Computation of Max–Plus Eigenvector Representations for Nonlinear H_∞ Value Functions

Michelle Horton * William M. McEneaney †

Abstract

We consider the H_∞ problem for a nonlinear system. The corresponding dynamic programming equation takes the form of a nonlinear, first-order PDE – possessing a term which is quadratic in the gradient. Nevertheless, the associated semi-group is linear over the max-plus algebra, and the correct solution of the PDE (the available storage) is a fixed point of this semi-group. Also, the solution lies in the space of semi-convex functions, and one has a max-plus basis for this space. Combining this max-plus basis with the max-plus linearity of the semi-group leads to a reduction of the problem to that of finding a max-plus eigenvector corresponding to eigenvalue 0. That is, the nonlinear problem reduces to a max-plus linear problem.

1 Introduction

We consider the H_∞ problem for a nonlinear system. Nonlinear H_∞ control and its relation to the associated dynamic programming equation (DPE) has been studied extensively (see [1], [8], [13], [3] among many notable others). For continuous systems, this DPE takes the form of a partial differential equation (PDE). Unfortunately, the H_∞ DPE is nonlinear – possessing a term which is quadratic in the gradient. In the Robust/H_∞ Filter problem, a new numerical method using the max–plus (to be defined below) linearity of the solution operator for a similar DPE was recently obtained [5]. This linearity was also noticed earlier in [9]. We show that the solution of the H_∞ PDE is given by a max–plus eigenvector corresponding to a max–plus eigenvalue of 0 (the max–plus multiplicative identity) for a particular matrix. This matrix is associated with the solution operator of the PDE. That is, we are turning the nonlinear H_∞ problem into a *max–plus eigenvector computation* although one must still compute (or approximate) the matrix defining this problem. This algorithm is analogous to a spectral method for a linear problem (but in the max–plus sense) as opposed to finite difference methods. We remind the reader that the max–plus algebra is a commutative semi–field on $\mathbf{R} \cup \{-\infty\}$ given by

$$\begin{aligned} a \oplus b &= \max\{a, b\}, \\ a \otimes b &= a + b \end{aligned} \tag{1}$$

Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. This can be extended to a field [2], but we do not need that here.

We consider mainly the design case where one fixes a control and a value of the disturbance attenuation parameter, γ, and desires to compute the solution to the PDE. (If the PDE has a nonnegative solution which is zero at the origin, then one has an H_∞ controller.) We will also consider a class of active control cases where one replaces the max–plus algebra above by a min-plus algebra.

In Section 2 it will be shown that the available storage is the unique viscosity solution of the PDE in a certain class. Further, this solution will be shown to satisfy a certain steady–state dynamic programming principle (DPP), or in other words, that it is a fixed point of the semi-group. It will also be shown that this semi-group is linear in the max–plus algebra. In Section 3, we show that the solution lies in the space of semi–convex functions, indicate a max–plus basis for this space, and obtain the connection with a max-plus eigenvector problem for eigenvalue 0 (the max-plus multiplicative identity). In Section 4, we demonstrate convergence of one particular method (the power method) for the solution of the eigenvector problem. We also prove that the eigenvector is unique up to a max-plus multiplicative constant. In Section 5, we briefly consider a class of "active" control problems where the DPE has a concave Hamiltonian, and consequently, one can use a similar technique to that above but with the min–plus algebra.

*Dept. of Mathematics, North Carolina State University, Raleigh, NC 27695–8205, USA, mhiggin@eos.ncsu.edu, Research partially supported by ONR grant N0014-96-1-0648.

†Dept. of Mathematics, North Carolina State University, Raleigh, NC 27695–8205, USA, http://www4.ncsu.edu/~wmm/, wmm@eos.ncsu.edu, Research partially supported by AFOSR grant F08671-98-0-1098.

2 Solution operator and max–plus linearity

Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{2}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T]\ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are probably not necessary but are sufficient for the results to follow. We will assume that all the functions f, σ and l (given below) are smooth, although that is not required for the results. We will assume that there exist $K, c \in (0,\infty)$ such that

$$\begin{array}{l} |f(x) - f(y)| \leq K|x-y| \qquad \forall\, x,y \in \mathbf{R}^n \\ (x-y)^T(f(x) - f(y)) \leq -c|x-y|^2 \qquad \forall\, x,y \in \mathbf{R}^n \\ f(0) = 0 \end{array} \tag{A1}$$

Note that this automatically implies the closed–loop stability criterion of H_∞ control. We assume that

$$\begin{array}{l} |\sigma(x) - \sigma(y)| \leq K|x-y| \qquad \forall\, x,y \in \mathbf{R}^n \\ |\sigma(x)| \leq M \qquad \forall\, x \in \mathbf{R}^n \end{array} \tag{A2}$$

for some $M < \infty$. Let $l(x)$ be the running cost (to which the L_2–norm disturbance penalty will be added). We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{array}{l} |l_{xx}(x)| \leq \beta \qquad \forall\, x \in \mathbf{R}^n \\ 0 \leq l(x) \leq \alpha|x|^2 \qquad \forall\, x \in \mathbf{R}^n. \end{array} \tag{A3}$$

(There is a reason for allowing β to be greater than 2α, which one might notice below.)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T<\infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt. \tag{3}$$

The corresponding DPE is

$$0 = \frac{1}{2\gamma^2}\nabla W^T \sigma(x)\sigma^T(x)\nabla W + f^T(x)\nabla W + l(x). \tag{4}$$

It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \tag{5}$$

We note that there are linear examples where (5) exactly determines the minimum disturbance attenuation parameter. One has the following result ([10],[11]). The proof can be found in the references.

Theorem 2.1 *There exists a unique continuous viscosity solution of (4) in the class*

$$0 \leq W(x) \leq c\frac{(\gamma-\delta)^2}{2M^2}|x|^2 \tag{6}$$

for some $\delta > 0$, and this is the "correct" solution given by (3).

We would like to show that W is a fixed point of the solution operator to the DPE, where this solution operator has the representation

$$S_\tau[W(\cdot)](x) = \sup_{w \in \mathcal{W}} \left\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt + W(X(\tau)) \right\}.$$

By a modification of the proof of the previous theorem, one can show the following. The proof can be found in [7].

Theorem 2.2 *For any $\tau \in [0,\infty)$, W given by (3) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class mentioned above.*

Note that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve. This will be discussed further below. The proof of the max–plus linearity of S_τ is trivial (see [7]).

Theorem 2.3 *The solution operator, S_τ, is linear in the max–plus algebra. That is, (for constant c and functions ϕ, ψ)*

$$\begin{array}{l} S_\tau[\phi \oplus \psi](x) = S_\tau[\phi](x) \oplus S_\tau[\psi](x) \\ S_\tau[c \otimes \phi](x) = c \otimes S_\tau[\phi](x) \end{array}$$

3 Max–plus basis and the eigenvector equation

Define the space of semi–convex functions, $\mathcal{S}$, as the set of $\psi : \mathbf{R}^n \to \mathbf{R}$ such that for any $R < \infty$ there exists $c_R < \infty$ such that $\psi(x) + \frac{c_R}{2}|x|^2$ is convex over $B_R = \{x \in \mathbf{R}^n : |x| \leq R\}$ (see [5]). We refer to such a c_R as a semi–convexity constant for ψ. The proof of the following is rather technical.

Theorem 3.1 *W lies in $\mathcal{S}$.*

The following max–plus basis can be derived from convex duality; see [5] for details. Let $\phi \in \mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over B_R, and let $C > c_R I$ (see [5] for definition) be a positive definite matrix. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x-x_i)^T C(x-x_i)$$

for each i. Then,

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \qquad \forall\, x \in B_R \tag{7}$$

where $a_i \doteq -\max_{x\in B_R}[\psi_i(x) - \phi(x)]$. This is a max–plus basis expansion.

We assume C can be chosen so that $S_\tau(\psi_i(x))$ has an expansion

$$S_\tau(\psi_i(x)) = \bigoplus_{i=1}^{\infty} B_{j,i} \otimes \psi_j \tag{A4}$$

where

$$B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$$

for each i (we discuss this a little more in the next section). (This is equivalent to an assumption that $S_\tau(\psi_i(x))$ is semi-convex with some constant c'_R on B_R where $C > c'_R I$.) Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation. In order to reduce complexity, we suppose throughout the next two sections that W has a max–plus basis expansion with a finite number of terms. Let $W(x) = \bigoplus_{i=1}^n a_i \otimes \psi_i$ and $a^T = (a_1\ a_2 \cdots a_n)$. Also assume that the expansions of the $S_\tau(\psi_i(x))$ terminate at n terms. The full error analysis (where we show that the errors introduced by truncation go to 0 as $n \to \infty$) will be delayed to a later paper. Lastly, let us assume that it is required that

$$a_j > -\infty \qquad \forall\, j \le n, \tag{A5}$$

that is, each basis function must be active. The key result is:

Theorem 3.2 *$S_\tau[W] = W$ if and only if $a = B \otimes a$ where $B \otimes a$ represents max–plus matrix multiplication.*

Proof.

First assume $S_\tau[W] = W$. We know that

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes \psi_i(x).$$

Applying S_τ and using Theorem 2.2 yields

$$S_\tau(W(\cdot))(x) = \bigoplus_{i=1}^{n} a_i \otimes S_\tau\psi_i(x)$$

which by (A4)

$$= \bigoplus_{i=1}^{n}\{a_i \otimes [\bigoplus_{j=1}^{n} B_{j,i} \otimes \psi_j(x)]\}$$
$$= \bigoplus_{j=1}^{n}\Big[\bigoplus_{i=1}^{n} B_{j,i} \otimes a_i\Big] \otimes \psi_j(x).$$

From $S_\tau(W(\cdot))(x) = W(x)$, we get that

$$\bigoplus_{j=1}^{n} a_j \otimes \psi_j(x) = \bigoplus_{j=1}^{n}\Big[\bigoplus_{i=1}^{n} a_i \otimes B_{j,i}\Big] \otimes \psi_j(x).$$

But this implies that

$$\begin{aligned} &\max\ \{a_1 + \psi_1(x), \cdots, a_n + \psi_n(x)\} \\ &= \max\{(\max_i(a_i + B_{1,i}) + \psi_1(x)), \\ &\quad \cdots, (\max_i(a_i + B_{n,i}) + \psi_n(x))\}. \end{aligned}$$

For any x we know we can find an interval $[x_0, y_0]$ containing this x such that we obtain one particular $a_j + \psi_j(x)$ over the entire interval from the left hand side of the above equation and one particular $\max_i(a_i + B_{k,i}) + \psi_k(x)$ from the right hand side of the equation. Hence,

$$a_j + \psi_j(x) = \max_i(a_i + B_{k,i}) + \psi_k(x)$$

for all $x \in [x_0, y_0]$. But the only way the function on the right can match the function on the left over the whole interval is if $\psi_j \equiv \psi_k$ which implies $j = k$. Therefore,

$$a_j = \max_i(a_i + B_{j,i})$$

for this j. But by (A5), for any component j, $a_j + \psi_j(x)$ dominates somewhere and so this holds for all j. And, so $a = B \otimes a$.

Now start with the fact that if $a = B \otimes a$ then

$$\begin{aligned} \bigoplus_{j=1}^{n} a_j \otimes \psi_j(x) &= \bigoplus_{j=1}^{n}[\bigoplus_{i=1}^{n} B_{j,i} \otimes a_i] \otimes \psi_j(x) \\ &= \bigoplus_{i=1}^{n}[a_i \otimes \bigoplus_{j=1}^{n}(B_{j,i} \otimes \psi_j(x))] \\ &= \bigoplus_{i=1}^{n} a_i \otimes S_\tau\psi_i(x) \end{aligned}$$

Now since $W(x) = \bigoplus_{j=1}^n a_j \otimes \psi_j(x)$, we get that

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes S_\tau\psi_i(x)$$

which by Theorem 2.2

$$= S_\tau\Big[\bigoplus_{i=1}^{n} a_i \otimes \psi\Big](x).$$

Hence $W(x)$ is a solution to $S_\tau(W(\cdot))(x) = W(x)$. $\square$

4 Approximating B and eigenvector computations

Recall that (for any specific τ)

$$B_{j,i} = -\max_{x \in B_R}(\psi_j(x) - \sup_{w \in \mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \psi_i(X(\tau))\}).$$

where we suppress the dependence of B on τ to reduce notation. Consider the approximation

$$\widetilde{B}_{j,i} \doteq -\max_x(\psi_j(x) - (l(x)\tau - \frac{\gamma^2}{2}|w_o|^2\tau + \psi_i(x) + \tau\nabla\psi_i \cdot [f(x) + \sigma(x)w_o])).$$

with

$$w_o = argmax\{\tau[l(x) - \frac{\gamma^2}{2}|w|^2] + \psi_i(x) + \tau\nabla\psi_i \cdot [f(x) + \sigma(x)w]\}.$$

Then $|B_{j,i} - \widetilde{B}_{j,i}| \le \overline{K}_2\tau + \overline{K}_3\tau^2$. Although the error in this approximation goes to zero as τ goes to zero, there appears to be a countervailing sensitivity to truncation errors in using a finite number of basis vectors, thereby preventing any "free lunch". Note that if we substitute our choice of w_o and ψ_i we get that

$$\widetilde{S}_\tau(\psi_i) \doteq -\frac{1}{2}(x - x_i)^T C(x - x_i) + \tau[l(x) + (x - x_i)^T C\sigma(x)\sigma^T(x)C(x - x_i) + f^T(x)C(x - x_i)]$$

where $\widetilde{S}_\tau$ is our approximation of S_τ. Hence when we choose our basis functions $\psi_i(x)$ we have to choose C such that $\psi_j(x) - \widetilde{S}_\tau(\psi_i)$ has a finite maximum $\forall$ i and j. This is simply a re-statement of (A4) under our approximate propagator $\widetilde{S}_\tau$. In practice, this is checked for each example given the particular dynamics. We will use $\widetilde{B}$ to compute vectors $\tilde{a}$ such that $\tilde{a} = \widetilde{B} \otimes \tilde{a}$. Define the norm $||a|| = \sup_i |a_i|$. We have not yet determined the size of $||a - \tilde{a}||$ (where $a = Ba$) given the above bound on $|B_{j,i} - \widetilde{B}_{j,i}|$. However, note that $||\widetilde{W}(x) - W(x)||_{L_\infty} \le ||a - \tilde{a}||$.

Once one has an approximation to B, one must compute the max–plus eigenvector. We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero (ignoring errors due to approximation). Research on conversion of eigenvector computation methods to the max–plus algebra is just beginning. One reference is [4]. We have not yet made use of such results. The method we currently use is the power method. That is, one computes an eigenvector, a by

$$a = \lim_{N \to \infty} B^N \otimes \vec{0}$$

where the power is to be understood in the max–plus sense and $\vec{0}$ is the zero vector. We have the following preliminary results.

Lemma 4.1 $\sum_{i=1}^m B_{k_i,k_{i+1}} \le 0$ *for any finite sequence* $\{k_i\}_{i=1}^{m+1}$ *such that* $k_1 = k_{m+1}$.

PROOF. Let $\overline{x_j} = argmax\{\psi_j(x) - W(x)\}$ and let w^ε be ε–optimal with corresponding state X_t^ε. Then

$$S_\tau(\psi_j(\cdot))(\overline{x_j}) \le \int_0^\tau l(X_t^\varepsilon(t)) - \frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt + \psi_j(X_\tau^\varepsilon) + \varepsilon \quad \forall \varepsilon > 0$$

Recalling Theorem 2.1 and the argmax property above we get $S_\tau(\psi_j(\cdot))(\overline{x_j}) - \psi_j(\overline{x_j}) \le 0$ and hence $B_{j,j} \le 0$. Longer loops follow from the semi-group property, $S_\tau(S_\tau(\psi_j)) = S_{2\tau}(\psi_j)$, and the fact that

$$B_{j,i_1} + \psi_j(x) \le S_\tau(\psi_{i_1}(x)). \tag{8}$$

Repeatedly applying the solution operator to both sides of (8) will eventually give $\sum_{i=1}^m B_{k_i,k_{i+1}} \le 0$. □

Let ψ_1 be the basis function centered at the origin. If we can choose C such that $l(x) + x^T C\sigma(x)\sigma^T(x)Cx + f^T(x)Cx \ge 0$ $\forall x$ then using Lemma 4.1 we obtain $B_{1,1} = 0$. Also, we believe with sufficient effort we can show that there exists a δ such that the sum from Lemma 4.1 is $\le -\delta$ (for any other sequence than the sequence $\{1, 1\}$), rather than simply ≤ 0 as asserted in the proof. Using these assumptions we obtain

Theorem 4.2 $\lim_{N\to\infty} B^N \otimes \vec{0}$ *exists, converges in a finite number of steps, and satisfies* $a = B \otimes a$.

PROOF. One can associate a directed graph with n nodes (recall B is $n \times n$) where the transition cost from node i to node j is $B_{i,j}$. Then the elements of $B^N \otimes \vec{0}$ correspond to the optimal costs of transition paths of length N along the graph, that is, $[B^N \otimes \vec{0}]_i$ is the maximum cost over all paths of length N starting at node i. The condition above on loop sums implies that the cost for traveling around any closed loop (other than the trivial loop at node 1) is $\le -\delta$. Hence, for sufficiently large N, the optimal paths merely sit at node 1 for long lengths of time, and adding another time–step merely leads to the optimal path spending one more time–step at node 1. Therefore, $B^N \otimes \vec{0}$ is constant for N sufficiently large. This proves the convergence result. The fact that the limit is an eigenvector then follows directly. □

Not only is the eigenvalue unique, but we can also show that

Corollary 4.3 *There is a unique eigenvector up to a max–plus multiplicative constant, and of course, this is the output of the above power method.*

PROOF. Consider $[B^N \otimes e]_i$ when e is any eigenvector and N is very large. It is the maximum cost of starting at node i and ending at node 1 plus the maximum cost of starting at node 1 and ending at any node with a penalty for ending at that node. But the latter cost is constant no matter which row we are in. Hence, $[\lim_{N\to\infty}(B^N \otimes e)]_i = \alpha + [\lim_{N\to\infty}(B^N \otimes \vec{0})]_i$ where $\alpha \in \mathbf{R}$ is independent of i. But, since e is an eigenvector, $B^N e = e$ for all N. Therefore, $e = \alpha \otimes a$. □

5 Active control and the min–plus algebra

We briefly discuss the case where one chooses a value of the disturbance attenuation parameter, and attempts to find the value function for this "active" control problem. In this case, the problem takes the form of a game. Let the dynamics be

$$\dot{X} = A(x) + D(X)u + \sigma(X)w, \qquad X(0) = x \qquad (9)$$

where we have now added the control, $u(\cdot)$, taking values in $\mathbf{R}^l$. Let the payoff be

$$J(T, x, u, w) = \int_0^T [l(x) + \frac{\eta^2}{2}|u|^2 - \frac{\gamma^2}{2}|w|^2]\, dt \qquad (10)$$

where the control is trying to minimize this, and the disturbance is maximizing it. We do not include the definition of value for the game in order to conserve space. We will assume that the controller dominates the disturbance in that

$$\frac{QQ^T}{2} \doteq \frac{DD^T}{2\eta^2} - \frac{\sigma\sigma^T}{2\gamma^2}$$

is bounded and uniformly elliptic.

Then analogous results to those above hold, and we find that this reduces to solving a *min–plus* (as opposed to max–plus) eigenvector equation, $a = B \otimes a$.

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