Idempotent Method for Deception Games and Complexity Attenuation ⋆

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Abstract: We extend the applicability of idempotent methods to deterministic dynamic games through the min-max distributive property. This induces a very high curse-of-complexity. A representation of the space of max-plus hypo-convex functions as a min-max linear space is used to obtain a method for attenuating this complexity growth. This is applied in a game of search and deception. The problem is formulated as a dynamic game over a max-plus probability simplex. Keywords: game theory, nonlinear control, numerical methods.

1. INTRODUCTION

In recent years, idempotent methods have been developed for solution of nonlinear control problems. (Note that idempotent algebras are those for which $a \oplus a = a$ for all a.) Most notably, max-plus methods have been applied to deterministic optimal control problems. These consist of max-plus basis methods, exploiting the max-plus linearity of the associated semigroup [1, 6, 13], and max-plus curse-of-dimensionality-free methods which exploit the max-plus additivity and the invariance of the set of quadratic forms under the semigroup operator [12, 13]. These methods achieved truly exceptional computational speeds on some classes of problems.

In this paper, we use some similar, but more abstract, tools which bring deterministic dynamic game problems into the realm under which curse-of-dimensionality-free idempotent methods will be applicable. We will first recall how one may apply the min-max distributive property to develop curse-of-dimensionality-free methods for discrete-time, deterministic dynamic games (as indicated in [10]). There are two main topics in the current paper: complexity attenuation and deception games.

The difficulty with idempotent curse-of-dimensionality-free methods for game problems is an extreme curse-of-complexity. In particular, the solution complexity grows exponentially as one propagates backward in time via the idempotent distributed dynamic programming principle (IDDPP). An approach for attenuating that complexity growth extends from developments in max-plus convex analysis. Using the IDDPP, one has a representation of the value function at each time-step as a pointwise minimum of max-plus affine functionals. We will see that this implies that our value function representation is an element of the space of max-plus hypo-convex functions. Using the results in [9], one may show that optimal complexity attenuation is achieved by pruning of the existing expansion. We note here that one may think of this step as optimal projection onto a min-max subspace of specified dimension. This will allow us to determine a surprisingly simple means by which this may be achieved.

Once we have these tools in hand, we consider the deception game. We suppose Player 1 is employing one or more sensing entities (e.g., UAVs, UUVs, humans) to search for certain assets. Player 2 may employ deception to hinder that search. Specifically, at a certain cost, Player 2 may choose to alter Player 1's observation. We suppose that the search domain consists of a finite set of locations. In this case, the appropriate state-space is the space of max-plus probability vectors over the set of possible asset configurations. We will indicate the dynamic program for solution of this deception game problem, the associated IDDPP, and the corresponding computations.

2. IDEMPOTENT METHOD FOR GAMES

We briefly describe the idempotent approach. All control spaces will be finite. We suppose the dynamics are

$$\xi_{t+1} = h(\xi_t, u_t, w_t), \quad \xi_s = x \in G \subseteq I\!R^I, \tag{1}$$

where s is the initial time. We suppose $u_t \in \mathcal{U}$ and $w_t \in \mathcal{W}$ for all t, with $W \doteq \#\mathcal{W}$ (the cardinality of $\mathcal{W}$) and $U = \#\mathcal{U}$. We assume $h(\cdot, u, w)$ maps G into G for all $u \in \mathcal{U}$, $w \in \mathcal{W}$. Time is discrete with $t \in]s, T[\doteq \{s, s+1, s+2, \dots T\}$, where for integers $a \le b$, we use $]a, b[$ to denote $\{a, a+1, \dots b\}$ throughout. Also for simplicity, we assume only a terminal cost, which will be $\phi : G \to I\!R$. We let $\mathcal{U}$ be the minimizing player's control set, and $\mathcal{W}$ be the maximizing player's control set. The payoff, starting from any $(t, x) \in]s, T[\times G$ will be

$$J_t(x, u_{]t,T-1[}, w_{]t,T-1[}) = \phi(\xi_T), \tag{2}$$

where $u_{]t,T-1[}$ denotes a sequence of controls, $\{u_t, u_{t+1}, \dots u_{T-1}\}$, with similar meaning for $w_{]t,T-1[}$. We will work with the upper value. At any time $t \in]s, T-1[$, this is

$$V_t(x) = \max_{\tilde{w}^t \in \widetilde{W}^t} \min_{u_{]t,T-1[} \in \mathcal{U}^{T-t}} J_t(x, u_{]t,T-1[}, \tilde{w}(u_{]t,T-1[})) \tag{3}$$

where $\widetilde{W}^t = \{\tilde{w}^t : \mathcal{U}^{T-t} \to \mathcal{W}^{T-t}, \text{ nonanticipative}\}$. The associated dynamic programming equation is

$$V_t(x) = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} V_{t+1}(h(x, u, w)). \tag{4}$$

⋆ Research partially supported by NSF grant DMS-0808131 and AFOSR Grant FA9550-10-1-0499.

Suppose ϕ takes the form $\phi(x) = \min_{z_T \in \mathcal{Z}_T} g_T(x; z_T)$, where we let $Z_T = \#\mathcal{Z}_T < \infty$. Then,

$$V_T(x) = \min_{z_T \in \mathcal{Z}_T} g_T(x; z_T). \tag{5}$$

Combining (4) and (5), one has

$$V_{T-1}(x) = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \min_{z_T \in \mathcal{Z}_T} g_T(h(x, u, w); z_T). \tag{6}$$

We now introduce the relevant idempotent algebras. The max-plus algebra (i.e., semifield) is given by

$$a \oplus b \doteq \max\{a, b\}, \qquad a \otimes b \doteq a + b,$$

operating on $I\!R^- \doteq I\!R \cup \{-\infty\}$. In the min-max algebra (i.e., semiring), the operations are defined as

$$a \wedge b \doteq \min\{a, b\}, \qquad a \vee b \doteq \max\{a, b\},$$

operating on $\overline{I\!R} \doteq I\!R \cup \{-\infty\} \cup \{+\infty\}$, where we note that $+\infty \wedge b = b$ for all $b \in \overline{I\!R}$ and $+\infty \vee b = +\infty$ for all $b \in \overline{I\!R}$ (c.f., [7]). We suppose each $g_T(\cdot; z_T)$ is max-plus affine. In other words, ϕ will be formed as the lower envelope of a finite set of max-plus affine functions. In fact, we are going to think of ϕ as a max-plus hypo-convex function. (We will have reason to reverse the ordering on the range space, and so our definition of max-plus hypo-convex functions will look directly analogous to the definition of standard-sense convex functions.) We may write these max-plus affine $g_T(\cdot; z_T)$ as

$$\begin{aligned} g_T(x; z_T) &= \alpha^{T,z_T} \odot x \oplus \beta^{T,z_T} \\ &= \left[\bigoplus_{i \in \mathcal{I}} \alpha_i^{T,z_T} \otimes x_i \right] \oplus \beta^{T,z_T} \end{aligned}$$

where $\mathcal{I} =]1, I[$. We will assume that the $h(\cdot, u, w)$ are max-plus linear. Specifically, we let

$$h(x, u, w) = A(u, w) \otimes x,$$

where here we use $\otimes$ to emphasize that this is max-plus matrix-vector multiplication. We see that

$$V_{T-1}(x) = \bigwedge_{u \in \mathcal{U}} \bigvee_{w \in \mathcal{W}} \bigwedge_{z_T \in \mathcal{Z}_T} \Big[\beta^{T,z_T} \oplus \alpha^{T,z_T} \odot A(u, w) \otimes x \Big]. \tag{7}$$

Define, for any $t \in]s+1, T[$, $\widehat{\mathcal{Z}}_t = \{\hat{z}_t : \mathcal{W} \to \mathcal{Z}_t\}$. Applying the min-max distributive property to (7) (and noting that $\oplus \equiv \vee$),

$$V_{T-1}(x) = \bigwedge_{u \in \mathcal{U}} \bigwedge_{\hat{z}_T \in \widehat{\mathcal{Z}}_T} \bigoplus_{w \in \mathcal{W}} \Big[\beta^{T,\hat{z}_T(w)} \oplus \alpha^{T,\hat{z}_T(w)} \odot A(u, w) \otimes x \Big]. \tag{8}$$

With a bit more work [10], one obtains the following.

Theorem 2.1. For any $t \in]s+1, T[$,

$$V_{t-1}(x) = \bigwedge_{z_{t-1} \in Z_{t-1}} \Big[\beta^{t-1,z_{t-1}} \oplus \alpha^{t-1,z_{t-1}} \odot x \Big],$$

where

$$\alpha_j^{t-1,z_{t-1}} \doteq \bigoplus_{w \in \mathcal{W}} \bigoplus_{i \in \mathcal{I}} \alpha_i^{t,\hat{z}_t(w)} \otimes A_{i,j}(u, w) \quad \forall j \in \mathcal{I},$$

$$\beta^{t-1,z_{t-1}} \doteq \bigoplus_{w \in \mathcal{W}} \beta^{t,\hat{z}_t(w)},$$

where $(u, \hat{z}_t) = \Gamma_{t-1}^{-1}(z_{t-1})$ for all $x \in I\!R^I$, $z_{t-1} \in \mathcal{Z}_{t-1}$, and Γ_{t-1} is a one-to-one, onto mapping from $\mathcal{U} \times \widehat{\mathcal{Z}}_t$ to $\mathcal{Z}_{t-1} \doteq]1, Z_{t-1}[$, with $Z_{t-1} = U(Z_t)^W$.

This is our IDDPP. The difficulty emerges through the iteration $Z_{t-1} = U(Z_t)^W$; in a naive application of this approach, the number of max-plus affine functions defining the value would grow extremely rapidly. This implies that the second piece of the algorithm must be complexity reduction in the representation at each step.

3. GENERAL COMPLEXITY REDUCTION PROBLEM AND CONTEXT

Certain function spaces may be spanned by infima of max-plus affine functions, that is, any element of the space may be represented as an infimum of a set of max-plus affine functions. By definition, any function in such a space as the above has an expansion, $f(x) = \inf_{\lambda \in \Lambda} \psi_\lambda(x)$, for some index set Λ, where the ψ_λ are max-plus affine. If the expansion is guaranteed to be countably infinite, we would write

$$f(x) = \inf_{i \in \mathbf{N}} \psi_i(x) = \bigwedge_{i \in \mathbf{N}} \psi_i(x) \doteq \bigwedge_{i \in \mathbf{N}} [a_i \oplus \psi_i'(x)],$$

where the ψ_i' are max-plus linear. We will refer to this as a min-max basis expansion, or simply a min-max expansion, and we think of the set of such ψ_i' as a min-max basis for the space.

Now we indicate the complexity reduction problem in a general form. Suppose we are given $f : \mathcal{X} \to I\!R$ with representation

$$f(x) = \bigwedge_{m \in \mathcal{M}} t_m(x) = \min_{m \in \mathcal{M}} t_m(x) = \min_{m \in]1,M[} t_m(x), \tag{9}$$

where $\mathcal{X}$ will be a partially ordered vector space. Except where noted, we will take $\mathcal{X} = I\!R^I$ for clarity. We are looking for $\{a_n : \mathcal{X} \to I\!R \,|\, n \in]1, N[\}$ with $N < M$, such that

$$g(x) \doteq \bigwedge_{n \in \mathcal{N}} a_n(x) = \min_{n \in \mathcal{N}} a_n(x) = \min_{n \in]1,N[} a_n(x) \tag{10}$$

approximates $f(x)$ from above. Note that throughout the paper, we will let $\mathcal{M} =]1, M[= \{1, 2, \ldots M\}$, $\mathcal{N} =]1, N[$ and $\mathcal{I} =]1, I[$.

3.1 Min-max spaces

As indicated earlier, it is well-known that it is useful to apply max-plus basis expansions to solve certain HJB PDEs and their corresponding control problems. In particular, the solutions are represented as max-plus sums of affine or quadratic functions. In fact, the spaces of standard-sense convex and semiconvex functions have max-plus bases (more properly, max-plus spanning sets) consisting of linear and quadratic functions, respectively,

We will be applying the analogous concept, where the standard algebra will be replaced by the max-plus, and the max-plus will be replaced by the min-max. On $I\!R^I$, we will define the partial order $x \preceq y$ if $x_i \le y_i$ for all $i \in \mathcal{I}$. Let $\mathcal{O}^I = [0, \infty)^I \doteq \{x \in I\!R^I \,|\, x \ge 0\}$. For $\delta \in \mathcal{O}^I$, let $\|\delta\|^{\oplus} \doteq \max_{i\in\mathcal{I}} \delta_i = \bigoplus_{i\in\mathcal{I}} \delta_i$. Let $\mathbf{1}$ denote a generic-length vector all of whose elements are 1's. Let

$$\mathcal{S}^{\mathbf{1}}(I\!\!R^I) \doteq \left\{ f : I\!\!R^I \to \overline{I\!\!R} \,\middle|\, 0 \le f(x+\delta) - f(x) \right. \tag{11}$$
$$\left. \le \|\delta\|^{\oplus}, \forall x \in I\!\!R^I, \delta \in \mathcal{O}^I \right\}.$$

It is not difficult to show that $\mathcal{S}^{\mathbf{1}}(I\!\!R^I)$ is exactly the space of sub-topical functions [15] from $I\!\!R^I$ to $\overline{I\!\!R}$. We will refer to a space as a min-max vector space if it satisfies the standard conditions (c.f. [13]). The definition of a max-plus vector space is analogous.

Theorem 3.1. $\mathcal{S}^{\mathbf{1}}$ is a min-max vector space.

One of the most useful aspects of looking at the spaces of convex and semiconvex spaces as max-plus vector spaces was that these spaces had countable max-plus bases, c.f., [13]. We are interested in analogous results here.

We take $\psi(x,z) : I\!\!R^I \times I\!\!R^I \to I\!\!R$ to be

$$\psi(x,z) \doteq z \odot x \doteq \bigoplus_{i\in\mathcal{I}} z_i \otimes x_i = \max_{i\in\mathcal{I}}\{z_i + x_i\}. \tag{12}$$

We will be taking $\phi_k(x) = \psi(x, z^k)$ where the z^k will form a countable dense subset of $I\!\!R^I$. The result will follow if we have

$$f(x) = \inf_{z\in I\!\!R^I} \{\max[c(z), \psi(x,z)]\},$$

where c has sufficient continuity properties. Note that this would imply that f was the lower envelope of a set of functions. Further, note that

$$c(z) \vee \psi(x,z) = c(z) \oplus \psi(x,z)$$

where the $\psi(\cdot, z)$ are max-plus linear. In other words, f would be an infimum of max-plus affine functions.

3.2 Min-max basis representation and max-plus convexity

Given $\overline{x} \in I\!\!R^I$, let $z^{\bar{x}} \in I\!\!R^I$ and $c(z^{\bar{x}})$ be given by

$$z_i^{\bar{x}} = f(\overline{x}) - \overline{x}_i \quad \forall\, i \in \mathcal{I}, \text{ and } c(z^{\bar{x}}) \doteq f(\overline{x}).$$

Note that this may not define c on all of $I\!\!R^I$. However, the composite mapping $\overline{x} \mapsto c(z^{\bar{x}})$ is defined on all of $I\!\!R^I$. See [9] for the proofs of the following set of results.

Theorem 3.2. Let $\{x_k\}_{k\in\mathbf{N}}$ be a countable dense subset of $I\!\!R^I$. Let $\phi_k(x) \doteq \psi(x, \hat{z}^k)$, where $\hat{z}^k \doteq z^{x_k}$, for all $x \in I\!\!R^I$ and all $k \in \mathbf{N}$. For any $f \in \mathcal{S}^{\mathbf{1}}$,

$$f(x) = \bigwedge_{k\in\mathbf{N}} c_k \vee \phi_k(x) \quad \forall\, x \in I\!\!R^I, \tag{13}$$

where $c_k \doteq c(\hat{z}^k)$ for all $k \in \mathbf{N}$.

We now show that $\mathcal{S}^{\mathbf{1}}$ has a very nice interpretation. We begin with the definition of max-plus convex sets. A set, $\mathcal{C} \subseteq I\!\!R^I$ is *max-plus convex* if

$$\lambda_1 \otimes x^1 \oplus \lambda_2 \otimes x^2 \in \mathcal{C}$$

for all $x^1, x^2 \in \mathcal{C}$ and all $\lambda_1, \lambda_2 \in [-\infty, 0]$ such that $\lambda_1 \oplus \lambda_2 = 0$. See [3], [15]. We now turn to max-plus hypo-convex functions. We would like the set of such functions to form a min-max vector space. Consequently, we define the ordering on the range space, $\overline{I\!\!R}$, by $y_1 \preceq^R y_2$ if $y_1 \ge y_2$, and $y_1 \prec^R y_2$ if $y_1 > y_2$; relations $\succeq^R$ and $\succ^R$ are defined analogously. We henceforth refer to this as the *range order.* Suppose $f : I\!\!R^I \to \overline{I\!\!R}$, and define the max-plus epigraph

$$\mathrm{epi}^{\oplus} f \doteq \{(x,y) \in I\!\!R^I \times \overline{I\!\!R} \,|\, y \succeq^R f(x)\}. \tag{14}$$

Alternatively, f may be referred to as the hypograph [15], but due to the natural reversal of order in the range space here, the term max-plus epigraph is more appropriate in this context. Lastly, we say f is *max-plus hypo-convex* if $\mathrm{epi}^{\oplus} f$ is max-plus convex. With some work, one obtains:

Theorem 3.3. $f \in \mathcal{S}^{\mathbf{1}}$ if and only if f is max-plus hypo-convex.

3.3 Complexity reduction and abstract formulation

Recall that our originating problem was complexity reduction in a min-max expansion; see (9),(10). The a_n and t_m will now be selected from a specified class of functions, such as the affine functions. We take $\mathcal{X} \doteq I\!\!R^I$ throughout. We will use a measure of approximation quality which is monotonic and max-plus hypo-convex. Specifically, we wish to minimize (i.e., range-order maximize)

$$J(A) \doteq \int_G^{\oplus} \left\{ \left[\bigwedge_{n\in\mathcal{N}} \alpha^n \odot x \right] - \left[\bigwedge_{m\in\mathcal{M}} \tau^m \odot x \right] \right\} dx \tag{15}$$

where $A \doteq \{\alpha^n\}_{n\in\mathcal{N}}$, subject to the constraints

$$\alpha^n \cdot x \ge \bigwedge_{m\in\mathcal{M}} \tau^m \odot x \quad \forall\, x \in I\!\!R^I, \forall n \in \mathcal{N}. \tag{16}$$

Embedding this problem in a larger class of problems, and using a technique similar to that in [11], we find [9]:

Theorem 3.4. There exists A^* minimizing (i.e., range-order maximizing) J subject to constraints (16). Further, there exist $\{m_n\}_{n\in\mathcal{N}} \subset \mathcal{M}$ such that $A^* = \{\tau^{m_n}\}_{n\in\mathcal{N}}$.

Remark 3.5. Note that the above result covers only the max-plus linear case. We may extend this to the affine case on $G' \subseteq \overline{I\!\!R}^{I'}$ with $I' \doteq I - 1$ by letting $G = G' \times \{0\}$. Then with $\alpha^n = ([\alpha']^n, \beta) \in \overline{I\!\!R}^I$, for any $x' \in G'$ there exists unique $x \in G$ given by $x = (x', 0)$ such that

$$[\alpha']^n \odot x' \oplus \beta = \alpha^n \odot (x', 0) = \alpha^n \odot x.$$

With this equivalence, one extends our result to affine functionals.

3.4 Pruning Evaluation

Now that the complexity reduction problem has been reduced to a pruning operation, we investigate means for pruning. In any pruning algorithm, one will need to determine the loss due to a potential pruning. That is, one needs to compute $J(A)$ for the unpruned remaining subsets, A of T. In this section, we will find that the computation of $J(A)$ is remarkably easy. We modify our notation from that of the previous section slightly. Suppose our affine functionals are given by

$$\begin{aligned} t_m(x) &= \tilde{\tau}^m \odot x \oplus \beta^m \\ &= \hat{\tau}^m \odot \hat{x}(x) = \hat{\tau}^m \odot \hat{x} \end{aligned}$$

where $\hat{\tau}^m, \hat{x}^m(x) \in I\!\!R^{I+1}$ are given by $\hat{\tau}^m = ((\tilde{\tau}^m)^T, \beta^m)^T$ and $\hat{x} = \hat{x}(x) = (x^T, 0)^T$. Note that we will freely use the shorthand $\hat{x}$ in place of $\hat{x}(x)$.

Definition 3.6. Given affine functional t_m defined by $\hat{\tau}^m$ as above, the crux of $\hat{\tau}^m$ is $x^c = x^c(\hat{\tau}^m) \in I\!R^I$ given by $x_i^c = \hat{\tau}_{I+1}^m - \hat{\tau}_i^m = \beta^m - \tilde{\tau}_i^m$ for all $i \in \mathcal{I}$.

Loosely speaking, $x^c = x^c(\hat{\tau}^m)$ is the projection down into the domain of the point where all the hyperplanes of the graph of $t_m(\cdot)$ come together.

Definition 3.7. At any $\overline{x} \in I\!R^I$ we let

$$\mathcal{R}_0(\overline{x}) \doteq \{x \in I\!R^I \,|\, x_j \le \overline{x}_j \;\forall j \in \mathcal{I}\},$$

and further, for each $i \in \mathcal{I}$, we let

$$\mathcal{R}_i(\overline{x}) \doteq \left\{ x \in I\!R^I \setminus \bigcup_{j=0}^{i-1} \mathcal{R}_j(\overline{x}) \,\middle|\, x_i - \overline{x}_i \ge x_j - \overline{x}_j, \right. \\ \left. \forall j \in \mathcal{I} \setminus \{i\} \right\}.$$

Definition 3.8. For all $m \in \mathcal{M}$, we let $\mathcal{R}_0^m = \mathcal{R}_0(x^c(\hat{\tau}^m))$ and $\mathcal{R}_i^m = \mathcal{R}_i(x^c(\hat{\tau}^m))$ for all $i \in \mathcal{I}$.

Lemma 3.9. For any $m \in \mathcal{M}$,

$$t_m(x) = \hat{\tau}_{I+1}^m = t_m(x^c) = t_m(x^c(\hat{\tau}^m)) \quad \forall\, x \in \mathcal{R}_0^m,$$

and

$$t_m(x) = \hat{\tau}_{I+1}^m + (x_i - x_i^c(\hat{\tau}^m)) = \hat{\tau}_i^m + x_i \;\forall x \in \mathcal{R}_i^m, \forall i \in \mathcal{I}.$$

Lemma 3.10. Let

$$f(x) = \bigwedge_{m \in \mathcal{M}} t_m(x) = \bigwedge_{m \in \mathcal{M}} \hat{\tau}^m \odot \hat{x}.$$

Let $\overline{x} \in I\!R^I$. For any $x \in \mathcal{R}_0(\overline{x})$, $f(x) \le f(\overline{x})$. Further, for any $i \in \mathcal{I}$ and any $x \in \mathcal{R}_i(\overline{x})$,

$$f(x) \le f(\overline{x}) + (x_i - \overline{x}_i).$$

Proof. Using Theorem 3.3, one finds $f \in \mathcal{S}^{\mathbf{1}}$. The result follows from the definition of $\mathcal{S}^{\mathbf{1}}$. □

Theorem 3.11. Suppose $G \subseteq I\!R^I$ and $\mathcal{M}' \subset \mathcal{M}$. Suppose $x^c(\hat{\tau}^m) \in G$ for all $m \in \mathcal{M}'$, and that there exists $x \in G$ such that

$$\bigwedge_{m \in \mathcal{M} \setminus \mathcal{M}'} \hat{\tau}^m \odot \hat{x}(x) > \bigwedge_{m \in \mathcal{M}} \hat{\tau}^m \odot \hat{x}(x). \tag{17}$$

Then,

$$\int_G^{\oplus} \left\{ \left[\bigwedge_{m \in \mathcal{M} \setminus \mathcal{M}'} \hat{\tau}^m \odot \hat{x}(x) \right] - \bigwedge_{m \in \mathcal{M}} \hat{\tau}^m \odot \hat{x}(x) \right\} dx \\ = \bigoplus_{m' \in \mathcal{M}'} \left\{ \left[\bigwedge_{m \in \mathcal{M} \setminus \mathcal{M}'} \hat{\tau}^m \odot \hat{x}(x^c(\hat{\tau}^{m'})) \right] \right. \\ \left. - \hat{\tau}^{m'} \odot \hat{x}(x^c(\hat{\tau}^{m'})). \right\}. \tag{18}$$

We now see that under the conditions of Theorem 3.11, the evaluation of any pruning option is remarkably simple. The maximum difference over all of G is obtained by simply looking at difference at the crux of each element of $\mathcal{M}'$, thus only requiring function evaluations at $\#\mathcal{M}'$ points.

3.5 Application to the game problem

We will now see how this can be used for our game problem. Recall that using Theorem 2.1, if the value function for the game at any time, t, took the form

$$V_t(x) = \bigwedge_{z_t \in Z_t} \left[\beta^{t,z_t} \oplus \alpha^{t,z_t} \odot x \right], \tag{19}$$

then at time $t-1$, one had

$$V_{t-1}(x) = \bigwedge_{z_{t-1} \in Z_{t-1}} \left[\beta^{t-1,z_{t-1}} \oplus \alpha^{t-1,z_{t-1}} \odot x \right],$$

where the computation of the constants was given there. Although this avoided the curse-of-dimensionality, there was a very high "curse-of-complexity", where in particular, $Z_{t-1} = \#\mathcal{Z}_{t-1} = U(Z_t)^W$. Consequently, after each iteration of the algorithm, we approximate in order to reduce complexity. That is, given any V_t of the form (19), we seek a smaller set of max-plus affine functionals that yields the best approximation. Theorem 3.4 and Remark 3.5 tell us that this is optimally achieved by pruning of the $\mathcal{Z}_t$ set. *That is, in optimally reducing from $\mathcal{Z}_t$ to some smaller set of say, N, functionals, we do not need to search over all possible sets of size N of affine functionals, but only over subsets of $\mathcal{Z}_t$.* Further, (15) gives us a criterion by which we may measure the quality of any pruning option.

To generalize this to the affine case, we use Remark 3.5. For simplicity of notation, we replace $\mathcal{Z}_t =]1, Z_t[$ with $\mathcal{M} =]1, M[$, and pairs $(\alpha^{t,z_t}, \beta^{t,z_t})$ with (α^m, β^m). Also, we let $\tau^m \doteq (\alpha^m, \beta^m)$ for $m \in \mathcal{M}$. Let $\hat{G} \doteq G \times \{0\} \subseteq I\!R^I \times \{0\}$, and given $x \in I\!R^I$, let $\hat{x} = \hat{x}(x) \doteq (x, 0) \in \hat{G}$. Then,

$$\beta^{t,z_t} \oplus \alpha^{t,z_t} \odot x = \beta^m \oplus \alpha^m \odot x = \tau^m \odot \hat{x}.$$

Noting that, by Theorem 2.1, the optimal solution of (15),(16) is a subset of $\mathcal{M}$, which we will denote by $\mathcal{M}'$, optimization criterion (15) may be replaced by

$$\hat{J}(\mathcal{M}') \doteq \int_{\hat{G}}^{\oplus} \left\{ \left[\bigwedge_{m' \in \mathcal{M}'} \tau^{m'} \odot \hat{x} \right] - \left[\bigwedge_{m \in \mathcal{M}} \tau^m \odot \hat{x} \right] \right\} dx. \tag{20}$$

One may then apply Theorem 3.11 to determine which affine functionals to prune. As noted above, the evaluation of the right-hand side of (18) is quite straight-forward, requiring only simple linear algebra and maximization and minimization over finite sets. This is used to attenuate the complexity in our value function representations, (19), i.e., to reduce the number of affine functionals.

4. A DECEPTION GAME

We now consider a deception game. Player 1 will search for what we will refer to as the assets of Player 2 over a series of time-steps, $\mathcal{T}^- \doteq]0, T-1[\doteq \{0, 1, \dots T-1\}$. Then, at time T, Player 1 takes an action, $a \in \mathcal{A} =]1, A[$. The true Player 2 asset configuration is $x \in \mathcal{X} =]1, X[$. (In the case of a single asset hidden among L possible locations, one would take $\mathcal{X} =]1, L[$.) Given true asset configuration x, Player 2 would receive a loss, $c(x, a)$. Here, we use the convention that Player 2 wishes to maximize (make less negative) the loss, $c(x, a)$. We assume a zero-sum game. Let $C(a)$ be the vector of length X with components $c(x, a)$. It is natural to use the max-plus probability structure (c.f., [2, 5, 14] and the references therein) for deterministic games.

Suppose that Player 1's knowledge of the true asset configuration is described by max-plus probability distribution, $q \in S^{\oplus X}$, where

$$S^{\oplus X} \doteq \Big\{ q = \in [-\infty, 0]^X \,\Big|\, \bigoplus_{x \in \mathcal{X}} q_x = 0 \Big\},$$

where $[-\infty, 0]$ denotes $(-\infty, 0] \cup \{-\infty\}$ and the X superscript indicates outer product X times. (Recall that $[-\infty, 0]$ is analogous to $[0, 1]$ in the standard algebra.) We may interpret each component, q_x, as the (relative) cost to Player 2 to cause Player 1 to believe that the asset configuration is x. This will be become more clear below. The expected payoff for action $a \in \mathcal{A}$ given max-plus distribution q at terminal time T, is as follows. Letting max-plus random variable ξ be distributed according to q, and $\mathbf{E}_q^{\oplus}$ denote max-plus expectation according to this q, the expected payoff is

$$\hat{J}(q, a) = \mathbf{E}_q^{\oplus}[c(a, \xi)] = \bigoplus_{x \in \mathcal{X}} c(a, x) \otimes q_x = C(a) \odot q. \quad (21)$$

Given that Player 1 wants to minimize (make more negative) the loss to Player 2, the value of information q at time T is

$$\phi(q) \doteq \min_{a \in \mathcal{A}} J(q, a) = \bigwedge_{a \in \mathcal{A}} [C(a) \odot q]. \quad (22)$$

We see that if information is represented by a max-plus probability distribution over a finite set (and one has a finite set of controls), then *the value of information takes the form of a min-max sum of max-plus linear functionals over a max-plus probability simplex.*

We will view ϕ as the terminal payoff in the deception game. Now we describe the actual deception game. At each time, $t \in \mathcal{T}^-$, Player 1 may task sensing entities. The possible Player 1 sensing controls at each time step are denoted by $u \in \mathcal{U} =]1, U[$. Each sensing step results in an observation (or set of observations) denoted by $y \in \mathcal{Y}$. Again, in the max-plus probability structure, one may associate max-plus probabilities with costs. Let the max-plus probability of observing y given sensing control u and true asset state x be denoted by $p^{\oplus}(y|x; u) \in [-\infty, 0]$. These may be associated with Player 2's deception actions. We suppose that at each time step, Player 2 may use a combination of decoys, stealth and "no action". Here, each use of a decoy or stealth will have associated costs. (Note that the use of stealth may be associated with a cessation of activity which would otherwise be benefiting Player 2.) That is, we may interpret $p^{\oplus}(y|x; u)$ as the (non-positive) cost to Player 2 to cause Player 1 to observe y given true state x and sensing control u.

Suppose $q(t)$ is the max-plus probability distribution after observation at time t. Suppose that at time $t+1$, Player 1 employs control $u(t+1) = \hat{u} \in \mathcal{U}$ with resulting observation $y \in \mathcal{Y}$ (which we recall may be at least partially controlled by Player 2). The resulting cost for any true state $x \in \mathcal{X}$ would be

$$\hat{q}_x(t+1) = p^{\oplus}(y|x; \hat{u}) + q_x(t) = p^{\oplus}(y|x; \hat{u}) \otimes q_x(t).$$

In solving the optimization problem, we are concerned only with the relative costs, and so we may normalize so that the max-plus sum over $x \in \mathcal{X}$ is zero. Let $q(t+1)$ denote the normalized cost, where we want $\bigoplus_{x \in \mathcal{X}} q_x(t+1) = 0$. The normalized cost is

$$\begin{aligned} q_x(t+1) &= p^{\oplus}(y|x; \hat{u}) \otimes q_x(t) \\ &\quad - \Big\{ \bigoplus_{\zeta \in \mathcal{X}} \big[p^{\oplus}(y|\zeta; \hat{u}) \otimes q_\zeta(t) \big] \Big\} \qquad (23) \\ &= p^{\oplus}(y|x; \hat{u}) \otimes q_x(t) \oslash \Big\{ \bigoplus_{\zeta \in \mathcal{X}} \big[p^{\oplus}(y|\zeta; \hat{u}) \otimes q_\zeta(t) \big] \Big\}, \end{aligned}$$

where $\oslash$ indicates max-plus division (standard-sense subtraction). One sees that this is directly analogous to Bayes rule in standard-algebra probability. We may interpret each component of the resulting max-plus probability at time r, $q_x(r)$, as the maximal (least negative) relative cost to Player 2 for modification of the observation process to yield observed sequence $\{y(0), y(1), \ldots y(r)\}$ given true state x.

For simplicity, we assume that the sensing entities can move from any sensing control to any other in one time-step. Consequently, the state process for the game is simply $q(t)$. One may also easily include a second controller for Player 2 which allows the assets to change configuration, with an associated cost (analogous to a Markov chain transition matrix), but we do not include this here. The payoff will be the terminal value of information, ϕ, obtained above plus the deception costs. Let the value function for this game at time t be denoted by $V :]0, T[\times S^{\oplus X} \to \overline{I\!R}$. At terminal time,

$$V(T, q) = \phi(q) = \bigwedge_{a \in \mathcal{A}} [C(a) \odot q].$$

We now develop the specific IDDPP for this problem. Although the dynamics of q do not quite fit the general form given in Section 2, we nonetheless obtain a similar IDDPP due to the max-plus expectation operation. The dynamic programming principle takes the form

$$\begin{aligned} V(t, q) &= \bigwedge_{u \in \mathcal{U}} \mathbf{E}^{\oplus} V(t+1, q(t+1)) \\ &= \bigwedge_{u \in \mathcal{U}} \bigvee_{y \in \mathcal{Y}} [V(t+1, q(t+1)) \otimes p_{t+1}^{\oplus}(y)], \quad (24) \end{aligned}$$

where $p_{t+1}^{\oplus}(y)$ denotes the max-plus probability of observation y at time $t+1$ and $q(t+1)$ is generated from q by dynamics (23). Suppose that at time $t+1$,

$$V(t+1, q) = \bigwedge_{z \in \mathcal{Z}_{t+1}} [d_{t+1}(z) \odot q], \quad (25)$$

for some finite index set $\mathcal{Z}_{t+1}$ and associated coefficient vectors $d_{t+1}(z)$. In other words, we are assuming that $V(t+1, \cdot)$ is a max-plus hypo-convex function defined by a finite number of max-plus linear functionals. Employing form (25) in (24) yields

$$\begin{aligned} V(t, q) = \bigwedge_{u \in \mathcal{U}} \bigvee_{y \in \mathcal{Y}} \Big\{ \bigwedge_{z \in \mathcal{Z}_{t+1}} & \big[[d_{t+1}(z) \odot q(t+1)] \\ & \otimes p_{t+1}^{\oplus}(y) \big] \Big\}. \quad (26) \end{aligned}$$

Noting that $p_{t+1}^{\oplus}(y) = \bigoplus_{\zeta \in \mathcal{X}} \big[p^{\oplus}(y|\zeta; \hat{u}) \otimes q_\zeta(t) \big]$, this becomes

$$V(t,q) = \bigwedge_{u\in\mathcal{U}} \bigvee_{y\in\mathcal{Y}} \left\{ \bigwedge_{z\in\mathcal{Z}_{t+1}} \left[[d_{t+1}(z)\odot q(t+1)] \right.\right. \tag{27}$$

$$\left.\left. \otimes \bigoplus_{\zeta\in\mathcal{X}} \left[p^{\oplus}(y|\zeta;\hat{u})\otimes q_\zeta\right]\right]\right\},$$

Combining all components of (23) into a single equation yields

$$q(t+1) = D^{y,\hat{u}} \otimes q(t) \oslash \left\{ \bigoplus_{\zeta\in\mathcal{X}} \left[p^{\oplus}(y|\zeta;\hat{u})\otimes q_\zeta(t)\right]\right\}, \tag{28}$$

where $D^{y,\hat{u}}$ is the $X\times X$ diagonal matrix with diagonal terms $D^{y,\hat{u}}_{x,x} = p^{\oplus}(y|x;\hat{u})$, and the first $\otimes$ indicates max-plus matrix-vector multiplication. Substituting (28) into (27), and cancelling terms yields

$$V(t,q) = \bigwedge_{u\in\mathcal{U}} \bigvee_{y\in\mathcal{Y}} \left\{ \bigwedge_{z\in\mathcal{Z}_{t+1}} \left[d_{t+1}(z)\odot D^{y,u}\otimes q\right]\right\},$$

which upon letting $\hat{d}_t(z,y,u) \doteq [D^{y,u}]^T d_{t+1}(z) = D^{y,u} d_{t+1}(z)$, becomes

$$= \bigwedge_{u\in\mathcal{U}} \bigvee_{y\in\mathcal{Y}} \left\{ \bigwedge_{z\in\mathcal{Z}_{t+1}} \left[\hat{d}_t(z,y,u)\odot q\right]\right\}.$$

This is identical to the form of (7), but without the additive constant terms (β^{t,ζ_t}) that were present there. Therefore, following exactly the same steps, one has

$$V(t,q) = \bigwedge_{u\in\mathcal{U}} \bigwedge_{\tilde{z}\in\tilde{\mathcal{Z}}_t} \left[\tilde{d}_t(\tilde{z},u)\odot q\right], \tag{29}$$

where $\tilde{d}_t(\tilde{z},u) \doteq \bigoplus_{y\in\mathcal{Y}} \hat{d}_t(\tilde{z}_y,y,u)$ and $\tilde{\mathcal{Z}}_t \doteq \left\{\tilde{z} = \{\tilde{z}_y\}_{y\in\mathcal{Y}} \,\middle|\, z_y \in Z_{t+1}\,\forall y\in\mathcal{Y}\right\}$. Then, letting $\mathcal{Z}_t =]1, Z_t[$ where $Z_t = U[\#\mathcal{Z}_{t+1}]^Y$ and making the bijection identification between $\mathcal{Z}_t$ and $\mathcal{U}\times\tilde{\mathcal{Z}}$, one may rewrite (29) as

$$V(t,q) = \bigwedge_{z\in\mathcal{Z}_t} [d_t(z)\odot q], \tag{30}$$

where $d_t(z)$ is obtained from the corresponding value of $\tilde{d}_t(\tilde{z},u)$. Consequently, the form of the value as a min-max sum of max-plus linear functionals is inherited.

We have demonstrated that this class of deception games also falls into the class where the general backward IDDPP of Section 2 may be applied. Further, as the form of the value is a min-max sum of max-plus linear functionals, the results of Section 3 regarding the complexity attenuation step may be applied as well. That is, pruning at each step is optimal for complexity attenuation, and the calculations required for the pruning reduce to a quite small set of max-plus linear (in this case) function evaluations.

The algorithms have been coded and tested on some simple examples. A plot of a solution along a three-dimensional sub-manifold of the max-plus probability simplex appears in Figure 1. (Recall that the max-plus probability simplex is different in shape than a standard-algebra probability simplex; for obvious reasons, we truncate the figure, extending only to −10 in each component.) The value at each point is denoted by color.

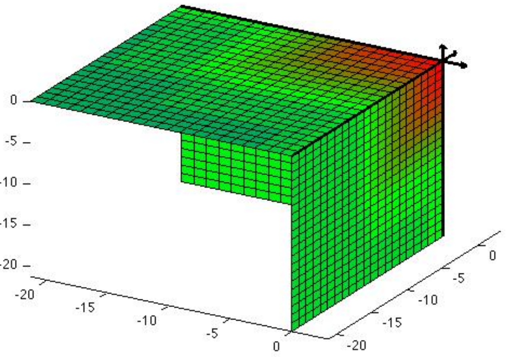

Fig. 1. Value on $S^{\oplus 3}$ sub-manifold.

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