Max-Plus Eigenvector Representations for Solution of Nonlinear H_∞ Problems: Basic Concepts

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Abstract

The H_∞ problem for a nonlinear system is considered. The corresponding dynamic programming equation is a fully nonlinear, first-order, steady-state partial differential equation (PDE), possessing a term which is quadratic in the gradient. The solutions are typically nonsmooth, and further, there is non-uniqueness among the class of viscosity solutions. In the case where one tests a feedback control to see if it yields an H_∞ controller, the PDE is a Hamilton-Jacobi-Bellman equation. In the case where the "optimal" feedback control is being determined as well, the problem takes the form of a differential game, and the PDE is, in general, an Isaacs equation. The computation of the solution of a nonlinear, steady-state, first-order PDE is typically quite difficult. In this paper, we develop an entirely new class of methods for the obtaining the "correct" solution of such PDEs. These methods are based on the linearity of the associated semi-group over the max-plus (or, in some cases, min-plus) algebra. In particular, solution of the PDE is reduced to solution of a max-plus (or min-plus) eigenvector problem for known unique eigenvalue 0 (the max-plus multiplicative identity). It is demonstrated that the eigenvector is unique, and that the power method converges to it. An example is included.

Key words: Nonlinear H_∞ control, dynamic programming, numerical methods, partial differential equations, max-plus algebra.

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1 Introduction

We consider the H_∞ problem for a nonlinear system. The corresponding dynamic programming equation (DPE) is a fully nonlinear, first-order, steady-state partial differential equation (PDE), possessing a term which is quadratic in the gradient (for background, see [1], [3], [19], [36] among many notable others). The solutions are typically nonsmooth, and further, there are multiple viscosity solutions – that is, one does not even have uniqueness among the class of viscosity solutions (cf. [30], [31]). In the (design) case where one tests a feedback control to see if it yields an H_∞ controller, the PDE is a Hamilton-Jacobi-Bellman (HJB) equation. In the case where the "optimal" feedback control is being determined as well, the problem takes the form of a differential game, and the PDE is, in general, an Isaacs equation. The computation of the solution of a nonlinear, steady-state, first-order PDE is typically quite difficult, and possibly even more so in the presence of the non-uniqueness mentioned above. Some previous works in the general area of numerical methods for these problems are [4], [8], [9], [16], [17], [22], and the references therein. In this paper, we take the first steps toward an entirely new class of methods for the solution of such PDEs. The methods will be based on the linearity of the associated semi-group over the max-plus (or, in some cases, min-plus) algebra.

The Robust/H_∞ filter ([29], [13], [15], see also [5], [21], [6]) has an associated time-dependent, fully nonlinear, first-order PDE. The solution of this PDE is propagated forward in time by the associated semi-group. In [14], the linearity of this semi-group *over the max-plus algebra* was noted, and provided a key ingredient in the development of a numerical algorithm for this filter. This linearity had previously been noted in [23]. A second key ingredient was the development of an appropriate basis for the solution space over the max-plus algebra (i.e. with the max-plus algebra replacing the standard underlying field). This reduced the problem of propagation of the solution of the PDE forward in time to max-plus matrix-vector multiplication (with dimension being that of the number of basis functions being used).

Returning to the H_∞ problem, let us focus on the design case – where a feedback control and disturbance attenuation parameter are chosen, and the associated PDE is solved to determine whether this is indeed an H_∞ controller with that disturbance attenuation level. (If there is a non-negative, locally bounded solution which is zero at the origin, then it is such an H_∞ controller; see for instance [1], [35].) As noted above, the PDE is fully nonlinear, steady-state and first-order, possessing a term which is quadratic in the gradient. In fact, the Hamiltonian is concave in the gradient variable. For instance, the PDE might take the form

$$0 = H(x, \nabla W) = -\left[\frac{1}{2\gamma^2}(\nabla W)^T a(x) \nabla W + (f(x))^T \nabla W + l(x)\right]$$

where the notation will be described further below. There are typically multiple solutions of such PDEs. In the linear-quadratic case, two of these solutions correspond to the stable and anti-stable manifolds associated with the Hamiltonian. The "correct" solution (i.e.

the one corresponding to the available storage or value) was characterized in [35] as the smallest, non-negative viscosity solution which is zero at the origin. In [31], [30], for the class of problems considered here, a specific quadratic growth bound was given which isolated this correct solution as the unique, non-negative solution satisfying $0 \leq W(x) \leq C|x|^2$ for a specific C depending on the problem data.

In this paper, we will notice that the solutions of this PDE are fixed points of the associated semi-group, that is

$$W = S_\tau[W] \tag{1}$$

where S_τ is the semi-group with time-step τ. In fact, one can view the PDE as the infinitesimal version of this fixed-point equation.

The max-plus algebra is a commutative semi-field over $\mathbf{R} \cup \{-\infty\}$ with the addition and multiplication given by

$$\begin{aligned} a \oplus b &= \max\{a, b\}, \\ a \otimes b &= a + b \end{aligned} \tag{2}$$

where the operations are defined for $-\infty$ in the obvious way. Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. Note that it is not a field since the additive inverses are missing. Roughly speaking, it can be extended to include additive inverses [2], but we do not need that here. Note that since 0 is the multiplicative identity, we can rewrite (1) as

$$0 \otimes W = S_\tau[W]. \tag{3}$$

We will also show that S_τ is linear over the max-plus algebra. With this in mind, we can then think of W as an infinite-dimensional eigenvector (or eigenfunction) for S_τ corresponding to eigenvalue 0. If one approximates W by some finite-dimensional vector of coefficients in a max-plus basis expansion, then (3) can be re-cast as a finite-dimensional max-plus eigenvector equation (approximating the true solution). Thus, the nonlinear PDE problem is reduced to the solution of a (max-plus) linear eigenvector problem. This will be the key result which will be demonstrated here. The more general result is as follows. One can take a finite number of basis functions and solve the above problem with this finite set. Further, one can show that as the number of basis functions used increases (in a reasonable way), the solutions of these finite approximate problems converge to the true value function, W. The technical arguments supporting this convergence result are quite substantial, and would make this paper overly long. Consequently, this analysis will be delayed to a later paper [28]; see also [25], [24] for the essential points.

We will also examine briefly, a sub-class of problems with active control, that is where the feedback is unknown. In this case, the problem takes the form of a differential game, and the corresponding PDE is, in general, an Isaacs equation. Here, we will consider the case where the controller dominates the disturbance in the sense that the Hamiltonian will be convex in the gradient variable. In this case, there will be an equivalent control formulation, and the semi-group will be linear over the min-plus algebra. (In this case, the maximization in (2) is replaced by minimization, and the algebra is defined over $\mathbf{R} \cup \{+\infty\}$.) Analogous results to the max-plus case will hold.

The purpose of this paper is the rapid dissemination, in journal article form, of the main ideas in this new approach. Consequently, as discussed above, further results regarding details of the numerical methods, convergence proofs and error bounds are delayed to the forthcoming companion paper [28] (although one may note that [24], [25] contain the main points).

Portions of this paper have appeared previously in [18], [25], [26] and [27].

2 Background on existence and uniqueness

In this section, we define the system and provide some basic results which we will need in the main part of the paper. The results in this section are not the focus of the paper, but are simply some particular previous results that can be used as a precursor to the new material to follow in later sections.

We consider the system

$$\begin{aligned} &\dot{X} = f(X) + \sigma(X)w \\ &X(0) = x \end{aligned} \tag{4}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T] \ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are not necessary but are sufficient for the results to follow. No attempt has been made at this point to formulate tight assumptions. In particular, in order to provide some clear sketches of proofs, we will assume that all the functions f, σ and l (given below) are smooth, although that is not required for the results. We will assume that there exist $K, c, M \in (0,\infty)$ such that

$$\begin{aligned} &(x-y)^T(f(x)-f(y)) \le -c|x-y|^2 \qquad \forall\, x,y \in \mathbf{R}^n \\ &f(0) = 0 \\ &|f_x(x)| \le K \qquad \forall\, x \in \mathbf{R}^n \end{aligned} \tag{$A1$}$$

Note that this automatically implies the closed–loop stability criterion of H_∞ control. We assume that

$$\begin{aligned} &|\sigma(x)| \le M \qquad \forall\, x \in \mathbf{R}^n \\ &|\sigma^{-1}(x)| \le M \qquad \forall\, x \in \mathbf{R}^n \\ &|\sigma_x(x)| \le K \qquad \forall\, x \in \mathbf{R}^n \end{aligned} \tag{$A2$}$$

Here, we of course use σ^{-1} to indicate the Moore-Penrose pseudo-inverse, and it is implicit in the bound on $\sigma^{-1}(x)$ that σ is uniformly nondegenerate.

Let $l(x)$ be the running cost (to which the L_2–norm disturbance penalty will be added). We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{array}{ll} l_{xx}(x) \leq \beta & \forall\, x \in \mathbf{R}^n \\ 0 \leq l(x) \leq \alpha |x|^2 & \forall\, x \in \mathbf{R}^n. \end{array} \tag{A3}$$

(There is a reason for allowing β to be greater than 2α, which one might notice; see [30].)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function (again, also referred to as the value function in the sequel), $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T < \infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2} |w(t)|^2 \, dt. \tag{5}$$

The corresponding DPE is

$$\begin{aligned} 0 &= -\sup_{w \in \mathbf{R}^m} \left\{ [f(x) + \sigma(x) w]^T \nabla W + l(x) - \tfrac{\gamma^2}{2} |w|^2 \right\} \\ &= -\left[\tfrac{1}{2\gamma^2} (\nabla W)^T \sigma(x) \sigma^T(x) \nabla W + f^T(x) \nabla W + l(x) \right]. \end{aligned} \tag{6}$$

Since W itself does not appear in (6), one can always scale by an additive constant. It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \tag{A4}$$

We note that there are examples where $(A4)$ fails and the available storage is ∞. Then one has the following result ([31],[30]). The proof can be found in the references.

Theorem 2.1 *There exists a unique continuous viscosity solution of (6) in the class*

$$0 \leq \widetilde{W}(x) \leq c \frac{(\gamma - \delta)^2}{2M^2} |x|^2 \tag{7}$$

for sufficiently small $\delta > 0$.

We also have the following. Consider the finite time horizon control problem with dynamics (4) and payoff and value given by

$$\begin{aligned} J(T, x, w) &= \textstyle\int_0^T l(X(t)) - \frac{\gamma^2}{2} |w(t)|^2 \, dt \\ V(T, x) &= \sup_{w \in \mathcal{W}} J(T, x, w). \end{aligned} \tag{8}$$

Theorem 2.2 *The unique continuous viscosity solution of the previous theorem is given by*

$$\begin{aligned} W(x) &= \lim_{T\to\infty} V(T,x) \\ &= \sup_{T<\infty} V(T,x) \end{aligned} \tag{9}$$

which is also equivalent to representation (5).

The proof of this can also be found in the references ([31], [30]).

3 Max–Plus linearity of the semi–group

We will show that W is a fixed point of the corresponding semi–group (solution operator). The semi–group is defined directly by the dynamic programming principle (DPP) for the finite time–horizon problem. We begin with the following lemma which is a statement of the DPP for this particular problem.

Lemma 3.1 *Let V be given by (8). Then,*

$$V(T,x) = \sup_{w\in\mathcal{W}} \left\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + V(T-\tau, X(\tau)) \right\}$$

(where X satisfies (4)) for any $0 \le \tau \le T < \infty$ and any $x \in \mathbf{R}^n$.

Since the proof of Lemma 3.1 is quite straight–forward, we do not include it. Similar proofs can be found in many of the standard references of the field. The particular proof in this case appears in [27].

Define the space of semi–convex functions, $\mathcal{S}$, as the set of $\psi : \mathbf{R}^n \to \mathbf{R}$ such that for any $R < \infty$ there exists $C_R < \infty$ such that $\psi(x) + \frac{C_R}{2}|x|^2$ is convex over $B_R = \{x \in \mathbf{R}^n : |x| \le R\}$. We refer to such a C_R as a semi–convexity constant for ψ over B_R. Note that any $\psi \in \mathcal{S}$ is automatically locally Lipschitz [12].

Define the semi–group $S_\tau[W]$ for W in the domain of S_τ, $\mathrm{Dom}[S_\tau] \doteq \{W \in \mathcal{S} : S_\tau[W(\cdot)](x) < \infty\ \forall x \in \mathbf{R}^n\}$, by $S_\tau[W(\cdot)](x) = \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\}$ where X satisfies (4). We note that $\mathrm{Dom}[S_\tau]$ includes all semiconvex functions satisfying (7).

Theorem 3.2 *For any $\tau \in [0,\infty)$, W given by (9) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class (7).*

PROOF. We begin with the first assertion. From (9), one has

$$W(x) = \lim_{T\to\infty} V(T,x). \tag{10}$$

Let $T_2 > T_1$. Let w^ε be ε–optimal for (8) with time–horizon T_1. Let w be given by

$$w(t) = \begin{cases} w^\varepsilon(t) & \text{if } t \in [0,T_1] \\ 0 & \text{if } t \in (T_1,T_2] \end{cases}$$

Then, using $(A3)$

$$V(T_1,x) \le J(T_1,x,w^\varepsilon) + \varepsilon \le J(T_2,x,w) + \varepsilon \le V(T_2,x) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $V(\cdot,x)$ is monotonically increasing. Using this monotonicity and (10), one has

$$W(x) = \sup_{T<\infty} V(T,x)$$

which by Lemma 3.1

$$= \sup_{T<\infty}\sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + V(T-\tau,X(\tau))\}$$

where X satisfies (4)

$$= \sup_{w\in\mathcal{W}}\sup_{T}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + V(T-\tau,X(\tau))\}$$

which by (10) again

$$\begin{aligned} &= \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\} \\ &= S_\tau[W(\cdot)](x). \end{aligned}$$

The proof of uniqueness is nearly identical to the uniqueness proof of Theorem 2.5 in [30] (see also [31]), and so we only sketch the main points. Let $\overline{W}$ satisfy $\overline{W} = S_\tau[\overline{W}]$. One shows that $\overline{W}(x) \le W(x)$ and $\overline{W}(x) \ge W(x)$ for all $x \in \mathbf{R}^n$. The first of these two inequalities is more technical, and we address that first. To achieve the first inequality, one considers $S_\tau^n[\overline{W}](x) = S_{n\tau}[\overline{W}](x) = \sup_{w\in\mathcal{W}}\{\int_0^{n\tau} l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \overline{W}(\tau,X(\tau))\}$ for any n, and shows that for $\varepsilon \in (0,1)$, there is a bound on ε–optimal w^ε which is independent of n. The argument relies on the bounds in (7) and Assumption $(A1)$. Then using Assumption $(A1)$ again, one shows that one can find arbitrarily large terminal times, $n\tau$, and corresponding w^ε such that the state at termination, $X^\varepsilon(n\tau)$ is arbitrarily close to the origin. Using (7) again, one sees that the terminal cost in these cases, $\overline{W}(X^\varepsilon(n\tau))$ is arbitrarily close to zero. Using this, one shows that $\overline{W}(x) \le \sup_{w\in\mathcal{W}}\sup_{T<\infty}\{\int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt\} = W(x)$. Again, we refer the reader to [30] for the details.

For the reverse inequality, note that for arbitrarily large integer n,

$$\begin{aligned} V(n\tau, x) &= \sup_{w\in\mathcal{W}} \Big\{ \int_0^{n\tau} l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt \Big\} \\ &\le \sup_{w\in\mathcal{W}} \Big\{ \int_0^{n\tau} l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \overline{W}(X(\tau)) \Big\} \\ &= S_{n\tau}[\overline{W}](x) = S_\tau^n[\overline{W}](x) = \overline{W}(x). \end{aligned} \tag{11}$$

On the other hand, by Theorem 2.2,

$$V(T, x) \to W(x). \tag{12}$$

Then (11) and (12) imply $\overline{W}(x) \ge W(x)$ (otherwise one obtains a contradiction). Thus, one has the two inequalities which imply that any solution of $\overline{W} = S_\tau[\overline{W}]$ must be W. □

Note that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve. This is of interest in the actual construction of numerical algorithms, but we will not follow up on the point in this paper.

Now we will demonstrate that S_τ is linear in the max–plus algebra. Let $\psi, \phi \in \mathrm{Dom}[S_\tau]$. For $c \in \mathbf{R}$, the proof that max–plus multiplication passes through the operator is trivial:

$$S_\tau[c \otimes \psi](x) = S_\tau[c + \psi](x) = c + S_\tau[\psi](x) = c \otimes S_\tau[\psi](x) \tag{13}$$

for all $x \in \mathbf{R}^n$. In the case that $c = -\infty$, one has

$$\begin{aligned} S_\tau[-\infty \otimes \psi](x) &= \sup_{w\in\mathcal{W}} \Big\{ \int_0^{\tau} l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \psi(X(\tau)) - \infty \Big\} \\ &= \sup_{w\in\mathcal{W}} \{-\infty\} = -\infty = -\infty \otimes S_\tau[\psi](x) \end{aligned}$$

for all $x \in \mathbf{R}^n$. Now considering max–plus addition, we note that

$$\begin{aligned} S_\tau[\phi \oplus \psi](x) &= \sup\nolimits_{w\in\mathcal{W}} \Big\{ \max[\phi(X(\tau)), \psi(X(\tau))] + \textstyle\int_0^\tau l(X(r)) - \frac{\gamma^2}{2}|w(r)|^2\,dr \Big\} \\ &\ge \sup\nolimits_{w\in\mathcal{W}} \Big\{ \phi(X(\tau)) + \textstyle\int_0^\tau l(X(r)) - \frac{\gamma^2}{2}|w(r)|^2\,dr \Big\} \\ &= S_\tau[\phi](x) \end{aligned} \tag{14}$$

By symmetry, one obtains an analogous inequality for ψ. Consequently,

$$S_\tau[\phi \oplus \psi](x) \ge \max\{S_\tau[\phi](x), S_\tau[\psi](x)\} = S_\tau[\phi](x) \oplus S_\tau[\psi](x). \tag{15}$$

Now let w^ε be ε–optimal in the first equation of (14) (with corresponding X^ε). Then

$$S_\tau[\phi \oplus \psi](x) \le \max[\phi(X^\varepsilon(\tau)), \psi(X^\varepsilon(\tau))] + \int_0^\tau l(X^\varepsilon(r)) - \frac{\gamma^2}{2}|w^\varepsilon(r)|^2\,dr + \varepsilon \tag{16}$$

and so, in the case where $\phi(X^\varepsilon(\tau)) \geq \psi(X^\varepsilon(\tau))$

$$\begin{aligned} S_\tau[\phi \oplus \psi](x) &\leq \phi(X^\varepsilon(\tau)) + \textstyle\int_0^\tau l(X^\varepsilon(r)) - \frac{\gamma^2}{2}|w^\varepsilon(r)|^2\, dr + \varepsilon \\ &\leq S_\tau[\phi](x) + \varepsilon \end{aligned} \tag{17}$$

Considering also the case where $\phi(X^\varepsilon(\tau)) < \psi(X^\varepsilon(\tau))$ and taking $\varepsilon > 0$ arbitrarily small, one obtains

$$S_\tau[\phi \oplus \psi](x) \leq S_\tau[\phi](x) \oplus S_\tau[\psi](x).$$

Combining this with (15) yields

$$S_\tau[\phi \oplus \psi](x) = S_\tau[\phi](x) \oplus S_\tau[\psi](x). \tag{18}$$

By (13) and (18), one has

Theorem 3.3 *The semi–group, S_τ, is linear in the max–plus algebra.*

4 Semi–convexity of the value and a max–plus basis

In this section, we first show that the value, W, lies in the space of semi–convex functions, and then we describe the max–plus basis for this space that we will use in the following sections. We need the following lemma; for a proof, see [30], [31].

Lemma 4.1 *Given $\varepsilon \in (0,1)$, $R < \infty$, and $|x| < R$, let $w^{\varepsilon,T}$ be ε–optimal for problem (8). Then, there exists $M_R < \infty$ (where this is independent of ε, $|x| \leq R$ and $T \in (0,\infty)$) such that*

$$\|w^{\varepsilon,T}\|_{L_2(0,T)} \leq M_R.$$

Lemma 4.2 *Given $\varepsilon \in (0,1)$, $R < \infty$, and $|x| < R$, let $w^{\varepsilon,T}$ be ε–optimal for problem (8). Then, there exists $D_R < \infty$ (where this is independent of ε, $|x| \leq R$ and $T \in (0,\infty)$) such that*

$$|X(t)| \leq D_R \qquad \forall\, t \in [0,T].$$

Proof. The proof follows directly from Lemma 4.1 and the assumptions, particularly the contractivity of f. Specifically, let X satisfy (4) with initial $|X(0)| = |x| \leq R$ and ε–optimal disturbance, $w^{\varepsilon,T}$. Then,

$$\frac{d}{dt}[|X(t)|^2] = 2X^T f(X) + 2X^T \sigma(X) w^{\varepsilon,T}$$

which by Assumptions $(A1)$ and $(A2)$

$$\leq -2c|X|^2 + 2M|X||w^{\varepsilon,T}|$$
$$\leq -c|X|^2 + \frac{M^2}{c}|w^{\varepsilon,T}|^2.$$

Integrating, one finds

$$|X(t)|^2 \leq |x|^2 e^{-ct} + \frac{M^2}{c}\int_0^t e^{c(r-t)}|w^{\varepsilon,T}(r)|^2\,dr$$

which by Lemma 4.1

$$\leq |x|^2 + \frac{M^2}{c}M_R^2$$

which yields the result with $D_R \doteq [R^2 + (M^2/c)M_R^2]^{\frac{1}{2}}$.

Theorem 4.3 *$V(T,x)$ is semi-convex with constants independent of $T \in [0,\infty)$.*

PROOF. Let $x \in B_R(0)$, $u \in \mathbf{R}^n$ and $|u| = 1$. It is sufficient to show that the second differences of $V(T,\cdot)$ are bounded from below on any $B_R(0)$ by some $-C_R$.

Let $\delta, \varepsilon \in (0,1)$. Let w be ε–optimal for (8) with corresponding $X_0(\cdot)$ satisfying (4) with $X_0(0) = x$. Then

$$\begin{aligned} V(T,x-\delta u) - 2V(T,x) + V(T,x+\delta u) \geq{}& J(T,x-\delta u, w+\Delta) - 2J(T,x,w) \\ &+J(T,x+\delta u,w) - 2\varepsilon \end{aligned} \tag{19}$$

where Δ will be given below. Let $X_{-\delta}$, X_δ satisfy the dynamics of (4) with initial conditions $X_{-\delta}(0) = x - \delta u$ and $X_\delta(0) = x + \delta u$. Also let the corresponding disturbance processes be $w + \Delta$ and w, respectively. Then,

$$\dot{X}_\delta - \dot{X}_0 = f(X_\delta) - f(X_0) + (\sigma(X_\delta) - \sigma(X_0))w,$$

and

$$\dot{X}_0 - \dot{X}_{-\delta} = f(X_0) - f(X_{-\delta}) + \sigma(X_0)w - \sigma(X_{-\delta})(w+\Delta).$$

Now we choose

$$\begin{aligned} \Delta &\doteq -\sigma^{-1}(X_{-\delta})\{f(X_\delta) - f(X_0) + (\sigma(X_\delta) - \sigma(X_0))w - (f(X_0) - f(X_{-\delta})) \\ &\qquad -(\sigma(X_0) - \sigma(X_{-\delta}))w\} \\ &= -\sigma^{-1}(X_{-\delta})\{f(X_\delta) - 2f(X_0) + f(X_{-\delta}) + (\sigma(X_\delta) - 2\sigma(X_0) + \sigma(X_{-\delta}))w\} \end{aligned} \tag{20}$$

(Although Δ is defined by the above feedback formula, the corresponding Δ as a function of t is used.) By substitution, we see

$$\dot{X}_\delta - \dot{X}_0 = \dot{X}_0 - \dot{X}_{-\delta},$$

and since $X_\delta(0) - X_0(0) = \delta u$ and $X_0(0) - X_{-\delta}(0) = \delta u$,

$$X_\delta(t) - X_0(t) = X_0(t) - X_{-\delta}(t) \qquad \forall\, t \geq 0. \tag{21}$$

Consequently, by (19) and (21)

$$\begin{aligned}
V(T, x-\delta u) - 2V(T,x) + V(T, x+\delta u) &\geq \int_0^T l(X_{-\delta}) - 2l(X_0) + l(X_\delta)\, dt \\
&\qquad -\frac{\gamma^2}{2}\int_0^T |w+\Delta|^2 - 2|w|^2 + |w|^2\, dt - 2\varepsilon \\
&= \int_0^T l(X_0(t) - [X_\delta(t) - X_0(t)]) - 2l(X_0) + l(X_0(t) + [X_\delta(t) - X_0(t)])\, dt \\
&\qquad -\frac{\gamma^2}{2}\int_0^T 2w^T\Delta + |\Delta|^2\, dt - 2\varepsilon.
\end{aligned}$$

By Assumption $(A3)$, the second differences in l are bounded, and so there exists $Q_l < \infty$ such that

$$\begin{aligned}
V(T, x-\delta u) - 2V(T,x) + V(T, x+\delta u) \;&\geq\; -\int_0^T Q_l |X_\delta(t) - X_0(t)|^2\, dt \\
&\qquad -\frac{\gamma^2}{2}\int_0^T 2w^T\Delta + |\Delta|^2\, dt - 2\varepsilon.
\end{aligned} \tag{22}$$

Now,

$$\begin{aligned}
\frac{d}{dt}|X_\delta(t) - X_0(t)|^2 &= 2[X_\delta(t) - X_0(t)]^T[\dot{X}_\delta(t) - \dot{X}_0(t)] \\
&= 2[X_\delta(t) - X_0(t)]^T[f(X_\delta(t)) - f(X_0(t)) + (\sigma(X_\delta(t)) - \sigma(X_0(t)))w(t)]
\end{aligned}$$

which by $(A1)$, $(A2)$ (and using the general inequality $2ab \leq ca^2 + b^2/c$)

$$\begin{aligned}
&\leq -2c|X_\delta(t) - X_0(t)|^2 + 2K|X_\delta(t) - X_0(t)|^2|w(t)| \\
&\leq |X_\delta(t) - X_0(t)|^2\left[-c + \frac{K^2}{c}|w(t)|^2\right].
\end{aligned}$$

By separation of variables and Lemma 4.1,

$$\begin{aligned}
|X_\delta(t) - X_0(t)|^2 \;&\leq\; |(x+\delta u) - x|^2 e^{-ct + \frac{K^2}{c}\int_0^t |w|^2\, dt} \\
&\leq\; \delta^2 e^{-ct} e^{\frac{K^2}{c}{M_R}^2}.
\end{aligned}$$

Hence there exists $\overline{c}$ (independent of T) such that

$$|X_\delta(t) - X_0(t)|^2 \leq \delta^2 \overline{c} e^{-ct}. \tag{23}$$

Substituting (23) into (22), we get

$$V(T, x-\delta u) - 2V(T,x) + V(T, x+\delta u) \geq -\int_0^T Q_l \delta^2 \overline{c} e^{-ct}\, dt - \frac{\gamma^2}{2}\int_0^T 2w^T\Delta + |\Delta|^2\, dt - 2\varepsilon. \tag{24}$$

Now, by Lemma 4.2, $|X_0(t)| \le D_R$ for all $t \ge 0$, and so by the smoothness of f, σ, there exist $Q_{f,R}, Q_{\sigma,R} < \infty$ such that $|f_{xx}(x)| \le Q_{f,R}$, $|\sigma_{xx}(x)| \le Q_{\sigma,R}$ on $|x| \le D_R + \delta$. Thus, by (20), (21) and $(A3)$,

$$|\Delta(t)| \le M \left\{ Q_{f,R}|X_\delta(t) - X_0(t)|^2 + Q_{\sigma,R}|X_\delta(t) - X_0(t)|^2|w(t)| \right\} \qquad \forall\, t \in [0,T]$$

where the bound is independent of $T < \infty$. Employing (23), this yields

$$|\Delta(t)| \le M\tilde{c}_R\delta^2\Big[e^{-ct}(1 + |w(t)|)\Big] \qquad \forall\, 0 \le t \le T < \infty \tag{25}$$

for appropriate choice of $\tilde{c}_R$ dependent on R.

Substituting (25) into (24) yields

$$\begin{aligned} V(T, x - \delta u) - 2V(T,x) + V(T, x + \delta u) \ge & -\int_0^T Q_l\delta^2\overline{c}e^{-ct}\, dt \\ & -\frac{\gamma^2}{2}\int_0^T \Big[2M\tilde{c}_R\delta^2(|w(t)| + |w(t)|^2)e^{-ct} \\ & \qquad +2M^2\tilde{c}_R^2\delta^4(1 + |w(t)|^2)e^{-2ct}\Big]\, dt - 2\varepsilon \\ \ge & -\widehat{M}_R\delta^2 - 2\varepsilon \qquad \forall\, 0 \le t \le T < \infty \end{aligned}$$

for appropriate choice of $\widehat{M}_R$. Since $\varepsilon > 0$ was arbitrary, this implies semi-convexity. $\square$

Remark 4.4 *Note that if one supposes uniformly bounded second derivatives of f and σ, then the use of Lemma 4.2 (which requires the contractivity of f) can be avoided. However, one still needs bounded $\|w\|$, and the proof of Lemma 4.1 also made use of the contractivity of f. Consequently, although the addition of assumptions of uniformly bounded second derivatives would shorten the proof, it is not clear that it would be useful* **unless** *one needed to obtain a tighter bound on the growth of $\widehat{M}_R$ as a function of R.*

Corollary 4.5 *$W(x)$ is semi-convex.*

We now turn to the max–plus basis over $\mathcal{S}$. The following theorem is a minor variant of the semi-convex duality result given in [14]. It is derived from convex duality [33], [34] in a straight–forward manner. Refer to [14] for a proof; the only change is the replacement of a constant $c > C_R$ there by a symmetric matrix C such that $C - C_R I > 0$ where I is the (usual algebra) identity matrix. This replacement allows more freedom in the actual numerical implementation.

Theorem 4.6 *Let $\phi \in \mathcal{S}$. Let C be a symmetric matrix such that $C - C_R I > 0$ where $C_R > 0$ is a semi–convexity constant for ϕ, and let L_R be the Lipschitz constant for ϕ over $B_R(0)$. Then for all $x \in B_R(0)$,*

$$\phi(x) = \max_{\tilde{x} \in B_{L_R/C_R}} \Big[-\frac{1}{2}(x - \tilde{x})^T C(x - \tilde{x}) + a_{\tilde{x}}\Big] = \max_{\tilde{x} \in \mathbf{R}^n}\Big[-\frac{1}{2}(x - \tilde{x})^T C(x - \tilde{x}) + a_{\tilde{x}}\Big]$$

where

$$a_{\tilde{x}} = -\max_{x \in B_R}\Big[-\frac{1}{2}(x-\tilde{x})^T C(x-\tilde{x}) - \phi(x)\Big].$$

Let $\phi \in \mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over $B_{L_R/C_R}(0)$, and let symmetric $C - C_R I > 0$ where (again) $C_R > 0$ is a semi–convexity constant for ϕ over $B_R(0)$. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x-x_i)^T C(x-x_i)$$

for each i. Then, using Theorem 4.6, one finds (see [14] for details)

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{26}$$

where

$$a_i \doteq -\max_{x \in B_R}[\psi_i(x) - \phi(x)]. \tag{27}$$

This is a countable max–plus basis expansion for ϕ. More generally, the set $\{\psi_i\}$ forms a max–plus basis for the space of semi–convex functions over $B_R(0)$ with semi–convexity constant, C_R, and one might denote this space as $\mathcal{S}_{R,C_R}$. We now have the following.

Theorem 4.7 *Given $R < \infty$, there exists semi-convexity constant $C_R < \infty$ for W over $B_R(0)$, and a corresponding Lipschitz constant, L_R. Let $C - C_R I > 0$ and $\{x_i\}$ be dense over $B_{L_R/C_R}(0)$, and define the basis $\{\psi_i\}$ as above. Then*

$$W(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{28}$$

where

$$a_i \doteq -\max_{x \in B_R}[\psi_i(x) - W(x)]. \tag{29}$$

5 The eigenvector equation

In order to reduce complexity, we suppose throughout the next two sections that W has a max–plus basis expansion with a finite number of terms. Of course, one must consider the error introduced by truncating the expansion at a finite number of terms in the numerical computations. However, this requires a large number of technical estimates which would push this paper well beyond standard page limits. Consequently, this analysis is delayed to a second paper [28] (but see [25], [24] for the essential points).

Let $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i$, $a^T = (a_1, a_2, \cdots, a_n)$, and B be the matrix with entries $B_{i,j} = -\max_{x \in B_R}(\psi_i(x) - S_\tau(\psi_j(x)))$. Note that B actually depends on τ, but for this section we fix any value $\tau \leq \tau_R$, and suppress the dependence in the notation, where

$S_\tau[\psi_i](x)$ is C^2 on $[0,\tau)\times B_R(0)$ for all i. (See, for instance [11] for existence of τ_R.) Here, we also choose C such that the semi–convexity of the basis functions is increasing for small time; this ensures that the elements of B are finite. (In practice this is done by a pre-processing check using local Riccati equations.) By the semi–convexity preseving property of S_τ for the given choice of C, one notes that

$$S_\tau[\psi_j](x) = \bigoplus_{i=1}^n B_{i,j} \otimes \psi_i(x) \tag{30}$$

where again we truncate the expansion, and postpone analysis of the errors introduced by truncation to the second paper.

We will use the notation $B \otimes a$ for max–plus multiplication of matrix B and vector a.

Theorem 5.1

1. *Suppose W is a solution to $S_\tau[W] = W$, and that any expansion $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$ on $B_R(0)$ requires $a_i > -\infty$ for $\forall i$. Then $a = B \otimes a$.*
2. *Conversely, suppose $a = B \otimes a$ and that $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$ on $B_R(0)$. Then $S_\tau[W] = W$. on $B_R(0)$*

PROOF.

We begin with the first assertion. We have

$$\bigoplus_{i=1}^n a_i \otimes \psi_i(x) \;=\; W(x)$$

which by assumption

$$= S_\tau[W(\cdot)](x) = S_\tau\Big[\bigoplus_{i=1}^n a_i \otimes \psi_i(\cdot)\Big](x),$$

and then by Theorem 3.3

$$= \bigoplus_{j=1}^n a_j \otimes S_\tau \psi_j(x)$$

which by (30)

$$\begin{aligned}
&= \bigoplus_{j=1}^n \Big\{ a_j \otimes \Big[\bigoplus_{i=1}^n B_{i,j} \otimes \psi_i(x)\Big]\Big\} \\
&= \bigoplus_{j=1}^n \Big\{ \bigoplus_{i=1}^n (a_j \otimes B_{i,j} \otimes \psi_i(x))\Big\} \\
&= \bigoplus_{i=1}^n \Big\{ \Big[\bigoplus_{j=1}^n (B_{i,j} \otimes a_j)\Big] \otimes \psi_i(x)\Big\}.
\end{aligned}$$

But this implies that

$$\max\{a_1+\psi_1(x),\cdots,a_n+\psi_n(x)\} \;=\; \max\{(\max_j(B_{1,j}+a_j)+\psi_1(x)),\cdots \\ \cdots,(\max_j(B_{n,j}+a_j)+\psi_n(x))\}. \tag{31}$$

Since it is necessary that any particular $a_{i_0} > -\infty$ (i.e. that this term is needed in the expansion), there exists $y_{i_0} \in B_R(0)$ such that $a_{i_0}+\psi_{i_0}(y_{i_0}) > a_j+\psi_j(y_{i_0})$ for all $j \neq i_0$. Then, by continuity, there exists a neighborhood $B_\delta(y_{i_0}) \subset B_R(0)$ such that the left–hand side of (31) is simply $a_{i_0}+\psi_{i_0}(x)$ for all $x \in B_\delta(y_{i_0})$. Consequently,

$$a_{i_0}+\psi_{i_0}(x) = \max_i\{\max_j[B_{i,j}+a_j]+\psi_i(x)\} \qquad \forall\, x \in B_\delta(y_{i_0}). \tag{32}$$

However, the only way in which a maximum over a finite set of quadratics can be exactly identical to another quadratic over an open neighborhood is if one element of the set of quadratics is exactly the quadratic being matched. Consequently, (32) yields

$$a_{i_0} = \max_j[B_{i_0,j}+a_j].$$

Since this must hold for any choice of i_0, one has

$$a = B \otimes a$$

where a is the vector $[a_i]$ and B is the matrix $[B_{i,j}]$.

For the second assertion of the theorem, note that if $a = B \otimes a$, then

$$\begin{aligned} \bigoplus_{i=1}^n a_i \otimes \psi_i(x) \;&=\; \bigoplus_{i=1}^n \Big\{\bigoplus_{j=1}^n (B_{i,j}\otimes a_j)\Big\} \otimes \psi_i(x) \\ &= \bigoplus_{j=1}^n \Big\{a_j \otimes \bigoplus_{i=1}^n (B_{i,j}\otimes \psi_i(x))\Big\} \\ &= \bigoplus_{j=1}^n a_j \otimes S_\tau[\psi_j](x) \\ &= S_\tau\Big[\bigoplus_{j=1}^n a_j \otimes \psi_j\Big](x). \end{aligned}$$

Now since $W(x) = \bigoplus_{j=1}^n a_j \otimes \psi_j(x)$, one sees that this last expression is $W(x) = S_\tau[W](x)$. □

6 The power method

The goal then is to solve the eigenvector equation

$$e = B \otimes e, \qquad \text{or, equivalently,} \qquad 0 \otimes e = B \otimes e. \tag{33}$$

There are two main steps. The first is to compute (an approximation to) B, and the second is to solve the eigenvector equation given B. We will address the latter step in this section; the former step will be discussed in the following section.

Note that since the eigenvalue is known, the solution of (33) reduces to the solution of a linear system. One obvious approach would be to solve this linear system directly. However, without the expansion of the max-plus algebra in some way so as to compensate for the lack of additive inverses (see [2]), one cannot proceed in that manner. Consequently, we instead have used the power method to obtain the eigenvector. Not only does this yield a numerical algorithm for the solution of (33), but interestingly, it has also led to a proof of the fact that in this special case, there is a unique eigenvector (corresponding to the unique eigenvalue of 0 – see [2] for the uniqueness of max-plus eigenvalues). Of course, this uniqueness is in marked contrast to the state of affairs over the usual field. To be more specific regarding the power method, note that it will be shown that the eigenvector may be computed via the iteration

$$e = \lim_{m\to\infty} B^m \otimes 0$$

where 0 represents the (usual) zero vector, and the B^m represents the max–plus product repeated m times. This method is referred to as the power method.

Let W be the available storage (given by (9)). For the remainder of the section, fix any $\tau \in (0,\infty)$. Define

$$H(x,y) \doteq S_\tau[W](x) - \sup_{w\in\mathcal{W}_y} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau)) \Big\} \tag{34}$$

where $X(0)=x$ and $\mathcal{W}_y = \{w \in \mathcal{W} :\ X(\tau)=y\}$.

Lemma 6.1 *There exist $M_x, M_{x,y} < \infty$ such that*

$$S_\tau[W](x) = \sup_{w\in\mathcal{W}_{M_x}} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau)) \Big\}$$

$$\sup_{w\in\mathcal{W}_y} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau)) \Big\} = \sup_{w\in\mathcal{W}_{y,M_{x,y}}} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau)) \Big\}$$

where

$$\mathcal{W}_{M_x} \doteq \{w\in\mathcal{W} :\ \|w\| \le M_x\} \quad \text{and} \quad \mathcal{W}_{y,M_{x,y}} \doteq \{w\in\mathcal{W}_y :\ \|w\| \le M_{x,y}\}.$$

More exactly, ε–optimal w (for $\varepsilon \le 1$) are bounded by M_x and $M_{x,y}$ for each of the above problems, respectively.

Proof. The proof of the first assertion (for $S_\tau[W]$) can be found in [30].

For the second assertion, one has

$$\sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt+W(X(\tau))\Big\}=\sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t))\\ -\frac{\gamma^2}{2}|w(t)|^2\,dt\Big\}+W(y).$$

As in [30], one finds (for proper choice of C_1, $C_{2,\gamma}$)

$$\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt+W(y)\le C_1|x|^2-C_{2,\gamma}\|w\|^2\,dt+W(y)$$

which is

$$\le -1 \tag{35}$$

if

$$\|w\|^2\ge(1/C_{2,\gamma})[1+C_1|x|^2+W(y)]\doteq M_{x,y}. \tag{36}$$

On the other hand, taking $w_0\doteq 0$ with corresponding trajectory $X_0(\cdot)$, one has

$$\int_0^\tau l(X_0(t))-\frac{\gamma^2}{2}|w_0(t)|^2\,dt+W(y)=\int_0^\tau l(X_0(t))\,dt+W(y)\ge 0. \tag{37}$$

Comparing (35) and (37) yields the result. □

The following lemma follows directly from the definition of H and Lemma 6.1, and so we do not include a proof.

Lemma 6.2 *H is continuous.*

Lemma 6.3 *$H(0,0)=0$, and $H(x,x)>0$ if $x\neq 0$.*

PROOF. Noting that

$$H(x,x)=\sup_{w\in\mathcal{W}}\Big\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt+W(X(\tau))\Big\}\\ -\sup_{w\in\mathcal{W}_x}\Big\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt+W(X(\tau))\Big\}, \tag{38}$$

it is obvious that $H(x,x)\ge 0$ for any x. (It might be worthwhile to note that by the assumptions on the dynamics, $\mathcal{W}_y\neq\emptyset$ for any $y\in\mathbf{R}^n$, and so $H(x,x)<\infty$.)

We consider the case $x\neq 0$. To prove $H(x,x)>0$, suppose $H(x,x)\le 0$. Then by (38) and Theorem 3.2, this implies

$$W(x)-\sup_{w\in\mathcal{W}_x}\Big\{\int_0^\tau l(X(t))-\frac{\gamma^2}{2}|w(t)|^2\,dt+W(x)\Big\}\le 0$$

or,

$$\sup_{w\in\mathcal{W}_x}\Big\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt\Big\} \geq 0. \tag{39}$$

As in the proof of Lemma 4.2, one has

$$\frac{d}{dt}|X(t)|^2 \leq -c|X(t)|^2 + \frac{M^2}{c}|w(t)|^2,$$

which, using $X(0)=x$, and solving the differential inequality (as before) yields

$$|X(t)|^2 \leq |x|^2 e^{-ct} + \frac{M^2}{c}\|w\|^2_{L_2(0,t)}.$$

Since we require $X(\tau)=x$ (i.e. $w\in\mathcal{W}_x$), this implies

$$\|w\|^2_{L_2(0,\tau)} \geq \frac{c}{M^2}|x|^2(1-e^{-c\tau}) \geq \frac{c}{M^2}|x|^2. \tag{40}$$

Now, by $(A4)$,

$$\gamma^2 > \frac{2M^2\alpha}{c^2} \doteq \overline{\gamma}^2.$$

Let

$$\delta \doteq \gamma^2 - \overline{\gamma}^2, \tag{41}$$

and let

$$\varepsilon \doteq \frac{\delta c}{8M^2}|x|^2. \tag{42}$$

Let w^ε be ε–optimal for (39) with corresponding trajectory X^ε. Then $X^\varepsilon(\tau)=x$ and

$$\int_0^\tau l(X^\varepsilon(t)) - \frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt \geq -\varepsilon. \tag{43}$$

Let

$$\tilde{\gamma}^2 \doteq \gamma^2 - \frac{\delta}{2} \tag{44}$$

(so that $\tilde{\gamma}^2 = \overline{\gamma}^2 + \delta/2 > \overline{\gamma}^2$). Then,

$$\int_0^\tau l(X^\varepsilon(t)) - \frac{\tilde{\gamma}^2}{2}|w^\varepsilon(t)|^2\,dt = \int_0^\tau l(X^\varepsilon(t)) - \frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt + \int_0^\tau \Big(\frac{\gamma^2-\tilde{\gamma}^2}{2}\Big)|w^\varepsilon(t)|^2\,dt$$

which by (43) and (44)

$$\geq -\varepsilon + \frac{\delta}{4}\|w^\varepsilon\|^2_{L_2(0,\tau)}$$

which by (40) and (42)

$$\geq -\frac{\delta c}{8M^2}|x|^2 + \frac{\delta c}{4M^2}|x|^2 = \frac{\delta c}{8M^2}|x|^2. \tag{45}$$

Then, since $X^\varepsilon(\tau) = x$, one may loop over this trajectory repeatedly by employing the disturbance

$$\widetilde{w}(t) \doteq w^\varepsilon(t - n\tau) \qquad \forall\, t \in [n\tau, (n+1)\tau), \ \ \forall n \in \{0, 1, 2, \ldots\}.$$

Letting $\widetilde{X}$ be the corresponding trajectory, and employing (45), this yields

$$\int_0^{n\tau} l(\widetilde{X}(t)) - \frac{\tilde{\gamma}^2}{2}|\widetilde{w}(t)|^2\, dt \geq n\frac{\delta c}{8M^2}|x|^2.$$

This implies that

$$\lim_{T\to\infty} \sup_{w\in\mathcal{W}} \int_0^T l(X(t)) - \frac{\tilde{\gamma}^2}{2}|w(t)|^2\, dt = +\infty$$

which contradicts Theorems 2.1 and 2.2 (since $\tilde{\gamma} > \overline{\gamma}$). Therefore $H(x, x) > 0$.

Lastly, we turn to the proof that $H(0, 0) = 0$. Since $H(0, 0) \geq 0$ was noted above, it is only required to show $H(0, 0) \leq 0$. However, as in (38),

$$H(0, 0) = -\sup_{w\in\mathcal{W}_0} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt \Big\}$$

where $X(0) = X(\tau) = 0$. But taking $w \equiv 0$ (and using $l \geq 0$) implies that

$$\sup_{w\in\mathcal{W}_0} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt \Big\} \geq 0.$$

Consequently $H(0, 0) \leq 0$. □

The proof of convergence of the power method relies on the following two lemmas.

Lemma 6.4 *Let $w \in \mathcal{W}$.*

$$\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt \leq W(x) - W(X(\tau)) - H(x, X(\tau)) \tag{46}$$

where $X(0) = x$.

Proof. By (34) and the fact that $W = S_\tau[W]$,

$$W(x) = H(x, y) + \sup_{w\in\mathcal{W}_y} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + W(X(\tau)) \Big\}. \tag{47}$$

Fix any $\overline{w} \in \mathcal{W}$, and let $\overline{y} = X(\tau)$ where $X(\cdot)$ is the corresponding trajectory. Then

$$W(x) = H(x, X(\tau)) + \sup_{w\in\mathcal{W}_{\overline{y}}} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + W(X(\tau)) \Big\}.$$

But $\overline{w} \in \mathcal{W}_y$, so the result follows. □

Now let the $\{x_j\}$ be such that $x_1 = 0$, that is

$$\psi_1(x) = -\frac{1}{2}x^T C x.$$

Lemma 6.5 *$B_{1,1}=0$. Also, there exists $\delta>0$ such that for all $j\neq 1$, $B_{j,j}\leq -\delta$.*

PROOF. We prove the second assertion first. Let

$$\overline{x}_j=\operatorname{argmax}\{\psi_j(x)-W(x)\}, \tag{48}$$

the existence and uniqueness of which follows from the choice of the $\{\psi_j\}$ – in particular, the quadratic growth condition $C-C_RI>0$. In fact, there exists $\overline{K}>0$ such that

$$W(\overline{x}_j)-W(x)\leq \psi_j(\overline{x}_j)-\psi_j(x)-\overline{K}|x-\overline{x}_j|^2 \qquad \forall\, x\in\mathbf{R}^n. \tag{49}$$

Let w^ε be ε–optimal for $S_\tau[\psi_j]$ with corresponding process X^ε. Then

$$S_\tau[\psi_j](\overline{x}_j)\leq \int_0^\tau l(X^\varepsilon(t))-\frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt+\psi_j(X^\varepsilon(\tau))+\varepsilon$$

which by Lemma 6.4,

$$\leq W(\overline{x}_j)-W(X^\varepsilon(\tau))+\psi_j(X^\varepsilon(\tau))-H(\overline{x}_j,X^\varepsilon(\tau))+\varepsilon$$

which by (49),

$$\leq \psi_j(\overline{x}_j)-\overline{K}|X^\varepsilon(\tau)-\overline{x}_j|^2-H(\overline{x}_j,X^\varepsilon(\tau))+\varepsilon \tag{50}$$

and so, by Lemmas 6.2 and 6.3, there exists $\delta_j>0$ such that

$$\leq \psi_j(\overline{x}_j)-\delta_j+\varepsilon$$

which yields the assertion (with $\delta\doteq\min_{j\leq n}\delta_j$).

The proof of the first assertion is relatively straight-forward, and we simply sketch it. First we should perhaps note that in the case $j=1$, one sees by the definition of ψ_1 and (7) that $\overline{x}_j=\overline{x}_1=0$, the origin. Also, $w\equiv 0$ is the optimal disturbance for problem $S_\tau[\psi_1](0)$, in which case, the state remains $X(t)=0$ for all $t\in[0,\tau]$. Consequently, $S_\tau[\psi_1](\overline{x}_1)=\psi_1(\overline{x}_1)$. This yields only $B_{1,1}\leq 0$ rather than $B_{1,1}\leq-\delta$, so there is no contradiction. Now recall that $\psi_1(x)$ is the basis function centered at the origin. One notes that $W(0)=0$ and $\nabla W(0)=0$. Using the fact that $S_\tau[\psi_1]$ is smooth, one finds $S_\tau[\psi_1](0)=0$ and $\nabla\{S_\tau[\psi_1]\}(0)=0$. Since $S_\tau[\psi_1]$ is semi–convex with constant C, one obtains $S_\tau[\psi_1](0)=\psi_1(0)$ and $S_\tau[\psi_1](x)\geq\psi_1(x)$ for all x. This yields $B_{1,1}=0$. □

Theorem 6.6 *Let $N\in\{1,2,\ldots,n\}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1\leq k_i\leq n$ for all i and $k_{N+1}=k_1$. Suppose we are not in the case $k_i=1$ for all i. Then*

$$\sum_{i=1}^N B_{k_i,k_{i+1}}\leq-\delta.$$

PROOF. By Lemma 6.5, this is true for $N=1$. We first prove the case $N=2$. The proof of the general case will then be relatively clear, but we also will provide a sketch of the induction argument. First note the monotonicity of the semi–group in the sense that if $g_1(x) \le g_2(x)$ for all x, then

$$S_\tau[g_1](x) \le S_\tau[g_2](x) \qquad \forall\, x \in \mathbf{R}^n. \tag{51}$$

Suppose either $i \neq 1$ or $j \neq 1$. Without loss of generality, suppose $i \neq 0$. By definition $\psi_j(x) + B_{j,i} \le S_\tau[\psi_i](x)$ for all x. Using (51), this implies

$$S_\tau[\psi_j + B_{j,i}] \le S_\tau[S_\tau[\psi_i]] = S_{2\tau}[\psi_i],$$

and by the max-plus linearity of S_τ, this yields

$$S_\tau[\psi_j] + B_{j,i} \le S_{2\tau}[\psi_i]$$

which implies in particular that

$$S_\tau[\psi_j](\overline{x}_i) + B_{j,i} \le S_{2\tau}[\psi_i](\overline{x}_i) \tag{52}$$

where $\overline{x}_i$ is the argmax as given in (48).

On the other hand, by (50) (with X^ε generated by an ε–optimal w^ε)

$$S_{2\tau}[\psi_i](\overline{x}_i) \le \psi_i(\overline{x}_i) - \overline{K}|X^\varepsilon(2\tau) - \overline{x}_i|^2 - H(\overline{x}_i, X^\varepsilon(2\tau)) + \varepsilon,$$

and then by Lemmas 6.2, 6.3 (as in the proof of Lemma 6.5)

$$\le \psi_i(\overline{x}_i) - \delta_i + \varepsilon. \tag{53}$$

(Note here that one may need to reduce the size of δ_i and δ to allow for the finite number of time horizons $\{\tau, 2\tau, \ldots, n\tau\}$.) Combining (52) and (53) yields

$$S_\tau[\psi_j](\overline{x}_i) - \psi_i(\overline{x}_i) + B_{j,i} \le -\delta_i + \varepsilon \le -\delta + \varepsilon. \tag{54}$$

Then, using the definition of $B_{i,j}$, and letting $\varepsilon \downarrow 0$ yields $B_{i,j} + B_{j,i} \le -\delta$.

The case for $N > 2$ follows easily by induction, and we give only the main points. To obtain the result analogous to (52) for $N > 2$, suppose that for some $N \ge 2$, one has

$$S_\tau[\psi_j] + B_{j,k_1} + \sum_{l=1}^{N-2} B_{k_l,k_{l+1}} \le \S_{N\tau}[\psi_{k_{N-1}}] \tag{55}$$

which we already have for $N = 2$, Then

$$\begin{aligned} S_\tau[\psi_j] + B_{j,k_1} + \sum_{l=1}^{N-2} B_{k_l,k_{l+1}} + B_{k_{N-1},i} &\le \S_{N\tau}[\psi_{k_{N-1}}] + B_{k_{N-1},i} \\ &= \S_{N\tau}[\psi_{k_{N-1}} + B_{k_{N-1},i}] \\ &\le \S_{N\tau}[S_\tau[\psi_i]] = \S_{(N+1)\tau}[\psi_i]. \end{aligned}$$

Consequently, an induction argument yields this for all N. The remainder of the proof follows analogously to that for the case of $N = 2$. $\square$

We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero (ignoring errors due to approximation).

Theorem 6.7 $\lim_{N\to\infty} B^N \otimes 0$ *exists, converges in a finite number of steps, and satisfies* $e = B \otimes e$.

PROOF. Given the matrix B, one can associate a corresponding directed graph with n nodes (recall B is $n \times n$) where the transition cost from node i to node j is $B_{i,j}$. Let us refer to the node corresponding to ψ_1 as the origin node. The elements of $B^N \otimes 0$ correspond to the optimal costs of transition paths of length N, that is, $[B^N \otimes 0]_i$ is the maximum cost over all paths of length N starting at node i and ending at any node k. For instance,

$$[B^2 \otimes 0]_i = \bigoplus_j \bigoplus_k [B_{i,j} \otimes B_{j,k} \otimes 0] = max_{j,k}[B_{i,j} + B_{j,k} + 0].$$

A loop will be defined to be any path (or segment of a larger path) which starts and ends at the same node. Since the graph has only n nodes, any path of length greater than n must contain at least one loop. Let l_i be the maximal cost of a path from node i to the origin node. By Theorem 6.6, any loop other than the trivial loop (the loop of length one from the origin node back to itself) must have negative cost to travel the loop. Thus, the path corresponding to cost l_i must not have any loops (except possibly trivial loops which we will ignore). Consequently, the length of the path corresponding to l_i might as well be assumed to be no longer than n and have no loops. Similarly, let l^0 be the maximal cost of a path from the origin node to any other node. By the same analysis as for l_i, one may assume without loss of generality that the path has no loops and has length no longer than n. Then $\bar{l}_i \doteq l_i + l^0$ is the maximal cost of any path from node i through the origin node to any node k, and the maximal cost path will have no loops (ignoring trivial loops). This path has length $\leq 2n$.

Let the maximum cost path from any node to any other node without loops be C_B.

Let $N > 2n$. Suppose a path from node i to some other node has length N. The path must contain at least one loop. In fact, defining M_N to be the largest integer less than N/n, a path from node i to the origin node of length N must contain at least M_N loops. Thus, by Theorem 6.6, the path must have cost at most $C_B - M_N\delta$ unless at least one of the loops is the trivial loop. Suppose N is sufficiently large that $C_B - M_N\delta < \bar{l}_i$. Suppose the path does not contain the trivial loop. Then, the path yielding cost $\bar{l}_i$ with the addition of trivial loops until the length becomes N (recall the path passed through the origin node) has cost $\bar{l}_i > C_B - M_N\delta$, and so has larger cost. Alternatively, suppose the path does contain at least one trivial loop and at least one other loop. Let the other loop have length m. Then the path cost may be increased by at least δ by elimination of this loop followed by the addition of m trivial loops (so as to maintain total path length N). Thus, we find that any path of length sufficiently large so that $C_B - M_N/\delta < \bar{l}_i$ (and $N > 2n$) cannot be of maximal cost unless it is the loopless path of cost $\bar{l}_i$ with the addition of trivial loops until the length becomes N. Therefore, $[B^N \otimes 0]_i = \bar{l}_i$ for all N such that $C_B - M_N/\delta < \bar{l}_i$ and $N > 2n$.

Thus, we see that $B^N \otimes 0$ converges in a finite number of steps. Denote the limit as e. Note that

$$\begin{aligned} e = \lim_{N\to\infty} B^N \otimes 0 &= B \otimes \lim_{N\to\infty} B^{N-1} \otimes 0 \\ &= B \otimes \lim_{M\to\infty} B^M \otimes 0 = B \otimes e, \end{aligned}$$

and so we see that e is a max-plus eigenvector corresponding to max-plus eigenvalue 0. □

Not only is the max-plus eigenvalue unique [2], but one also obtains the following.

Corollary 6.8 *There is a unique max-plus eigenvector, e up to a max–plus multiplicative constant, and of course, this is the output of the above power method.*

PROOF. Let e be the eigenvector given by the power method (i.e. $e = \lim_{N\to\infty} B^N \otimes 0$). Consider $[B^N \otimes e']_i$ when e' is any eigenvector and N will be taken sufficiently large. This is the maximal cost of any path of length N starting at node i and ending at any node k with terminal cost e'_k. By a proof similar to that used in the proof of Theorem 6.7, for N sufficiently large, say $N \geq \overline{N}$, the maximal cost path will pass through the origin node, and the only loops will be trivial loops. Let l_i again be the maximal cost of a loopless path from node i to the origin node, and now let $\hat{l}^0_{e'}$ be the maximal cost of a loopless path from the origin node to any node, k, with terminal cost e_k. Then, the maximal cost of the path of length N will be given by $l_i + \hat{l}^0_{e'}$ independent of $N \geq \overline{N}$.

Since e' is an eigenvector, for each $i \in \{1, 2, \ldots, n\}$

$$e'_i = [\lim_{N\to\infty} B^N \otimes e']_i = l_i + \hat{l}^0_{e'} = l_i + l^0 + (\hat{l}^0_{e'} - l^0)$$

(where l^0 is given in the proof of Theorem 6.7)

$$\begin{aligned} &= [\lim_{N\to\infty} B^N \otimes 0]_i + (\hat{l}^0_{e'} - l^0) \\ &= e_i + (\hat{l}^0_{e'} - l^0) \\ &= (\hat{l}^0_{e'} - l^0) \otimes e_i. \end{aligned}$$

Thus, $e' = (\hat{l}^0_{e'} - l^0) \otimes e$ which proves the result. □

7 An approximation for B

As noted at the top of the previous section, the other main component of an algorithm based on the max-plus eigenvector representation is the computation, or more exactly approximation, of B. There are several possibilities for this computation. Again, a full discussion of these methods would lengthen the paper beyond acceptable page limits, and so we defer a full discussion to the later companion paper [28]. Instead, we outline one method here. However, it is perhaps interesting to at least note here that there are specific error tolerances for the computation of the elements of B beyond which one cannot guarantee convergence of the power method described above, and that these play a crucial role in the structure of the technical error analysis which appears in the companion paper [28].

Recall that (for any specific τ)

$$B_{i,j} = -\max_{x\in B_R}\Big\{\psi_i(x) - \sup_{w\in\mathcal{W}}\Big[\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \psi_j(X(\tau))\Big]\Big\}. \tag{56}$$

where we suppress the dependence of B on τ to reduce notation. Also, recall that (see, for instance [11]) given any $R<\infty$, there exists a $\tau_R<\infty$ such that $V(\tau,x) = S_\tau[\psi_j](x)$ is a C^2 solution of (6) on $[0,\tau_R)\times B_R(0)$. Consequently, for $\tau\in(0,\tau_R)$, one can use a first–order Taylor approximation. That is,

$$B_{i,j} \simeq -\max\Big\{\psi_i(x) - \psi_j(x) - \tau\Big[\frac{1}{2\gamma^2}\nabla\psi_j^T\sigma(x)\sigma^T(x)\nabla\psi_j + f^T(x)\nabla\psi_j + l(x)\Big]\Big\}.$$

Of course, one can proceed to higher order terms and/or other approaches as well; these will be discussed in the later paper [28] (but see [24] for some indications).

8 Active control and the min–plus algebra

The analysis in the above sections concentrated on the case where one is testing some (presumably relatively simple) control which is already included in dynamics $f(x)$ in order to determine the value function. (Recall that if this is locally bounded, then the system satisfies an H_∞ bound at that particular choice of disturbance attenuation parameter γ.)

Here we consider the case where one has not a priori chosen some particular control, but is instead attempting to compute the value function of the "active" control problem for some given value of disturbance attenuation parameter γ. In this case, the problem takes the form of a game. We specifically consider the case where the dynamics take the form

$$\begin{aligned}\dot X &= A(X) + D(X)u + \sigma(X)w\\ X(0) &= x\end{aligned} \tag{57}$$

where we have now added the control, $u(\cdot)$, taking values in $\mathbf{R}^l$, and D is an $n \times l$ matrix–valued multiplier on the control. In this section we will not include all of the details, but merely outline the manner in which one can use analogous ideas to those above in a specific class of problems. For more detail on the relation between the value function for this problem and the viscosity solution of the corresponding DPE, uniqueness proofs, and the corresponding assumptions, we refer the reader to [30]. Consider a payoff and value which now take the form

$$\begin{aligned} J(T,x,u,w) &= \textstyle\int_0^T [l(X(t)) + \frac{\eta^2}{2}|u(t)|^2 - \frac{\gamma^2}{2}|w(t)|^2]\,dt \\ W(x) &= \inf_{\lambda\in\Lambda} \sup_{T<\infty} \sup_{w\in\mathcal{W}} J(T,x,\lambda[w],w) \end{aligned} \tag{58}$$

where Λ is the set of non–anticipative strategies for the controller. More specifically, we define a non–anticipative strategy for the controller according to the Elliott–Kalton definition [10] as a mapping $\lambda : \mathcal{W} \to \mathcal{U}$ which is non–anticipative in the sense that if $w(r) = \widehat{w}(r)$ for a.e. $r \in [0,t]$, then $\lambda[w](r) = \lambda[\widehat{w}](r)$ for a.e. $r \in [0,t]$ for all $t > 0$ where in particular, we have taken $\mathcal{U} = \{u : [0,\infty) \to \mathbf{R}^l : \ u \in L_2[0,T] \ \ \forall T < \infty\}$.

The corresponding DPE is

$$\begin{aligned} 0 &= -A(x)\cdot\nabla W + l(x) \\ &\quad - \inf_{u\in\mathbf{R}^l}[(D(x)u)\cdot\nabla W + \tfrac{\eta^2}{2}|u|^2] \\ &\quad - \sup_{w\in\mathbf{R}^m}[(\sigma(x)w)\cdot\nabla W - \tfrac{\gamma^2}{2}|w|^2] \\ &= -\left[A(x)\cdot\nabla W + l(x) - (\nabla W)^T\left(\tfrac{DD^T}{2\eta^2} - \tfrac{\sigma\sigma^T}{2\gamma^2}\right)\nabla W\right] \end{aligned} \tag{59}$$

We will assume that the controller dominates the disturbance in that

$$\frac{QQ^T}{2} \doteq \frac{DD^T}{2\eta^2} - \frac{\sigma\sigma^T}{2\gamma^2}$$

is bounded and uniformly nondegenerate in the sense that there exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that

$$\alpha_1|\xi|^2 \le \xi^T \frac{Q(x)Q^T(x)}{2}\xi \le \alpha_2|\xi|^2 \qquad \forall\, x,\xi \in \mathbf{R}^n.$$

Then DPE (59) also corresponds to the pure control (as opposed to game) problem

$$\begin{aligned} &\dot{X} = A(x) + Q(x)v \\ &X(0) = 0 \\ &W(x) = \inf_{v\in\mathcal{U}} \textstyle\int_0^\infty l(X) + \frac{1}{2}|v|^2\,dt. \end{aligned} \tag{60}$$

when the long–run average cost per unit time is zero.

As in the previous sections we find that W satisfies the steady–state DPP

$$\begin{aligned} W(x) &= S_\tau[W](x) \\ &= \inf_{v\in\mathcal{U}} \left\{ \textstyle\int_0^\tau l(x) + \frac{1}{2}|v|^2\,dt + W(X(\tau)) \right\} \end{aligned} \tag{61}$$

for any $\tau > 0$. In this case, S_τ is linear in the min–plus algebra, where this is the commutative semi–field defined by the operations

$$\begin{aligned} a \oplus b &= \min\{a, b\} \\ a \otimes b &= a + b \end{aligned} \tag{62}$$

over $\mathbf{R} \cup \{+\infty\}$.

By a proof analogous to that in the max-plus case, W is now semi–concave. There is also an analogous semi–concave basis, and we may write

$$W(x) = \bigoplus_{i=1}^{\infty} a_i \otimes \phi_i(x) \simeq \bigoplus_{i=1}^{n} a_i \otimes \phi_i(x)$$

where $\phi_i(x) = \frac{1}{2}(x - x_i)^T C(x - x_i)$ for a dense set $\{x_i\}$ and proper choice of matrix C. Then we again obtain a relation similar to Theorem 5.1 relating the solution of (61) to the min–plus eigenvector equation

$$0 \otimes e = B \otimes e$$

where B is defined analogously.

9 Example

As an example, we consider the min–plus problem in two–dimensions with $X(t) = (X_1(t), X_2(t))^T$ and dynamics

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_2 \\ (3/4)\arctan(2X_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w. \tag{63}$$

Let γ, η be such that the reduction of the previous section yields

$$\frac{QQ^T}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Note that this example was chose so as to represent a second–order system of the form

$$\ddot{y} = (3/4)\arctan(2y) + u + w.$$

The running cost was simply $l(x) = x_1^2 + x_2^2$. One might note that the assumption of uniform non–degeneracy is violated in this example (although one still has controllability).

Use of the above approach yields the results depicted below. The computations were run at a coarseness level that allowed computation in less than 10 seconds on a Sun Ultra 10. Figure 1a depicts W. Figures 1b and 2a depict the partials of W; note the non–smoothness. Figure 2b depicts the approximate error computed by taking first–order differences (probably too coarse a method) on the grid to approximate ∇W, and then plugging this back into the PDE. The resulting values were scaled by dividing by $1 + |x|^2$. The purpose of this single, simple example is merely to indicate that the approach is feasible. A convergence study and error analysis, giving further insight into the choice of basis functions and time step, will appear in [28] (although the main points are discussed in [24], [25]).

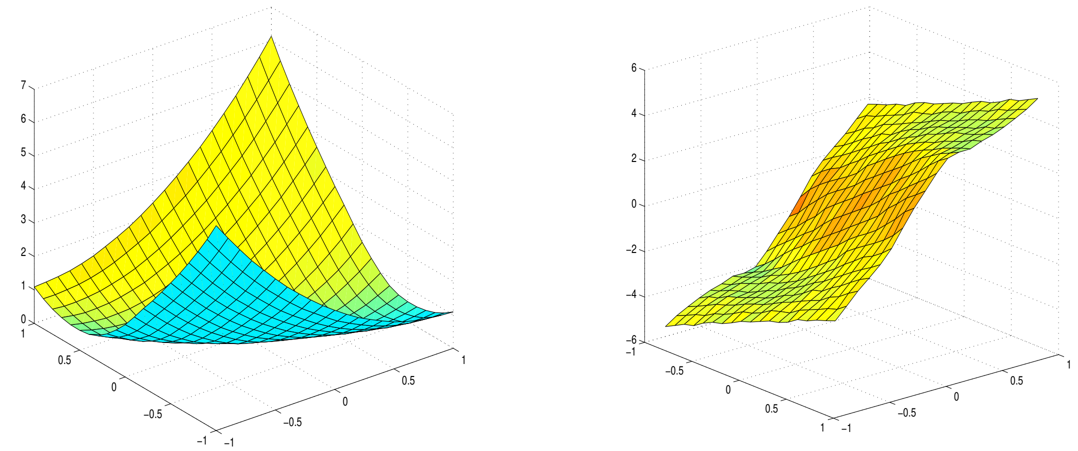

Figure 1: W and partial with respect to x_1

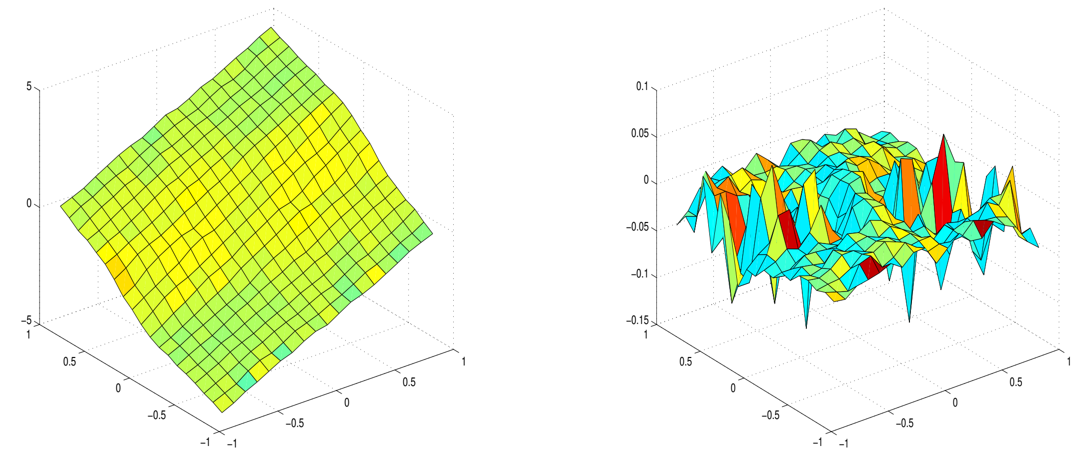

Figure 2: Partial with respect to x_2 and errors

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