forall k,$$
$$\text{and } \alpha_{[\tau_{k-1},\tau_k)} \in C([\tau_{k-1},\tau_k);\mathcal{A})\,\forall k \in]1,K[\,\}, \tag{3.37}$$

and we note that, of course, $C([0,\infty);\mathcal{A}) \subset \mathcal{A}^\infty$. Also, we replace the time-independent potential energy function with

$$-V^\alpha(r,x) \doteq -\hat{V}(x,\alpha(r)) \tag{3.38}$$
$$= \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i(\alpha_{i,j}(r) m_j)\left[1 - \frac{(\alpha_{i,j}(r)|x^i - x^j|)^2}{2}\right].$$

THEOREM 3.1. *For all $t \geq 0$ and all $x,z \in I\!\!R^n$,*

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty} \max_{\alpha(\cdot)\in\mathcal{A}^\infty} J^c(t,x,u,\alpha,z),$$

where

$$J^c(t,x,u,\alpha,z) \doteq \int_0^t T(u(r)) - V^\alpha(r,\xi(r))\,dr$$
$$+\psi^c(\xi(t),z).$$

Proof. Let $\epsilon, \bar{\delta}$ be as in Assumption $(A.N1)$. Let $\mathcal{U}_\epsilon^\infty$ denote the set of ϵ-optimal controls, $u^\epsilon \in \mathcal{U}^\infty$. Obviously,

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}_\epsilon^\infty} \bar{J}^c(t,x,u,z)$$
$$= \inf_{u\in\mathcal{U}_\epsilon^\infty} \int_0^t T(u(r)) \tag{3.39}$$
$$+ \max_{\alpha\in\mathcal{A}} \sum_{(i,j)\in\mathcal{I}^\Delta} \widehat{G}\, m_i(\alpha_{i,j} m_j)\left[1 - \frac{(\alpha_{i,j}|\xi^i(r) - \xi^j(r)|)^2}{2}\right] dr$$
$$+\psi^c(\xi(t),z).$$

Note that, as in the proof of Lemma 3.2, for $\rho > 0$

$$\operatorname*{argmax}_{\alpha_{i,j}\in[0,\infty)} \widehat{G}\, m_i(\alpha_{i,j} m_j)\left[1 - \frac{(\alpha_{i,j}\rho)^2}{2}\right] = \sqrt{\frac{2}{3}}\frac{1}{\rho}. \tag{3.40}$$

Let $\bar{\alpha}^*(x)$ be given by

$$\bar{\alpha}^*_{i,j}(x^i,x^j) \doteq \sqrt{\frac{2}{3}}\frac{1}{|x^i - x^j|}, \tag{3.41}$$

for all $(i,j) \in \mathcal{I}^\Delta$, $\forall |x^i - x^j| > 0$, where the domain is implicitly clear. Further, given $u \in \mathcal{U}_\epsilon^\infty$ and corresponding ξ, let $\alpha^*(r) = \alpha^*(r;u(\cdot)) = \{\alpha^*_{i,j}(r) \,|\, (i,j) \in \mathcal{I}^\Delta\} \in \mathcal{A}^\infty$ be given by

$$\alpha^*_{i,j}(r) = \bar{\alpha}^*_{i,j}(\xi^i(r),\xi^j(r)), \quad \forall r \in [0,t). \tag{3.42}$$

Note that by $(A.N1)$, for $u \in \mathcal{U}_\epsilon^\infty$, $\alpha^* \in \mathcal{A}^\infty$ (by which we mean there exists an extension of α^* past the terminal time, t, in $\mathcal{A}^\infty$).

By (3.39),

$$W^c(t,x,z) \geq \inf_{u\in\mathcal{U}^\infty} \max_{\alpha(\cdot)\in\mathcal{A}^\infty} J^c(t,x,u,\alpha,z).$$

On the other hand, combining (3.39)–(3.42), one has

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}_\epsilon^\infty} J^c(t,x,u,\alpha^*,z),$$

which completes the proof. □

COROLLARY 3.1. *For all $t \geq 0$ and all $x,z \in I\!\!R^n$,*

$$W^c(t,x,z) = \inf_{u\in\mathcal{U}^\infty} \max_{\alpha(\cdot)\in\bar{\mathcal{A}}^\infty} J^c(t,x,u,\alpha,z),$$

where $\bar{\mathcal{A}}^\infty \doteq L_\infty([0,\infty);\mathcal{A})$.

We specifically note that the problem of finding the fundamental solution of the TPBVP for the N-body problem has been converted to a differential game. In a heuristic sense, one may think of the problem now as not only a search over possible world lines of the bodies, but as also including a search over negotiated potentials between the bodies. Again heuristically, one may think of the potentials, not as fields existing throughout space but as the opposing player in a game interpretation. The first player minimizes the action at each moment, with immediate effect on the kinetic term and integrated effect on the other terms, while the second player maximizes the potential term at each moment. The analytical gain obtained through the use of this viewpoint is that it allows one to express the potential energy as a quadratic form.

We note that (3.39) is a non-standard form for dynamic games. The inf / sup is neither in terms of non-anticipative strategies (c.f., [2, 7]), nor in terms of state feedback controls. This is due to the very simple form of the maximizing player, which is only a representation for the running cost.

LEMMA 3.3. *$W^c(t,x,z) \in [0, \bar{D}t + \psi^c(x,z)]$ for all $t \geq 0$ and all $x,z \in I\!\!R^n$, where $\bar{D} = (G/\bar{\delta})\sum_{(i,j)\in\mathcal{I}^\Delta} m_i m_j$.*

LEMMA 3.4. *For ϵ-optimal $u^\epsilon \in \mathcal{U}^\infty$, with $\epsilon \in (0,1]$, we have $\|u\|_{L_2(0,t)} \leq \frac{2}{\bar{m}}(\bar{D}t + \psi^c(x,z) + 1)$, where $\bar{m} \doteq \min_{i\in\mathcal{N}} m_i$.*

LEMMA 3.5. *For any $t_0 > 0$, $W^c(t,x,z)$ is semiconvex in x, uniformly in $(t,x,z,c) \in [t_0,\infty) \times I\!\!R^n \times I\!\!R^n \times [0,\infty]$.*

The HJB PDE associated with our problem here is

$$
\begin{aligned}
0 &= -\tfrac{\partial}{\partial t}W(t,x,z) - H(x, \nabla_x W(t,x,z)) \\
&\doteq -\tfrac{\partial}{\partial t}W(t,x,z) \\
&\quad + \inf_{v\in I\!R^n} \sup_{\alpha\in\mathcal{A}} \Big\{ \tfrac{1}{2} v'\mathcal{M}v - \hat{V}(x,\alpha) + v'\nabla_x W(t,x,z) \Big\}. \qquad (3.43)
\end{aligned}
$$

Note that the right-hand side of (3.43) is separated (and in fact, the Isaacs condition is satisfied). Consequently, we may write (3.43) as

$$
\begin{aligned}
0 &= -\tfrac{\partial}{\partial t}W(t,x,z) \\
&\quad - \tfrac{1}{2}\big(\nabla_x W(t,x,z)\big)'\mathcal{M}^{-1}\nabla_x W(t,x,z) \\
&\quad + \sup_{\alpha\in\mathcal{A}}\{-\hat{V}(x,\alpha)\}. \qquad (3.44)
\end{aligned}
$$

Let

$$
D^{\bar{\delta}} \doteq \big\{ x \in I\!R^n \,\big|\, |x^i - x^j| > \bar{\delta}\, \forall (i,j) \in \mathcal{I}^\Delta \big\}, \qquad (3.45)
$$

where $\bar{\delta}$ is as indicated in Assumption $(A.N1)$. Also, for $t > 0$, let

$$
\begin{aligned}
\mathcal{D}_t^{\bar{\delta}} &\doteq \big\{ W : [0,t]\times \bar{D}^{\bar{\delta}} \to I\!R \,\big| \\
&\qquad W \in C([0,t]\times\bar{D}^{\bar{\delta}}) \cap C((0,t)\times D^{\bar{\delta}}) \big\}. \qquad (3.46)
\end{aligned}
$$

Theorem 3.2. *Let $c \in (0,\infty)$, $t > 0$ and $z \in D^{\bar{\delta}}$. Suppose $W \in \mathcal{D}_t^{\bar{\delta}}$ satisfies (3.43) on $(0,t)\times D^{\bar{\delta}}$, and initial condition*

$$
W(0,x,z) = \psi^c(x,z), \qquad x \in D^{\bar{\delta}}. \qquad (3.47)
$$

Then, $W(t,x,z) = W^c(t,x,z)$ for all $x \in D^{\bar{\delta}}$. In particular, for any ϵ-optimal u^ϵ with $\epsilon \in (0,\bar{\epsilon}]$, $W(t,x,z) \le \bar{J}^c(t,x,u^\epsilon,z)$, and further, with the controller $u^*(s)$ given by $u^*(s) = \tilde{u}(s,\tilde{\xi}(s))$ where $\tilde{\xi}(s)$ is generated by feedback $\tilde{u}(s,x) \doteq \nabla_x W(t-s,x,z)$, one has $W(t,x,z) = \bar{J}^c(t,x,u^*,z)$.

We now proceed to consider the game where the order of infimum and supremum are reversed. Due to the very simple form of this particular game, with the α controller acting only on the running cost and that being in a separated form, an unusual equivalence can be obtained. Let

$$
\underline{W}^c(t,x,z) \doteq \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \qquad (3.48)
$$

By the usual reordering inequality, one immediately has

$$
\underline{W}^c(t,x,z) \le W^c(t,x,z) \quad \forall (t,x,z) \in [0,\infty)\times I\!R^n \times I\!R^n. \qquad (3.49)
$$

It will be helpful to introduce more notation. For any $\alpha \in \bar{\mathcal{A}}^\infty$, we let

$$
\mathcal{W}^{\alpha,c}(t,x,z) \doteq \inf_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,z). \qquad (3.50)
$$

The corresponding Hamiltonian will be

$$
H^\alpha(r,x,p) \doteq V^\alpha(t,x) + \frac{1}{2}p'\mathcal{M}^{-1}p. \qquad (3.51)
$$

Of course, one immediately sees that

$$
\underline{W}^c(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z) \qquad (3.52)
$$

for all $(t,x,z) \in [0,\infty)\times I\!R^n \times I\!R^n$.

In a similar fashion to the above verification results, we have the following.

Theorem 3.3. *Let $c \in (0,\infty)$, $z \in D^{\bar{\delta}}$ and $\alpha \in \bar{\mathcal{A}}^\infty$. In particular, suppose that α is piecewise continuous, with possible discontinuities only at $0 = \tau_0 < \tau_1 < \tau_2 < \dots \tau_K = t$ with $K < \infty$. Let $\tau_0 = 0$, $\tau_K = t$ and $\mathcal{O}^t \doteq \bigcup_{k\in]0,K-1[}(\tau_k,\tau_{k+1})$. Suppose $W^\alpha \in C((0,t)\times \overline{D}^{\bar{\delta}}; I\!R) \cap C^1(\mathcal{O}^t \times D^{\bar{\delta}}; I\!R)$ satisfies*

$$
\begin{aligned}
0 &= -\tfrac{\partial}{\partial r}W^\alpha(r,x,z) - H^\alpha(t-r,x,\nabla_x W^\alpha(r,x,z)) \\
&\qquad (r,x) \in \mathcal{O}^t \times D^{\bar{\delta}}, \qquad (3.53) \\
W^\alpha(0,x,z) &= \psi^c(x,z), \qquad x \in D^{\bar{\delta}}. \qquad (3.54)
\end{aligned}
$$

Then, $W^\alpha(t,x,z) \le J^c(t,x,u,\alpha,z)$ for all $x \in I\!R^n$, $u \in \mathcal{U}^\infty$. Furthermore, $W^\alpha(t,x,z) = J^c(t,x,u^*,\alpha,z)$ for the input $u^*(s) \doteq -\mathcal{M}^{-1}\nabla_x W^\alpha(t-s,x,z)$, $s \in [0,t]$. Consequently $W^\alpha(t,x,z) = \mathcal{W}^{\alpha,c}(t,x,z)$.

Now let u^* be the optimal controller for our original problem (with potential energy function, $V(\cdot)$), that is $u^*(s) = \tilde{u}(s,\tilde{\xi}(s))$ where $\tilde{\xi}(s)$ is generated by feedback $\tilde{u}(s,x) \doteq \nabla_x W^c(t-s,x,z)$. Let $\xi^*(s)$ be the resulting trajectory, where of course, $\xi^* \equiv \tilde{\xi}$. For $s \in [0,t]$, let $\alpha^*(s)$ be given by (3.42), that is, $\alpha^*_{i,j}(s) = \alpha^*_{i,j}(s,u^*(\cdot)) \doteq \bar{\alpha}^*_{i,j}([\xi^*]^i(s),[\xi^*]^j(s))$, where we remind the reader that $\bar{\alpha}^*$ is given in (3.41).

Lemma 3.6. *Let $t \in (0,\infty)$ and $x,z \in D^{\bar{\delta}}$. Let $u^\dagger$ be a critical point of $\overline{J}^c(t,x,\cdot,z)$, and let the corresponding state trajectory be denoted by $\xi^\dagger$. Let $\alpha^*(r) \doteq \bar{\alpha}^*(\xi^\dagger(r))$ for all $r \in [0,t)$ where $\bar{\alpha}^*$ is given by (3.41). Then, $u^\dagger$ is a critical point of $J^c(t,x,\cdot,\alpha^*,z)$.*

By the choice of u^* as a minimizer and Lemma 3.6, we immediately have

Lemma 3.7. *Let $t \in (0,\infty)$ and $x,z \in D^{\bar{\delta}}$. Then, u^* is a critical point of $J^c(t,x,\cdot,\alpha^*,z)$.*

Lemma 3.8. *Let $x,z \in D^{\bar{\delta}}$. Let $\bar{t} = \bar{t}(\bar{\delta}) \doteq \sqrt{\frac{2\bar{\delta}^3}{3G\max_{i\ni]1,n[}(\sum_{j>i} m_j)}} = \sqrt{\frac{\sqrt{3}\bar{\delta}^3}{\sqrt{2}\widehat{G}\max_{i\ni]1,n[}(\sum_{j>i} m_j)}}$. If $t \in (0,\bar{t})$, then $J^c(t,x,\cdot,\alpha^*,z)$ is strictly convex, and further, u^* is the minimizer of $J^c(t,x,\cdot,\alpha^*,z)$.*

Theorem 3.4. *If $t \in [0, \bar{t})$, then*

$$\underline{W}^c(t,x,z) = W^c(t,x,z) \quad \forall (x,z) \in D^{\bar{\delta}} \times D^{\bar{\delta}}.$$

By Theorem 3.4 and (3.52), one also has:

Corollary 3.2. *If $t \in [0, \bar{t})$, then*

$$\underline{W}^c(t,x,z) = W^c(t,x,z) = \sup_{\alpha \in \bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,c}(t,x,z)$$

for all $(x,z) \in D^{\bar{\delta}} \times D^{\bar{\delta}}$.

Recall that the fundamental solution of interest is obtained in the $c \to \infty$ limit.

Theorem 3.5. $W^\infty(t,x,z) = \sup_{\alpha \in \bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,\infty}(t,x,z)$ *for all $t \in [0, \bar{t})$ and $x, z \in D^{\bar{\delta}}$.*

3.1 Fundamental Solution as Set of Riccati Solutions We will find that the fundamental solution may be given in terms of a set of solutions of Riccati equations.

We look for a solution, $\breve{\mathcal{W}}^{\alpha,c}$, of the form

$$\breve{\mathcal{W}}^{\alpha,c}(t,x,z) = \tfrac{1}{2}\left[\, x' P_t^c\, x + 2x' Q_t^c z + \tfrac{1}{2}\, z' R_t^c\, z + \gamma_t^c \right], \tag{3.55}$$

where $P_\cdot, Q_\cdot, R_\cdot, \gamma_\cdot$ implicity depend on the choice of $\alpha \in \bar{\mathcal{A}}^\infty$. In particular, we suppose that P_t has the form

$$P_t^c = \begin{bmatrix} p_t^{c,1,1} I_3 & p_t^{c,1,2} I_3 & \dots & p_t^{c,1,N} I_3 \\ p_t^{c,2,1} I_3 & p_t^{c,2,2} I_3 & \dots & p_t^{c,2,N} I_3 \\ \vdots & \vdots & \ddots & \vdots \\ p_t^{c,N,1} I_3 & p_t^{c,N,2} I_3 & \dots & p_t^{c,N,N} I_3 \end{bmatrix} \tag{3.56}$$

where each of the $p_t^{i,j}$ are scalars and I_3 denotes the 3×3 identity matrix. We also suppose analogous forms for Q_t and R_t. Lastly, we suppose P_t, R_t are symmetric. Recall from Theorem 3.3 that $\breve{\mathcal{W}}^{\alpha,c}$ will need to satisfy (3.53) and (3.54). The initial condition (3.54) implies

$$\begin{gathered} p_0^{c,i,i} = r_0^{c,i,i} = -q_0^{c,i,i} = c, \quad \forall i \in \mathcal{N}, \\ p_0^{c,i,j} = r_0^{c,i,j} = -q_0^{c,i,j} = 0, \quad \forall i \neq j, \\ \gamma_0 = 0. \end{gathered}$$

From (3.55) and (3.56), we have

$$\tfrac{\partial}{\partial t} \breve{\mathcal{W}}^{\alpha,c}(t,x,z) = \tfrac{1}{2}\Bigg\{ \sum_{i,j \in \mathcal{N}} \Big[\, \dot{p}_t^{c,i,j} [x^i]' x^j + 2\dot{q}_t^{c,i,j} [x^i]' z^j + \dot{r}_t^{c,i,j} [z^i]' z^j \Big] + \dot{\gamma}_t \Bigg\}. \tag{3.57}$$

Next, note that

$$\frac{1}{2} [\nabla_x \breve{\mathcal{W}}^{\alpha,c}]' \mathcal{M}^{-1} \nabla_x \breve{\mathcal{W}}^{\alpha,c} = \frac{1}{2} \sum_{i \in \mathcal{N}} \frac{1}{m_i} \left| \nabla_{x^i} \breve{\mathcal{W}}^{\alpha,c} \right|^2.$$

Also,

$$\nabla_{x^i} \breve{\mathcal{W}}^{\alpha,c}(t,x,z) = \sum_{j \in \mathcal{N}} \left[p_t^{c,i,j} x^j + \frac{q_t^{c,i,j} + p_t^{c,j,i}}{2} z^j \right]$$

which implies,

$$\begin{aligned} \left| \nabla_{x^i} \breve{\mathcal{W}}^{\alpha,c}(t,x,z) \right|^2 = \sum_{j,k \in \mathcal{N}} & p_t^{c,i,j} p_t^{c,i,k} [x^j]' x^k \\ & + p_t^{c,i,j} q_t^{c,i,k} [x^j]' z^k + q_t^{c,i,j} p_t^{c,i,k} [z^j]' z^k \\ & + q_t^{c,i,j} q_t^{c,i,k} [z^j]' z^k. \end{aligned} \tag{3.58}$$

Substituting (3.57)–(3.58) into (3.53), and collecting like terms, we find

$$\begin{aligned} \dot{p}_r^{c,i,j} &= \begin{cases} -\sum_{k \in \mathcal{N}} \frac{1}{m_k} (p_r^{c,i,k})^2 \\ \quad -\sum_{k \neq i} \widehat{G} m_i m_k (\alpha_{i,j}(t-r))^3 & \text{if } i = j, \\ -\sum_{k \in \mathcal{N}} \frac{1}{m_k} p_t^{c,i,k} p_r^{c,k,j} \\ \quad +\widehat{G} m_i m_j (\alpha_{i,j}(t-r))^3 & \text{if } i \neq j, \end{cases} \\ \dot{q}_r^{c,i,j} &= -\sum_{k \in \mathcal{N}} \frac{1}{m_k} p_r^{c,i,k} q_r^{c,k,j}, \\ \dot{r}_r^{c,i,j} &= -\sum_{k \in \mathcal{N}} \frac{1}{m_k} q_r^{c,k,i} q_r^{c,k,j}, \\ \dot{\gamma}_r^c &= 2 \sum_{i \neq j} \widehat{G} m_i m_j \alpha_{i,j}(t-r). \end{aligned}$$

Now let $\bar{P}_t^c$, $\bar{Q}_t^c$ and $\bar{R}_t^c$ denote the $N \times N$ matrices consisting of $p_t^{c,i,j}$, $q_t^{c,i,j}$ and $r_t^{c,i,j}$, respectively, where we note that by our above assumptions, $\bar{P}_t^c$ and $\bar{R}_t^c$ are symmetric. Also let $\bar{\nu}_t$ denote the $N \times N$ matrix of terms given by

$$\bar{\nu}_t^{i,j} = \begin{cases} -\sum_{k \neq i} \widehat{G} m_i m_k (\alpha_{i,j}(r))^3 & \text{if } i = j, \\ \widehat{G} m_i m_j (\alpha_{i,j}(r))^3 & \text{if } i \neq j. \end{cases} \tag{3.59}$$

Then, we have the Riccati system

$$\dot{\bar{P}}_t^c = -\bar{P}_t^c \mathcal{M}_0^{-1} \bar{P}_t^c + \bar{\nu}_t \tag{3.60}$$

$$\dot{\bar{Q}}_t^c = -\bar{P}_t^c \mathcal{M}_0^{-1} \bar{Q}_t^c \tag{3.61}$$

$$\dot{\bar{R}}_t^c = -\left[\bar{Q}_t^c\right]' \mathcal{M}_0^{-1} \bar{Q}_t^c \tag{3.62}$$

$$\dot{\gamma}_t = 2 \sum_{i \neq j} \widehat{G} m_i m_j \alpha_{i,j}(r). \tag{3.63}$$

where $\mathcal{M}_0$ is the diagonal $N \times N$ matrix with diagonal elements $(m_1, m_2 \dots m_N)$. As the above Riccati system has a solution (up to the time of the vertical asymptote), this validates the assumption of solution form (3.55).

Now, note that by Theorem 3.5

$$W^\infty(t,x,z) = \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \mathcal{W}^{\alpha,\infty}(t,x,z)$$
$$= \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \lim_{c\to\infty} \mathcal{W}^{\alpha,c}(t,x,z)$$

which by (3.55),

$$= \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \lim_{c\to\infty} \tfrac{1}{2}\left[x'P_t^c\,x + 2x'Q_t^c z + \tfrac{1}{2}\,z'R_t^c\,z + \gamma_t^c\right],$$
$$= \sup_{\alpha\in\bar{\mathcal{A}}^\infty} \tfrac{1}{2}\left[x'P_t^\infty\,x + 2x'Q_t^\infty z + \tfrac{1}{2}\,z'R_t^\infty\,z + \gamma_t^\infty\right].$$

(3.64)

Let $\mathcal{G}(t) = \mathcal{G}(t; m_1, m_2, \ldots m_N)$ be given by

$$\mathcal{G}(t) = \left\{ \left(P_t^\infty, Q_t^\infty, R_t^\infty, \gamma_t^\infty\right) \,\middle|\, \alpha\in\bar{\mathcal{A}}^\infty \right\}. \tag{3.65}$$

We see that

$$W^\infty(t,x,z) = \sup_{(P,Q,R,\gamma)\in\mathcal{G}(t)} \tfrac{1}{2}\left[x'Px + 2x'Qz + \tfrac{1}{2}\,z'Rz + \gamma\right]$$

which by the linearity in (P,Q,R,γ) of the expression inside the supremum,

$$= \sup_{(P,Q,R,\gamma)\in\widehat{\mathcal{G}}(t)} \tfrac{1}{2}\left[x'Px + 2x'Qz + \tfrac{1}{2}\,z'Rz + \gamma\right], \tag{3.66}$$

where $\widehat{\mathcal{G}}(t) \doteq \langle\mathcal{G}(t)\rangle$. Consequently, we will see that the set $\hat{\mathcal{G}}(t) = \hat{\mathcal{G}}(t; m_1, m_2, \ldots m_N)$ will represent the general solution of the N-body TPBVP. With the symmetry of P, R, one can see that this set lies in $I\!R^{2N^2+N+2}$, a finite-dimensional space.

Of course, one may be concerned about computation of the suprema in (3.64) and (3.66). Specifically, one would like to know whether the object inside the supremum in (3.64) is concave. In this regard, it is helpful to define

$$\mathcal{P}_t^c = \mathcal{P}_t^c(\alpha) \doteq \left(P_t^c \quad Q_t^c(Q_t^c)' \quad R_t^c \right). \tag{3.67}$$

We will say that a matrix-valued function, say $\mathcal{P} : \bar{\mathcal{A}}^\infty \to \mathcal{L}(I\!R^{2n}; I\!R^{2n})$ is concave if its domain, $\bar{\mathcal{A}}^\infty$, is convex and $\mathcal{P}(\alpha^0+\delta\hat\alpha) - 2\mathcal{P}(\alpha^0) + \mathcal{P}(\alpha^0-\delta\hat\alpha) \succeq 0$ for all $\alpha^0\in\bar{\mathcal{A}}^\infty$, $\hat\alpha\in L_\infty([0,\infty); I\!R^{I^\Delta})$ and $\delta\in I\!R$ such that $\alpha^0+\delta\hat\alpha\in\bar{\mathcal{A}}^\infty$, where we find it useful to include δ in this definition. Here, we use the standard partial order given by $\mathcal{P}\preceq\hat{\mathcal{P}}$ if and only if $\hat{\mathcal{P}}-\mathcal{P}$ is non-negative definite.

LEMMA 3.9. *$\mathcal{P}_t^c$ is a concave function of $\alpha\in\bar{\mathcal{A}}^\infty$.*

THEOREM 3.6. *For all $t\ge 0$, $c>0$ and $x,z\in I\!R^n$, $\mathcal{W}^{\alpha,c}(t,x,z)$ and $\mathcal{W}^{\alpha,\infty}(t,x,z)$ are concave in α.*

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