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GRAVITY GRADIENT MEASUREMENTS

D. Sonnabend and W. M. McEneaney

Navigation Systems Section
Jet Propulsion Laboratory, MS 301–125J
California Institute of Technology
4800 Oak Grove Dr.
Pasadena, CA 91109

ABSTRACT

The problem of rotation corrections for hard mounted gravity gradiometers is widely regarded as intractable. In large part this is due to the unavailability of sufficiently accurate star sensors, gyros, and angular accelerometers. It will be shown here that, for floated satellite gradiometers, subject to very low levels of unmodeled forces and torques, dynamic estimation can dramatically reduce rotation correction errors. The filter structure is developed, and the value of factorization techniques is examined. Numerical examples are given for a few practical cases employing plausible instrument ensembles. To achieve these early results, great simplifications have been made, and important error sources have been suppressed.

1. NOTATION

Uppercase bold roman or greek letters are two dimensional arrays; e.g.: $\mathbf{T}$, $\mathbf{\Gamma}$, $\mathbf{M}$.

Lowercase bold roman or greek letters are column vectors; e.g.: $\mathbf{a}$, $\boldsymbol{\omega}$, $\mathbf{z}$.

Overdots signify time derivatives.

T superscripts denote transposes.

$\mathbf{a}$ = accelerometer inertial position vector
$\mathbf{b}$ = geometrical gradient function
$\mathbf{f}$ = unmodeled force vector on floated instrument
$G = 6.670 \times 10^{-11}$ N - m^2/kg^2 = universal gravitational constant
$\mathbf{g}$ = earth gradient noise
h = satellite altitude
$\mathbf{I}_n = n \times n$ identity tensor or matrix
$\mathbf{J}$ = floated instrument inertia tensor
J = floated instrument inertia scalar
k_f = force correlation constant
k_g = gradient correlation constant
m = mass of floated body
m_o = maximum mountain mass
$\mathbf{M}$ = state covariance after state update
$\mathbf{P}$ = state covariance after measurement update
q = number of scalar measurements
$\mathbf{Q}$ = process noise covariance
$\mathbf{R}$ = measurement covariance
$\mathbf{r}$ = fixed moment arm of $\mathbf{f}$ from center of mass
$R_e = 6.367 \times 10^6$ m = earth radius
$\mathbf{T}$ = intrinsic tensor
$\mathbf{x}$ = state vector
$\mathbf{y} = \mathbf{r}/J$
$\mathbf{z}$ = measurement vector
$\mathbf{0}$ = zero matrix or vector
$\mathbf{\Gamma}$ = gravity gradient tensor
$\boldsymbol{\gamma}$ = vector constructed from $\mathbf{\Gamma}$
$\boldsymbol{\theta}$ = inertial orientation of floated instrument
$\boldsymbol{\zeta}$ = process noise
$\mathbf{\Lambda}$ = gradient covariance
$\mathbf{\Lambda}_g$ = gradient process noise covariance
$\mathbf{\Lambda}_w$ = drag process noise covariance
$\mu = G \times$ mass of earth
σ_{aa} = angular accelerometer standard deviation
σ_f = drag standard deviation
σ_g = gyro standard deviation
σ_{la} = linear accelerometer standard deviation
$\sigma_s t$ = star tracker standard deviation
σ_T = gradiometer standard deviation
τ_f = force correlation time
τ_g = gradient correlation time
$\boldsymbol{\omega}$ = floated instrument angular velocity
$\mathcal{E}$ = 3 index permutation operator
$\{\mathbf{X}\}_i$ = ith row of $\mathbf{X}$

As no coordinate reflections are needed here, no distinctions will be made between vectors and pseudovectors.

2. PROBLEM FORMULATION

A fixed (non-rotating) gravity gradiometer measures some or all of the components of the space rate of change of the gravity field vector. The problem discussed here arises because rotating instruments also see effects corresponding to centrifugal and Coriolis terms in accelerometers. It is not hard to show that the overall measurement, which we will call the intrinsic tensor $\mathbf{T}$, is given by

$$\mathbf{T} = \mathbf{\Gamma} + \omega^2 \mathbf{I_3} - \boldsymbol{\omega}\boldsymbol{\omega}^T + \mathcal{E}\dot{\boldsymbol{\omega}} \tag{1}$$

where all error modeling has been suppressed. Here, $\mathcal{E}$ is the 3–index permutation tensor, whose elements ε_{ijk} are 1, -1, or 0; if i, j, and k are an even, odd, or no permutation of 1, 2, 3. The last term in (1) then works out to

$$\mathcal{E}\dot{\boldsymbol{\omega}} = \begin{bmatrix} 0 & \dot{\omega}_3 & -\dot{\omega}_2 \\ -\dot{\omega}_3 & 0 & \dot{\omega}_1 \\ \dot{\omega}_2 & -\dot{\omega}_1 & 0 \end{bmatrix} \tag{2}$$

an antisymmetric tensor. $\mathcal{E}$ will also be used for cross products below.

If you want $\mathbf{\Gamma}$, the most direct route is to measure $\boldsymbol{\omega}$, and possibly $\dot{\boldsymbol{\omega}}$, and solve (1) for $\mathbf{\Gamma}$. Unfortunately, available instruments can't measure these quantities to adequate accuracy for satellite applications.

Since $\mathbf{\Gamma}$ is both symmetric and traceless (the latter is not warranted for all applications),

$$\mathbf{\Gamma}^T = \mathbf{\Gamma} \qquad ; \qquad Tr(\mathbf{\Gamma}) = 0 \tag{3}$$

and the remaining terms in (1) are either symmetric or antisymmetric, various tricks have been proposed to relax the requirements for supplementary inertial measurements. Putting aside engineering considerations that may make such approaches difficult, we will only point out that any estimation structure based on (1), that enforces the conditions (3), will gain the full benefits of these constraints.

Further relief is possible in cases where the instrument ensemble is floated inside a satellite. By carefully protecting the floated body from unmodeled forces and torques, the rotational and translational equations of motion can be used to augment the estimator, and greatly improve the attitude estimates. This paper will lay out the estimator structure for a few plausible instrument combinations, and attempt to show the improvements possible from adding the dynamics. In the interest of getting answers with only minimal effort, the effects of self gravity and vibration rectification will be ignored; and grossly oversimplified dynamical and kinematical models will be assumed.

OVERSIMPLIFIED DYNAMICS

The floated instrument ensemble will be treated as a single rigid body, capable of translation and rotation; i.e., a six degree of freedom system. Since control forces and torques will be employed to deal with modelable disturbances and perceived motions, and are generally well understood, neither the control, nor the modelable disturbances will be included in the present model. Thus the only forcing terms will be a stochastic force $\mathbf{f}$, and the corresponding torque due to applying $\mathbf{f}$ at a moment arm $\mathbf{r}$, assumed known, since a protected floated body should have only a few exposed locations. The effects of an uncertain $\mathbf{r}$ will require further study.

The translational motion of the center of mass of the floated body is simply:

$$m\ddot{\mathbf{a}} = \mathbf{f} \tag{4}$$

while the rotational motion is given by:

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{r} \times \mathbf{f} = -(\mathcal{E}\mathbf{r})\mathbf{f} \tag{5}$$

These will be used to model the outputs of accelerometers located at the center of mass. For accelerometers not so favorably located, centrifugal and Coriolis terms would have to be added; but these complications will also be swept under the rug here. A considerable simplification in (5) is possible by assuming the floated body to have a spherical inertia tensor, i.e., $\mathbf{J} = J\mathbf{I}_3$; when (5) becomes:

$$\dot{\boldsymbol{\omega}} = -(\mathcal{E}\mathbf{y})\mathbf{f} \quad ; \quad \mathbf{y} = \mathbf{r}/J \tag{6}$$

This simplification would hardly be justified in an analysis of a real system; but our only intent in this paper is to examine the value of dynamic estimation.

Kinematical equations are necessary because attitude measurements will tend to strengthen the dynamics. Here, we'll assume a star tracker measures inertial attitude. A full treatment would relate the rates of change of the variables used to represent attitude to $\boldsymbol{\omega}$, a set of nonlinear equations, no matter what representation is chosen. To simplify, we'll suppose the floated body is controlled to some nominal value of $\boldsymbol{\omega}$, when the kinematics can be taken as:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} \tag{7}$$

where $\boldsymbol{\theta}$ is a set of angles relating instrument coordinates to inertial space.

EXTERNAL FORCE

The force $\mathbf{f}$ is primarily the result of air drag; and, as this is correlated over some distance D_f, there will be a correlation time

$$\tau_f = D_f/v \tag{8}$$

where:

$$v^2 = \mu/(h + R_e) \tag{9}$$

Here we have assumed the orbit to be circular around a spherical earth. We are thus led to model $\mathbf{f}$ as a Gauss–Markov process:

$$\mathbf{f}(t + \Delta t) = k_f \mathbf{f}(t) + \mathbf{w} \tag{10}$$

where:

$$k_f = e^{-\Delta t/\tau_f} \tag{11}$$

Assuming $\mathbf{f}$ to be a stationary isotropic Gaussian process (solar and oblateness effects, which would add orbital and twice orbital variations, are ignored), with zero mean and covariance $\sigma_f^2\mathbf{I}_3$; then from (10), and the independence of $\mathbf{w}$ and $\mathbf{f}(t)$, the covariance of $\mathbf{w}$ is

$$\boldsymbol{\Lambda}_w = E(\mathbf{w}\mathbf{w}^T) = \left(1 - k_f^2\right)\sigma_f^2\mathbf{I}_3 \tag{12}$$

and σ_f is the standard deviation of a large ensemble of independent samples of $\mathbf{f}$. A more accurate model would probably use an altitude dependent σ_f.

EARTH GRADIENT

Consider a spherical or point mass m on the earth, a distance y off the ground track of the satellite, to the right. Then, its position vector relative to the instrument is

$$\mathbf{r} = -[h,\ vt,\ y]^T$$

where t is the time relative to closest approach. The gradient due to m is

$$\boldsymbol{\Gamma} = \frac{Gm}{r^3}\left(\frac{3\mathbf{r}\mathbf{r}^T}{r^2} - \mathbf{I}_3\right) \tag{13}$$

The parenthetical tensor is dimensionless, and of order unity at any t. Thus, the time behavior is essentially that of r^{-3}. The correlation time τ_g then is, more or less, the time when r^{-3} drops to $1/e$ of its peak value. Since the average value of $r^3(0)$ occurs at $y^2 = h^2/3$, this works out to

$$\tau_g \simeq \frac{h}{v}\sqrt{\frac{4}{3}\left(e^{2/3} - 1\right)} = \frac{1.124h}{v} \tag{14}$$

Next, consider an infinite uniform flat plate. The gravity field caused by the plate must be everywhere normal to the plate. Thus, all components except the normal–normal component of $\boldsymbol{\Gamma}$ must vanish. As $\boldsymbol{\Gamma}$ is traceless, the latter must vanish too, and $\boldsymbol{\Gamma} = \mathbf{0}$, regardless of the orientation of the instrument. A useful result may be derived from this. If $\mathbf{r}$ is of the form $[h,\ x,\ y]^T$, and m is the mass of the element $dA = dx\,dy$ of the plate, then by integrating (13), we find:

$$\iint_{\Re^2} r^{-5}(3\mathbf{r}\mathbf{r}^T - r^2\mathbf{I}_3)\,dx\,dy = \mathbf{0} \tag{15}$$

where $\Re^2$ denotes an infinite plane. It will be useful to express this in cylindrical coordinates:

$$\begin{aligned} \mathbf{r} &= [h,\ uc\theta,\ us\theta]^T & (16)\\ r^2 &= h^2 + u^2 & (17) \end{aligned}$$

when the area element becomes $dA = u\,du\,d\theta$.

To continue the analysis, we will find it helpful to stretch $\boldsymbol{\Gamma}$ out into a vector, using the scheme:

$$\boldsymbol{\gamma} = [\Gamma_{11},\ \Gamma_{12},\ \Gamma_{13},\ \Gamma_{22},\ \Gamma_{23}] \tag{18}$$

As Γ_{33} is not independent of the rest, it is not included. For a single mass point, we can write this as:

$$\boldsymbol{\gamma} = Gm\mathbf{b}(\mathbf{r}) \tag{19}$$

where

$$\begin{aligned} \mathbf{b}(\mathbf{r}) &= r^{-5}[3h^2 - r^2,\ 3hx,\ 3hy,\ 3x^2 - r^2,\ 3xy]^T & (20)\\ &= r^{-5}[3h^2 - r^2,\ 3huc\theta,\ 3hus\theta,\ 3u^2c^2\theta - r^2,\ 3u^2s\theta\, c\theta]^T \end{aligned}$$

With these definitions, the above theorem reduces to:

$$\iint_{\Re^2} \mathbf{b}(\mathbf{r})\,dA = \mathbf{0} \tag{21}$$

We are now ready to consider the gradient due to random geology. In another sweeping oversimplification, we suppose the flat earth below the satellite to be littered with N masses m_k, drawn independently from some distribution $p(m)$. The gradient due to this ensemble is then:

$$\boldsymbol{\gamma} = G\sum_{k=1}^{N} m_k\mathbf{b}(\mathbf{r}_k)$$

where the $\mathbf{r}_k$ are selected uniformly over the earth's surface. If we truncate this distribution over some huge circle: $0 \leq u \leq U$, then $p(u,\ \theta) = 1/(\pi U^2)$. Thus, the mean gradient, found by repeating this experiment many times, is

$$\begin{aligned} \boldsymbol{\mu} &= E(\boldsymbol{\gamma}) = GNE(m)E[\mathbf{b}(\mathbf{r})]\\ &= \frac{GN\mu_m}{\pi U^2}\int_{-\pi}^{\pi}\int_0^U \mathbf{b}(\mathbf{r})u\,du\,d\theta \end{aligned}$$

where μ_m is the mean of the $p(m)$ distribution. Clearly, from (21), by letting $U \to \infty$, the above theorem yields $\boldsymbol{\mu} = \mathbf{0}$, for any $p(m)$.

We now turn to the covariance $\boldsymbol{\Lambda}$. By definition:

$$\Lambda_{ij} = E(\gamma_i\gamma_j) = G^2E\left\{\left[\sum_{k=1}^{N} m_k b_i(\mathbf{r}_k)\right]\left[\sum_{l=1}^{N} m_l b_j(\mathbf{r}_l)\right]\right\}$$

Since $\boldsymbol{\mu} = \mathbf{0}$, and the selections are independent, the product terms all vanish unless $k = l$. Thus:

$$\begin{aligned} \Lambda_{ij} &= G^2NE[m^2b_i(\mathbf{r})b_j(\mathbf{r})]\\ &= \frac{G^2N\left(\mu_m^2 + \sigma_m^2\right)}{\pi U^2}\int_{-\pi}^{\pi}\int_0^U b_i(\mathbf{r})b_j(\mathbf{r})u\,du\,d\theta \end{aligned}$$

where σ_m is the standard deviation of $p(m)$. Now, the average areal mass density is

$$\rho = N\mu_m/(\pi U^2) \tag{22}$$

so that, letting $U \to \infty$, and after a fair amount of algebra, we have

$$\boldsymbol{\Lambda} = \frac{3\pi G^2\rho}{32h^4}\left(\mu_m + \frac{\sigma_m^2}{\mu_m}\right)\begin{bmatrix} 8 & 0 & 0 & -4 & 0\\ 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 4 & 0 & 0\\ -4 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Note the nonzero elements Λ_{14} and Λ_{41}. Also note that $\boldsymbol{\Lambda}$ drops off with h^4, rather than h^6, as might have been guessed.

In working out an example, we will suppose that $p(m)$ falls linearly to zero at some value m_o. With a little algebra, this works out to:

$$\mu_m + \frac{\sigma_m^2}{\mu_m} = \frac{m_o}{2} \tag{24}$$

Clearly, adopting a different $p(m)$ would affect only the scalar coefficient in (23), similar to ρ or h. However, a more general $p(m, u, \theta)$, allowing for laterally correlated geology, could change the matrix elements as well.

Combining these ideas, we will model the dynamic $\boldsymbol{\gamma}$ as a Gauss–Markov process:

$$\boldsymbol{\gamma}(t + \Delta t) = k_g \boldsymbol{\gamma}(t) + \mathbf{g} \tag{25}$$

where

$$k_g = e^{-\Delta t / \tau_g} \tag{26}$$

and τ_g is from (14). As with air drag, we take $\boldsymbol{\gamma}$ to be a stationary Gaussian process; and the same analysis leads to

$$\boldsymbol{\Lambda}_g = \left(1 - k_g^2\right) \boldsymbol{\Lambda} \tag{27}$$

OVERSIMPLIFIED MEASUREMENTS

A fairly general form for a measurement model is

$$\mathbf{z}(t) = \mathcal{H}[\mathbf{x}(t)] + \mathbf{bias} + \mathbf{v}(t)$$

where $\mathbf{z}$ is a vector of measurements, $\mathbf{x}$ is the state vector, $\mathcal{H}$ is a function modeling the measurements, and $\mathbf{v}$ is instrument noise. The choice for the state is determined by the final goal of obtaining $\boldsymbol{\Gamma}$, and by the fact that $\ddot{\mathbf{a}}$ is measured (and appears only in (4)). Thus $\mathbf{x}$ is constructed from $\mathbf{f}$, $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, and $\boldsymbol{\gamma}$; more specifically, we let

$$\mathbf{x} = \left[\mathbf{f}^T, \boldsymbol{\omega}^T, \boldsymbol{\theta}^T, \boldsymbol{\gamma}^T\right]^T$$

Bias is usually defined in such a way that the noise has zero mean. When it is observable, it is often included in $\mathbf{x}$, removed from the measurement equations; and equations of the form

$$\textit{rate of change of bias} = 0$$

are included in the dynamics. Here, for simplicity, we'll merely ignore bias entirely. Also for simplicity, all noise will be taken as Gaussian, with diagonal covariance. Finally, we will assume that all measurements are either linear (as many here are), or that the errors are small enough so that a non-iterative, linearized measurement update will be sufficient. Thus, all measurements will take the form

$$\mathbf{z}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{v}(t)$$

and by the above, they may be broken up into their scalar components

$$z_i(t) = \{\mathbf{H}\}_i \mathbf{x}(t) + v_i(t) \tag{28}$$

The measurement devices may include a linear accelerometer, an angular accelerometer, a gyro, a star tracker, and the gradiometer itself. The specific measurement equations for each of these will now be listed.

The linear accelerometer will measure all three components of the acceleration with the same error variance, σ_{la}^2. Thus, for each component, the measurement form is

$$z_{lai} = \frac{1}{m} f_i + v_{lai} \tag{29}$$

For the angular accelerometer we find, by breaking up (6), that we have the measurement form

$$z_{aai} = -\{\mathcal{E}y\}_i \mathbf{f} + v_{aai} \tag{30}$$

with each v_{aai} having the same error variance σ_{aa}^2.

For the three-component gyro and star tracker measurements, we have the simple forms

$$z_{gi} = \omega_i + v_{gi} \tag{31}$$

$$z_{sti} = \theta_i + v_{sti} \tag{32}$$

We again let $E(v_{gi}^2) = \sigma_g^2$ and $E(v_{sti}^2) = \sigma_{st}^2$.

Finally, we come to the gradiometer itself. We will assume that we can measure all 9 elements of $\mathbf{T}$; although omitting some measurements will prove to be easy. First, we must string out $\mathbf{T}$ into a vector $\mathbf{t}$, so that the measurement vector has the form

$$\mathbf{z}_T = \mathbf{t}(\mathbf{x}) + \mathbf{v}_T \tag{33}$$

To construct this function from (1), we apply (6) and (18):

$$\mathbf{t}(\mathbf{x}) = \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} = \begin{bmatrix} \gamma_1 + \omega_2^2 + \omega_3^2 \\ \gamma_2 - \omega_1\omega_2 - \{\mathcal{E}\mathbf{y}\}_3\mathbf{f} \\ \gamma_3 - \omega_1\omega_3 + \{\mathcal{E}\mathbf{y}\}_2\mathbf{f} \\ \gamma_2 - \omega_1\omega_2 + \{\mathcal{E}\mathbf{y}\}_3\mathbf{f} \\ \gamma_4 + \omega_1^2 + \omega_3^2 \\ \gamma_5 - \omega_2\omega_3 - \{\mathcal{E}\mathbf{y}\}_1\mathbf{f} \\ \gamma_3 - \omega_1\omega_3 - \{\mathcal{E}\mathbf{y}\}_2\mathbf{f} \\ \gamma_5 - \omega_2\omega_3 + \{\mathcal{E}\mathbf{y}\}_1\mathbf{f} \\ -\gamma_1 - \gamma_4 + \omega_1^2 + \omega_2^2 \end{bmatrix} \tag{34}$$

As (34) is a nonlinear function of $\mathbf{x}$, we must linearize:

$$\mathbf{z}_T - \mathbf{z}_{T_o} = (\mathbf{x} - \mathbf{x}_o)\frac{\partial \mathbf{t}(\mathbf{x}_o)}{\partial \mathbf{x}} + \mathbf{v}_T$$

from which, with the help of (2):

$$\mathbf{H} = \frac{\partial \mathbf{t}(\mathbf{x}_o)}{\partial \mathbf{x}} \tag{35}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 2\omega_2 & 2\omega_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -y_2 & y_1 & 0 & -\omega_2 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -y_3 & 0 & y_1 & -\omega_3 & 0 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ y_2 & -y_1 & 0 & -\omega_2 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\omega_1 & 0 & 2\omega_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -y_3 & y_2 & 0 & -\omega_3 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ y_3 & 0 & -y_1 & -\omega_3 & 0 & -\omega_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & y_3 & -y_2 & 0 & -\omega_3 & -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2\omega_1 & 2\omega_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

For each component, we again let $E(v_{Ti}^2) = \sigma_T^2$. We will assume this, although in a practical gradiometer it may be more realistic to distinguish between the diagonal and off-diagonal elements of $\mathbf{T}$. Further realism might even call for correlated measurement error, leading to a non-diagonal measurement error covariance matrix.

FILTER STRUCTURE

Two filtering algorithms were employed: the standard Kalman filter and the U–D factored version of the Kalman filter. Prior to describing each in more detail, a few comments are necessary. Since $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are controlled to nominal values, we are not concerned with the evolution of $\mathbf{x}$; only the history of the state covariance. Thus we will limit our discussion to covariance propagation. Also, although we have discussed the dynamics, we have not actually written the equations in a form suitable for a filter. We look for a form like

$$\mathbf{x}(t + \Delta t) = \boldsymbol{\Phi}\mathbf{x}(t) + \mathbf{G}\boldsymbol{\zeta}(t) \tag{36}$$

In deriving $\boldsymbol{\Phi}$, the $\mathbf{f}$ rows come directly from (10). As for $\boldsymbol{\omega}$, for a sufficiently small Δt, we have from (6):

$$\boldsymbol{\omega}(t + \Delta t) = \boldsymbol{\omega}(t) - \Delta t(\mathcal{E}\mathbf{y})\mathbf{f} \tag{37}$$

while (7) yields

$$\boldsymbol{\theta}(t + \Delta t) = \boldsymbol{\theta}(t) + \Delta t\, \boldsymbol{\omega}(t) \tag{38}$$

Finally, the $\boldsymbol{\gamma}$ rows come directly from (25). Combining these results we get:

$$\boldsymbol{\Phi} = \begin{bmatrix} k_f\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\Delta t\mathcal{E}\mathbf{y} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta t\mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & k_g\mathbf{I}_5 \end{bmatrix} \tag{39}$$

$$\boldsymbol{\zeta}(t) = \left[\mathbf{w}^T(t), \mathbf{g}^T(t)\right]^T \tag{40}$$

and $\mathbf{G}$ is a 14×8 matrix of 0's, with $\mathbf{I}_3$ and $\mathbf{I}_5$ in the upper left and lower right corners respectively.

Now, we discuss the equations employed in our standard Kalman filter software. Each of the measurements of the previous section had the form (28), with z_i a scalar and $E(v_i^2) = \sigma^2$. Using a well known matrix inversion lemma, the covariance is updated by

$$\mathbf{P} = \mathbf{M} - \frac{(\{\mathbf{H}\}_i\mathbf{M})^T(\{\mathbf{H}\}_i\mathbf{M})}{\{\mathbf{H}\}_i\mathbf{M}\{\mathbf{H}\}_i^T + \sigma^2}$$

where $\mathbf{M}$ and $\mathbf{P}$ are the pre- and post-measurement covariances.

It should be noted that for the linear accelerometer, gyro, and star tracker, one could form an $\{\mathbf{H}\}_i$ which consisted entirely of 0's except for, say, the k^{th} element which would be a 1, when the above reduces to

$$\mathbf{P} = \mathbf{M} - \frac{\{\mathbf{M}\}_k^T\{\mathbf{M}\}_k}{M_{kk} + \sigma^2}$$

In these cases, the update computations require essentially only an outer product of two vectors, a very fast operation.

As for the state update, this is realized through

$$\mathbf{M} = \mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{41}$$

where

$$\mathbf{Q} = E(\zeta\zeta^T) = \begin{bmatrix} \mathbf{\Lambda}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_g \end{bmatrix} \tag{42}$$

and $\mathbf{P}$ and $\mathbf{M}$ are the covariances prior to and after the state update.

The U–D factored filter is more complex. We will not fully describe each of the filter steps, but refer to the relevant theorems in Bierman[1].

As a reminder, we note that in the U–D factored filter, the state covariance is factored as $\mathbf{P} = \mathbf{U}\mathbf{D}\mathbf{U}^T$, where $\mathbf{D}$ is diagonal, and $\mathbf{U}$ is upper triangular, with 1's along the diagonal. The numerical stability of the U–D filter is well known.

First, it is necessary to factor the initial state covariance as $\mathbf{P}_0 = \mathbf{U}_0\mathbf{D}_0\mathbf{U}_0^T$. The mechanization of this follows from the discussion of square root free upper triangular factoring in Bierman[2].

The measurement update mechanics follow from Theorem V.3.1[3]. In order to perform the state update, it is necessary for $\mathbf{Q}$ of (41) to be diagonal; thus we factor (41) as

$$\begin{aligned} \mathbf{M} &= \mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^T + \mathbf{G}\mathbf{U}_Q\mathbf{D}_Q\mathbf{U}_Q^T\mathbf{G}^T \\ &= \mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^T + \widehat{\mathbf{G}}\mathbf{D}_Q\widehat{\mathbf{G}}^T \end{aligned} \tag{43}$$

Since $\mathbf{Q}$ is constant, $\widehat{\mathbf{G}}$ is constant, and factoring need only be done once. With this form for the state update, the mechanization follows from Theorem VI.4.1 (Modified Weighted Gram–Schmidt Orthogonalization and Matrix Factorization)[4]

COVARIANCE PROGRAM

The above theory is the basis for two sets of computer programs, written in APL, for watching the evolution of the post measurement state covariance, using Kalman and U–D theory. The first program, common to both sets, is an interactive module for entering all the parameters. These are h, m, J, $\mathbf{r}$, ω, σ_{la}, σ_{aa}, σ_g, σ_{st}, σ_T, D_f, σ_f, m_o, ρ, Δt, the number of time steps to be calculated, and logical vectors indicating which components of each instrument are measured.

Next, each set has a program for computing the parameters and matrices which are needed for the measurement and state updates, and which don't vary with time. In the order of computation these are:

1) v, from (9).
2) $\mathbf{k}_f$, from (8) and (11).
3) The scalar factor in (12).
4) $\mathbf{k}_g$, from (14) and (26).
5) The scalar coefficient in (23), using (24).
6) $\mathcal{E}\mathbf{y}$, from (2) and (6).
7) $\mathbf{G}$ and $\mathbf{Q}$, from (12), (23), (27), and (42) (or $\widehat{\mathbf{G}}$ and $\mathbf{D}_Q$ from (43).
8) $\mathbf{\Phi}$, from (39).
9) The angular accelerometer and gradiometer $\mathbf{H}$ matrices, from (6), (30), and (38). Rows are stripped from them, corresponding to unmeasured components.

[1] G. J. Bierman, *Factorization Methods for Discrete Sequential Estimation*, Academic Press, 1977
[2] Ibid, P. 52
[3] Ibid, P. 77
[4] Ibid, P. 127

10) An initial pessimistic value of $\mathbf{M}$, based on the measurement standard deviations, and the gradient process noise. The upper left 9×9 is diagonal, consisting of 10 times triplets of the measurement variances; while the lower right 5×5 is just $\mathbf{\Lambda}$ from (23). From the U–D filter, $\mathbf{M}$ is factored as discussed in the previous section.

The filters themselves are essentially loops, in which the measurement and state updates of the state covariance are performed at each Δt. The actual steps are:

1) The linear accelerometer update is performed according to (29), for each measured acceleration component.
2) The angular accelerometer update is performed according to (30), for each measured acceleration component.
3) The gyro update is performed according to (31), for each measured angular velocity component.
4) The star tracker update is performed according to (32), for each measured attitude component.
5) The gradiometer update is performed according to (35), for each measured intrinsic tensor component. This yields the post measurement $\mathbf{P}$.
6) Except for the force components, an output array is augmented with the square root of the trace of $\mathbf{P}$. This stores the results for printing.
7) The state update is performed, using (41) or (43).
8) Return to 1) for the number of steps requested.

A common output program prints the chosen parameter values, the square root of the trace of the initial pre-measurement state covariance, and the later covariance evolution. To avoid excessive length, a selection routine removes most of the time steps, in ever increasing jumps. An optional title is permitted. A sample run is shown in Fig. 1.

RESULTS

Both covariance programs have been run for a number of cases, concentrating mainly on the intended NASA mission, using the University of Maryland gradiometer. Fig. 1 shows what we have called the Baseline case; although most of the numbers are speculative. Of course, a main intention of the program is to be able to test the value of pushing the system design in one direction or another.

The U. Md. instrument resides in a dewar, cooled by liquid helium. It measures the three diagonal components of $\mathbf{\Gamma}$, and incorporates a six axis accelerometer, which measures both $\ddot{\mathbf{a}}$ and $\dot{\omega}$. Gyros and star trackers are assumed to be attached to the outside of the dewar. The whole dewar is assumed to be floated inside a larger satellite, to avoid vibration and air drag. Thus, the mass properties are guesses of those of a representative floated dewar. An altitude of 200 km is chosen, close to current project thinking; and the nominal ω is obtained by assuming the instrument to be earth pointing at that altitude.

Instrument noise levels have been obtained by assuming independent measurements every 5 sec. With a gradiometer spectrum of 3×10^{-4} E/Hz$^{-1/2}$ this comes to a σ_T of 1.342×10^{-4} E in each axis. The linear and angular accelerometers have been specified at 10^{-11} m/sec^2-Hz$^{1/2}$ and 10^{-11} rad/sec^2 -Hz$^{1/2}$, respectively; leading to $\sigma_{la} = 4.472 \times 10^{-12}$ m/sec^2 and $\sigma_{aa} = 4.472 \times 10^{-12}$ rad/sec^2, respectively. A good, but not great gyro package is assumed, capable of measuring ω to $\sigma_g = 10^{-7}$ rad/sec. Finally, a set of star trackers measure θ to $\sigma_s = 5 \times 10^{-6}$ rad, or about 1 arc second.

For process noise, air drag is quite arbitrarily taken as $\sigma_f = 10^{-6}$ N, with a correlation distance $D_f = 20$ km. Real data on this is actually quite sketchy; but, as will be seen below, air drag doesn't much matter. The gravity signal due to topography below the satellite is assumed to be made up of mountains of a maximum mass $m_o = 3 \times 10^{14}$ kg, or about 100 km^3, randomly scattered at an average density of $\rho = 3 \times 10^6$ kg/m^2, or one km^3 mountain per km^2. With these values, the coefficient of the $\mathbf{\Lambda}$ matrix in (23) is $(.0192\ \mathrm{E})^2$.

Several interesting features are apparent in Fig. 1. First, from (1) and (3), and that only $\omega_3 \neq 0$:

$$Tr(\mathbf{T}) = 2\omega_3^2 = 1.41 \times 10^{-6}\,\mathrm{rad}^2/\mathrm{sec}^2 = 1410\,\mathrm{E}$$

Reading this backwards, the gradiometer is determining ω_3^2 to better than a part in 10^7, accounting for the extraordinarily fine recovery of ω_3. In some sense, the gradiometer in this case is a better gyro than the gyro. This is one of many examples in the results below, where the instruments support each other through the filter.

Now, look at the Γ components in Fig. 1. The unmeasured off-diagonal components are seen to be fixed at their a priori values. This is the natural result of making no measurements of a stationary process. On the other hand, the diagonal components are seen to settle quickly to values about 17% below the gradiometer measurement accuracy. The settling time correlates well with when the ω errors have dropped out of sight in (1), and also with the correlation time τ_g of the earth gradient. Which of these is the dominant effect is not yet clear. On the other hand, getting below the measurement errors is due to enforcing (3).

Finally, there is the slow settling of ω and θ. The possibility that this might be caused by the gradiometer was eliminated by making a run equivalent to the baseline, but without the gradiometer, when nearly the same behavior was noted. Theoretical studies, not completed at this writing, indicate that the settling time and final values of the covariance are not easy to predict, given several measurements and 9 degrees of freedom. Understanding this will require more work.

The first main variation from the Baseline was a set of runs made with the baseline parameters, except that the altitude was varied from 160 – 5000 km, with a more or less commensurate variation in σ_f. The evolution of the error in ω_1 and ω_2 looks just the same as the Baseline for all of these runs; but ω_3 worsens somewhat with altitude, and takes longer to settle. This is because the nominal ω_3 is dropping with altitude, thus reducing the strength of the gradiometer in measuring this component. The increase in τ_g with altitude undoubtedly is the cause of the increased settling time. The gradient process noise drops rapidly with increasing altitude, which directly reduces the uncertainty of the unmeasured components. Above about 1000 km, this also begins to reduce the uncertainty of the *measured* components, and the difference between the Γ_{11} and Γ_{22} process noise levels begins to show up. A separate run showed that increasing σ_f alone had no effect, so that with accelerometers of this quality, drag uncertainties are not a problem. Note however, that this may be optimistic, as scale factor errors have been ignored.

A series of experiments were made in which one of the four attitude instruments (linear and angular accelerometers, gyro, and star tracker) were removed. Except for the angular accelerometer, none of the removals had any effect on the recovery of the gradient; and the angular accelerometer removal only worsened the Γ_{11} and Γ_{22} recovery by about 2%. As far as the attitude recovery is concerned, the linear accelerometer removal worsened ω by about 2%, and θ hardly at all. The angular accelerometer removal doubled the ω error, with only a very small effect on θ. Removing the gyro had next to no effect. Finally, deleting the star tracker worsened ω_1 and ω_2 by about 30%, with no effect on ω_3 (which is really determined by the gradiometer). Of course, the θ error slowly diverges, as it is not observable by any of the other instruments. In practice, a star tracker is essential, in order to transform Γ into earth coordinates; but the present analysis can't get at that issue.

Next, a run similar to the Baseline was made, except that the gradiometer was a full tensor instrument (see Fig. 2). First, the filter is a long way from settling, even after 2975 sec, when our patience ran out. Since we already see a considerable improvement in ω, it is evident that the extra gradiometer components strengthen attitude. On the other hand, the continuing improvement in Γ_{13} and Γ_{23} is obviously the direct result of corresponding improvements in ω_1 and ω_2; another example of cooperation in the filter. Note the asymmetry between ω_1 and ω_2. Lots more work of this sort will be needed to determine the most cost effective ensemble of instruments for a given scientific enquiry.

Our last, and possibly most curious experiment of the set was to look at an inertially fixed gradiometer, i.e., the nominal $\omega = 0$ (see Fig. 3). The run is otherwise the same as the Baseline. The ω recovery is now generally worse, particularly ω_3, which is no longer strengthened by the gradiometer. Conversely (perhaps perversely), Γ_{11} and Γ_{22} are now slightly *better*. This is probably due to the vanishing of the ω terms in (1). This run may be somewhat misleading, in that the effect of errors in removing the main earth gradient terms should be much worse in this case, but are not included anywhere in the analysis.

A number of test cases were also run to determine the relative benefits of using the U–D formulation over using the original Kalman formulation. In particular, we looked for two effects: catastrophic failure due to nearly perfect measurements and divergence due to round–off errors.

The first type of failure was generated by keeping all the parameters as described above, turning on all measurement components, and increasing the a priori covariance of the state errors. When the a priori error standard deviation reached a factor of 10^6 above the nominal value, the Kalman formulation failed due to catastrophic subtraction. The U–D formulation did not fail even when the standard deviation multiplication factor was raised to 10^{23}. One could note that there was no failure in the Kalman program until the initial noise levels were unrealistically high, but a failure at a factor of only 10^6 above the nominal reduces one's confidence in the filter's ability to perform reliably.

The second type of failure was generated by making the following changes from the above nominal inputs. The gradiometer measurements were reduced to the diagonal elements only. The angular accelerometer, gyro, and star tracker were restricted to measuring only the first and third components of their respective vectors, and the linear accelerometer measurements were restricted to only the second and third components. The final change was increasing the drag standard deviation by a factor of 10^6.

When this was done, the Kalman formulation had approximately a 2 % error relative to the U–D formulation in the standard deviation of Γ_{11} and Γ_{22} after 225 seconds. (The U–D results were assumed to be correct rather than the Kalman results, since the Kalman estimate standard deviation actually decreased as the noise levels were increased.) Again, the inputs needed to produce the effect were somewhat unrealistic, however, the same argument as before applies, and in fact, is even more compelling since an error such as this is harder to detect. An additional run was made similar to the above, in which the maximum mountain mass was increased by a factor of 10^6, instead of increasing the drag. This time both filters gave results essentially identical to the case with neither the drag nor the mountains increased, indicating that process noise is much more important to the filter performance than the gradient signal level. We should also note that the input space has not been fully explored, and these effects might be produced by other, possibly more realistic, sets of input parameters.

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GRADIOMETER DYNAMIC ESTIMATION - GRADIENT IN EOTVOS, ALL OTHER UNITS SI

Figure 1 - Baseline

ALTITUDE: 2E5
MASS + MOMENT OF INERTIA OF SPHERICAL BODY: 100 50
FORCE APPLIED AT: 1 1 1
ANGULAR VELOCITY: 0 0 1.1864E¯3

MEASUREMENT STANDARD DEVIATIONS
LINEAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
ANGULAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
GYROS: 1E¯7 ACTIVE COMPONENTS: 1 1 1
STAR TRACKERS: 5E¯6 ACTIVE COMPONENTS: 1 1 1
GRADIOMETER: 1.342E¯4 ACTIVE COMPONENTS: 1 0 0 0 1 0 0 0 1

DRAG STANDARD DEVIATION + CORRELATION DISTANCE: 1E¯6 20000
MOUNTAIN AREAL DENSITY + MAXIMUM MASS: 3E6 3E14
TIME STEP DURATION + NUMBER OF STEPS: 5 153

INITIAL ESTIMATE STANDARD DEVIATIONS:
FORCE, ANGULAR VELOCITY, ATTITUDE: 1.41E¯009 3.16E¯007 1.58E¯005
GRADIENT: 5.43E¯002 3.84E¯002 3.84E¯002 3.33E¯002 1.92E¯002

POST MEASUREMENT STANDARD DEVIATIONS

TIME	ω1	ω2	ω3	θ1	θ2	θ3	Γ11	Γ12	Γ13	Γ22	Γ23
5	9.53E¯008	9.53E¯008	4.90E¯011	4.77E¯006	4.77E¯006	4.77E¯006	.000116	.038394	.038394	.000116	.019197
10	6.89E¯008	6.89E¯008	3.57E¯011	3.46E¯006	3.46E¯006	3.45E¯006	.000113	.038394	.038394	.000113	.019197
15	5.66E¯008	5.66E¯008	3.09E¯011	2.85E¯006	2.85E¯006	2.84E¯006	.000112	.038394	.038394	.000112	.019197
20	4.91E¯008	4.91E¯008	2.87E¯011	2.50E¯006	2.50E¯006	2.47E¯006	.000112	.038394	.038394	.000112	.019197
30	3.99E¯008	3.99E¯008	2.73E¯011	2.09E¯006	2.09E¯006	2.02E¯006	.000112	.038394	.038394	.000112	.019197
50	3.02E¯008	3.02E¯008	2.68E¯011	1.72E¯006	1.72E¯006	1.57E¯006	.000112	.038394	.038394	.000112	.019197
75	2.36E¯008	2.36E¯008	2.68E¯011	1.53E¯006	1.53E¯006	1.29E¯006	.000112	.038394	.038394	.000112	.019197
105	1.86E¯008	1.86E¯008	2.68E¯011	1.43E¯006	1.43E¯006	1.09E¯006	.000112	.038394	.038394	.000112	.019197
140	1.47E¯008	1.47E¯008	2.68E¯011	1.37E¯006	1.37E¯006	9.43E¯007	.000112	.038394	.038394	.000112	.019197
180	1.15E¯008	1.15E¯008	2.68E¯011	1.31E¯006	1.31E¯006	8.32E¯007	.000112	.038394	.038394	.000112	.019197
225	9.07E¯009	9.07E¯009	2.68E¯011	1.25E¯006	1.25E¯006	7.45E¯007	.000112	.038394	.038394	.000112	.019197
275	7.17E¯009	7.17E¯009	2.68E¯011	1.18E¯006	1.18E¯006	6.74E¯007	.000112	.038394	.038394	.000112	.019197
330	5.71E¯009	5.71E¯009	2.68E¯011	1.11E¯006	1.11E¯006	6.15E¯007	.000112	.038394	.038394	.000112	.019197
390	4.59E¯009	4.59E¯009	2.68E¯011	1.05E¯006	1.05E¯006	5.66E¯007	.000112	.038394	.038394	.000112	.019197
455	3.73E¯009	3.73E¯009	2.68E¯011	9.89E¯007	9.89E¯007	5.24E¯007	.000112	.038394	.038394	.000112	.019197
525	3.06E¯009	3.06E¯009	2.68E¯011	9.32E¯007	9.32E¯007	4.88E¯007	.000112	.038394	.038394	.000112	.019197
600	2.53E¯009	2.53E¯009	2.68E¯011	8.80E¯007	8.80E¯007	4.56E¯007	.000112	.038394	.038394	.000112	.019197
680	2.12E¯009	2.12E¯009	2.68E¯011	8.33E¯007	8.33E¯007	4.29E¯007	.000112	.038394	.038394	.000112	.019197
765	1.79E¯009	1.79E¯009	2.68E¯011	7.90E¯007	7.90E¯007	4.04E¯007	.000112	.038394	.038394	.000112	.019197

GRADIOMETER DYNAMIC ESTIMATION - GRADIENT IN EOTVOS, ALL OTHER UNITS SI

Figure 2 - Full Gradiometer

ALTITUDE: 2E5
MASS + MOMENT OF INERTIA OF SPHERICAL BODY: 100 50
FORCE APPLIED AT: 1 1 1
ANGULAR VELOCITY: 0 0 1.1864E¯3

MEASUREMENT STANDARD DEVIATIONS
LINEAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
ANGULAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
GYROS: 1E¯7 ACTIVE COMPONENTS: 1 1 1
STAR TRACKERS: 5E¯6 ACTIVE COMPONENTS: 1 1 1
GRADIOMETER: 1.342E¯4 ACTIVE COMPONENTS: 1 1 1 1 1 1 1 1 1

DRAG STANDARD DEVIATION + CORRELATION DISTANCE: 1E¯6 20000
MOUNTAIN AREAL DENSITY + MAXIMUM MASS: 3E6 3E14
TIME STEP DURATION + NUMBER OF STEPS: 5 153

INITIAL ESTIMATE STANDARD DEVIATIONS:
FORCE, ANGULAR VELOCITY, ATTITUDE: 1.41E¯009 3.16E¯007 1.58E¯005
GRADIENT: 5.43E¯002 3.84E¯002 3.84E¯002 3.33E¯002 1.92E¯002

POST MEASUREMENT STANDARD DEVIATIONS

TIME	ω1	ω2	ω3	θ1	θ2	θ3	Γ11	Γ12	Γ13	Γ22	Γ23
5	3.06E¯008	1.60E¯008	4.90E¯011	4.77E¯006	4.77E¯006	4.77E¯006	.000116	.000095	.036357	.000116	.018926
10	2.83E¯008	1.51E¯008	3.46E¯011	3.45E¯006	3.45E¯006	3.45E¯006	.000113	.000095	.033585	.000113	.017968
15	2.64E¯008	1.44E¯008	2.83E¯011	2.84E¯006	2.84E¯006	2.84E¯006	.000112	.000095	.031351	.000112	.017139
20	2.49E¯008	1.38E¯008	2.45E¯011	2.48E¯006	2.47E¯006	2.47E¯006	.000111	.000095	.029498	.000111	.016413
30	2.24E¯008	1.28E¯008	2.00E¯011	2.04E¯006	2.03E¯006	2.02E¯006	.000111	.000095	.026558	.000111	.015193
50	1.89E¯008	1.13E¯008	1.55E¯011	1.63E¯006	1.59E¯006	1.57E¯006	.000110	.000095	.022453	.000110	.013361
75	1.60E¯008	9.88E¯009	1.27E¯011	1.40E¯006	1.33E¯006	1.29E¯006	.000110	.000095	.018985	.000110	.011727
105	1.35E¯008	8.67E¯009	1.07E¯011	1.28E¯006	1.17E¯006	1.09E¯006	.000110	.000095	.016004	.000110	.010283
140	1.13E¯008	7.58E¯009	9.33E¯012	1.21E¯006	1.07E¯006	9.43E¯007	.000110	.000095	.013401	.000110	.008998
180	9.38E¯009	6.61E¯009	8.27E¯012	1.17E¯006	1.01E¯006	8.32E¯007	.000110	.000095	.011127	.000110	.007841
225	7.73E¯009	5.73E¯009	7.45E¯012	1.13E¯006	9.76E¯007	7.45E¯007	.000110	.000095	.009172	.000110	.006795
275	6.34E¯009	4.93E¯009	6.80E¯012	1.09E¯006	9.48E¯007	6.74E¯007	.000110	.000095	.007526	.000110	.005854
330	5.20E¯009	4.23E¯009	6.29E¯012	1.05E¯006	9.23E¯007	6.15E¯007	.000110	.000095	.006169	.000110	.005017
390	4.27E¯009	3.61E¯009	5.87E¯012	9.99E¯007	8.97E¯007	5.66E¯007	.000110	.000095	.005068	.000110	.004284
455	3.52E¯009	3.08E¯009	5.54E¯012	9.51E¯007	8.69E¯007	5.24E¯007	.000110	.000095	.004183	.000110	.003651
525	2.93E¯009	2.62E¯009	5.27E¯012	9.04E¯007	8.39E¯007	4.88E¯007	.000110	.000095	.003473	.000110	.003113
600	2.45E¯009	2.24E¯009	5.06E¯012	8.59E¯007	8.08E¯007	4.56E¯007	.000110	.000095	.002904	.000110	.002658
680	2.06E¯009	1.92E¯009	4.89E¯012	8.17E¯007	7.77E¯007	4.29E¯007	.000110	.000095	.002445	.000110	.002276
765	1.75E¯009	1.65E¯009	4.75E¯012	7.77E¯007	7.46E¯007	4.04E¯007	.000110	.000095	.002074	.000110	.001957

GRADIOMETER DYNAMIC ESTIMATION - GRADIENT IN EOTVOS, ALL OTHER UNITS SI

Figure 3 - Inertial Orientation

ALTITUDE: 2E5
MASS + MOMENT OF INERTIA OF SPHERICAL BODY: 100 50
FORCE APPLIED AT: 1 1 1
ANGULAR VELOCITY: 0 0 0

MEASUREMENT STANDARD DEVIATIONS
LINEAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
ANGULAR ACCELEROMETERS: 4.472E¯12 ACTIVE COMPONENTS: 1 1 1
GYROS: 1E¯7 ACTIVE COMPONENTS: 1 1 1
STAR TRACKERS: 5E¯6 ACTIVE COMPONENTS: 1 1 1
GRADIOMETER: 1.342E¯4 ACTIVE COMPONENTS: 1 0 0 0 1 0 0 0 1

DRAG STANDARD DEVIATION + CORRELATION DISTANCE: 1E¯6 20000
MOUNTAIN AREAL DENSITY + MAXIMUM MASS: 3E6 3E14
TIME STEP DURATION + NUMBER OF STEPS: 5 153

INITIAL ESTIMATE STANDARD DEVIATIONS:
FORCE, ANGULAR VELOCITY, ATTITUDE: 1.41E¯009 3.16E¯007 1.58E¯005
GRADIENT: 5.43E¯002 3.84E¯002 3.84E¯002 3.33E¯002 1.92E¯002

POST MEASUREMENT STANDARD DEVIATIONS

TIME	$\omega 1$	$\omega 2$	$\omega 3$	$\theta 1$	$\theta 2$	$\theta 3$	$\Gamma 11$	$\Gamma 12$	$\Gamma 13$	$\Gamma 22$	$\Gamma 23$
5	9.53E¯008	9.53E¯008	9.53E¯008	4.77E¯006	4.77E¯006	4.77E¯006	.000110	.038394	.038394	.000110	.019197
10	6.89E¯008	6.89E¯008	6.89E¯008	3.46E¯006	3.46E¯006	3.46E¯006	.000110	.038394	.038394	.000110	.019197
15	5.66E¯008	5.66E¯008	5.66E¯008	2.85E¯006	2.85E¯006	2.85E¯006	.000110	.038394	.038394	.000110	.019197
20	4.91E¯008	4.91E¯008	4.91E¯008	2.50E¯006	2.50E¯006	2.50E¯006	.000110	.038394	.038394	.000110	.019197
30	3.99E¯008	3.99E¯008	3.99E¯008	2.09E¯006	2.09E¯006	2.09E¯006	.000110	.038394	.038394	.000110	.019197
50	3.02E¯008	3.02E¯008	3.02E¯008	1.72E¯006	1.72E¯006	1.72E¯006	.000110	.038394	.038394	.000110	.019197
75	2.36E¯008	2.36E¯008	2.36E¯008	1.53E¯006	1.53E¯006	1.53E¯006	.000110	.038394	.038394	.000110	.019197
105	1.86E¯008	1.86E¯008	1.86E¯008	1.43E¯006	1.43E¯006	1.43E¯006	.000110	.038394	.038394	.000110	.019197
140	1.47E¯008	1.47E¯008	1.47E¯008	1.37E¯006	1.37E¯006	1.37E¯006	.000110	.038394	.038394	.000110	.019197
180	1.15E¯008	1.15E¯008	1.15E¯008	1.31E¯006	1.31E¯006	1.31E¯006	.000110	.038394	.038394	.000110	.019197
225	9.07E¯009	9.07E¯009	9.07E¯009	1.25E¯006	1.25E¯006	1.25E¯006	.000110	.038394	.038394	.000110	.019197
275	7.17E¯009	7.17E¯009	7.17E¯009	1.18E¯006	1.18E¯006	1.18E¯006	.000110	.038394	.038394	.000110	.019197
330	5.71E¯009	5.71E¯009	5.71E¯009	1.11E¯006	1.11E¯006	1.11E¯006	.000110	.038394	.038394	.000110	.019197
390	4.59E¯009	4.59E¯009	4.59E¯009	1.05E¯006	1.05E¯006	1.05E¯006	.000110	.038394	.038394	.000110	.019197
455	3.73E¯009	3.73E¯009	3.73E¯009	9.89E¯007	9.89E¯007	9.89E¯007	.000110	.038394	.038394	.000110	.019197
525	3.06E¯009	3.06E¯009	3.06E¯009	9.32E¯007	9.32E¯007	9.32E¯007	.000110	.038394	.038394	.000110	.019197
600	2.53E¯009	2.53E¯009	2.53E¯009	8.81E¯007	8.81E¯007	8.81E¯007	.000110	.038394	.038394	.000110	.019197
680	2.12E¯009	2.12E¯009	2.12E¯009	8.33E¯007	8.33E¯007	8.33E¯007	.000110	.038394	.038394	.000110	.019197
765	1.79E¯009	1.79E¯009	1.79E¯009	7.90E¯007	7.90E¯007	7.90E¯007	.000110	.038394	.038394	.000110	.019197