PAYOFF SUBOPTIMALITY AND ERRORS IN VALUE INDUCED BY APPROXIMATION OF THE HAMILTONIAN *

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Abstract. Dynamic programming reduces the solution of optimal control problems to solution of the corresponding Hamilton-Jacobi-Bellman partial differential equations (HJB PDEs). In the case of nonlinear deterministic systems, the HJB PDEs are fully nonlinear, first-order PDEs. Standard, grid-based techniques to the solution of such PDEs are subject to the curse-of-dimensionality, where the computational costs grow exponentially with state-space dimension. Among the recently developed max-plus methods for solution of such PDEs, there is a curse-of-dimensionality-free algorithm. Such an algorithm can be applied in the case where the Hamiltonian takes the form of a pointwise maximum of a finite number of quadratic forms. In order to take advantage of this curse-of-dimensionality-free algorithm for more general HJB PDEs, we need to approximate the general Hamiltonian by a maximum of these quadratic forms. In doing so, one introduces errors. In this work, we obtain a bound on the difference in solution of two HJB PDEs, as a function of a bound on the difference in the two Hamiltonians. Further, we obtain a bound on the suboptimality of the controller obtained from the solution of the approximate HJB PDE rather than from the original.

1. Introduction. The use of dynamic programming to solve nonlinear control problems leads to the familiar dynamic programming equation. In the case of problems in continuous space/time governed by finite-dimensional "deterministic" (or max-plus stochastic, c.f., [17]) dynamics, the dynamic programming equation takes the form of a Hamilton-Jacobi-Bellman partial differential equation (HJB PDE). For instance, in the infinite time-horizon case, this is typically a PDE over a region in a space whose dimension is the dimension of the state variable in the control problem. We remark that the solutions are generally nonsmooth, and the theory of viscosity solutions yields the appropriate solution definition (c.f., [6], [10], [11], [19]).

The difficulty lies in computing the solution of the HJB PDE. The most intuitive, and commonly applied, approaches are grid-based (c.f., [6], [7], [13], [14], [15], [19] among many others), and are subject to the curse-of-dimensionality (whereby the computational cost growth is very roughly on the order of $(2D)^n$ where D is the required number of grid points per dimension, and more importantly, n is the space dimension).

A recent development is the discovery of the curse-of-dimensionality-free methods exploiting semiconvex dual operators and max-plus linearity ([26], [27], [28]). Using convex-programming based pruning, a problem over $I\!R^6$ was solved on a desktop machine [24]. This approach has, so far, only been developed for steady-state problems over the entire space, although the class could be enlarged. (For other max-plus-based methods developed for larger classes of problems, see [1], [2], [18], [28], [29].) The curse-of-dimensionality-free approach currently handles HJB PDE problems of form

$$0 = -\widetilde{H}(x, \nabla V) \;\; \forall x \in I\!R^n \setminus \{0\}, \quad V(0) = 0 \tag{1.1}$$

where

$$\widetilde{H}(x, \nabla V) = \max_{m \in \mathcal{M}} \{H^m(x, \nabla V)\}, \tag{1.2}$$

$\mathcal{M} = \{1, 2 \dots M\}$, and the H^m have computationally simpler forms. In particular, the H^m considered to date in the curse-of-dimensionality-free methods have been

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quadratic functions of x and p. Also, note that by boundary condition, $V(0)=0$, we mean that the solution is zero at the origin.

In [27], [28], the method was developed for infinite time-horizon problems, and the curse-of-dimensionality-free nature was made clear. In [25], [26], the convergence rate for the algorithm was obtained. In particular, it was shown that there were two parameters, τ (the time-step size) and $T = N\tau$ (the approximating finite time-horizon) such that the errors go to zero as $T = N\tau \to \infty$ and $\tau \downarrow 0$. Further, a required relation between the relative T and τ rates was indicated. The errors in the pre-limit solution approximation are bounded in a form $0 \le \widetilde{V} - V^a \le \varepsilon(1+|x|^2)$ where $\widetilde{V}$ is the true solution and V^a is the computed approximation. Additionally, one has $T = N\tau \propto \varepsilon^{-1}$ and $\tau \propto \varepsilon^2$, and so $N \propto \varepsilon^{-3}$. The computational cost growth with (space dimension) n is only on the order of n^3 (due to some matrix inverses). However, the approach is subject to a curse-of-complexity, where the computational cost can grow like M^N. Attenuation of this curse-of-complexity growth through pruning, using semidefinite programming, is an active area of research [24].

Although the PDEs of (1.1) are certainly nontrivial nonlinear PDEs, we would like to solve more general HJB PDEs. A function, say $F(y)$, is semiconvex if given any $R < \infty$, there exists $C_R < \infty$ such that $F(y) + \frac{C_R}{2}|y|^2$ is convex over $B_R(0)$. (Note that the space of semiconvex functions certainly contains both the space of twice continuously differentiable functions and the space of convex functions as subspaces.) It is well known that one can approximate any semiconvex function as the pointwise maximum of quadratic forms. In fact, this is simply a max-plus basis expansion over the max-plus vector space, or moduloid, of semiconvex functions (c.f., [28]). With this in mind, we see that one could approximate any semiconvex Hamiltonian by a Hamiltonian, $\widetilde{H}$, of the form (1.2) with quadratic H^m. One could then solve the HJB PDE problem (1.1) with a curse-of-dimensionality-free method, thereby yielding an approximate solution of the HJB PDE with the original semiconvex Hamiltonian. Such a procedure would induce two error sources. The first consists of the errors in the solution of (1.1) generated by the curse-of-dimensionality-free algorithm. These are briefly discussed in the previous paragraph, and fully discussed in [25], [26]. The second source are those induced by the approximation of the original Hamiltonian by $\widetilde{H}$. This latter error source is under discussion here. Although the analysis to follow is specifically oriented toward approximation by $\widetilde{H}$ of the above form, the general concepts may be more widely applicable. Further, in addition to obtaining bounds on the difference between the solution of the original and approximating HJB PDE problems (Theorem 3.6), we also obtain a lower bound on the suboptimality of the controller obtained by use of the solution of (1.1) in the controller computation (Theorem 4.15). This latter question, while substantially more difficult, is, of course, of significant practical value.

It is worth noting that there is existing literature on the problem of bounding differences between the solutions of two HJB PDEs (although this does not typically include estimates of resulting control-problem payoff suboptimality), but for somewhat different classes of systems, c.f., [5, 21]. As the related numerical methods there are in the class of grid-based methods (again c.f., [6], [7], [13], [14], [15], [19] among many others), the goals in those efforts are also somewhat different. For problems on a finite time-horizon, one might see [21], while for infinite time-horizon problems (including second-order problems), [5] is relevant. In [5] the control spaces are compact, and the running cost is bounded; here, those conditions are not present, but instead there are conditions which imply a particular kind of stability for ε–optimal trajectories. Because of the difference in assumptions, direct comparison between the results there and the results of Section 3 here are not currently possible.

2. Problem Statement, Assumptions and Preliminary Results. We will consider HJB PDE problem

$$0 = -H(x, \nabla V) = -\sup_{w \in \mathbb{R}^k} \left[f'(x,w)\nabla V + l(x,w) \right], \tag{2.1}$$
$$V(0) = 0$$

where $x \in \mathbb{R}^n$. More specifically, we are seeking the particular viscosity solution of (2.1) which is the value function of the following optimal control problem. The dynamics are given by

$$\dot{\xi}_t = f(\xi_t, w_t) \doteq g(\xi_t) + \sigma(\xi_t) w_t, \qquad \xi_0 = x, \tag{2.2}$$

and the running cost is

$$l(\xi_t, w_t) \doteq L(\xi_t) - \frac{\gamma^2}{2}|w_t|^2. \tag{2.3}$$

It is worth noting, that with the above forms for f and l,

$$H(x,p) = g(x)'p + L(x) + \tfrac{1}{2\gamma^2} p'\sigma(x)\sigma'(x)p.$$

The value function we seek, which is the supremum of the payoff over controls $w \in \mathcal{W} \doteq L_2((0,\infty); \mathbb{R}^k)$, is

$$\widehat{V}(x) = \sup_{w \in \mathcal{W}} \sup_{T < \infty} \int_0^T l(\xi_t, w_t)\, dt. \tag{2.4}$$

We assume, $\exists K, c, d_\sigma, C_L, \alpha \in (0,\infty)$ such that the following hold. $g(x)$ is globally Lipschitz continuous with constant K, $(x-y)'(g(x)-g(y)) \leq -c|x-y|^2$ for all x, y, and $g(0) = 0$. $\sigma(x)$ is globally Lipschitz continuous with constant K, and $|\sigma(x)|, |\sigma^{-1}(x)| \leq d_\sigma$ for all $x \in \mathbb{R}^n$ (where σ^{-1} denotes the Moore-Penrose pseudoinverse). $L \in C^2(\mathbb{R}^n)$, with $|L_x(x)| \leq C_L(1+|x|)$, $|L_{xx}(x)| \leq C_L$ and $0 \leq L(x) \leq \alpha|x|^2$ for all $x \in \mathbb{R}^n$. Finally, we assume $\gamma^2/(2d_\sigma^2) > \alpha/c^2$. $\quad (A.V)$

Although the dynamics are written in the standard form (2.2), perhaps it should be noted that we say ξ is a solution of (2.2) on $[0,\infty)$ if it satisfies the integral form

$$\xi_t = x + \int_0^t f(\xi_r, w_r)\, dr = x + \int_0^t g(\xi_r) + \sigma(\xi_r) w_r\, dr, \tag{2.5}$$

for $t \in [0,\infty)$. The existence of a unique, absolutely continuous solution to (2.2) (equivalently, (2.5)) is essentially standard; the only unusual aspect is that here one allows $w \in L_2([0,\infty); \mathbb{R}^k)$ rather than restricting the input to be bounded.

In [28], Theorems 3.19 and 3.20, it was demonstrated that the above assumptions guarantee the following:

THEOREM 2.1. *$\widehat{V}$ (given by (2.4)) is a continuous viscosity solution of (2.1), and is the unique such solution within the class*

$$\mathcal{G}_\delta \doteq \left\{ \phi : \phi \textit{ is semiconvex, } 0 \leq \phi(x) \leq c\frac{\gamma^2 - \delta^2}{2d_\sigma^2}|x|^2 \right\} \tag{2.6}$$

for $\delta > 0$ sufficiently small.

Recall that the overall approach here is the approximate computation of $\widehat{V}$ by approximation of H with an $\widetilde{H}$ taking the form (1.2) with quadratic H^m, and then solution of (1.1) with a curse-of-dimensionality-free method [27], [28]. In particular, we assume that H and $\widetilde{H}$ are *close* in following sense.

Assume that there exists $\theta > 0$ such that, for all $x, p \in I\!R^n$ such that $\widetilde{H}(x,p) \le 0$, one has

$$\widetilde{H}(x,p) \le H(x,p) \le \widetilde{H}(x,p) + \theta\left[|x|^2 + |p|^2\right]. \tag{A.c}$$

Note that the coefficient θ parameterizes the degree of closeness between H and $\widetilde{H}$. As we are dealing with max-plus vector spaces, $\widetilde{H}$ approximates H from below (c.f. [28]), and so this approximation assumption is one-sided.

Let $D^-V(x)$ denote the subdifferential of V at x, i.e.,

$$D^-V(x) = \left\{ p \in I\!R^n \;\middle|\; \liminf_{y\to x} \frac{V(y) - V(x) - p\cdot(y-x)}{|y-x|} \ge 0 \right\}.$$

Remark 2.2. If $\widetilde{V}$ is a viscosity solution of (1.1), and $p \in D^-\widetilde{V}(x)$, then by the definition of viscosity solutions, $\widetilde{H}(x,p) \le 0$. Consequently, the inequalities of Assumption ($A.c$) hold for all x, p such that $p \in D^-\widetilde{V}(x)$.

We will suppose that the H^m are general quadratic forms, with parameters meeting certain conditions which guarantee existence and uniqueness within a certain function class. The H^m take the form

$$H^m(x,p) = \tfrac{1}{2}x'D^m x + \tfrac{1}{2}p'\Sigma^m p + (A^m x)'p + (l_1^m)'x + (l_2^m)'p + \alpha^m, \tag{2.7}$$

where each $\Sigma^m = (1/\gamma^2)\sigma^m(\sigma^m)'$ for appropriate matrices σ^m. The control problem associated to our HJB PDE problem (1.1),(1.2),(2.7) is given by

$$\widetilde{V}(x) \doteq \sup_{T<\infty} \sup_{\mu\in\mathcal{D}_\infty} \sup_{w\in\mathcal{W}} \int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2\,dt, \tag{2.8}$$

where

$$L^m(x) = \tfrac{1}{2}x'D^m x + (l_1^m)'x + \alpha^m, \tag{2.9}$$

$$\dot{\xi} = A^{\mu_t}\xi_t + l_2^{\mu_t} + \sigma^{\mu_t}w_t, \qquad \xi_0 = x, \tag{2.10}$$

and

$$\mathcal{D}_\infty = \{\mu : [0,\infty) \to \mathcal{M} \mid \text{ measurable } \}. \tag{2.11}$$

In regards to the problem data, we make the following assumptions.

Assume there exists $c_A \in (0,\infty)$ such that $x'A^m x \le -c_A|x|^2$ for all $x \in I\!R^n$ and all $m \in \mathcal{M}$. Also, assume that all D^m are symmetric.
Assume $H^1(x,p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha^1 = 0$; D^1 is positive definite; $\Sigma^1 > 0$; $x'A^1x \le -c_{A,1}|x|^2$ $\forall x \in I\!R^n$; and $\gamma^2/(2d_{\sigma,1}^2) > c_D/c_{A,1}^2$, where c_D is such that $x'D^1x \le c_D|x|^2$ $\forall x \in I\!R^n$ and $d_{\sigma,1} \doteq |\sigma^1|$.
Assume that system $\dot{\xi}^{\mu_t} = A^{\mu_t}\xi^{\mu_t} + l_2^{\mu_t} + \sigma^{\mu_t}w$ is controllable in the sense that given $x, y \in I\!R^n$ and $T > 0$, there exist processes $w \in \mathcal{W}$ and μ measurable with range in $\mathcal{M}$, such that $\xi_T = y$ when $\xi_0 = x$ and one applies controls w, μ. $(A.m)$
Assume there exists $c_1 < \infty$ such that for any $\varepsilon \in (0,1]$ and any $\varepsilon-$optimal pair, $\mu^\varepsilon, w^\varepsilon$ for the $\widetilde{H}$ problem, one has

$$\|w^\varepsilon\|^2_{L_2[0,T]} \le c_1(1+|x|^2)$$

for all $T < \infty$ and all $x \in I\!R^n$.

The first assumption in $(A.m)$ is not restrictive, as without this nominal stability, sensible problems with positive definite running cost would have unbounded value. The second of the assumptions assures that at least one of the Hamiltonians has a purely quadratic structure, and this one typically "looks like" H near the origin. The controllability assumption is (currently) needed for technical reasons. The last assumption is due to some technical issues that arise when one allows possibly nonzero l_1^m, l_2^m, α^m. At the end of Appendix B, some specific conditions implying this last assumption are given. However, as our focus is on value function and payoff errors induced by approximation errors in the Hamiltonian, rather than on algorithms for construction of such Hamiltonians, we do not follow this further here.

In order to indicate that the assumptions are not unreasonable, we include a simple example. First, note that we are working with a class of systems which are nominally stable. That is, the undisturbed system, $\dot{\xi} = g(\xi)$ is exponentially stable (by $(A.V)$), where g may already contain a proposed feedback controller, as for example $g(x) = \hat{g}(x, \bar{u}(x))$, and $w.$ represents a disturbance process. One may then think of the value function as an "available storage" (c.f., [20]), although that is not the topic of interest here. Consider the one-dimensional example with nominal dynamics

$$g(x) = \begin{cases} -k_0 x & \text{if } |x| \le x_0, \\ -k_1 x + (k_1 - k_0)x_0 & \text{if } x > x_0, \\ -k_1 x + (k_0 - k_1)x_0 & \text{if } x < -x_0, \end{cases}$$

where, in particular, we let $k_0 = 2$, $k_1 = 3$ and $x_0 = 1$. We also let $\sigma(x) \equiv 2$, $\gamma^2 = 1$ and $\alpha = 3/4$. This is a system where the level of stability shifts at $x = \pm x_0$. One can easily verify that conditions $(A.V)$ are satisfied. Now we indicate the approximating constituent Hamiltonians. We specifically indicate an H_1; Hamiltonians H_m for $m > 1$ are not so restricted as H_1, and so we do not include examples. One may take $H_1(x,p) = (D_1/2)x^2 + (\Sigma_1/2)p^2 - \tilde{k}xp$. That is, one takes $l_1^1 = l_2^1 = 0$ and $\alpha^1 = 0$. In particular, suppose

$$D_1 = \frac{1}{4}\left[\frac{\alpha}{2}\right] = 3/8 \quad \text{and} \quad \Sigma_1 = \frac{1}{4}\left[\frac{\sigma^2}{2\gamma^2}\right] = 1.$$

Then, we may take $\tilde{k} = (11/8)k_0 = 11/4$. One can verify that conditions $(A.c)$ and

$(A.m)$ are satisfied.

The existence and uniqueness of the absolutely continuous solution of (2.10) (or the integral form thereof) is standard under the given assumptions.

THEOREM 2.3. *$\widehat{V}$ and $\widetilde{V}$ are semiconvex.*

A proof of Theorem 2.3 is provided in Appendix A. Lastly, we will need the following, where the proof also appears in Appendix A.

THEOREM 2.4. *$\widetilde{V}$ is a continuous viscosity solution of (1.1). Further, $\widetilde{V}(x) \in [0, \widehat{V}(x)]$ for all $x \in I\!R^n$.*

We will now obtain a lemma which will be helpful in the error estimate. Let $T \in (0, \infty)$, and let $W : I\!R^n \times [0, \infty) \to [0, \infty)$ be the finite time-horizon value function given by

$$W(x, T) = \sup_{w \in \mathcal{W}} \int_0^T l(\xi_t, w_t) dt, \tag{2.12}$$

where ξ satisfies (2.2) with $\xi_0 = x$. Also define

$$\bar{\delta} \doteq \gamma^2 - \frac{2d_\sigma^2 \alpha}{c^2}. \tag{2.13}$$

LEMMA 2.5. *Let w_t^ε be ε–optimal for problem (2.12), and let ξ_t^ε denote the corresponding state process. Then,*

$$\int_0^T \tfrac{1}{2} |w_t^\varepsilon|^2 \, dt \leq \frac{\varepsilon}{\bar{\delta}} + \frac{\alpha}{\bar{\delta} c} |x|^2$$

and

$$\int_0^T |\xi_t^\varepsilon|^2 dt \leq \frac{2d_\sigma^2 \varepsilon}{\bar{\delta} c} + \left[\frac{1}{c} + \frac{2d_\sigma^2 \alpha}{\bar{\delta} c^2} \right] |x|^2.$$

The proof of Lemma 2.5 is given in Appendix A.

3. Error in the Value Function. As noted in Theorem 2.4, $0 \leq \widetilde{V}(x) \leq \widehat{V}(x)$ for all $x \in I\!R^n$. Now we obtain an upper bound on $\widehat{V} - \widetilde{V}$. The main result will be Theorem 3.6. Prior to this we obtain some technical results.

LEMMA 3.1. *There exists $K_g < \infty$ such that for any $x \in I\!R^n$,*

$$|p| \leq K_g |x| \qquad \forall \, p \in D^- \widetilde{V}(x).$$

Proof. By Theorem 2.4, Remark 2.2, and (1.2), for all $p \in D^- \widetilde{V}(x)$, one has

$$H^1(x, p) \leq \widetilde{H}(x, p) \leq 0.$$

Using (2.7) and Assumption $(A.m)$, this implies

$$\tfrac{1}{2} x' D^1 x + \tfrac{1}{2} p' \Sigma^1 p + (A^1 x)' p \leq 0 \qquad \forall \, p \in D^- \widetilde{V}(x).$$

Rearranging this, and dropping superscripts for convenience, yields

$$\left(p + \Sigma^{-1} A x\right)' \Sigma \left(p + \Sigma^{-1} A x\right) \leq x' (A' \Sigma^{-1} A - D) x.$$

Thus,

$$\left|p + \Sigma^{-1} A x\right|^2 \lambda_{\min}[\Sigma] \le |x|^2 \lambda_{\max}[A'\Sigma^{-1}A - D],$$

where, $\lambda_{\min}[X] = \min_i(\lambda_i[X])$ and $\lambda_{\max}[X] = \max_i(\lambda_i[X])$ with the $\lambda_i[X]$ being the eigenvalues of X. By Assumption $(A.m)$, $\lambda_{\min}[\Sigma] = \lambda_{\min}[\Sigma^1] > 0$. With a little calculation, this implies the desired result. □

REMARK 3.2. Using the above proof, a specific value of the bound, K_g, can be explicitly computed as:

$$K_g = \left(\lambda_{\max}\left[(A^1)'(\Sigma^1)^{-2}A^1\right]\right)^{1/2} + \sqrt{\frac{\lambda_{\max}[A^1{\Sigma^1}^{-1}A^1 - D^1]}{\lambda_{\min}[\Sigma^1]}}. \tag{3.1}$$

Fix $R < \infty$, and let $x \in B_R$. Let $\varepsilon \in (0,1]$, and let w^ε be an ε–optimal controller for (2.12). Also, let ξ^ε denote the corresponding state process.

LEMMA 3.3. *For any $T \in [0,\infty)$, $\widetilde{V}(\xi_t^\varepsilon)$ is absolutely continuous on $[0,T]$, and*

$$\widetilde{V}(\xi_T^\varepsilon) - \widetilde{V}(x) = \int_0^T \frac{d}{dt}\widetilde{V}(\xi_t^\varepsilon)\,dt,$$

where the time-derivative exists almost everywhere.

Proof. The semiconvexity of $\widetilde{V}$ (given in Theorem 2.3) implies local Lipschitz behavior (c.f., [16]). Further, by the (absolute) continuity of solutions of (2.2)/(2.5), and the finiteness of T, ξ_t^ε remains in a bounded set. Then, combining the absolute continuity of ξ^ε with the Lipschitz property of $\widetilde{V}$ over the bounded set, one immediately obtains the absolute continuity of $\widetilde{V}(\xi_t^\varepsilon)$. The remaining assertion is a direct result of the absolute continuity. □

Before proceeding further, we need a technical definition and lemma. Given $V : I\!R^n \to I\!R$ and $x, u \in I\!R^n$ with $u \ne 0$, let $D_u V(x)$ denote the one-sided directional derivative of V at x in direction u, if it exists. Also, let $D_u^{sd} V(x)$ denote the (one-sided) semiderivative (c.f., [31]) given by

$$D_u^{sd} V(x) = \lim_{\delta\downarrow 0,\, \hat{u}\to u} \frac{V(x+\delta\hat{u}) - V(x)}{\delta}, \tag{3.2}$$

if it exists.

LEMMA 3.4. *Let $V : I\!R^n \to I\!R$ and $x, u \in I\!R^n$. Suppose V is semiconvex. Then, $D_u^{sd} V(x)$ and $D_u V(x)$ exist, and further, $D_u^{sd} V(x) = D_u V(x)$.*

Proof. Let V, x, u be as indicated, and let $R > |x|$. By the semiconvexity of V, there exists $C_R < \infty$ such that on $B_R(0)$, the function $V^c(x) \doteq V(x) + C_R|x|^2$ is convex. Therefore, by [31], Example 7.27, V^c is semidifferentiable on $B_R(0)$. Also, by [31], Corollary 7.22, $-C_R|\cdot|^2$, being differentiable, is semidifferentiable. Consequently, as the finite sum of semidifferentiable functions is semidifferentiable ([31], Exercise 10.27(a)), we see that V is semidifferentiable on $B_R(0)$, and in particular, at x. By definition (3.2), it is immediate that because $D_u^{sd} V(x)$ exists, $D_u V(x)$ exists and $D_u^{sd} V(x) = D_u V(x)$. □

LEMMA 3.5. *For any $T \in [0,\infty)$,*

$$\widetilde{V}(\xi_T^\varepsilon) - \widetilde{V}(x) = \int_0^T \max_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot f(\xi_t^\varepsilon, w_t^\varepsilon)\,dt.$$

Proof. Let $0 < t < t+\delta < T$. It is helpful to write

$$\frac{1}{\delta}[\widetilde{V}(\xi^\varepsilon_{t+\delta}) - \widetilde{V}(\xi^\varepsilon_t)] = \frac{1}{\delta}\left[\widetilde{V}\left(\xi^\varepsilon_t + \delta\left[\frac{\xi^\varepsilon_{t+\delta} - \xi^\varepsilon_t}{\delta}\right]\right) - \widetilde{V}(\xi^\varepsilon_t)\right]. \tag{3.3}$$

By the absolute continuity of ξ^ε, for a.e. $t \in (0,T)$,

$$\lim_{\delta\to 0} \frac{\xi^\varepsilon_{t+\delta} - \xi^\varepsilon_t}{\delta} = f(\xi^\varepsilon_t, w^\varepsilon_t). \tag{3.4}$$

Recall from Theorem 2.3 that $\widetilde{V}$ is semiconvex. Then, we see from (3.3), (3.4) and Lemma 3.4 that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}[\widetilde{V}(\xi^\varepsilon_{t+\delta}) - \widetilde{V}(\xi^\varepsilon_t)] = D^{sd}_{f(\xi^\varepsilon_t, w^\varepsilon_t)}\widetilde{V}(\xi^\varepsilon_t) = D_{f(\xi^\varepsilon_t, w^\varepsilon_t)}\widetilde{V}(\xi^\varepsilon_t) \quad \text{a.e. } t \in (0,T), \tag{3.5}$$

(and similarly with $\lim_{\delta\uparrow 0}$). Now, by [6], Proposition II.4.7 (where we remark that the proposition is trivially generalized to include semiconvex V and directions u such that $|u| \neq 1$, and we do not include the details),

$$D_{f(\xi^\varepsilon_t, w^\varepsilon_t)}\widetilde{V}(\xi^\varepsilon_t) = \max_{p\in D^-\widetilde{V}(\xi^\varepsilon_t)} p \cdot f(\xi^\varepsilon_t, w^\varepsilon_t) \quad \text{a.e. } t \in (0,T). \tag{3.6}$$

By (3.5) and (3.6), $\frac{d}{dt}\widetilde{V}(\xi^\varepsilon_t)$ exists and is given by

$$\frac{d}{dt}\widetilde{V}(\xi^\varepsilon_t) = \max_{p\in D^-\widetilde{V}(\xi^\varepsilon_t)} p \cdot f(\xi^\varepsilon_t, w^\varepsilon_t) \quad \text{a.e. } t \in (0,T).$$

Combining this with Lemma 3.3 yields the desired result. □

We now proceed to obtain the main result of the section. For any $t \in [0,T]$, let

$$v^\varepsilon_t \doteq \max_{p\in D^-\widetilde{V}(\xi^\varepsilon_t)} p \cdot f(\xi^\varepsilon_t, w^\varepsilon_t) \quad \text{and} \quad p^\varepsilon_t \in \operatorname*{argmax}_{p\in D^-\widetilde{V}(\xi^\varepsilon_t)} p \cdot f(\xi^\varepsilon_t, w^\varepsilon_t).$$

By the ε-optimality of w^ε, one has

$$W(x,T) \leq \int_0^T [l(\xi^\varepsilon_t, w^\varepsilon_t) + v^\varepsilon_t]\, dt - \int_0^T v^\varepsilon_t\, dt + \varepsilon,$$

(where existence of the integrals follows from Lemma 3.5). Then, by Lemma 3.5,

$$W(x,T) \leq \widetilde{V}(x) - \widetilde{V}(\xi^\varepsilon_T) + \int_0^T [l(\xi^\varepsilon_t, w^\varepsilon_t) + v^\varepsilon_t]\, dt + \varepsilon. \tag{3.7}$$

Next, note that

$$l(\xi^\varepsilon_t, w^\varepsilon_t) + v^\varepsilon_t = l(\xi^\varepsilon_t, w^\varepsilon_t) + p^\varepsilon_t \cdot f(\xi^\varepsilon_t, w^\varepsilon_t) \leq H(\xi^\varepsilon_t, p^\varepsilon_t),$$

which by Assumption $(A.c)$,

$$\leq \widetilde{H}(\xi^\varepsilon_t, p^\varepsilon_t) + \theta(|\xi^\varepsilon_t|^2 + |p^\varepsilon_t|^2). \tag{3.8}$$

However, by Remark 2.2 and the fact that $p^\varepsilon_t \in D^-\widetilde{V}(\xi^\varepsilon_t)$, $\widetilde{H}(\xi^\varepsilon_t, p^\varepsilon_t) \leq 0$, and so, (3.8) implies

$$l(\xi^\varepsilon_t, w^\varepsilon_t) + v^\varepsilon_t \leq \theta(|\xi^\varepsilon_t|^2 + |p^\varepsilon_t|^2)$$

which by Lemma 3.1,

$$\leq \theta(1+K_g^2)|\xi_t^\varepsilon|^2. \tag{3.9}$$

Substituting (3.9) into (3.7), one obtains

$$W(x,T) \leq \widetilde{V}(x) - \widetilde{V}(\xi_T^\varepsilon) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$

and noting $\widetilde{V} \geq 0$,

$$\leq \widetilde{V}(x) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$

which, by Lemma 2.5,

$$\leq \widetilde{V}(x) + \theta(1+K_g^2)[\varepsilon C_1 + C_2|x|^2] + \varepsilon, \tag{3.10}$$

where

$$C_1 = 2d_\sigma^2/(\bar{\delta}c) \quad \text{and} \quad C_2 = \frac{1}{c} + \frac{2d_\sigma^2\alpha}{\bar{\delta}c^2} \tag{3.11}$$

Since this is true for all $\varepsilon \in (0,1]$, we have

$$W(x,T) \leq \widetilde{V}(x) + \theta(1+K_g^2)C_2|x|^2. \tag{3.12}$$

Then, noting (c.f., [28]) that $W(x,T) \to \widehat{V}(x)$ as $T \to \infty$, (3.12) (along with Theorem 2.4) yields the value approximation result:

THEOREM 3.6. *For all $x \in I\!R^n$,*

$$\widehat{V}(x) - \theta(1+K_g^2)C_2|x|^2 \leq \widetilde{V}(x) \leq \widehat{V}(x),$$

where C_2 is given by (3.11), and θ is as given in Assumption (A.c).

Thus, we see that $\widetilde{V}$ approximates $\widehat{V}$ arbitrarily well if $\widetilde{H}$ is sufficiently close to H, this closeness being parameterized by θ.

4. Degree of Suboptimality of the Controller. In the previous section, it was shown that if the approximating Hamiltonian is close to the Hamiltonian of the originating problem in a certain sense, then the corresponding viscosity solutions will be close in an appropriate sense. However, recall that we are specifically concerned with a case where we can efficiently solve the HJB PDE corresponding to the approximating Hamiltonian, and would like to use this solution to generate a controller for the originating problem. Consequently, we would like to know whether an (approximate) optimal control generated from the solution of the approximate HJB PDE, will perform well when applied to the true system, which is described by the originating Hamiltonian. We begin with some preparatory results, which are minor variations of well-known properties of viscosity solutions and semiconvexity. Between Lemma 4.5 and Lemma 4.7, the optimal control approximation will be introduced. The main development will begin with Theorem 4.10.

LEMMA 4.1. *Suppose V is a semiconvex viscosity solution of $0 = \widehat{H}(x,\nabla V)$, where $\widehat{H}$ is continuous. For any $x, q \in I\!R^n$, there exists $\bar{p} \in D^-V(x)$ such that*

$$\bar{p} \cdot q = \max_{p \in D^-V(x)} p \cdot q \tag{4.1}$$

and

$$\widehat{H}(x,\bar{p}) = 0. \tag{4.2}$$

Proof. Let

$$D^*V(x) \doteq \left\{ p \in I\!R^n \;\middle|\; \begin{array}{l} \exists\{x_n\} \subseteq \mathcal{A}, \text{ such that} \\ x_n \to x \text{ and } p = \lim_{n\to\infty} \nabla V(x_n) \end{array} \right\} \tag{4.3}$$

where $\mathcal{A} = \{a \in I\!R^n \mid \nabla V(x) \text{ exists}\}$. We note that by Rademacher's Theorem (c.f., [33]), the Lebesgue measure of $\mathcal{A}^c$ is zero, due to the fact that V is locally Lipschitz, which follows from the semiconvexity (c.f., [16]). The generalized gradient (c.f., [8]) is the convex hull of $D^*V(x)$, denoted by $\langle D^*V(x)\rangle$. Then, by the semiconvexity of V and [6], Proposition II.4.7,

$$D^-V(x) = \langle D^*V(x)\rangle. \tag{4.4}$$

Obviously, for any $q \in I\!R^n$, $p \cdot q$ is linear as a function of p, and so it takes its maximum over a convex hull at a point in the generating set. Using this observation and (4.4), we see that

$$\max_{p\in D^-V(x)} p\cdot q = \max_{p\in\langle D^*V(x)\rangle} p\cdot q = \max_{p\in D^*V(x)} p\cdot q. \tag{4.5}$$

Let $\bar p \in \operatorname{argmax}_{p\in D^*V(x)} p\cdot q$. By (4.4) and (4.5), $\bar p$ achieves the maximum in (4.1). Since $\bar p \in D^*V(x)$, by (4.3) there exists $x_n \to x$ with $\nabla V(x_n) \to \bar p$. However, $\widehat H(x_n, \nabla V(x_n)) = 0$ for all n, and so, by the continuity of $\widehat H$, $\widehat H(x,p) = 0$. $\square$

It will be helpful to make the following definition. Let

$$\mathcal{P}(x;\widetilde V) \doteq \operatorname{argmax}\bigl\{ f(x,w)\cdot p + l(x,w) \mid (w,p) \in I\!R^k \times D^-\widetilde V(x) \bigr\}.$$

REMARK 4.2. As is well-known, the subdifferential is closed (and convex). Recalling from Theorem 2.3, that $\widetilde V$ is semiconvex, and hence locally Lipschitz, we see that $D^-\widetilde V(x)$ is compact. Combining this with the fact that $f(x,w)\cdot p + l(x,w)$ is concave quadratic in w, one easily obtains the existence of $\mathcal{P}(x;\widetilde V)$.

Also, let

$$\mathcal{W}^0(x;\widetilde V) = \operatorname*{argmax}_{w\in I\!R^k} \max_{p\in D^-\widetilde V(x)} \left[f(x,w)\cdot p + l(x,w) \right],$$

and

$$\begin{aligned} \mathcal{P}^0(x;\widetilde V) &= \operatorname*{argmax}_{p\in D^-\widetilde V(x)} \max_{w\in I\!R^k} \left[f(x,w)\cdot p + l(x,w) \right] \\ &= \operatorname*{argmax}_{p\in D^-\widetilde V(x)} \Bigl[g(x)\cdot p + L(x) + \tfrac{1}{2\gamma^2} p'\sigma(x)\sigma'(x)p \Bigr]. \end{aligned}$$

It will also be handy to note some simple relations.

LEMMA 4.3. *If $\hat w \in \mathcal{W}^0(x;\widetilde V)$, then there exists $\hat p \in D^-\widetilde V(x)$ such that $(\hat w, \hat p) \in \mathcal{P}(x;\widetilde V)$. On the other hand, $(\hat w,\hat p) \in \mathcal{P}(x;\widetilde V)$ implies that $\hat w \in \mathcal{W}^0(x;\widetilde V)$.*

Proof. To simplify the notation, let $G(x,w,p) \doteq f(x,w)\cdot p + l(x,w)$. Suppose $\hat w \in \mathcal{W}^0(x;\widetilde V)$. Then

$$\max_{p\in D^-\widetilde V(x)} G(x,\hat w,p) = \max_{w\in I\!R^k} \max_{p\in D^-\widetilde V(x)} G(x,w,p),$$

where, recalling (from Remark 4.2) the compactness of $D^-\widetilde V(x)$ and noting the continuity of G, we see that there exists $\hat p \in D^-\widetilde V(x)$ such that $(\hat w,\hat p) \in \mathcal{P}(x;\widetilde V)$.

Alternatively, let $(\hat{w},\hat{p}) \in \mathcal{P}(x;\widetilde{V})$. If $\hat{w} \notin \mathcal{W}^0(x;\widetilde{V})$, then

$$\begin{aligned} \max_{p\in D^-\widetilde{V}(x)} G(x,\hat{w},p) &< \max_{w\in I\!R^k} \max_{p\in D^-\widetilde{V}(x)} G(x,w,p) \\ &= G(x,\hat{w},\hat{p}) \le \max_{p\in D^-\widetilde{V}(x)} G(x,\hat{w},p), \end{aligned}$$

which is a contradiction. □

Similarly, one has:

Lemma 4.4. *If $\hat{p} \in \mathcal{P}^0(x;\widetilde{V})$, then there exists $\hat{w} \in I\!R^k$ such that $(\hat{w},\hat{p}) \in \mathcal{P}(x;\widetilde{V})$. On the other hand, $(\hat{w},\hat{p}) \in \mathcal{P}(x;\widetilde{V})$ implies that $\hat{p} \in \mathcal{P}^0(x;\widetilde{V})$.*

We now get a simple representation for $\hat{w}$, which will be useful in bounding the control effort.

Lemma 4.5. *Suppose $\hat{p} \in \mathcal{P}^0(x;\widetilde{V})$, and let $\hat{w} = \hat{w}(x,\hat{p}) = \frac{1}{\gamma^2}\sigma'(x)\hat{p}$. Then, $(\hat{w},\hat{p}) \in \mathcal{P}(x;\widetilde{V})$, and $\hat{w} \in \mathcal{W}^0(x;\widetilde{V})$. On the other hand, if $\bar{w} \in \mathcal{W}^0(x;\widetilde{V})$, then there exists $\bar{p} \in D^-\widetilde{V}(x)$ such that $(\bar{w},\bar{p}) \in \mathcal{P}(x;\widetilde{V})$ and $\bar{w} = \frac{1}{\gamma^2}\sigma'(x)\bar{p}$.*

Proof. Using (2.2) and (2.3), we have

$$f(x,\hat{w})\cdot\hat{p} + l(x,\hat{w}) = [g(x)+\sigma(x)\hat{w}]\cdot\hat{p} + L(x) - \frac{\gamma^2}{2}|\hat{w}|^2$$

which by a simple calculation of the maximum of a quadratic,

$$= \max_{w\in I\!R^k}[f(x,w)\cdot\hat{p} + l(x,w)],$$

which, since $\hat{p} \in \mathcal{P}^0(x;\widetilde{V})$,

$$= \max_{p\in D^-\widetilde{V}(x)} \max_{w\in I\!R^k}[f(x,w)\cdot p + l(x,w)],$$

which implies $(\hat{w},\hat{p}) \in \mathcal{P}(x;\widetilde{V})$. The fact that $\hat{w} \in \mathcal{W}^0(x;\widetilde{V})$ then follows from Lemma 4.3.

On the other hand, suppose $\bar{w} \in \mathcal{W}^0(x;\widetilde{V})$. Then, by Lemma 4.3, there exists $\bar{p} \in D^-\widetilde{V}(x)$ such that $(\bar{w},\bar{p}) \in \mathcal{P}(x;\widetilde{V})$, and further, by Lemma 4.4, $\bar{p} \in \mathcal{P}^0(x;\widetilde{V})$. Now, $(\bar{w},\bar{p}) \in \mathcal{P}(x;\widetilde{V})$ implies

$$f(x,\bar{w})\cdot\bar{p} + l(x,\bar{w}) = \max_{p\in D^-\widetilde{V}(x)} \max_{w\in I\!R^k}[f(x,w)\cdot p + l(x,w)],$$

and since $\bar{p} \in \mathcal{P}^0(x;\widetilde{V})$,

$$\begin{aligned} &= \max_{w\in I\!R^k}[f(x,w)\cdot\bar{p} + l(x,w)] \\ &= \max_{w\in I\!R^k}\left[g(x)\cdot\bar{p} + L(x) + w'\sigma'(x)\bar{p} - \frac{\gamma^2}{2}|w|^2\right], \end{aligned}$$

which by simple calculation of the maximum of a quadratic function, implies $\bar{w} = \frac{1}{\gamma^2}\sigma'(x)\bar{p}$. □

We now deal with a technical issue related to existence of solutions. We will make an assumption, and then indicate a class of systems meeting the assumption. Let

$$\begin{aligned} F^s(x) &\doteq \left\{ g(x) + \frac{1}{\gamma^2}\sigma(x)\sigma'(x)p \,\middle|\, p \in \mathcal{P}^0(x;\widetilde{V}) \right\} \\ &= \left\{ g(x) + \sigma(x)w \,\middle|\, w = \frac{1}{\gamma^2}\sigma'(x)p,\ p \in \mathcal{P}^0(x;\widetilde{V}) \right\} \end{aligned}$$

which by Lemma 4.5,

$$= \left\{ g(x) + \sigma(x) w \,\middle|\, w \in \mathcal{W}^0(x; \widetilde{V}) \right\}.$$

Consider the differential inclusion

$$\dot{\xi} \in F^s(\xi), \qquad \xi_0 = x. \tag{4.6}$$

We assume there exists a locally Lipschitz solution of (4.6). $(A.s)$

We denote this solution of (4.6) as $\bar{\xi}$. Note that in the case where $\widetilde{V}$ is smooth, (4.6) reduces to an ordinary differential equation, and there is no technical issue. However, in general, existence proofs for differential inclusions are less trivial than those for differential equations. One class of problems where it is known that $(A.s)$ holds is as follows.

THEOREM 4.6. *Suppose that $-F^s$ is monotone in the sense that $(u-v)\cdot(x-y) \geq 0$ for all $u \in -F^s(x)$, $v \in -F^s(y)$, and all $x, y \in I\!R^n$. Then for any $x \in I\!R^n$, there exists a locally Lipschitz solution of (4.6), and further, $\|\xi_t\|$ is monotonically decreasing.*

Proof. With a possible linear rescaling of the time variable, $I - F^s$ is onto (where I indicates the identity mapping). By [3], Theorem 3.1.1, this implies that $-F^s$ is maximal monotone. Then, by [3], Theorem 3.2.1, one obtains the result. □

LEMMA 4.7. *For any $T \in [0, \infty)$,*

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \frac{d}{dt} \widetilde{V}(\bar{\xi}_t)\, dt$$

where $\frac{d}{dt}\widetilde{V}(\bar{\xi}_t)$ exists a.e.

Proof. By the existence and continuity of $\bar{\xi}$ on $[0,T]$ (for any T), there exists $R_T < \infty$ such that $|\bar{\xi}_t| \leq R_T$ for all $t \in [0,T]$. Also, by the Lipschitz continuity of $\widetilde{V}$ (implied by the semiconvexity, c.f., [16]), there exists $K_T < \infty$ such that

$$|\widetilde{V}(x) - \widetilde{V}(y)| \leq K_T(x-y) \qquad \forall\, x, y \in B_{R_T}, \tag{4.7}$$

and this implies $|p| \leq K_T$ for all $p \in D^-\widetilde{V}(x)$ for all $x \in B_{R_T}$. By (4.7) and the Lipschitz behavior of $\bar{\xi}$ on $[0,T]$, $\widetilde{V}(\bar{\xi}_\cdot)$ is Lipschitz on $[0,T]$, which implies absolute continuity. Therefore, $\frac{d}{dt}\widetilde{V}(\bar{\xi}_t)$ exists a.e. on $[0,T]$, and $\widetilde{V}(\bar{\xi}_t) - \widetilde{V}(\bar{\xi}_s) = \int_s^t \frac{d}{dr}\widetilde{V}(\bar{\xi}_r)\, dr$ for all $0 \leq s \leq t \leq T$. □

As noted in the proof just above, there exists $R_T < \infty$ such that $\bar{\xi}_t \in \bar{B}_{R_T}$ for all $t \in [0,T]$. Then, again using the local Lipschitz nature of $\widetilde{V}$, there exists $K_T < \infty$ such that $D^-\widetilde{V}(x) \subseteq \bar{B}_{K_T}(0)$ for all $x \in \bar{B}_{R_T}(0)$, that is $D^-\widetilde{V}(\bar{B}_{R_T}(0)) \subseteq \bar{B}_{K_T}(0)$. Consequently, with a slight abuse of notation, $\mathcal{W}^0(\bar{B}_{R_T}(0); \widetilde{V}) \subseteq \frac{d_\sigma}{\gamma^2}\bar{B}_{K_T}(0)$. Then, by for example, [3] Corollary 1.14.1,

LEMMA 4.8. *There exists a measurable selection, $\bar{w}_\cdot \in \mathcal{W}$, such that $\bar{w}_t \in \mathcal{W}^0(\bar{\xi}_t; \widetilde{V})$ for a.e. $t \in [0,T]$, where $\dot{\bar{\xi}} = g(\bar{\xi}_t) + \sigma(\bar{\xi}_t)\bar{w}_t$ with $\bar{\xi}_t = x$. Of course, where $\nabla\widetilde{V}(\bar{\xi}_t)$ exists, the inclusion reduces to $\bar{w}_t = \frac{1}{\gamma^2}\sigma'(\bar{\xi}_t)\nabla\widetilde{V}(\bar{\xi}_t)$.*

The proof of the following lemma is essentially identical to the proof of Lemma 3.5, and so we do not repeat it.

LEMMA 4.9. *For any $T \in [0, \infty)$ and $x \in I\!R^n$,*

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \max_{p \in D^-\widetilde{V}(\bar{\xi}_t)} p \cdot f(\bar{\xi}_t, \bar{w}_t)\, dt.$$

It will be necessary to show that solutions driven by our feedback control are well-behaved, i.e., staying bounded and eventually decaying to the origin. This step is comprised of the material from Theorem 4.10 through Lemma 4.13.

THEOREM 4.10. *For any $T \in [0,\infty)$ and $x \in I\!R^n$,*

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

Proof. Let $T \in [0,\infty)$ and $x \in I\!R^n$. We have

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt = \int_0^T \left[l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p \right] dt$$

$$- \int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p\, dt, \tag{4.8}$$

where the integrability follows from Lemma 4.9. Define

$$\mathcal{H}_0(x; \widetilde{H}) \doteq \{ p \in I\!R^n \mid \widetilde{H}(x,p) = 0 \}.$$

Then, note that

$$l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p = \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, \bar{w}_t) + f(\bar{\xi}_t, \bar{w}_t) \cdot p \right],$$

and, since $\bar{w}_t \in \mathcal{W}^0(\bar{\xi}_t; \widetilde{V})$,

$$= \max_{w \in I\!R^k} \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right]$$

$$= \max_{w \in I\!R^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\},$$

which by Lemma 4.1,

$$= \max_{w \in I\!R^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\}$$

$$= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \max_{w \in I\!R^k} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right]$$

$$= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} H(\bar{\xi}_t, p),$$

which by Assumption $(A.c)$,

$$\geq \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \widetilde{H}(\bar{\xi}_t, p),$$

which since $p \in \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})$,

$$= 0.$$

Integrating this over time, we see that,

$$\int_0^T \left[l(\bar{\xi}_t, \bar{w}_t) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p \right] dt \geq 0. \tag{4.9}$$

Substituting (4.9) into (4.8), one finds

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq - \int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}_t) \cdot p\, dt,$$

which by Lemma 4.9,

$$= \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

□

COROLLARY 4.11. *For any $x \in I\!R^n$, and any $T \in [0, \infty)$,*

$$\int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt + \widehat{V}(\bar{\xi}_T) \geq \widehat{V}(x) - K_x$$

where $K_x \doteq \widehat{V}(x) - \widetilde{V}(x)$.

Proof. The result follows immediately from the theorem by noting that $\widetilde{V}(\bar{\xi}_T) \leq \widehat{V}(\bar{\xi}_T)$. □

COROLLARY 4.12. *Given any $R < \infty$, there exists $\widetilde{M}_R, M_R < \infty$ such that for all $|x| \leq R$ and all $T \in [0, \infty)$,*

$$\int_0^T |\bar{w}_t|^2\, dt \leq \widetilde{M}_R \quad \text{and} \quad \int_0^T |\bar{\xi}_t|^2\, dt \leq M_R.$$

Proof. Let $R < \infty$, $|x| \leq R$ and $T \in [0, \infty)$. Consider the finite time-horizon problem given by dynamics (2.2) with payoff and value

$$\widehat{J}^f(x, T, w) = \int_0^T l(\xi_t, w_t)\, dt + \widehat{V}(\xi_T),$$
$$\widehat{V}^f(x, T) = \sup_{w \in \mathcal{W}} \widehat{J}^f(x, T, w),$$

where l is given by (2.3). By Corollary 4.11, $\bar{w}$ is ε–optimal with $\varepsilon = \widehat{V}(x) - \widetilde{V}(x)$. Using Theorems 2.1 and 2.4, we see that this implies $\bar{w}$ is ε–optimal for any

$$\varepsilon \in \left[c\frac{\gamma^2}{2d_\sigma^2}|x|^2, \infty\right) \subseteq \left[c\frac{\gamma^2}{2d_\sigma^2}R^2, \infty\right). \tag{4.10}$$

Then, by [30] Lemma 2.2,

$$\|\bar{w}\|^2_{L_2(0,T)} \leq \frac{\varepsilon}{\bar{\delta}} + \frac{1}{\bar{\delta}}\left[\frac{c\gamma^2}{d_\sigma^2}e^{-cT} + \frac{\alpha}{c}\right]R^2, \tag{4.11}$$

where we recall $\bar{\delta} > 0$ is given by (2.13). Taking the lower bound for ε in (4.10), and substituting that into (4.11), we see that

$$\|\bar{w}\|^2_{L_2(0,T)} \leq \frac{1}{\bar{\delta}}\left[\frac{3c\gamma^2}{2d_\sigma^2} + \frac{\alpha}{c}\right]R^2 \doteq \widetilde{M}_R. \tag{4.12}$$

Then with this same value of ε, by [30] Lemma 2.3, one finds that there exists $C_1 < \infty$ (independent of $R, T < \infty$) such that $\int_0^T |\bar{\xi}_t|^2\, dt \leq \frac{C_1}{\bar{\delta}} R^2$, which completes the proof.
□

LEMMA 4.13. *Given $\varepsilon \in (0, 1]$, $x \in I\!R^n$ and $\overline{T} < \infty$, there exists $T > \overline{T}$ such that*

$$0 \leq \widetilde{V}(\bar{\xi}_T) \leq \widehat{V}(\bar{\xi}_T) < \varepsilon.$$

Proof. As the other inequalities are already proven, we prove only the rightmost. Using Corollary 4.12, it is easy to show that given $\bar{\varepsilon} > 0$ and $\overline{T} < \infty$, there exists $T \in [\overline{T}, \infty)$ such that

$$|\bar{\xi}_T|^2 < \bar{\varepsilon}. \tag{4.13}$$

However, by Theorem 2.1, there exists $C_V < \infty$ such that $\widehat{V}(x) \leq C_V |x|^2$, and consequently,

$$\widehat{V}(\bar{\xi}_T) \leq C_V |\bar{\xi}_T|^2. \tag{4.14}$$

Combining (4.13) and (4.14) yields the result. □

We now begin the development leading to the main result of the section. By Corollary 4.12, we see that given $\hat{\varepsilon} > 0$, there exists $\widehat{T} < \infty$ such that

$$\|\bar{\xi}\|^2_{L_2(\widehat{T},\infty)}, \|\bar{w}.\|^2_{L_2(\widehat{T},\infty)} < \hat{\varepsilon},$$

which implies that given $\tilde{\varepsilon} > 0$, there exists $\widetilde{T} < \infty$ such that

$$\int_{\widetilde{T}}^{\infty} \left| l(\bar{\xi}_t, \bar{w}_t) \right| dt < \tilde{\varepsilon}, \tag{4.15}$$

which implies that $\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt$ exists. In particular, given $\tilde{\varepsilon} > 0$,

$$\left| \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt - \lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \right| < \tilde{\varepsilon}, \tag{4.16}$$

for all $T \geq \widetilde{T}$. By (4.16) and Theorem 4.10, given $\tilde{\varepsilon} > 0$,

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x) - \tilde{\varepsilon} - \widetilde{V}(\bar{\xi}_T) \qquad \forall T \geq \widetilde{T}. \tag{4.17}$$

Combining (4.17) and Lemma 4.13 (with $\overline{T}$ replacing $\widetilde{T}$), one sees that given $\tilde{\varepsilon} > 0$,

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x) - 2\tilde{\varepsilon}.$$

Lastly, since this is true for all $\tilde{\varepsilon} > 0$, we obtain:

THEOREM 4.14.

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widetilde{V}(x).$$

Combining Theorem 4.14 and Theorem 3.6, we have:

THEOREM 4.15. *For any $x \in \mathbb{R}^n$,*

$$\lim_{T\to\infty} \int_0^T l(\bar{\xi}_t, \bar{w}_t)\, dt \geq \widehat{V}(x) - \theta(1 + K_g^2) C_2 |x|^2,$$

where C_2 is as given in (3.11), and K_g is indicated in Lemma 3.1, with an explicit bound given in Remark 3.2.

In other words, the payoff obtained with control $\bar{w}.$, based on solution of the approximating problem, will be arbitrarily close to the optimal payoff, $\widehat{V}(x)$. Further, the bound on the difference, $\theta C_3(1+|x|^2)$, goes to zero as $\theta \to 0$, where θ parameterizes the closeness of $\widetilde{H}$ to the originating Hamiltonian, H.

5. Concluding Remarks. We consider approximation of one HJB PDE by another, where the second PDE has a Hamiltonian given as a maximum of quadratic forms. The main results are Theorem 3.6, indicating the relative closeness of the solutions, and Theorem 4.15, indicating the nearness to optimality of the controller obtained from the solution of the approximating HJB PDE. We note that one may also be able to get a result analogous to Theorem 4.15, but working from the solution-approximation results of [5]. An underlying motivation is that we have numerical methods which are quite fast for HJB PDEs such that the Hamiltonian is given as a maximum of quadratic forms, i.e., the curse-of-dimensionality-free methods (c.f., [24, 25, 28]). Consequently, another avenue for future effort could be in development of numerical algorithms where at each succeeding step of the curse-of-dimensionality-free algorithm, one uses increasingly close approximations of the original Hamiltonian. Noting that a key step in application of curse-of-dimensionality-free algorithms is pruning, which may be viewed as optimal max-plus projection onto a subspace of given dimension, combining these steps – Hamiltonian approximation and propagation with projection – could be a promising future direction.

Appendix A.. First, we provide a proof of Theorem 2.3.

Proof. The proof that under Assumptions $(A.V)$, $\widehat{V}$ is semiconvex may be found in [28], Chapter 4. We now indicate the proof of the semiconvexity of $\widetilde{V}$. The proof is similar to the proof of [28], Theorem 7.8, where in that case all the l_1^m, l_2^m and α^m were zero. For completeness, we provide the proof in our case here.

Let $\widetilde{J}: I\!R^n \times [0,\infty) \times \mathcal{D}_\infty \times \mathcal{W} \to I\!R$ be given by

$$\widetilde{J}(x,T,\mu,w) \doteq \int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2\,dt. \tag{5.1}$$

Fix any $x,\eta \in I\!R^n$ with $|\eta| = 1$ and any $\delta > 0$. Let $\varepsilon > 0$, and let $\mu^\varepsilon \in \mathcal{D}_\infty$, $w^\varepsilon \in \mathcal{W}$ and T^ε be ε–optimal for $\widetilde{V}(x)$. Let $\xi^\delta, \xi^0, \xi^{-\delta}$ be solutions of dynamics (2.10), with initial conditions $\xi_0^\delta = x + \delta\eta$, $\xi_0^0 = x$ and $\xi_0^{-\delta} = x - \delta\eta$, respectively, where the inputs are μ^ε and w^ε for all three processes. Then,

$$\begin{aligned}&\widetilde{V}(x-\delta\eta) - 2\widetilde{V}(x) + \widetilde{V}(x+\delta\eta)\\ &\geq \widetilde{J}(x-\delta\eta,T^\varepsilon,w^\varepsilon,\mu^\varepsilon) - 2\widetilde{J}(x,T^\varepsilon,w^\varepsilon,\mu^\varepsilon) + \widetilde{J}(x+\delta\eta,T^\varepsilon,w^\varepsilon,\mu^\varepsilon) - 2\varepsilon.\end{aligned} \tag{5.2}$$

Also note that

$$\dot{\xi}^\delta - \dot{\xi}^0 = A^{\mu_t^\varepsilon}[\xi^\delta - \xi^0] \quad \text{and} \quad \dot{\xi}^0 - \dot{\xi}^{-\delta} = A^{\mu_t^\varepsilon}[\xi^0 - \xi^{-\delta}]. \tag{5.3}$$

Then, letting $\Delta_t^+ \doteq \xi_t^\delta - \xi_t^0$, one also has $\xi_t^0 - \xi_t^{-\delta} = \Delta_t^+$. Using this in (5.2), along with (2.8),(2.9), one finds

$$\begin{aligned}&\widetilde{V}(x-\delta\eta) - 2\widetilde{V}(x) + \widetilde{V}(x+\delta\eta)\\ &\geq \tfrac{1}{2}\int_0^{T_\varepsilon} \Big[(\xi_t^\delta)' D^{\mu_t^\varepsilon}\xi_t^\delta - 2(\xi_t^0)' D^{\mu_t^\varepsilon}\xi_t^0 + (\xi_t^{-\delta})' D^{\mu_t^\varepsilon}\xi_t^{-\delta}\Big]\,dt - 2\varepsilon\\ &\geq \int_0^{T_\varepsilon} (\Delta_t^+)' D^{\mu_t^\varepsilon}\Delta_t^+\,dt - 2\varepsilon.\end{aligned} \tag{5.4}$$

Also, by (5.3) one has $\dot{\Delta}_t^+ = A^{\mu_t^\varepsilon}\Delta_t^+$, and using Assumption $(A.m)$, this yields $\frac{d}{dt}|\Delta_t^+|^2 = 2(\Delta_t^+)' A^{\mu_t^\varepsilon}\Delta_t^+ \geq -2c_A|\Delta_t^+|^2$. This implies

$$|\Delta_t^+|^2 \geq e^{-2c_A t}\delta^2 \qquad \forall\, t \geq 0. \tag{5.5}$$

Also, by the symmetry of the D^m (from Assumption $(A.m)$) and finiteness of $\mathcal{M}$, there exists $\lambda_D > -\infty$ such that $\lambda_D \le \lambda_i^m$ for all eigenvalues λ_i^m of each D^m. Then, by (5.4) and (5.5)

$$\widetilde{V}(x-\delta\eta) - 2\widetilde{V}(x) + \widetilde{V}(x+\delta\eta) \ge \int_0^{T_\varepsilon} \lambda_D |\Delta_t^+|^2\,dt - 2\varepsilon \ge \frac{\lambda_D}{2c_A}\delta^2 - 2\varepsilon.$$

Because $\varepsilon > 0$ and $|\eta| = 1$ were arbitrary, one obtains the result. $\square$

Secondly, we provide a proof of Theorem 2.4.

Proof. Consider the finite time-horizon problem

$$\widetilde{V}^f(x,T) \doteq \sup_{\mu\in\mathcal{D}_\infty} \sup_{w\in\mathcal{W}} \widetilde{J}(x,T,\mu,w), \tag{5.6}$$

where $\widetilde{J}$ is given by (5.1). Noting that $\max_{m\in\mathcal{M}} L^m(x) \ge L^1(x) \ge 0$ for all x, one easily sees that $\widetilde{V}^f$ is monotonically increasing in T, and so $0 \le \widetilde{V}^f(x,T) \le \widetilde{V}(x)$. Combining this with [25], Theorem 4.1 yields

$$0 \le \widetilde{V}^f(x,T) \le \widetilde{V}(x) \le \widehat{V}(x) \qquad \forall\, x \in I\!R^n, \tag{5.7}$$

which is the second assertion of Theorem 2.4.

The proof of the first assertion of Theorem 2.4, i.e., that $\widetilde{V}$ is a continuous viscosity solution of (1.1), is rather standard. The existence of both an L_2 control and a discrete-valued control, move it slightly out of the space of proved results. There are likely multiple ways to extend existing results to cover this case. We sketch one approach, based on a similar proof appearing in [28], [30]. In particular, we demonstrate the continuity in a fair amount of detail. From that point onward, the proof is extremely standard, and in the interests of space, we do not provide details.

Let $T < \infty$, $x \in I\!R^n$ and $\varepsilon \in (0,1]$. Let $\mu^\varepsilon, w^\varepsilon$ be ε–optimal for problem $\widetilde{V}^f(x,T)$, and let the resulting solution of the dynamics given in (2.10), be denoted by ξ^ε. For $t \in [0,T]$, one has

$$|\xi_t^\varepsilon - x| \le \int_0^t |A^{\mu_r^\varepsilon}||\xi_r^\varepsilon| + |l_2^{\mu_r^\varepsilon}| + |\sigma^{\mu_r^\varepsilon}||w_r^\varepsilon|\,dr,$$

which upon letting $d_A \doteq \max_{m\in\mathcal{M}} |A^m|$ and $d_{l,2} \doteq \max_{m\in\mathcal{M}} |l_2^m|$,

$$\le d_A \int_0^t |\xi_r^\varepsilon - x|\,dr + (d_A|x| + d_{l,2})t + d_\sigma \int_0^t |w_r^\varepsilon|\,dr$$

$$\le d_A \int_0^t |\xi_r^\varepsilon - x|\,dr + (d_A|x| + d_{l,2})t + d_\sigma \|w\|_{L_2(0,\infty)}\sqrt{t}.$$

Employing Gronwall's inequality, one obtains

$$\begin{aligned} |\xi_t^\varepsilon - x| \le\, & (d_A|x| + d_{l,2})t + d_\sigma \|w^\varepsilon\|_{L_2(0,\infty)}\sqrt{t} \\ & + d_A e^{d_A t}\left[(d_A|x| + d_{l,2})\frac{t^2}{2} + \frac{2d_\sigma}{3}\|w^\varepsilon\|_{L_2(0,\infty)} t^{3/2}\right]. \end{aligned} \tag{5.8}$$

Employing Assumption $(A.m)$ to eliminate the w-dependence of the right-hand side yields

$$\begin{aligned} |\xi_t^\varepsilon - x| \le\, & (d_A|x| + d_{l,2})t + d_\sigma c_1(1+|x|)\sqrt{t} \\ & + d_A e^{d_A t}\left[(d_A|x| + d_{l,2})\frac{t^2}{2} + \frac{2d_\sigma c_1}{3}(1+|x|)t^{3/2}\right] \\ \le\, & C_1(1+|x|)\sqrt{t} \qquad \forall\, t \in [0,1], \end{aligned} \tag{5.9}$$

for proper choice of $C_1 < \infty$.

Next suppose $y \in I\!R^n$, and let ζ denote the solution of dynamics given in (2.10), driven by the same $\mu^\varepsilon, w^\varepsilon$ as above, but with initial condition $\zeta_0 = y$. One immediately sees that

$$\frac{d}{dt}|\xi_t^\varepsilon - \eta_t|^2 = (\xi_t^\varepsilon - \zeta_t)' A^{\mu_t^\varepsilon}(\xi_t^\varepsilon - \zeta_t) \le -c_A|\xi_t^\varepsilon - \eta_t|^2.$$

Integrating, one has

$$|\xi_t^\varepsilon - \zeta_t|^2 \le e^{-c_A t}|x-y|^2 \qquad \forall\, t \in [0,\infty) \qquad \forall\, t \in [0,\infty),$$

which obviously implies

$$|\xi_t^\varepsilon - \zeta_t| \le e^{-c_A t/2}|x-y| \qquad \forall\, t \in [0,\infty). \tag{5.10}$$

Next, note that by [25], Lemma 4.3,

$$|\xi_t^\varepsilon|^2 \le |x|^2 + \frac{2}{c_A^2} d_{l,2}^2 t + \frac{2\bar{d}_\sigma^2}{c_A}\|w^\varepsilon\|_{L_2(0,\infty)}^2 \qquad \forall\, t \in [0,\infty),$$

where $\bar{d}_\sigma \doteq \max_{m\in\mathcal{M}}|\sigma^m|$. Using Assumption $(A.m)$, this implies

$$|\xi_t^\varepsilon| \le C_2(1+|x|)(1+\sqrt{t}) \qquad \forall\, t \in [0,\infty), \tag{5.11}$$

for proper choice of $C_2 < \infty$. Combining (5.10) and (5.11), one has

$$|\zeta_t| \le C_3(1+|x|+|y|)(1+\sqrt{t}) \qquad \forall\, t \in [0,\infty), \tag{5.12}$$

for proper choice of $C_3 < \infty$.

We have

$$\begin{aligned}
\widetilde{V}^f(x,T) - \widetilde{V}(y,T) &\le \widetilde{J}(x,T,\mu^\varepsilon,w^\varepsilon) - \widetilde{J}(y,T,\mu^\varepsilon,w^\varepsilon) + \varepsilon \\
&= \int_0^T \tfrac{1}{2}\Big[(\xi_t^\varepsilon)' D^{\mu_t^\varepsilon}\xi_t^\varepsilon - \zeta_t' D^{\mu_t^\varepsilon}\zeta_t\Big] + l_1^{\mu_t^\varepsilon}\cdot(\xi_t^\varepsilon - \zeta_t)\,dt \\
&\le \int_0^T \bar{d}_D(|\xi_t^\varepsilon| + |\zeta_t|)|\xi_t^\varepsilon - \zeta_t| + d_{l,1}|\xi_t^\varepsilon - \zeta_t|\,dt.
\end{aligned} \tag{5.13}$$

where $\bar{d}_D \doteq \max_{m\in\mathcal{M}}|D^m|$. Substituting (5.10)–(5.12) into (5.13), one finds that for proper choice of $C_4 < \infty$,

$$\widetilde{V}^f(x,T) - \widetilde{V}(y,T) \le C_4(1+|x|+|y|)|x-y|,$$

for all $T \in [0,\infty)$ and all $x, y \in I\!R^n$. By symmetry, we have

$$|\widetilde{V}^f(x,T) - \widetilde{V}(y,T)| \le C_4(1+|x|+|y|)|x-y|, \tag{5.14}$$

for all $T \in [0,\infty)$ and all $x, y \in I\!R^n$. This is the first part of our continuity proof; next we turn to continuity in T.

Let $x \in I\!R^n$ and $0 \le T \le T+\tau < \infty$. Let $\mu^\varepsilon, w^\varepsilon$ be ε–optimal for problem $\widetilde{V}^f(x,T+\tau)$. Recalling from the top of the proof that $\widetilde{V}^f(x,\cdot)$ is monotonically increasing, one has

$$\widetilde{V}^f(x,T+\tau) - \widetilde{V}^f(x,T) \ge 0. \tag{5.15}$$

On the other hand,

$$\widetilde{V}^f(x,T+\tau)-\widetilde{V}^f(x,T)\leq \int_T^{T+\tau} L^{\mu_t^\varepsilon}(\xi_t^\varepsilon)\,dt+\varepsilon$$
$$\leq \int_T^{T+\tau} \tfrac{1}{2}d_D|\xi_t^\varepsilon|^2+d_{l,1}|\xi_t^\varepsilon|+d_\alpha\,dt$$

where $d_{l,1}\doteq \max_{m\in\mathcal{M}}|l_1^m|$ and $d_\alpha\doteq\max_{m\in\mathcal{M}}|\alpha^m|$. By (5.11), this implies

$$\widetilde{V}^f(x,T+\tau)-\widetilde{V}^f(x,T)\leq \int_T^{T+\tau}\Big[d_D C_2^2(1+|x|)^2(1+t)$$
$$+d_{l,1}C_2(1+|x|)(1+\sqrt{t})+d_\alpha\Big]\,dt,$$

which for proper choice of $C_5<\infty$ (independent of $x\in I\!R^n$ and $T\in[0,\infty)$),

$$\leq C_5(1+|x|^2)(1+T)\tau, \tag{5.16}$$

for all $T,\tau\in[0,\infty$ and $x\in I\!R^n$.

Combining (5.14), (5.15) and (5.16), we see that $\widetilde{V}^f$ is locally Lipschitz continuous over $I\!R^n\times[0,\infty)$. Further, the Lipschitz continuity constant for the space variable variation is independent of $T\in[0,\infty)$. The remaining tasks are straight-forward modifications of existing proofs. First, note that by a proof similar to that for [28], Theorem 3.11, $\widetilde{V}^f$ is a viscosity solution of

$$0=V_T-\widetilde{H}(x,\nabla V),\qquad x\in I\!R^n,\ \ T>0,$$
$$V(0,x)=0\qquad x\in I\!R^n.$$

Further, $\widetilde{V}^f$ is monotonically increasing and bonded above by $\widetilde{V}$. With a proof similar to that of [28], Theorem 3.20, one finally finds that $\widetilde{V}$ is a continuous viscosity solution of (1.1). □

Lastly, we provide a proof of Lemma 2.5. The proof of the lemma is conceptually equivalent to the combined proofs of [28] Lemma 3.10 and Theorem 3.17, but with zero terminal cost. For completeness, the following proof is provided.

Proof. Let

$$Q_T\doteq\int_0^T \alpha|\xi_t|^2\,dt,$$

which by Assumption $(A.V)$,

$$\geq \int_0^T L(\xi_t)\,dt. \tag{5.17}$$

We have

$$\dot{Q}_T=\alpha|\xi_T|^2=\alpha\Big[|x|^2+2\int_0^T \xi_t'g(\xi_t)+\xi_t'\sigma(\xi_t)w_t\,dt\Big],$$

which by Assumption $(A.V)$,

$$\leq \alpha\Big[|x|^2-2c\int_0^T|\xi_t|^2\,dt+\int_0^T d_\sigma|\xi_t||w_t|\,dt\Big]$$
$$\leq \alpha\Big[|x|^2-c\int_0^T|\xi_t|^2\,dt+\frac{d_\sigma^2}{c}\int_0^T|w_t|^2\,dt\Big]$$
$$=-cQ_T+\alpha|x|^2+\frac{\alpha d_\sigma^2}{c}\int_0^T|w_t|^2\,dt.$$

Integrating this, one obtains

$$Q_T \le \frac{\alpha}{c}|x|^2(1-e^{-cT}) + \frac{\alpha d_\sigma^2}{c}\int_0^T e^{c(t-T)}\int_0^t |w_r|^2\,dr\,dt.$$

Applying integration by parts on the last term, with a little work one finds

$$Q_T \le \frac{\alpha}{c}|x|^2 + \frac{\alpha d_\sigma^2}{c^2}\int_0^T |w_t|^2\,dt. \tag{5.18}$$

Combining (5.17) and (5.18), one sees

$$\int_0^T L(\xi_t) - \frac{\gamma^2}{2}|w_t|^2\,dt \le \frac{\alpha}{c}|x|^2 + \Big(\frac{\alpha d_\sigma^2}{c^2} - \frac{\gamma^2}{2}\Big)\int_0^T |w_t|^2\,dt,$$

which, recalling (2.13),

$$= \frac{\alpha}{c}|x|^2 - \frac{\bar{\delta}}{2}\int_0^T |w_t|^2\,dt, \tag{5.19}$$

where, by Assumption $(A.V)$, $\bar{\delta} > 0$. Note that by taking $w_t = 0$ for all $t \in [0,T]$, one sees $W(x,T) \ge 0$. Therefore, for ε–optimal w^ε,

$$\int_0^T L(\xi_t) - \frac{\gamma^2}{2}|w_t^\varepsilon|^2\,dt \ge -\varepsilon. \tag{5.20}$$

Combining (5.19) and (5.20), one sees that for ε–optimal w^ε,

$$\tfrac{1}{2}\|w^\varepsilon\|^2_{L_2(0,T)} \le \frac{\varepsilon}{\bar{\delta}} + \frac{\alpha}{\bar{\delta} c}|x|^2, \tag{5.21}$$

which is the first asserted bound.

We now turn to the bound on $\int_0^T |\xi_t^\varepsilon|^2\,dt$. This bound is obtained in the same manner as the bound in [28] Theorem 3.17, with mainly a change in the bound on $\frac{1}{2}\|w^\varepsilon\|^2_{L_2(0,T)}$, which is used in the proof of the bound on $\int_0^T |\xi_t^\varepsilon|^2\,dt$. Because of this similarity, as well as a similarity to the above bound on Q_T, we provide only a rather terse proof. Let $R_T \doteq \int_0^T |\xi_t^\varepsilon|^2\,dt$. Proceeding in a similar manner to that above for Q_T, one finds

$$\dot{R}_T = |x|^2 + 2\int_0^T (\xi_t^\varepsilon)' g(\xi_t^\varepsilon) + (\xi_t^\varepsilon)'\sigma(\xi_t^\varepsilon) w_t^\varepsilon\,dt,$$

which upon applying Assumption $(A.V)$, and employing standard techniques,

$$\le -cR_T + |x|^2 + \frac{d_\sigma^2}{c}\int_0^T |w_t^\varepsilon|^2\,dt.$$

Integrating this, and applying integration by parts as above, one obtains

$$R_T \le \frac{1}{c}|x|^2 + \frac{d_\sigma^2}{c}\int_0^T |w_t^\varepsilon|^2\,dt.$$

Employing (5.21) in this yields

$$R_T \le \frac{2d_\sigma^2\varepsilon}{\bar{\delta}c} + \Big[\frac{1}{c} + \frac{2d_\sigma^2\alpha}{\bar{\delta}c^2}\Big]|x|^2,$$

which is the second assertion. □

Appendix B.. Conditions implying Assumption $(A.m)$**.:** Suppose all of Assumption Block $(A.m)$ with the exception of the last assumption there. Assume further, that

$$l_2^m = 0 \quad \forall m \in \mathcal{M}, \quad \text{and} \quad \gamma^2/(2\bar{d}_\sigma^2) > \alpha/c_A^2, \tag{A.B}$$

where $\bar{d}_\sigma \doteq \max_{m\in\mathcal{M}} |\sigma^m|$. We prove that these specific conditions are sufficient such that the last assumption in Assumption Block $(A.m)$ holds. We proceed very similarly to the proof of Lemma 2.5 just above, and so the steps are mainly only sketched.Taking $p = 0$ in Assumption $(A.c)$ implies $L^m(x) \le L(x)$ for all $x \in I\!R^n$ and $m \in \mathcal{M}$. Combining this with Assumption $(A.V)$ yields

$$L^m(x) \le \alpha|x|^2 \qquad \forall\, x \in I\!R^n,\ \forall m \in \mathcal{M}. \tag{5.22}$$

Fix $x \in I\!R^n$, and let the dynamics be driven by ε–optimal $\mu^\varepsilon, w^\varepsilon$. Let $Q_T \doteq \int_0^T \alpha|\xi_t^\varepsilon|^2\,dt$, and note that by (5.22),

$$Q_T \ge \int_0^T L^{\mu_t^\varepsilon}(\xi_t^\varepsilon)\,dt \qquad \forall\, T \in [0,\infty). \tag{5.23}$$

One finds

$$\dot{Q}_T = \alpha|x|^2 + 2\alpha \int_0^T (\xi_t^\varepsilon)'(A^{\mu_t^\varepsilon}\xi_t^\varepsilon + l_2^{\mu_t^\varepsilon} + \sigma^{\mu_t^\varepsilon} w_t^\varepsilon)\,dt,$$

which, using the assumptions,

$$\begin{aligned} &= \alpha|x|^2 + 2\alpha \int_0^T (\xi_t^\varepsilon)'(A^{\mu_t^\varepsilon}\xi_t^\varepsilon + \sigma^{\mu_t^\varepsilon} w_t^\varepsilon)\,dt \\ &\le -c_A Q_T + \alpha|x|^2 + \frac{\alpha\bar{d}_\sigma^2}{c_A}\int_0^T |w_t^\varepsilon|^2\,dt. \end{aligned}$$

Solving this differential inequality, and using integration by parts, one obtains

$$Q_T \le \frac{\alpha}{c_A}|x|^2 + \frac{\alpha\bar{d}_\sigma^2}{c_A^2}\int_0^T |w_t^\varepsilon|^2\,dt. \tag{5.24}$$

By (5.23) and (5.24), one has

$$\int_0^T L^{\mu_t^\varepsilon}(\xi_t^\varepsilon) - \frac{\gamma^2}{2}\int_0^T |w_t^\varepsilon|^2\,dt \le \frac{\alpha}{c_A}|x|^2 + \left[\frac{\alpha\bar{d}_\sigma^2}{c_A^2} - \frac{\gamma^2}{2}\right]\int_0^T |w_t^\varepsilon|^2\,dt. \tag{5.25}$$

Now, by Assumption $(A.m)$ (excluding the last component, of course), with $w^0 \equiv 0$ and any μ, one has $\int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}\int_0^T |w_t^0|^2\,dt \ge 0$, and consequently, for ε–optimal controls, one has

$$\int_0^T L^{\mu_t^\varepsilon}(\xi_t^\varepsilon) - \frac{\gamma^2}{2}\int_0^T |w_t^\varepsilon|^2\,dt \ge -\varepsilon. \tag{5.26}$$

Combining (5.25), (5.26) and Assumption $(A.B)$, and taking $\varepsilon \in (0,1]$ one obtains the last component of Assumption $(A.m)$.

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