Computational reduction for systems with low-dimensional nonlinearities via staticization-based duality

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***Abstract*— A finite-horizon nonlinear optimal control problem is considered. Stat-quad duality is used to generate an equivalent problem with linear dynamics and a modification term in the running cost and two auxiliary controls processes. This problem form is used to obtain a representation of the value function as a staticization problem over a set of quadratic functions, where the coefficients of the quadratics consists of the solution to a differential Riccati equation, a linear ODE and an integral. This representation allows the value function to be evaluated independently at any time and any point in the state space. A specialized numerical method is proposed for solving the resulting staticization problem, which is able to leverage the low dimensionality of nonlinearity. A numerical example with five-dimensional state space is included.**

I. INTRODUCTION

The most common approach to the solution of continuous-time/continuous-space nonlinear optimal control problems is dynamic programming. This converts the control problem into an associated nonlinear Hamilton-Jacobi partial differential equation (HJ PDE) problem. In the case of a deterministic-dynamics model, the HJ PDE is first-order. Classical methods for solution of the HJ PDE problem are subject to the curse-of-dimensionality. Although this difficulty has been greatly reduced in recent years through a variety of approaches, there still remains a very serious problem for the computational solution when the system state is of moderate dimension. On the other hand, in a wide variety of cases, the system dynamics might be largely linear, with only a few nonlinearities. (This notion will be rigorously defined in Section II.) Nonetheless, the dimension of the HJ PDE is that of the overall state space. Here, we present an approach where, in such cases, one can reduce the nonlinear problem to that of the "dimension" of the nonlinearity. In the specific approach here, we apply a method somewhat in the vein of a Pontryagin maximum principle approach to the resulting low-dimensional problem. This allows for a pointwise solution method that does not require solution over the entire space.

We now provide a bit more detail regarding the techniques developed herein. As suggested by the title, staticization is a critical element of the development. Staticization will be properly defined in Section III, but it is sufficient here to indicate that it maps functions to their values at stationary points (i.e., critical points); see [8], [9], [10] for the general theory and connections to extremization problems, cf., [4]. Its development is motivated by the study of conservative and quantum systems, where stationary trajectories of the action functional play a fundamental role, cf., [5], [6]. Subsequently, staticization has demonstrated utility in the development of tools for nonlinear control. In particular, the effort here represents a substantial extension to the results presented in [3], where the applicability was restricted by the use of minimization. Stat-quad duality (defined in Theorem 2) will be used to formulate an equivalent problem, where new auxiliary control inputs are introduced. The auxiliary controls come about due to relaxation. i.e. replacing the primal with the dual of the dual, which is exact if the primal is in the domain of the stat-quad transform. It might be noted that stat-quad duality is analogous to stat-duality [9] in a manner similar to that of the relationship between semi-convex duality and convex duality. The resulting problem has linear dynamics, and a running cost that is quadratic in the state and non-quadratic in the newly introduced control variables. One obtains the value function as the sum of a purely quadratic term and a nonlinear/nonquadratic term. The quadratic-term growth coefficient is obtained from solution of a differential Riccati equation (DRE), where the DRE is independent of these newly introduced controls. The nonlinear term is obtained through staticization over control-indexed affine functions. The linear terms in the affine functions are obtained from (control-dependent) linear ODEs, while the zeroth-order terms are obtained from integrals where the integrands contain a nonlinear function of the newly introduced controls. The dimension of the newly introduced controls is, roughly speaking, that of the low-dimensional nonlinearities, where, again, this dimensionality will be clarified in Section II. An efficient numerical method for solving the equivalent problem is proposed, exploiting the low dimensionality of the nonlinearity. Although the methods of dynamic programming are a major tool in the analysis herein, the resulting "low-dimensional" algorithm is such that the value function may be evaluated at any point in the state space independently. This is a key to the applicability of the specific numerical method generated below.

The subsequent sections of this paper are structured as follows. The optimal control problem of interest is specified in Section II, along with some notational conventions. The equivalent problem with linear dynamics is presented in Section III. This is reduced to a pointwise-in-space nonlinear integral equation in Section IV. A numerical method is proposed in Section V. That method is subsequently employed

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in the example that appears in Section VI.

II. Optimal control problem

Let $0 \le t < T < \infty$, $x \in \mathbb{R}^n$ and $\mathcal{U}(t,T) \doteq L^2((t,T);\mathbb{R}^j)$. We consider a control problem with state process given by

$$\dot{\xi}_s = A\xi_s + L^0 f(M^0\xi_s) + \sigma u_s, \quad \xi_t = x \in \mathbb{R}^n, \quad (1)$$

for $s \in (t,T)$, where $A \in \mathbb{R}^{n\times n}$, $\sigma \in \mathbb{R}^{n\times j}$, $L^0 \in \mathbb{R}^{n\times l}$, $f \in C^2(\mathbb{R}^k;\mathbb{R}^l)$, $\ell \in C^3(\mathbb{R}^k;\mathbb{R}_{\ge 0})$, $M^0 \in \mathbb{R}^{k\times n}$, $k,l \le n$, $u \in \mathcal{U}(t,T)$ denotes the control process, and in the interests of space, the time argument will often be indicated with subscript notation. The cost function is given by

$$J_t(x,u) \doteq \int_t^T \ell(M^0\xi_s) + \tfrac{1}{2}\xi_s' C\xi_s + \tfrac{1}{2}|u_s|^2\, ds \quad \forall\, x \in \mathbb{R}^n,$$

where C is a symmetric positive definite matrix.

Assumption 1: f and ℓ have bounded first and second derivatives.

We consider the value function

$$\tilde{W}_t(x) \doteq \inf_{u\in\mathcal{U}(t,T)} J_t(x,u). \quad (2)$$

The associated HJ PDE is given by

$$\begin{cases} 0 = -\partial_s U_s(x) + \tilde{H}(x, \nabla_x U_s(x)), \ (s,x) \in (t,T)\times\mathbb{R}^n, \\ U_T(x) = 0, \end{cases} \quad (3)$$

where

$$\begin{aligned} \tilde{H}(x,p) &\doteq -\min_{v\in\mathbb{R}^j}\{\tfrac{1}{2}x'Cx + \tfrac{1}{2}|v|^2 + \ell(M^0x) \\ &\qquad + p'[Ax + L^0 f(M^0x) + \sigma v]\} \\ &= -\{\overbrace{\tfrac{1}{2}x'Cx + p'Ax - \tfrac{1}{2}p'\sigma\sigma' p}^{\doteq \tilde{Q}(x,p)} \\ &\qquad + \underbrace{\ell(M^0x) + [(L^0)'p]' f(M^0x)}_{\doteq \tilde{N}(M^0x,(L^0)'p)}\}. \end{aligned}$$

By standard results (cf., [1]), the value function $\tilde{W}$ is the unique viscosity solution of (3).

III. Staticization and stat-quad representations

We begin by recalling some definitions regarding staticization [9]. Let V denote a normed vector space and $A \subset V$. Let $G : A \to \mathbb{R}$. We say $\bar{u} \in \operatorname{arg\,stat}_{u\in A} G(u)$ if $\bar{u} \in A$ and

$$\text{either} \limsup_{u\to\bar{u},\, u\in A\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{u-\bar{u}} = 0 \text{ or } \bar{u} \text{ is isolated.}$$

If $\operatorname{arg\,stat}_{u\in A} G(u) \ne \emptyset$, we define the set-valued staticization operator as follows:

$$\operatorname*{stat}_{u\in A}{}^s G(u) \doteq \{G(\bar{u}) | \bar{u} \in \operatorname*{arg\,stat}_{u\in A} G(u)\}.$$

Otherwise the $\operatorname{stat}^s_{u\in A} G(u)$ is undefined.

If $\operatorname{stat}^s_{u\in A} G(u) = \{a\}$ for some $a \in \mathbb{R}$, we define the single-valued stat operator as $\operatorname{stat}_{u\in A} G(u) = a$; otherwise it is undefined.

We have the following result regarding staticization in Banach spaces.

Theorem 1: Suppose V is a Banach space and $A \subset V$ is open. If G is Fréchet differentiable at $\bar{u} \in A$. Then $\bar{u} \in \operatorname{arg\,stat}_{u\in A} G(u)$ iff the Fréchet derivative at $\bar{u}$ is zero.

Let

$$\begin{aligned} Q^0(y^0,y^1,a,b) &\doteq -\tfrac{c_1}{2}|y^0 - a|^2 - \tfrac{c_2}{2}|y^1 - b|^2 \\ &= \tfrac{-1}{2}\big(y' - (a',b')\big)\hat{C}\big(y - (a',b')'\big), \end{aligned}$$

where $y' \doteq ((y^0)',(y^1)')$ and

$$\hat{C} \doteq \begin{pmatrix} c_1 I_k & 0_{k,\ell} \\ 0_{\ell,k} & c_2 I_\ell \end{pmatrix}, \quad (4)$$

in which $c_1, c_2 \ne 0$, and let I_m generically denote an m-dimensional identity matrix, and $0_{m,j}$ generically denote an $m \times j$ zero matrix.

Assumption 2: Assume $c_2 \in \mathbb{R}$ is such that $\tilde{\Gamma} \doteq \sigma\sigma' + c_2 L^0(L^0)'$ is positive definite.

Adapting [9, Theorem 4] to the current context, one has the following general result, which is an example of stat-quad duality.

Theorem 2: Let $\mathcal{B} \subseteq \mathbb{R}^{k+\ell}$ be open and $\tilde{\eta} : \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$, be given by $\tilde{\eta}(y) \doteq \tilde{N}_y(y) - ((c_1 y^0)', (c_2 y^1)')'$. Suppose $\tilde{\eta}^{-1} \in C^1(\mathcal{B};\mathbb{R}^{k+\ell})$ and onto $\mathbb{R}^{k+\ell}$. Then

$$\begin{aligned} &\tilde{N}(y^0,y^1) = \operatorname*{stat}_{(a,b)\in\mathcal{B}} \Big\{\tilde{\Theta}(a,b) + Q^0(y^0,y^1,a,b)\Big\} \\ &\forall (y^0,y^1) \in \mathbb{R}^{k+\ell}, \quad (5) \\ &\tilde{\Theta}(a,b) = \operatorname*{stat}_{(y^0,y^1)\in\mathbb{R}^{k+\ell}} \Big\{\tilde{N}(y^0,y^1) - Q^0(y^0,y^1,a,b)\Big\} \\ &\forall (a,b) \in \mathcal{B}. \quad (6) \end{aligned}$$

A case of special interest here is where there exists $c_{\tilde{N}} < \infty$ such that $|\nabla_{yy}\tilde{N}(y)| < c_{\tilde{N}}$ for all $y \in \mathbb{R}^{k+\ell}$. In this case, one obtains (cf., [9], [11], [3]) the following.

Lemma 3: There exists stat-quad duality (5)–(6) where in particular, $\mathcal{B} \doteq \mathbb{R}^{k+\ell}$, and each of the two associated argstat values is unique. Further, $|\nabla_{(a,b)(a,b)}\tilde{\Theta}(a,b)| \le 2c_{\tilde{N}}$ for all $(a,b) \in \mathbb{R}^{k+\ell}$, and if $\tilde{N} \in C^j(\mathbb{R}^{k+\ell})$, then $\tilde{\Theta} \in C^j(\mathbb{R}^{k+\ell})$.

We immediately obtain the following extension that applies stat-quad duality to the specific problem structure here.

Lemma 4: For all $(x,p) \in \mathbb{R}^{2n}$, $(a,b) \in \mathbb{R}^{k+\ell}$

$$\tilde{N}(M^0x,(L^0)'p) = \operatorname*{stat}_{(a,b)\in\mathbb{R}^{k+\ell}} \big\{\tilde{\Theta}(a,b) + Q^1(x,p,a,b)\big\} \quad (7)$$

$$\tilde{\Theta}(a,b) = \operatorname*{stat}_{(x,p)\in\mathbb{R}^{2n}} \big\{\tilde{N}(M^0x,(L^0)'p) - Q^1(x,p,a,b)\big\} \quad (8)$$

$$Q^1(x,p,a,b) \doteq Q^0(M^0x,(L^0)'p,a,b).$$

Applying Lemma 4 to (3) yields the equivalent HJ PDE:

$$0 = -\Big\{\partial_s U_s(x) + \tilde{Q}(x,\nabla U_s(x))$$

$$+ \operatorname*{stat}_{(a,b)\in\mathbb{R}^{k+l}} [\tilde{\Theta}(a,b) + Q^0(M^0x, (L^0)'\nabla U_s(x), a, b)] \Big\}$$

$$= -\Big\{ \partial_s U_s(x) + \tfrac{1}{2}x'Cx + (\nabla U_s(x))'Ax - \tfrac{1}{2}(\nabla U_s(x))'\tilde{\Gamma}(\nabla U_s(x)) + \operatorname*{stat}_{(a,b)\in\mathbb{R}^{k+l}} \big[\tilde{\Theta}(a,b) - \tfrac{c_1}{2}|M^0x - a|^2 - \tfrac{c_2}{2}|b|^2 + c_2(\nabla U_s(x))'L^0 b\big] \Big\}. \quad (9)$$

Then, by Assumption 2, there exists positive definite symmetric $\tilde{B}$ such that $\tilde{B}'\tilde{B} = \tilde{\Gamma}$, and thus,

$$-\tfrac{1}{2}p'\tilde{\Gamma}p = -\tfrac{1}{2}(\tilde{B}p)'(\tilde{B}p) = \operatorname*{stat}_{v\in\mathbb{R}^n}\{p'\tilde{B}v + \tfrac{1}{2}|v|^2\} \;\; \forall\, p \in \mathbb{R}^n.$$

Equation (9) can now be further rearranged as follows:

$$0 = -\big\{ \partial_s U_s(x) + \tfrac{1}{2}x'Cx + \nabla U_s(x)'Ax + \operatorname*{stat}_{v\in\mathbb{R}^n}\{\nabla U_s(x)'\tilde{B}v + \tfrac{1}{2}|v|^2\} + + \operatorname*{stat}_{(a,b)\in\mathbb{R}^{k+l}} \big[\tilde{\Theta}(a,b) - \tfrac{c_1}{2}|M^0x - a|^2 - \tfrac{c_2}{2}|b|^2 + c_2\nabla U_s(x)'L^0 b\big] \big\}, \quad (10)$$

Observe that (10), along with the original boundary condition $U_T \equiv 0$ is the attendant HJ PDE problem corresponding to a control problem with linear dynamics

$$\dot{\zeta}_s = A\zeta_s + \tilde{B}\nu_s + c_2 L^0 \beta_s, \quad \zeta_t = x \in \mathbb{R}^n, \qquad (11)$$

and cost function

$$\breve{J}_t(x,\nu,\alpha,\beta) \doteq \int_t^T \tfrac{1}{2}\zeta_s'C\zeta_s + \tfrac{1}{2}|\nu_s|^2 + \tilde{\Theta}(\alpha_s,\beta_s) - \tfrac{c_1}{2}|M^0\zeta_s - \alpha_s|^2 - \tfrac{c_2}{2}|\beta_s|^2\, ds, \quad (12)$$

where we staticize rather than minimize the cost in the value function given by

$$\breve{W}_t(x) \doteq \operatorname*{stat}_{(\nu,\alpha,\beta)\in L^2((t,T);\mathbb{R}^{n+k+l})} \breve{J}_t(x,\nu,\alpha,\beta). \qquad (13)$$

Staticization problem (11)–(13) belongs to the more general problem class given by

$$\dot{\hat{\xi}}_s \doteq \hat{A}\zeta_s + \hat{B}\nu_s + g_1(s), \quad \hat{\xi}(t) = \hat{x} \in \mathbb{R}^n, \qquad (14)$$

$$\bar{L}(s,x,\hat{u}) \doteq \tfrac{1}{2}\begin{bmatrix} x \\ \hat{u}\end{bmatrix}' \breve{C} \begin{bmatrix} x \\ \hat{u}\end{bmatrix} + \bar{\ell}(\hat{u}) + g_2(s)'x + g_3(s),$$

$$\bar{J}_t(\hat{x},\hat{u}) \doteq \int_t^T \bar{L}(s,\hat{\xi}_s,\hat{u}_s)\,ds + \bar{\psi}(\hat{\xi}_T),$$

$$\bar{W}_t(\hat{x}) \doteq \operatorname*{stat}_{\hat{u}\in\mathcal{U}(t,T)} \bar{J}_t(\hat{x},\hat{u}),$$

Under Assumption 1, the following verification theorem for staticizing optimal control problems will be used to establish the equivalence of between $\breve{W}$ and $\tilde{W}$.

Theorem 5: Let $\hat{A} \in \mathbb{R}^{n\times n}, \hat{B} \in \mathbb{R}^{n\times m}, \breve{C} \in \mathbb{R}^{(m+n)\times(m+n)}$ be constant matrices. Let $\bar{\ell} : \mathbb{R}^m \to \mathbb{R}$. Let $g_1, g_2, g_3 \in L^2([t,T])$. Denote the bottom right $m\times m$ block of $\breve{C}$ by $\breve{C}_{2,2}$. Suppose there exists some $K \in \mathbb{R}^+$ such that

$$|\bar{\ell}(\hat{v}_2) - \bar{\ell}(\hat{v}_1)| \le K(1 + |\hat{v}_1| + |\hat{v}_2|)|\hat{v}_2 - \hat{v}_1|,$$

and that $\eta : \hat{v} \mapsto \breve{C}_{2,2}v + \nabla_v\bar{\ell}(v)$ admits a globally Lipschitz inverse. If $W \in C^1$ satisfies

$$0 = -\partial_s W_s(\hat{x}) - \operatorname*{stat}_{\hat{v}\in\mathbb{R}^n}\{(\nabla_{\hat{x}} W_s(\hat{x}))'\bar{f}(s,\hat{x},\hat{v}) + \bar{L}(s,\hat{x},\hat{v})\}$$

for all $s \in [t,T]$, $\hat{x} \in \mathbb{R}^n$, $W_T(\cdot) = \bar{\psi}$, and there exists $K_2 \in \mathbb{R}_{\ge 0}$ such that $|\nabla_{\hat{x}} W_s(\hat{x})| \le K_2(1+|\hat{x}|)$ on $[t,T]\times\mathbb{R}^n$. Then $W = \bar{W}$ on $[t,T]\times\mathbb{R}^n$. Further, for $(s,\hat{x}) \in [t,T]\times\mathbb{R}^n$, let

$$\bar{H}^0(s,x,v) \doteq \nabla_{\hat{x}} W_s(\hat{x})'\bar{f}(s,\hat{x},v) + \bar{L}(s,\hat{x},v),$$

$$\hat{v}^*(s,\hat{x}) \in \operatorname*{arg\,stat}_{\hat{v}\in\mathbb{R}^m} \bar{H}^0(s,\hat{x},\hat{v}).$$

Then, letting $\hat{\xi}^*$ denote the trajectory generated by (14) with feedback $\hat{v}^*(s,\hat{\xi}^*_s)$ and $\tilde{v}^*_\cdot \doteq \hat{v}^*(\cdot,\hat{\xi}^*_\cdot)$, one has $\tilde{v}^* \in \mathcal{U}(t,T)$.

We note that the HJ PDEs (3), (9) and (10) are equivalent. Hence, Theorem 5 implies that $\breve{W} = \tilde{W}$.

The domain of staticization in (13) consists of three control processes, whereas we expect from the derivation of $\breve{W}$ that the staticization over ν may be performed independently from that of α, β. In order to achieve this, we first express $\breve{J}$ in a semi-quadratic form, and extend the results from [10] regarding the exchange of the order of staticization.

As the dynamics (11) is linear, the trajectory is given by

$$\zeta_s = [\mathcal{F}x]_s + [\mathcal{G}\nu]_s + c_2[\tilde{\mathcal{G}}\beta]_s \doteq \Phi_{s,t}x + \int_t^s \Phi_{s,\tau}\tilde{B}\nu_\tau\,d\tau + c_2\int_t^s \Phi_{s,\tau}L^0\beta_\tau\,d\tau, \qquad (15)$$

where $[\mathcal{F}x]_s \doteq \Phi_{s,t}x$ and for $s \in [t,T]$, $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are given by

$$[\mathcal{G}\nu]_s \doteq \int_t^s \Phi_{s,\tau}\tilde{B}\nu_\tau\,d\tau, \quad [\tilde{\mathcal{G}}\beta]_s \doteq \int_t^s \Phi_{s,\tau}L^0\beta_\tau\,d\tau.$$

At this point, it becomes helpful to compress the notation. Let $\mu \doteq (\alpha,\beta)$, $\mathcal{V}_{t,T} \doteq L^2((t,T);\mathbb{R}^n)$, $\mathcal{A}_{t,T} \doteq L^2((t,T);\mathbb{R}^k)$, $\mathcal{B}_{t,T} \doteq L^2((t,T);\mathbb{R}^l)$ and $\mathcal{M}_{t,T} \doteq \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$. We also suppress the subscript ${}_{t,T}$ when there is no risk of confusion. Using the explicit solution (15) in (12), we obtain the following result.

Theorem 6: Given $x \in \mathbb{R}^n$ and $t \le T$, cost $\breve{J}_t(x,\nu,\alpha,\beta)$ of (12) has the equivalent explicit semi-quadratic form

$$\breve{J}_t(x,\nu,\mu) = f_1(\mu;x) + \langle \bar{B}_2\mu, \nu\rangle_{\mathcal{V}} + \langle \hat{B}_2 x, \nu\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3\,\nu,\, \nu\rangle_{\mathcal{V}}, \qquad (16)$$

for all $x \in \mathbb{R}^n$, $\nu \in \mathcal{V}$, $\mu = (\alpha,\beta) \in \mathcal{M}$, in which $f_1(\mu;x)$, $\bar{B}_2$, $\hat{B}_2$, $\bar{B}_3$ are given by

$$f_1(\mu;x) \doteq \tfrac{1}{2}\langle x, \mathcal{F}'[C - c_1(M^0)'M^0]\mathcal{F}x\rangle_{\mathcal{V}} + c_2\langle \beta, \tilde{\mathcal{G}}'[C - c_1(M^0)'M^0]\mathcal{F}x\rangle_{\mathcal{B}} + c_1\langle \alpha, M^0\mathcal{F}x\rangle_{\mathcal{F}} + \tfrac{1}{2}\langle \mu, \hat{C}\mu\rangle_{\mathcal{V}} + \frac{c_2^2}{2}\langle \beta, \tilde{\mathcal{G}}'[C - c_1(M^0)'M^0]\tilde{\mathcal{G}}\beta\rangle_{\mathcal{B}} + c_1c_2\langle \alpha, M^0\tilde{\mathcal{G}}\beta\rangle_{\mathcal{A}} + \int_t^T \tilde{\Theta}(\mu_s)\,ds,$$

$$\bar{B}_2 \doteq \mathcal{G}'[c_1(M^0)', (C - c_1(M^0)'M^0)c_2\tilde{\mathcal{G}}],$$

$$\hat{B}_2 \doteq \mathcal{G}'(C - c_1(M^0)'M^0)\mathcal{F},$$

$$\bar{B}_3 \doteq \mathcal{G}'(C - c_1(M^0)'M^0)\mathcal{G} + I_n.$$

Moreover, $\bar{B}_3 \in \mathcal{L}(\mathcal{V};\mathcal{V})$, uniformly in $t \in [0,T]$.

Proof: We only rearrange $\breve{J}_t$ into the desired form, and omit the proof for the claimed properties of each operator. To this end, substituting (15) in (12) and performing some algebraic manipulations yields

$$\begin{aligned}
&\breve{J}_t(x,\nu,\alpha,\beta) - \int_t^T \tilde{\Theta}(\alpha_s,\beta_s)\,ds \\
&= \tfrac{1}{2}\langle \mathcal{F}x + \mathcal{G}\nu + c_2\tilde{\mathcal{G}}\beta, C[\mathcal{F}x + \mathcal{G}\nu + c_2\tilde{\mathcal{G}}\beta]\rangle_{\mathcal{V}_{t,T}} \\
&\quad + \tfrac{1}{2}\big|\nu\big|^2_{\mathcal{V}_{t,T}} - \tfrac{c_2}{2}|\beta|^2_{\mathcal{B}_{t,T}} \\
&\quad - \tfrac{c_1}{2}\big\langle M^0(\mathcal{F}x + \mathcal{G}\nu + c_2\tilde{\mathcal{G}}\beta) - \alpha, \\
&\qquad \big[M^0(\mathcal{F}x + \mathcal{G}\nu + c_2\tilde{\mathcal{G}}\beta) - \alpha\big]\big\rangle_{\mathcal{A}_{t,T}} \\
&= \tfrac{1}{2}\left\langle \begin{pmatrix} x \\ \hline \mu \\ \hline \nu \end{pmatrix}, \Pi \begin{pmatrix} x \\ \hline \mu \\ \hline \nu \end{pmatrix} \right\rangle_{\mathscr{X}},
\end{aligned}$$

in which $\mathscr{X} \doteq \mathbb{R}^n \times \mathcal{M}_{t,T} \times \mathcal{V}_{t,T}$, and

$$\Pi \doteq \left(\begin{array}{c|cc|c} \Pi^{11} & \Pi^{12} & \Pi^{13} & \Pi^{14} \\ \hline \star & \Pi^{22} & \Pi^{23} & \Pi^{24} \\ \star & \star & \Pi^{33} & \Pi^{34} \\ \hline \star & \star & \star & \Pi^{44} \end{array}\right),$$

has the compatibly defined operator-valued block entries

$$\begin{gathered}
\Pi^{11} \doteq \mathcal{F}'(C - c_1\,(M^0)'\,M^0\,\mathcal{F}, \\
\Pi^{12} \doteq c_1\,\mathcal{F}'\,(M^0)', \quad \Pi^{13} \doteq c_2\,\mathcal{F}'\,(C - (\mathcal{M}^0)'\,M^0)\,\tilde{\mathcal{G}}, \\
\Pi^{14} \doteq \mathcal{F}'\,(C - c_1\,(\mathcal{M}^0)'\,M^0)\,\mathcal{G}), \\
\Pi^{22} \doteq -c_1\,I_k, \quad \Pi^{23} \doteq c_1\,c_2\,M^0\,\tilde{\mathcal{G}}, \quad \Pi^{24} \doteq c_1\,M^0\,\mathcal{G}, \\
\Pi^{33} \doteq c_2^2\,\tilde{\mathcal{G}}'\,(C - c_1\,(M^0)'\,M^0)\,\tilde{\mathcal{G}} - c_2\,I_\ell, \\
\Pi^{34} \doteq c_2\,\tilde{\mathcal{G}}'\,(C - c_1\,(M^0)'\,M^0)\,\mathcal{G}, \\
\Pi^{44} \doteq \mathcal{G}'\,(C - c_1\,(M^0)'\,M^0)\,\mathcal{G} + I_m.
\end{gathered}$$

Equating like terms by inspection yields

$$\begin{aligned}
&f_1(\mu;x) - \int_t^T \tilde{\Theta}(\alpha_s,\beta_s)\,ds \\
&\quad = \tfrac{1}{2}\,x'\,\Pi^{11}\,x + x'\,\Pi^{12}\,\alpha + x'\,\Pi^{13}\,\beta \\
&\qquad + \tfrac{1}{2}\,\langle\alpha,\,\Pi^{22}\,\alpha\rangle_{\mathcal{A}_{t,T}} + \langle\alpha,\,\Pi^{23}\,\beta\rangle_{\mathcal{A}_{t,T}} \\
&\qquad + \tfrac{1}{2}\,\langle\beta,\,\Pi^{33}\,\beta\rangle_{\mathcal{B}_{t,T}} \\
&\quad = \tfrac{1}{2}\,x'\,\mathcal{F}'(C - c_1\,(M^0)'\,M^0\,\mathcal{F}\,x \\
&\qquad + x'\,c_1\,\mathcal{F}'\,(M^0)'\,\alpha \\
&\qquad + x'\,c_2\,\mathcal{F}'\,(C - (\mathcal{M}^0)'\,M^0)\,\tilde{\mathcal{G}}\,\beta \\
&\qquad - \tfrac{1}{2}\,\langle\mu,\,\hat{C}\,\mu\rangle_{\mathcal{M}_{t,T}} + c_1\,c_2\,\langle\alpha,\,M^0\,\tilde{\mathcal{G}}\,\beta\rangle_{\mathcal{A}_{t,T}} \\
&\qquad + c_2^2\,\langle\beta,\,\tilde{\mathcal{G}}'\,(C - c_1\,(M^0)'\,M^0)\,\tilde{\mathcal{G}}\,\beta\rangle_{\mathcal{B}_{t,T}},
\end{aligned}$$

and

$$\begin{aligned}
\bar{B}_2 &= \big(\ (\Pi^{24})' \quad (\Pi^{34})'\ \big) \\
&= \mathcal{G}'\,\big(\ c_1\,(M_0)' \quad c_2\,(C - c_1\,(M^0)'\,M^0)\,\tilde{\mathcal{G}}\ \big), \\
\hat{B}_2 &= (\Pi^{14})' = \mathcal{G}'\,(C - c_1\,(\mathcal{M}^0)'\,M^0)\,\mathcal{F},
\end{aligned}$$

$$\bar{B}_3 = \Pi^{44} = \mathcal{G}'\,(C - c_1\,(M^0)'\,M^0)\,\mathcal{G} + I_m,$$

as required. ∎

We proceed by following the development of §4.2 in [10]. The results there do not apply directly because of the extra term $\langle\hat{B}_2 x, \nu\rangle$ in (16). Similar results hold nonetheless, allowing one to split the staticization over $\mathcal{V}$ and $\mathcal{M}$. We note that [10] requires $\bar{B}_3$ be boundedly invertible, which is required here. The inverse can be found explicitly by interpreting $\bar{B}_3$ as solving a two-point boundary value problem.

Lemma 7: $y = \bar{B}_3 v$ is given by

$$y_s = \begin{bmatrix} 0_n & \tilde{B}' \end{bmatrix} \begin{pmatrix} \eta_s^1 \\ \eta_s^2 \end{pmatrix} + \nu_s,$$

where we recall that $\tilde{B}$ denotes a symmetric square-root of symmetric diagonalizable $\tilde{\Gamma}$, and η is the unique solution to the TPBVP

$$\begin{aligned}
\dot{\eta}_s &= \begin{pmatrix} A & 0_n \\ c_1(M^0)'M^0 - C & A' \end{pmatrix} \eta_s + \begin{pmatrix} \tilde{B} \\ 0_n \end{pmatrix} \nu_s, \\
\eta_t^1 &= 0 = \eta_T^2.
\end{aligned}$$

We make the following final assumption.

Assumption 3: Letting $\tilde{\Phi}_{s,t}$ denote the state transition matrix of the LTI system

$$\tfrac{d}{ds}\tilde{\Phi}_{s,t} = \begin{pmatrix} A & -\tilde{\Gamma} \\ c_1(M^0)'M^0 - C & A' \end{pmatrix} \tilde{\Phi}_{s,t},$$

assume that the bottom right $n \times n$ block of $\tilde{\Phi}_{s,t}$ is invertible for all $0 \le t \le s \le T$, and that this inverse is bounded uniformly in $s, t \in [0,T]$.

Assumption 3 ensures that $\bar{B}_3$ is boundedly invertible, which leads to the following result, which is a variation of [10, Theorem 4.11].

Theorem 8: Fix $(t,x) \in [t,T] \times \mathbb{R}^n$. Let

$$\breve{\mathcal{M}}_{t,T}^x \doteq \{\mu \in \mathcal{M}_{t,T} \mid \operatorname*{stat}_{\nu\in\mathcal{V}_{t,T}} \breve{J}_t(x,\nu,\mu) \text{ exists}\}.$$

Suppose $\mathrm{stat}_{(\nu,\mu)\in\mathcal{V}_{t,T}\times\mathcal{M}_{t,T}} \breve{J}_t(x,\nu,\mu)$ exists. Then,

$$\operatorname*{stat}_{\mu\in\breve{\mathcal{M}}_{t,T}^x} \operatorname*{stat}_{\nu\in\mathcal{V}_{t,T}} \breve{J}_t(x,\nu,\mu) = \operatorname*{stat}_{(\nu,\mu)\in\mathcal{V}_{t,T}\times\mathcal{M}_{t,T}} \breve{J}_t(x,\nu,\mu).$$

As a corollary, we conclude that

$$\tilde{W}_t(x) = \breve{W}(x) = \operatorname*{stat}_{\mu\in\breve{\mathcal{M}}_{t,T}^x} \operatorname*{stat}_{\nu\in\mathcal{V}_{t,T}} \breve{J}_t(x,\nu,\mu). \tag{17}$$

IV. Reduction of the staticization problem

Consider the inner stat in (17). Given $\mu \in \breve{\mathcal{M}}_{t,T}^x$, let

$$W_t^\mu(x) \doteq \operatorname*{stat}_{\nu\in\mathcal{V}_{t,T}} \breve{J}_t(x,\nu,\mu).$$

Note that W^μ corresponds to the HJ PDE problem given by letting the right-hand side of (9) be evaluated at the given $\mu = (\alpha,\beta)$. That is, W^μ corresponds to HJ PDE problem

$$\begin{aligned}
0 = -\Big\{&\partial_t W_t(x) + \tfrac{1}{2}x'Cx + \nabla_x W_t(x)'Ax \\
&- \tfrac{1}{2}\nabla_x W_t(x)'\,\tilde{\Gamma}\,\nabla_x W_t(x) + c_2\nabla_x W_t(x)'L^0\beta_t
\end{aligned}$$

$$+\,\tilde\Theta(\mu_t) - \tfrac{c_1}{2}|M^0 x - \alpha_t|^2 - \tfrac{c_2}{2}|\beta_t|^2\Big\}, \qquad (18)$$

$$W_T(x) = 0, \quad x \in \mathbb{R}^n. \qquad (19)$$

HJ problem (18)–(19) corresponds to an LQR problem, for which verification results are standard, implying existence of a unique viscosity solution. Consequently, in (17),

$$\breve{\mathcal{M}}_{t,T}^x \equiv \mathcal{M}_{t,T}. \qquad (20)$$

The value function W^μ corresponding to this LQR problem may be obtained as follows.

Theorem 9: Given any $\mu \in \mathcal{M}_{t,T}$,

$$W_t^\mu(x) = \tfrac12 x' P_t x + x' q_t^\mu + \tfrac12 r_t^\mu \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n,$$

where P, q^μ, r^μ satisfy the following ODEs,

$$\dot P_t = -(C - c_1 (M^0)' M^0) - A' P_t - P_t A + P_t \tilde\Gamma P_t, \quad (21)$$

$$\dot q_t^\mu = (P_t^\mu \tilde\Gamma - A') q_t^\mu - c_1 (M^0)'\alpha_t - c_2 P_t L^0 \beta_t, \qquad (22)$$

$$\dot r_t^\mu = (q_t^\mu)'\tilde\Gamma q_t^\mu + \mu' \hat C \mu - 2[c_2 (q_t^\mu)' L^0 \beta_t + \tilde\Theta(\mu_t)], \quad (23)$$

with terminal condition $P_T = 0_{n\times n}$, $q_T^\mu = 0_{n\times 1}$ and $r_T^\mu = 0$, and $\hat C$ as given in (4).

Proof: It is well known that the viscosity solution to HJ PDE associated to an LQR problem is unique. Here, we only show that W^μ as given above satisfies (18), given $\mu = (\alpha,\beta)$. In particular, recalling (18),

$$\begin{aligned}0 = -\Big\{&\partial_s W_s^\mu(x) + \tfrac12 x' C x + \nabla_x W_s^\mu(x)' A x \\ &- \tfrac12 \nabla_x W_s^\mu(x)'\,\tilde\Gamma\,\nabla_x W_s^\mu(x) + c_2 \nabla_x W_s^\mu(x)' L^0 \beta_s \\ &+ \tilde\Theta(\mu_s) - \tfrac{c_1}{2}|M^0 x - \alpha_s|^2 - \tfrac{c_2}{2}|\beta_s|^2\Big\}.\end{aligned}$$

Direct substitution yields

$$\begin{aligned}&\partial_s W_s^\mu(x) \\ &= \tfrac12 x'(-(C - c_1 (M^0)'M^0) - A'P_s - P_s A + P_s \tilde\Gamma P_s)x \\ &\quad + x'((P_s\tilde\Gamma - A')q_s^\mu - c_1 (M^0)'\alpha_s - c_2 P_s L^0 \beta_s) \\ &\quad + \tfrac12\Big[(q_s^\mu)'\tilde\Gamma q_s^\mu + \mu_s'\hat C \mu_s - 2[c_2 (q_s^\mu)' L^0\beta_s + \tilde\Theta(\mu_s)]\Big],\end{aligned}$$

and $\nabla_x W_s^\mu(x) = P_s x + q_s^\mu$. Hence,

$$\begin{aligned}&\partial_s W_s^\mu(x) + \tfrac12 x' C x + \nabla_x W_s^\mu(x)' A x \\ &\quad - \tfrac12 \nabla_x W_s^\mu(x)'\,\tilde\Gamma\,\nabla_x W_s^\mu(x) + c_2 \nabla_x W_s^\mu(x)' L^0\beta_s \\ &\quad + \tilde\Theta(\mu_s) - \tfrac{c_1}{2}|M^0 x - \alpha_s|^2 - \tfrac{c_2}{2}|\beta_s|^2 \\ &= \tfrac12 x'(-(C - c_1(M^0)'M^0) - A'P_s - P_s A + P_s\tilde\Gamma P_s)x \\ &\quad + x'((P_s\tilde\Gamma - A')q_s^\mu - c_1(M^0)'\alpha_s - c_2 P_s L^0\beta_s) \\ &\quad + \tfrac12\Big[(q_s^\mu)'\tilde\Gamma q_s^\mu + \mu_s'\hat C\mu_s - 2[c_2(q_s^\mu)'L^0\beta_s + \tilde\Theta(\mu_s)]\Big] \\ &\quad + \tfrac12 x' C x + \nabla_x W_s^\mu(x)' A x - \tfrac12 \nabla_x W_s^\mu(x)'\,\tilde\Gamma\,\nabla_x W_s^\mu(x) \\ &\quad + [\tilde\Theta(\mu_s) - \tfrac{c_1}{2}|M^0 x - \alpha_s|^2 - \tfrac{c_2}{2}|\beta_s|^2 \\ &\qquad\quad + c_2 \nabla_x W_s^\mu(x)' L^0 \beta_s] \\ &= \tfrac12 x'(c_1 (M^0)'M^0)x \\ &\quad + x'(-c_1(M^0)'\alpha_s - c_2 P_s L^0\beta_s) \\ &\quad + \tfrac12\Big[\mu_s'\hat C\mu_s - 2[c_2(q_s^\mu)'L^0\beta_s]\Big] \\ &\quad - \tfrac{c_1}{2}|M^0 x - \alpha_s|^2 - \tfrac{c_2}{2}|\beta_s|^2 + c_2(P_s x + q_s^\mu)'L^0\beta_s \\ &= 0.\end{aligned}$$

■

Applying Theorem 9 and (20), the value function $\tilde W$ of the original optimal control problem (2) has the representation

$$\tilde W_t(x) = \tfrac12 x' P_t x + \operatorname*{stat}_{\mu\in\mathcal{M}_{t,T}}\{x' q_t^\mu + \tfrac12 r_t^\mu\}. \qquad (24)$$

In order to find the stat in (24), it suffices to evaluate the arg stat, which is where the Fréchet derivative of the operand is zero. We remark that P is the solution to the DRE (21), which is independent of μ, q^μ is the solution to an linear ODE, whereas r^μ is simply an integral.

In the following discussion, we assume the solution P has been found and we aim to evaluate $\tilde W$ at some given (t,x). Observe that

$$q_s^\mu = \int_s^T \bar\Phi_{s,\tau}(c_1(M^0)'\alpha_\tau + c_2 P_\tau L^0\beta_\tau)\,d\tau, \qquad (25)$$

$$r_s^\mu = -\int_s^T (q_\tau^\mu)'\tilde\Gamma q_\tau^\mu + \mu_\tau'\hat C\mu_\tau - 2[c_2(q_\tau^\mu)'L^0\beta_\tau + \tilde\Theta(\mu_\tau)]\,d\tau,$$

where $\bar\Phi_{s,\tau}$ is the state transition matrix associated with the linear ODE

$$\tfrac{d}{ds}\bar\Phi_{s,\tau} = (P_s\tilde\Gamma - A')\bar\Phi_{s,\tau}.$$

Since (25) is linear in μ, the Fréchet differentials associated with r_t^μ and $x' q_t^\mu$ are given by

$$\begin{aligned}&\Big\langle \tfrac{dr_t^\mu}{d\mu}(\mu), \Delta^\mu\Big\rangle_{\mathcal V} \\ &\quad = -2\int_t^T [\hat C\mu_s - \nabla_{(a,b)}\tilde\Theta(\mu_s) - c_2(L^0\mathcal{I}^2)' q_s^\mu]'\Delta_s^\mu\,ds \\ &\qquad - 2\int_t^T [\tilde\Gamma q_s^\mu - c_2 L^0\mathcal{I}^2\mu_s]'\int_s^T \bar\Phi_{s,\sigma}\bar D_\sigma\hat C\Delta_\sigma^\mu\,d\sigma\,ds, \\ &\Big\langle \tfrac{d}{d\mu}[x' q_t^\mu](\mu), \Delta^\mu\Big\rangle_{\mathcal V} = \int_t^T x'\bar\Phi_{t,\tau}\bar D_\tau \hat C\Delta_\tau^\mu.\end{aligned}$$

for all $\Delta^\mu \in \mathcal{M}_{t,T}$, in which $\bar D_s \doteq [(M^0)', P_s L^0]$ and $\mathcal{I}^2 \doteq [0_{l\times k}, I_l]$. Consequently, by Theorem 1, we see that $\mu \in \operatorname{arg\,stat}_{\mu\in\mathcal{M}_{t,T}}\{\langle x, q_t^\mu\rangle + \tfrac12 r_t^\mu\}$ iff for all $\Delta^\mu\in\mathcal{M}_{t,T}$,

$$\begin{aligned}0 = &\int_t^T [-\hat C\mu_s + \nabla_{(a,b)}\tilde\Theta(\mu_s) + c_2(L^0\mathcal{I}^2)' q_s^\mu \\ &\qquad\qquad + \hat C(\bar\Phi_{t,s}\bar D_s)' x]'\Delta_s^\mu\,ds \\ &+ \int_t^T [-\tilde\Gamma q_s^\mu + c_2 L^0\mathcal{I}^2\mu_s]'\int_s^T \bar\Phi_{s,\sigma}\bar D_\sigma\hat C\Delta_\sigma^\mu\,d\sigma\,ds\end{aligned}$$

which, by considering the Hilbert adjoint in the double integral and rearranging, holds for all $\Delta^\mu\in\mathcal{M}_{t,T}$ iff

$$\begin{aligned}0 = &-\hat C\mu_s + \nabla_{(a,b)}\tilde\Theta(\mu_s) + c_2(L^0\mathcal{I}^2)' q_s^\mu + \hat C(\bar\Phi_{t,s}\bar D_s)' x \\ &+ \int_t^s \hat C(\bar\Phi_{\tau,s}\bar D_s)'(-\tilde\Gamma q_\tau^\mu + c_2 L^0\mathcal{I}^2\mu_\tau)\,d\tau, \qquad (26)\end{aligned}$$

a.e. $s\in(t,T)$.

V. A SPECIFIC NUMERICAL METHOD

We consider a reduced-complexity case for development of the numerical approach. Specifically take $L^0 = 0$, $\tilde{\Gamma} = I_n$, $\hat{C} = c_1 I_k$ and $t = 0$. Then, (26) reduces to

$$c_1\alpha_s - \nabla\tilde{\Theta}(\alpha_s) = c_1 M^0 \bar{\Phi}'_{0,s} x - c_1 M^0 \int_0^s \bar{\Phi}'_{\tau,s} q^\mu_\tau \, d\tau. \quad (27)$$

Denote $\eta(\alpha) \doteq \alpha - \frac{1}{c_1}\nabla\tilde{\Theta}(\alpha)$, $\sigma \wedge s \doteq \min(\sigma, s)$, and

$$\bar{\mathcal{D}}_{\sigma,s} \doteq \int_0^{\sigma\wedge s} \bar{\Phi}'_{\tau,s}\bar{\Phi}_{\tau,\sigma}\, d\tau,$$

Using $\eta, \bar{\mathcal{D}}$ along with (25), we may now rewrite (27) as

$$\begin{aligned}\eta(\alpha_s) &= M^0\bar{\Phi}'_{0,s}x - M^0\int_0^s \bar{\Phi}'_{\tau,s} q^\mu_\tau\, d\tau \\ &= M^0\bar{\Phi}'_{0,s}x - c_1 M^0 \int_0^s \bar{\Phi}'_{\tau,s}\int_\tau^T \bar{\Phi}_{\tau,\sigma}(M^0)'\alpha_\sigma\, d\sigma\, d\tau \\ &= M^0\bar{\Phi}'_{0,s}x - c_1 M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'\alpha_\sigma\, d\sigma. \quad (28)\end{aligned}$$

For sufficiently short time horizons, α can be found using fixed point iterations as follows. We state the following regularity result regarding η without proof.

Theorem 10: η is invertible and Lipschitz continuous with Lipschitz constant 2. The inverse of η is C^1 with Lipschitz constant 2. In particular, $(\nabla\eta(\mu))^{-1}$ is bounded by 2 and Lipschitz in μ with Lipschitz constant 4.

We now rearrange (28) as a fixed point iteration:

$$\alpha = \eta^{-1}\left(M^0\bar{\Phi}'_{0,s}x - c_1 M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'\alpha_\sigma\, d\sigma\right). \quad (29)$$

Theorem 11: For all sufficiently small T, there exists a unique solution $\alpha \in C^1([0,T];\mathbb{R}^k) \subset \mathcal{A}_{0,T}$ to (29).

Proof: Define $\mathcal{F} : C([0,T];\mathbb{R}^k) \to C([0,T];\mathbb{R}^k)$ by

$$\mathcal{F}(\alpha) \doteq \eta^{-1}\left(M^0\bar{\Phi}'_{0,s}x - c_1 M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'\alpha_\sigma\, d\sigma\right).$$

Then, for any $\alpha^1, \alpha^2 \in C([0,T];\mathbb{R}^k)$, by Thm. 10,

$$\begin{aligned}&\|\mathcal{F}(\alpha^2 - \alpha^1)\|_\infty \\ &\le 4\left\| c_1 M^0 \int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'(\alpha^2_\sigma - \alpha^1_\sigma)\, d\sigma \right\|_\infty \\ &\le 4|c_1| |M^0|^2 \sup_{\sigma,s\in[0,T]} |\bar{\mathcal{D}}_{\sigma,s}| \|\alpha^2 - \alpha^2\|_\infty T.\end{aligned}$$

We see that the Lipschitz constant depends linearly on T. By Banach fixed-point theorem, there exists some α such that $\alpha = \mathcal{F}[\alpha]$. Since the range of $\mathcal{F}$ is within $C^1([0,T];\mathbb{R}^k)$, it follows that $\alpha \in C^1([0,T];\mathbb{R}^k)$. ■

Although for longer time durations the fixed point iterations may not converge in general, we may use fixed point iteration to obtain an estimate solution to (29), and propagate the estimate to the true solution. Formally, suppose one would like to solve (29) for some large $\bar{T}$. We index α obtained by fixed point iteration by T and write α^T. Each α^T is still defined on $[0, \bar{T}]$.

Differentiating (28) w.r.t. T yields

$$\begin{aligned}&\nabla\eta(\alpha^T_s)\frac{\mathrm{d}\alpha^T_s}{\mathrm{d}T} \\ &= -c_1 M^0\bar{\mathcal{D}}_{T,s}(M^0)'\alpha^T_T - c_1 M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'\frac{\mathrm{d}\alpha^T_\sigma}{\mathrm{d}T}d\sigma,\end{aligned}$$

which may be rearranged to obtain

$$\begin{aligned}&\frac{\mathrm{d}\alpha^T_s}{\mathrm{d}T} + c_1[\nabla\eta(\alpha^T_s)]^{-1}M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'\frac{\mathrm{d}\alpha^T_\sigma}{\mathrm{d}T}d\sigma \\ &= -c_1[\nabla\eta(\alpha^T_s)]^{-1}M^0\bar{\mathcal{D}}_{T,s}(M^0)'\alpha^T_T. \quad (30)\end{aligned}$$

Define

$$K(g; c_1, \alpha) \doteq [\nabla\eta(\alpha^T_s)]^{-1}M^0\int_0^T \bar{\mathcal{D}}_{\sigma,s}(M^0)'g\, d\sigma.$$

We note that K depends on c_1 through $\bar{\mathcal{D}}$ (ultimately through P), whereas K depends on α through $\nabla\eta$, which is globally bounded.

One may interpret (30) as an ODE with state variable $\alpha^\cdot$, as in

$$(I + c_1 K(\cdot; c_1, \alpha))\frac{\mathrm{d}\alpha^T}{\mathrm{d}T} = -c_1 M^0\bar{\mathcal{D}}_{T,s}(M^0)'\alpha^T_T.$$

In particular, if c_1 is sufficiently small such that $|c_1|\sup_\alpha \|K(\cdot; c_1, \alpha)\| < 1$, then

$$\frac{\mathrm{d}\alpha^T}{\mathrm{d}T} = -(I + c_1 K(\cdot; c_1, \alpha))^{-1}c_1 M^0\bar{\mathcal{D}}_{T,s}(M^0)'\alpha^T_T. \quad (31)$$

Once an initial condition α is obtained (from fixed-point iteration for small T), the solution may then be propagated until the desired time T. Even though $I + c_1 K$ is an linear operator on functions, the inverse of which may not have a closed form representation, its inversion is tractable in practice if time is discretized. In particular, for small c_1, this simply requires inverting a diagonally dominant matrix computationally.

Hence we obtain the following algorithm for evaluating the value function at any point (t, x).

1) Discretize the time interval $[t, T]$; denote the grid points by $(t_k)_{k=1}^N$;
2) Compute (analytically or computationally) η (or η^{-1}) as a function;
3) Precompute Φ_{t_i,t_j} and $\bar{\mathcal{D}}_{t_i,t_j}$ for all $i, j = 1, \cdots, N$;
4) Choose some small integer $i \le N$. Initialize α^i to be the zero vector (or any vector) of length N;
5) Iterate using $\alpha^i = \mathcal{F}(\alpha^i)$ with t_i in place of T until desired accuracy;
6) Propagate using (31) from t_i to terminal time T with initial condition $\alpha^{t_i} = \alpha^i$;
7) Compute q and r using (25);
8) Compute $W(t, x)$ using (24).

If the value function needs to be evaluated at more than one point in the state space, we note that the computed results in 2) and 3) may be reused.

VI. Example

We briefly outline an example with state-space dimension five. The system is linear/quadratic with the exception of a one-dimensional nonlinearity in running cost. The dynamics are given by

$$A = \begin{pmatrix} 0 & 1.5 & 0 & 1 & 0 \\ -1 & 0 & 0 & -0.5 & 0.5 \\ 1.5 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $\sigma = I_5$, and $T - t = 0.5$. The quadratic term in the running cost is defined by $C = 0.6 I_5$. The nonlinearity direction is specified by $M^0 = (0, 1, 0, 0, 0)$, and the stat-quad dual of the nonlinearity is $\tilde{\Theta}(\alpha) = 0.5 \sin(2\alpha)$, where the dualizing coefficient is $c_1 = -3$. This corresponds to an ℓ term as shown in Fig. 1.

Using the approach described here, one may compute the solution on any plane (using approximately 1500 grid points) in under a minute with a typical laptop. This should be compared with the extreme computational cost of solving an HJ PDE over five-dimensional space. Figure 2 depicts the value function over the first two axes. Figure 3 contains the PDE back-substitution errors divided by the sum of the absolute values of the terms in the PDE.

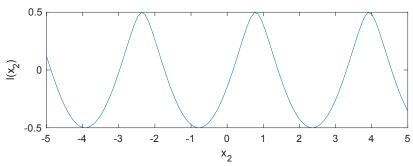

Fig. 1. Plot of $\ell(M^0 x)$.

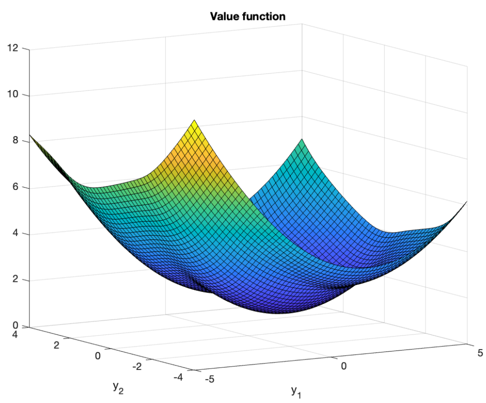

Fig. 2. Value function.

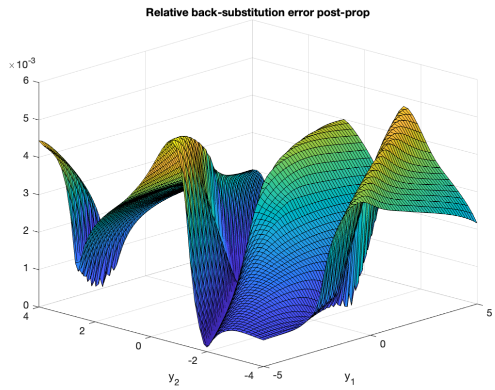

Fig. 3. Relative back-substitution errors.

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