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The Max–Plus Eigenvector Algorithm for Nonlinear H_∞ Control

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Abstract

We consider the H_∞ problem for a nonlinear system. The corresponding dynamic programming equation takes the form of a nonlinear, first-order PDE – possessing a term which is quadratic in the gradient. Nevertheless, the associated semi-group is linear over the max-plus algebra, and the correct solution of the PDE (the available storage) is a fixed point of this semi-group. Combining a max-plus basis with the max-plus linearity of the semi-group leads to a reduction of the problem to that of finding a max-plus eigenvector corresponding to eigenvalue 0. That is, the nonlinear problem reduces to a max-plus linear problem. We employ the power method to compute the eigenvector, and can guarantee convergence. Computationally efficient approximations for the matrix are considered. An example is displayed.

1 Introduction

We consider the H_∞ problem for a nonlinear system. Nonlinear H_∞ control and its relation to the associated dynamic programming equation (DPE) has been studied extensively (see [1], [11], [16], [3] among many notable others). For continuous systems, this DPE takes the form of a partial differential equation (PDE). Unfortunately, the H_∞ DPE is nonlinear – possessing a term which is quadratic in the gradient. A significant recent development has been the discovery of new numerical methods exploiting the max–plus (to be defined below) linearity of the associated solution operator [5], [9], [10], [7]. These algorithms employ a max-plus basis expansion for the space of semi–convex functions. The max–plus linearity was also noticed earlier in [12]. In this approach, the solution of the H_∞ PDE is given by a max–plus eigenvector corresponding to a max–plus eigenvalue of 0 (the max–plus multiplicative identity) for a particular matrix. This matrix is associated with the solution operator of the PDE. Note that the nonlinear H_∞ problem is transformed into a *max–plus eigenvector computation* although one must still compute (or approximate) the matrix defining this problem. This algorithm is analogous to a spectral method for a linear problem (but in the max–plus sense) as opposed to finite difference methods. We remind the reader that the max–plus algebra is a commutative semi–field on $\mathbf{R} \cup \{-\infty\}$ given by

$$\begin{aligned} a \oplus b &= \max\{a, b\}, \\ a \otimes b &= a + b \end{aligned} \tag{1}$$

Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. This can be extended to a field [2], but we do not need that here.

We display most of the analysis in the design case where one fixes a control and a value of the disturbance attenuation parameter, γ, and desires to compute the solution to the PDE. (If the PDE has a nonnegative solution which is zero at the origin, then one has an H_∞ controller.) We also consider a class of active control cases where one replaces the max–plus algebra above by a min-plus algebra.

In Section 2 we review the fact that the available storage satisfies a certain steady–state dynamic programming principle (DPP), or in other words, that it is a fixed point of the semi-group. It will also be shown that this semi-group is linear in the max–plus algebra. In Section 3, we review the max–plus basis for the space of semi–convex functions, and obtain the connection with a max-plus eigenvector problem for eigenvalue 0. In Section 4, we indicate the proof of convergence of the power method for the solution of the eigenvector problem. We also prove that the eigenvector is unique up to a max-plus multiplicative constant. In Section 5, we discuss an efficient method for approximation of the matrix. In Section 6, we briefly consider a class of "active" control problems where the DPE has a concave Hamiltonian, and consequently, one can use a similar technique to that above but with the min–plus algebra. In Section 7, we provide a two–dimensional example.

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2 Solution operator and max–plus linearity

Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{2}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T]\ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are probably not necessary but are sufficient for the results to follow. We will assume that all the functions f, σ and l (given below) are smooth. We will assume that there exist $K, c \in (0,\infty)$ such that

$$\begin{array}{l} |f(x) - f(y)| \le K|x-y| \qquad \forall\, x,y \in \mathbf{R}^n \\ (x-y)^T(f(x)-f(y)) \le -c|x-y|^2 \qquad \forall\, x,y \in \mathbf{R}^n \\ f(0) = 0 \end{array} \tag{A1}$$

Note that the second inequality automatically implies the closed–loop stability criterion of H_∞ control. (The second inequality would not be needed in the active control case of Section 6.) We assume that

$$\begin{array}{l} |\sigma(x) - \sigma(y)| \le K|x-y| \qquad \forall\, x,y \in \mathbf{R}^n \\ |\sigma(x)| \le M \qquad \forall\, x \in \mathbf{R}^n \end{array} \tag{A2}$$

for some $M < \infty$. Let $l(x)$ be the running cost. We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{array}{l} |l_{xx}(x)| \le \beta \qquad \forall\, x \in \mathbf{R}^n \\ 0 \le l(x) \le \alpha|x|^2 \qquad \forall\, x \in \mathbf{R}^n. \end{array} \tag{A3}$$

(There is a reason for allowing β to be greater than 2α, which one might notice below.)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T<\infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt. \tag{3}$$

The corresponding DPE is

$$0 = \frac{1}{2\gamma^2}\nabla W^T \sigma(x)\sigma^T(x)\nabla W + f^T(x)\nabla W + l(x). \tag{4}$$

It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \tag{5}$$

We note that there are linear examples where (5) exactly determines the minimum disturbance attenuation parameter. One has the following result ([13],[14]). The proof can be found in the references.

Rather than dwelling on the PDE representation (4), we would like to show that W is a fixed point of the solution operator to the DPE, where this solution operator has the representation

$$S_\tau[W(\cdot)](x) = \sup_{w \in \mathcal{W}} \left\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + W(X(\tau)) \right\}.$$

By a modification of the proof of the previous theorem, one can show the following. The proof can be found in [9].

Theorem 2.1 *For any $\tau \in [0,\infty)$, W given by (3) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class*

$$0 \le W(x) \le c\frac{(\gamma-\delta)^2}{2M^2}|x|^2 \tag{6}$$

for some $\delta > 0$, and this is the "correct" solution given by (3).

Note that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve. This will be discussed further below. The proof of the max–plus linearity of S_τ is trivial (see [9]).

Theorem 2.2 *The solution operator, S_τ, is linear in the max–plus algebra. That is, (for constant c and functions ϕ, ψ)*

$$\begin{array}{l} S_\tau[\phi \oplus \psi](x) = S_\tau[\phi](x) \oplus S_\tau[\psi](x) \\ S_\tau[c \otimes \phi](x) = c \otimes S_\tau[\phi](x) \end{array}$$

3 Max–plus basis and the eigenvector equation

Define the space of semi–convex functions, $\mathcal{S}$, as the set of $\psi : \mathbf{R}^n \to \mathbf{R}$ such that for any $R < \infty$ there exists $c_R < \infty$ such that $\psi(x) + \frac{c_R}{2}|x|^2$ is convex over $B_R = \{x \in \mathbf{R}^n : |x| \le R\}$ (see [5]). We refer to such a c_R as a semi–convexity constant for ψ. The proof of the following is rather technical.

Theorem 3.1 *W lies in $\mathcal{S}$.*

The following max–plus basis can be derived from convex duality; see [5] for details. Let $\phi \in \mathcal{S}$. Fix

$R < \infty$. Then ϕ is Lipschitz continuous with some constant L_R over $B_R(0)$ (cf.[6]). Let $\{x_i\}$ be a countable, dense set over $B_{L_R/(2c_R)}(0)$, and let $C > c_R I$ (see [5] for definition) be a positive definite matrix. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x-x_i)^T C(x-x_i)$$

for each i. Then,

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{7}$$

where $a_i \doteq -\max_{x\in B_R}[\psi_i(x) - \phi(x)]$. This is a max–plus basis expansion.

We assume C can be chosen so that $S_\tau(\psi_i(x))$ has an expansion

$$S_\tau(\psi_i(x)) = \bigoplus_{i=1}^{\infty} B_{j,i} \otimes \psi_j \tag{A4}$$

where

$$B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$$

for each i (we discuss this a little more in the next section). (This is equivalent to an assumption that $S_\tau(\psi_i(x))$ is semi-convex with some constant c_R' on B_R where $C > c_R' I$. In the case where $\sigma\sigma^T$ is uniformly nondegenerate, the existence of c_R' can be proven, but we do not include that here.) Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation. In order to reduce complexity, we suppose throughout the next two sections that W has a max–plus basis expansion with a finite number of terms. Let $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i$ and $a^T = (a_1\ a_2 \cdots a_n)$. Also assume that the expansions of the $S_\tau(\psi_i(x))$ terminate at n terms. The full error analysis (where we show that the errors introduced by truncation go to 0 as $n \to \infty$) will be delayed to a later paper. Lastly, let us assume that it is required that

$$a_j > -\infty \qquad \forall\, j \le n, \tag{A5}$$

that is, each basis function must be active. The key result is Theorem 3.2. The proof appears in [10].

Theorem 3.2 $S_\tau[W] = W$ *if and only if* $a = B \otimes a$ *where* $B \otimes a$ *represents max–plus matrix multiplication.*

PROOF. *We include the proof here for the sake of the reviewer; it will not appear in the final version.* First assume $S_\tau[W] = W$. We know that

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes \psi_i(x).$$

Applying S_τ and using Theorem 2.2 yields

$$S_\tau(W(\cdot))(x) = \bigoplus_{i=1}^{n} a_i \otimes S_\tau \psi_i(x)$$

which by (A4)

$$\begin{aligned} &= \bigoplus_{i=1}^{n}\{a_i \otimes [\bigoplus_{j=1}^{n} B_{j,i} \otimes \psi_j(x)]\} \\ &= \bigoplus_{j=1}^{n}\Big[\bigoplus_{i=1}^{n} B_{j,i} \otimes a_i\Big] \otimes \psi_j(x). \end{aligned}$$

From $S_\tau(W(\cdot))(x) = W(x)$, we get that

$$\bigoplus_{j=1}^{n} a_j \otimes \psi_j(x) = \bigoplus_{j=1}^{n}\Big[\bigoplus_{i=1}^{n} a_i \otimes B_{j,i}\Big] \otimes \psi_j(x).$$

But this implies that

$$\begin{aligned} &\max\ \{a_1 + \psi_1(x), \cdots, a_n + \psi_n(x)\} \\ &= \max\{(\max_i(a_i + B_{1,i}) + \psi_1(x)), \\ &\quad \cdots, (\max_i(a_i + B_{n,i}) + \psi_n(x))\}. \end{aligned}$$

By (A5), for any x we know we can find an interval $[x_0, y_0]$ containing this x such that we obtain one particular $a_j + \psi_j(x)$ over the entire interval from the left hand side of the above equation and one particular $\max_i(a_i + B_{k,i}) + \psi_k(x)$ from the right hand side of the equation. Hence,

$$a_j + \psi_j(x) = \max_i(a_i + B_{k,i}) + \psi_k(x)$$

for all $x \in [x_0, y_0]$. Therefore,

$$a_j = \max_i(a_i + B_{j,i})$$

for this j. Since j was arbitrary, we have proven the one direction.

Now suppose $a = B \otimes a$. Then

$$\begin{aligned} \bigoplus_{j=1}^{n} a_j \otimes \psi_j(x) &= \bigoplus_{j=1}^{n}[\bigoplus_{i=1}^{n} B_{j,i} \otimes a_i] \otimes \psi_j(x) \\ &= \bigoplus_{i=1}^{n}[a_i \otimes \bigoplus_{j=1}^{n}(B_{j,i} \otimes \psi_j(x))] \\ &= \bigoplus_{i=1}^{n} a_i \otimes S_\tau \psi_i(x) \end{aligned}$$

Now since $W(x) = \bigoplus_{j=1}^n a_j \otimes \psi_j(x)$, we get that

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes S_\tau \psi_i(x)$$

which by Theorem 2.2

$$= S_\tau\Big[\bigoplus_{i=1}^{n} a_i \otimes \psi\Big](x) = S_\tau[W]. \quad \square$$

4 Convergence of the eigenvector computations

Recall that we need to solve the eigenvector problem

$$0 \otimes e = B \otimes e, \quad \text{or,} \quad e = B \otimes e. \tag{8}$$

This requires two steps: computing (approximately) B, and then solving (8) given this B. In this section, we address the second step; in the next section, we will address the first step.

We will use the power method to compute the eigenvector. That is, we will demonstrate that

$$e = \lim_{m\to\infty} B^m \otimes 0$$

where 0 represents the zero vector, and the B^m represents the max–plus product repeated m times.

For the remainder of the section, fix any $\tau \in (0, \infty)$. Define

$$H(x,y) \doteq S_\tau[W](x) - \sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\Big\} \tag{9}$$

where $X(0) = 0$ and $\mathcal{W}_y = \{w \in \mathcal{W} : X(\tau) = y\}$. We note the following, but do not include the straightforward proof (although the machinery is in [14])

$$H(\cdot,\cdot) \quad \text{is continuous,} \tag{10}$$

$$H(0,0) = 0 \quad \text{and} \quad H(x,x) > 0 \text{ if } x \neq 0. \tag{11}$$

Lemma 4.1 *Let $w \in \mathcal{W}$.*

$$\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt \le W(x) - W(X(\tau)) - H(x, X(\tau))$$

where $X(0) = x$.

PROOF. By (9) and the fact that $W = S_\tau[W]$,

$$W(x) = H(x,y) + \sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\Big\}.$$

Fix any $w \in \mathcal{W}$, and let $y = X(\tau)$. The result follows. $\square$

Now let the $\{x_j\}$ be such that $x_1 = 0$.

Lemma 4.2 $B_{1,1} = 0$. *Also, there exists $\delta > 0$ such that for all $j \neq 1$, $B_{j,j} \le -\delta$.*

PROOF. In the interests of space, we prove only the second assertion. Let

$$\overline{x}_j = \operatorname{argmax}\{\psi_j(x) - W(x)\}, \tag{12}$$

the existence and uniqueness of which follows from the choice of the $\{\psi_j\}$. Further, there exists $K > 0$ such that

$$W(\overline{x}_j) - W(x) \le \psi_j(\overline{x}_j) - \psi_j(x) - K|x - \overline{x}_j|^2 \qquad \forall\, x \in \mathbf{R}^n. \tag{13}$$

Let w^ε be ε–optimal for $S_\tau[\psi_j]$ with corresponding process X^ε. Then

$$S_\tau[\psi_j](\overline{x}_j) \le \int_0^\tau l(X^\varepsilon(t)) - \frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt + \psi_j(X^\varepsilon(\tau)) + \varepsilon$$

which by Lemma 4.1,

$$\le W(\overline{x}_j) - W(X^\varepsilon(\tau)) + \psi_j(X^\varepsilon(\tau)) - H(x, X^\varepsilon(\tau)) + \varepsilon$$

which by (13),

$$\le \psi_j(\overline{x}_j) - K|X^\varepsilon(\tau) - \overline{x}_j|^2 - H(x, X^\varepsilon(\tau)) + \varepsilon$$

and so, by (11), there exists $\delta_j > 0$ such that

$$\le \psi_j(\overline{x}_j) - \delta_j + \varepsilon$$

which yields the assertion (with $\delta = \min_{j\le n}\delta_j$). $\square$

Theorem 4.3 *Let $N \in \mathcal{N}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \le k_i \le n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $N = 1$, $k_1 = k_2 = 1$. Then*

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \le -\delta.$$

PROOF. By Lemma 4.2, this is true for $N = 1$. We prove the case $N = 2$. The proof of the general case will then be obvious. First note the monotonicty of the semi–group in the sense that if $g_1(x) \le g_2(x)$ for all x, then

$$S_\tau[g_1](x) \le S_\tau[g_2](x) \qquad \forall\, x \in \mathbf{R}^n. \tag{14}$$

Suppose either $i \neq 1$ or $j \neq 1$. By definition $\psi_j(x) + B_{j,i} \le S_\tau[\psi_i](x)$ for all x. Using (14) and the max–plus linearity of the semi–group yields

$$S_\tau[\psi_j](x) + B_{j,i} \le S_{2\tau}[\psi_i](x)$$

which implies in particular that

$$S_\tau[\psi_j](\overline{x}_i) + B_{j,i} \le S_{2\tau}[\psi_i](\overline{x}_i).$$

Employing the last inequality of the proof of Lemma 4.2 (with having taken $\varepsilon \downarrow 0$ and 2τ replacing τ) on the right hand side of this yields

$$S_\tau[\psi_j](\overline{x}_i) + B_{j,i} \le \psi_i(\overline{x}_i) - \delta.$$

Using the definition of $B_{i,j}$, this implies $B_{j,i}+B_{i,j} \le -\delta$. □

We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero (ignoring errors due to approximation).

Theorem 4.4 $\lim_{N\to\infty} B^N \otimes 0$ *exists, converges in a finite number of steps, and satisfies* $e = B \otimes e$.

Proof. One can associate a directed graph with n nodes (recall B is $n \times n$) where the transition cost from node i to node j is $B_{i,j}$. Then the elements of $B^N \otimes 0$ correspond to the optimal costs of transition paths of length N along the graph, that is, $[B^N \otimes 0]_i$ is the maximum cost over all paths of length N starting at node i. Lemma 4.3 implies that the cost for traveling around any closed loop (other than the trivial loop at node 1) is $\le -\delta$. Hence, for sufficiently large N, the optimal paths merely sit at node 1 for long lengths of time, and adding another time–step merely leads to the optimal path spending one more time–step at node 1. Therefore, $B^N \otimes 0$ is constant for N sufficiently large. This proves the convergence result. The fact that the limit is an eigenvector then follows directly. □

Not only is the eigenvalue unique, but we can also show that

Corollary 4.5 *There is a unique eigenvector up to a max–plus multiplicative constant, and of course, this is the output of the above power method.*

Proof. Consider $[B^N \otimes e']_i$ when e' is any eigenvector and N is very large. It is the maximum cost of starting at node i and ending at node 1 plus the maximum cost of starting at node 1 and ending at any node with a penalty for ending at that node. But the latter cost is constant no matter which row we are in. Hence, $[\lim_{N\to\infty}(B^N \otimes e')]_i = \alpha + [\lim_{N\to\infty}(B^N \otimes 0)]_i$ where $\alpha \in \mathbf{R}$ is independent of i. But, since e' is an eigenvector, $B^N \otimes e' = e'$ for all N. Therefore, $e' = \alpha \otimes e$. □

5 Approximation of B

The feasibility of the algorithm is dependent upon a feasibile approximation algorithm for B. One approach is a Taylor series (in t) approximation to $S_t[\psi_i](x)$. More specifically, letting $V(t,x) = S_t[\psi_i](x)$, so that V satisfies

$$\begin{aligned} V_t &= f \cdot \nabla V + l + \nabla V^T \sigma\sigma^T \nabla V \\ V(0,x) &= \psi_i(x) \end{aligned} \tag{15}$$

one may approximate V as

$$V(t,x) = V_0(x) + V_1(x)t + \frac{1}{2}V_2(x)t^2 + \ldots . \tag{16}$$

Here $V_0(x) = \psi_i(x)$ and V_1 is the right hand side of (15) with ψ_i replacing V. The higher order terms are computed by differentiating (15), and we do not include them. Then

$$B_{j,i} = -\max_x \left\{ \psi_j(x) - [V_0 + V_1\tau + \frac{1}{2}V_2\tau^2 + \ldots](x) \right\}.$$

This method was applied but suffers from a problem which we describe only briefly. Note that the approximation of V via, say three terms, in the Taylor series at x out to time τ can be improved by reducing τ. However, the argmax moves off toward "∞" as $\tau \downarrow 0$, and the Taylor series approximation degrades as x moves off to ∞! Consequently, the approximation of $B_{j,i}$ does not tend to improve as $\tau \downarrow 0$, and so this approach was abandoned.

In its stead we are using a means of approximately tracking the argmax for small time intervals. Let

$$\overline{\psi}_j(t,x) = -\frac{1}{2}(x-\xi(t))^T C(x-\xi(t)$$

where

$$\xi(t) = x_i + (t/\tau)(x_j - x_i).$$

Let $V(t,x)$ be as above (with ψ_i as initial condition). Let

$$\overline{X}(t) = \operatorname{argmax}\left\{\overline{\psi}_j(t,x) - V(t,x)\right\}.$$

Note that

$$\overline{X}(\tau) = \operatorname{argmax}\left\{\psi_j(x) - V(\tau,x)\right\}$$

which is the desired quantity. The replacement of $\psi_j(\cdot)$ by $\overline{\psi}_j(t,\cdot)$ prevents the argmax from "blowing up" at $t \downarrow 0$. We use the Taylor expansions

$$\begin{aligned} V(t,x) &= V_0(t) + V_1(t)(x - \overline{X}(t)) \\ &\quad + \frac{1}{2}(x-\overline{X}(t))^T V_2(t)(x-\overline{X}(t)) + \ldots \\ f_k(x) &= f_{k\,0} + f_{k\,1}(x - \overline{X}(t)) + \ldots \end{aligned}$$

(where the k subscripts in the f expansion indicate components). One then obtains the following sequence of ordinary differential equations for the propagation of $\overline{X}$ leading to a computation of $B_{j,i}$:

$$\dot{\overline{X}}_k = (V_2 - C)^{-1}\Big\{\frac{1}{\tau}\sum_l C_{k,l}(x_{il} - x_{jl})$$
$$-\big[\sum_l ((f_{l1})_k V_{1x_l} + f_{l0} V_{2x_k,x_l})$$
$$-2\sum_{l,m}(V_{1x_l} a_{l,m} V_{2x_l,x_k}) + 2\overline{X}_k\big]\Big\}$$
$$\dot{V}_0 = \sum_k \Big[\overline{\psi}_{j\,x_k}(f_{k0} + \overline{X}_k - \sum_l (a_{k,l}\overline{\psi}_{j\,x_l}))\Big] + |\overline{X}|^2$$

plus higher order equations which we do not include here. It will be shown that the error induced by cutting off this series at a finite number of differential equations can be bounded for small time. We also note that the initial conditions for this propagation are easily obtained, although we do not include them here.

6 Active control and the min–plus algebra

We briefly discuss the case where one chooses a value of the disturbance attenuation parameter, and attempts to find the value function for this "active" control problem. In this case, the problem takes the form of a game. Let the dynamics be

$$\dot{X} = A(x) + D(X)u + \sigma(X)w, \qquad X(0) = x \tag{17}$$

where we have now added the control, $u(\cdot)$, taking values in $\mathbf{R}^l$. Let the payoff be

$$J(T,x,u,w) = \int_0^T [l(x) + \frac{\eta^2}{2}|u|^2 - \frac{\gamma^2}{2}|w|^2]\,dt \tag{18}$$

where the control is trying to minimize this, and the disturbance is maximizing it. We do not include the definition of value for the game in order to conserve space. We will assume that the controller dominates the disturbance in that

$$\frac{QQ^T}{2} \doteq \frac{DD^T}{2\eta^2} - \frac{\sigma\sigma^T}{2\gamma^2}$$

is bounded and uniformly elliptic.

Then analogous results to those above hold, and we find that this reduces to solving a *min–plus* (as opposed to max–plus) eigenvector equation, $a = B \otimes a$.

7 Example

As an example, we consider the min–plus problem in two–dimensions with $X(t) = (x_1(t), x_2(t))^T$ and dynamics

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ (3/4)\arctan(2x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w. \tag{19}$$

Let γ, η be such that the reduction of the previous section yields

$$\frac{QQ^T}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Note that this example was chose so as to represent a second–order system of the form

$$\ddot{y} = (3/4)\arctan(2y) + u + w.$$

The running cost was simply $l(X) = x_1^2 + x_2^2$.

The computations were run to obtain the results depicted in the figures below. The computations were run at a coarseness level that allowed computation in less than 10 seconds on a Sun Ultra 10. Figure 1 depicts W. Figures 2 and 3 depict the partials of W; note the non–smoothness. Figure 4 depicts the approximate error computed by taking first–order differences (probably too coarse a method) on the grid to approximate ∇W, and then plugging this back into the PDE. The resulting values were scaled by dividing by $1 + |x|^2$.

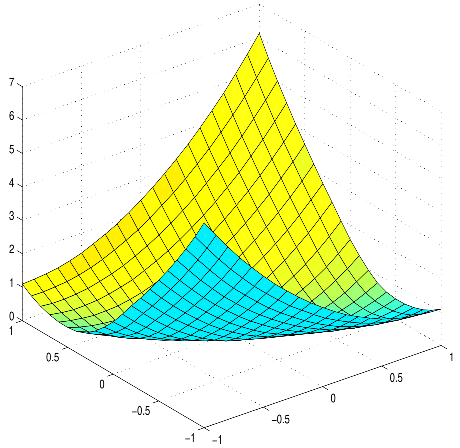

Figure 1: Storage function

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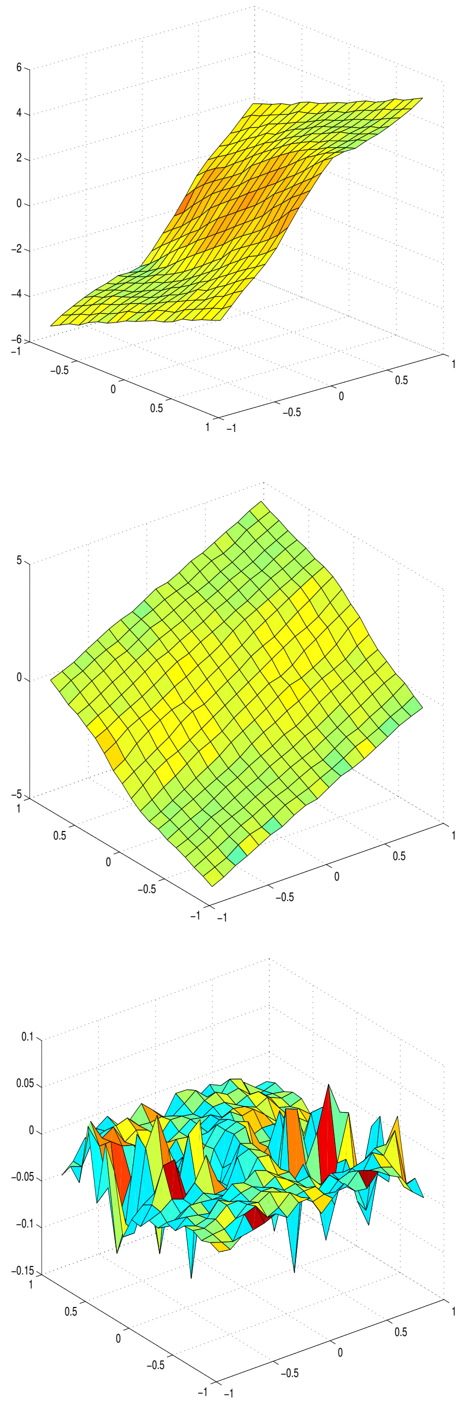

Figure 4: Scaled back-substitution error

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