Error Analysis of a Max-Plus Algorithm for a First-Order HJB Equation

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Abstract

We consider max-plus based algorithms for the solution of nonlinear H_∞ problems. This class of algorithms has been described for several problem types such as nonlinear H_∞ filtering, nonlinear H_∞ control and nonlinear H_∞ control under partial information. The corresponding error analysis is just beginning. In this paper, we both approach the error analysis for such an algorithm, and demonstrate convegence. The errors are due to both the truncation of the basis expansion and computation of the matrix whose eigenvector one computes.

1 Introduction

A class of max–plus algebra based algorithms has been proposed as a means for reducing computational costs in the solution of nonlinear H_∞ problems. It has been applied to nonlinear Robust/H_∞ filtering problems [4], [3] [10], nonlinear H_∞ control problems [12], [11], [9], and nonlinear H_∞ control under partial information [6]. We will concentrate on the nonlinear H_∞ control problem here as the application of the algorithm, although most of the material corresponds to the case of partial information as well. It has previously been shown that if the H_∞ value function had a max–plus basis expansion consisting of a finite number of basis functions, then the computation of the value function reduces to a max–plus eigenvector computation for a max–plus eigenvalue of zero It was also demonstrated that for the matrix corresponding to the H_∞ problem, there is only one eigenvalue (zero) and a unique eigenvector corresponding to this eigenvalue. Further, this unique eigenvector yields the "correct" viscosity solution of the HJB PDE (among many).

However, in reality, the value function would not have a *finite* max–plus expansion in any but the most unusual case. In the previous papers (with the exception of [6]), this point was not addressed since the main goal was to present this new class of algorithms. In [6], the question was addressed in a broad sense. In [8], it was shown that as the number of basis functions increased, the approximation obtained by the algorithm converged to the true value function (assuming perfect compuation of the matrix whose eigenvector one wants). We will now obtain some simple error estimates for the size of the errors introduced by this basis truncation. We will also consider errors introduced by the approximation of the elements of the matrix corresponding to the H_∞ problem. Finally, we consider the relative rates at which the spacing between the basis functions and the time discretization must go to zero together.

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2 Review of the Max–Plus Based Algorithm

We will consider the infinite time–horizon H_∞ problem in the fixed–feedback case where the control is built into the choice of dynamics. The case of active control computation (i.e. the game case) is discussed in [9], [12], [12]. Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{1}$$

where X is the state taking values in $\mathbf{R}^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^m : w \in L_2[0,T]\ \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance. Let $l(x)$ be the running cost.

We will not waste space in the abstract with detailed assumptions. The following are sufficient. We will assume that all the functions f, σ and l (given below) are smooth. We assume f, σ are globally Lipschitz, and that σ is bounded. We will assume that l is locally Lipschitz, zero at the origin, and satisfies a quadratic growth bound. We will assume that there exist $c \in (0,\infty)$ such that

$$(x-y)^T(f(x)-f(y)) \le -c|x-y|^2 \qquad \forall\, x,y \in \mathbf{R}^n \quad (A1)$$

Note that this assumption automatically implies the closed–loop stability criterion of H_∞ control.

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function (the value function), $W(x)$, such that

$$W(x) = \sup_{w\in\mathcal{W}} \sup_{T<\infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt. \qquad (2)$$

The corresponding PDE is

$$0 = -\Big[\frac{1}{2\gamma^2}\nabla W^T\sigma(x)\sigma^T(x)\nabla W + f^T(x)\nabla W + l(x)\Big]. \qquad (3)$$

We are looking for a solution satisfying $W(0)=0$. Under a certain assumption on the size of γ [12], one can show that there is a unique solution of the DPE in the class of (viscosity) solutions which do not grow faster than a certain rate. See [14].

One can show that W is a fixed point of the solution operator to the DPE, where this solution operator has the representation $S_\tau[W(\cdot)](x) = \sup_{w\in\mathcal{W}}\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\}$. More specifically, one obtains

Theorem 2.1 *For any $\tau\in[0,\infty)$, W given by (2) satisfies $S_\tau[W]=W$, and further, it is the unique solution in the class mentioned above.*

Note that we will not provide proofs for the results in this review section. The proofs can be found in the references given earlier.

Theorem 2.2 *The solution operator, S_τ, is linear in the max–plus algebra.*

A function ϕ is called **semiconvex** if for every $R<\infty$, there exists C_R such that

$$\widehat{\phi}(x) \doteq \phi(x) + \frac{C_R}{2}|x|^2$$

is convex on the ball $B_R \doteq \{x\in\mathbf{R}^n : |x|\le R\}$. The infimum over such C_R will be known as the semiconvexity constant for ϕ over B_R. We denote the space of semiconvex functions by $\mathcal{S}$, and the space of semiconvex functions which have semiconvexity constant C_R over B_R by $\mathcal{S}_{C_R}$

Theorem 2.3 *W lies in $\mathcal{S}$.*

The following max–plus basis can be derived from convex duality; see [4] for details. Let $\phi\in\mathcal{S}$. Let $\{x_i\}$ be a countable, dense set over B_R, and let $c>C_R$. Define

$$\psi_i(x) \doteq \frac{-c}{2}|x-x_i|^2 \qquad (4)$$

for each i. Then,

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i\otimes\psi_i(x)] \qquad \forall\, x\in B_R \qquad (5)$$

where $a_i \doteq -\max_{x\in B_R}[\psi_i(x)-\phi(x)]$. This is a max–plus basis expansion.

For the remainder of the section, fix any $\tau\in(0,\infty)$. We assume throughout that one may choose the $c>C_R$ such that

$$S_\tau[\psi_i]\in\mathcal{S}_c \text{ for all } i. \qquad (A2)$$

We will not discuss this assumption here, but simply note that we have verified that this assumption holds for several problems where candidate values of c were obtained by requiring solution of a certain Riccati inequality everywhere in B_R.

Under the assumption that W had a max–plus basis expansion with a finite number of terms, we obtained the following result in the referenced papers. Let $W(x) = \bigoplus_{i=1}^n a_i\otimes\psi_i$, $a^T = (a_1\ a_2\cdots a_n)$, and $B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$. Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation.

Theorem 2.4 *$S_\tau[W]=W$ if and only if $a = B\otimes a$ where $B\otimes a$ represents max–plus matrix multiplication.*

Now, suppose that one has computed B (the construction of B will be discussed in a later section). Once one has an approximation to B, one must compute the max–plus eigenvector. We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero. We now show that one can compute the eigenvector via the power method; this has the added benefit that the convergence analysis to follow is performed in an analogous way. In the power method, one computes an eigenvector, a by

$$a = \lim_{N\to\infty} B^N\otimes\vec{0}$$

where the power is to be understood in the max–plus sense and $\vec{0}$ is the zero vector. Along the way, we will obtain a bound that will be needed in the error analysis to follow further below.

Define

$$H(x,y) \doteq S_\tau[W](x) - \sup_{w\in\mathcal{W}_y}\Big\{\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + W(X(\tau))\Big\} \qquad (6)$$

where $X(0)=0$ and $\mathcal{W}_y = \{w\in\mathcal{W} : X(\tau)=y\}$.

Lemma 2.5 *Let $w\in\mathcal{W}$.*

$$\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt \le W(x) - W(X(\tau)) - H(x,X(\tau))$$

where $X(0)=x$.

Now let the $\{x_j\}$ be such that $x_1 = 0$, i.e. so that the first basis function is centered at the origin. By a modification of the proof in [9], one obtains

Lemma 2.6 *$B_{1,1} = 0$. Also, there exists $M < \infty$ such that for all $j \neq 1$, $B_{j,j} \leq -M|x_j|^2\tau^3$.*

Theorem 2.7 *Let $N \in \mathcal{N}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \leq k_i \leq n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $N = 1$, $k_1 = k_2 = 1$. Then*

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \leq -M \max_{k_i} |x_{k_i}|^2 N^3 \tau^3.$$

Theorem 2.8 *$\lim_{N\to\infty} B^N \otimes 0$ exists, converges in a finite number of steps, and satisfies $e = B \otimes e$.*

Not only is the eigenvalue unique, but one also has

Corollary 2.9 *There is a unique eigenvector up to a max–plus multiplicative constant, and of course, this is the output of the above power method.*

3 Convergence and Truncation Errors

In this section we consider the approximation due to using only a finite number of functions in the max–plus basis expansion. It will be shown that as the number of functions increases (in a reasonable way), the approximate solution obtained by the eigenvector computation of Section 2 converges from below to the value function, W. Error bounds will also be obtained.

First, we establish some notation. Let us have sets of basis functions indexed by n, that is the *sets* are indexed by n. Let the cardinality of the n^{th} set be $\mathcal{I}^{(n)}$. For each n, let $\mathcal{X}^{(n)} \doteq \{x_i^{(n)}\}_{i=1}^{\mathcal{I}^{(n)}}$ and $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$. For instance, in the one–dimensional case, one might have $\mathcal{X}^{(1)} = \{0\}$, $\mathcal{X}^{(2)} = \{-1/2, 0, 1/2\}$, $\mathcal{X}^{(3)} = \{-3/4, -1/2, -1/4, 0, 1/4, 1/2, 3/4\}$, and so on. Further, we will let the basis functions be given by $\psi_i^{(n)} \doteq \frac{-c}{2}|x - x_i^{(n)}|^2$, and consider the sets of basis functions $\Psi^{(n)} \doteq \{\psi_i^{(n)} : i \in \mathcal{I}^{(n)}\}$. Then define the approximations to the semigroup operator, S_τ by

$$S_\tau^{(n)}[\phi](x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x) \tag{7}$$

where

$$a_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - S_\tau[\phi](x)\right]. \tag{8}$$

In other words, $S_\tau^{(n)}$ is the result of the application of the S_τ followed by the truncation due to a finite number of basis functions.

Lastly, we use the notation ${S_\tau}^N$ to indicate repeated application of S_τ N times. (Of course, by the semigroup property, ${S_\tau}^N = S_{N\tau}$.) Correspondingly, we use the notation ${S_\tau^{(n)}}^N$ to indicate the application of $S_\tau^{(n)}$ N times.

Define $\phi_0(x) \equiv 0$ and $\phi_0^{(n)}(x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x)$ with $a_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - \phi_0(x)\right]$. It is well-known that (see [11], [14] among many others) that

$$\lim_{N\to\infty} {S_\tau}^N[\phi_0] = W. \tag{9}$$

Also, note that since $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$, one has

$${S_\tau^{(n)}}^N[\phi_0^{(n)}](x) \leq {S_\tau}^{(n+1)^N}[\phi_0^{(n+1)}](x) \leq {S_\tau}^N[\phi_0](x) \tag{10}$$

for all $x \in B_R$.

By Theorem 2.8, for each n, there exists $\overline{N}(n)$ such that

$${S_\tau^{(n)}}^N[\phi_0^{(n)}] = {S_\tau^{(n)}}^{\overline{N}(n)}[\phi_0^{(n)}] \qquad \forall\, N \geq \overline{N}(n). \tag{11}$$

Defining

$$W^{(n)^\infty} \doteq {S_\tau^{(n)}}^{\overline{N}(n)}[\phi_0^{(n)}], \tag{12}$$

we further find that the limit is the fixed point. That is,

$$S_\tau^{(n)}[W^{(n)^\infty}] = W^{(n)^\infty}. \tag{13}$$

Then, by (9), (10) and (12), we find that

$$W^{(n)^\infty} \text{ is monotonically increasing in } n \tag{14}$$

and

$$W^{(n)^\infty} \leq W. \tag{15}$$

Therefore, there exists $W^{\infty\infty} \leq W$ such that

$$W^{(n)^\infty} \uparrow W^{\infty\infty}. \tag{16}$$

Under Assumption $(A2)$, it is not difficult to show that given $\varepsilon > 0$, there exists $n_\varepsilon < \infty$ such that

$$W^{(n)^\infty}(x) = S_\tau^{(n)}[W^{(n)^\infty}](x) \geq S_\tau[W^{(n)^\infty}](x) - \varepsilon$$

for all $x \in B_R$. On the other hand, one always has

$$S_\tau^{(n)}[\phi] \leq S_\tau[\phi].$$

Combining these last two inequalities, one obtains

$$\begin{aligned} W^{(n)^\infty} &= S_\tau^{(n)}[W^{(n)^\infty}] \leq S_\tau[W^{(n)^\infty}] \\ &\leq S_\tau^{(n)}[W^{(n)^\infty}] - \varepsilon = W^{(n)^\infty} - \varepsilon. \end{aligned}$$

Combining this with (16), one finds

Theorem 3.1

$$W^{\infty\infty} = S_\tau[W^{\infty\infty}], \tag{17}$$

or in other words, $W^{\infty\infty}$ is a fixed point of S_τ.

Then, with some more work (see [8]), one obtains a convergence theorem.

Theorem 3.2

$$W^{\infty\infty}(x) = W(x) \qquad \forall\, x \in B_R.$$

The previous theorem demonstrated convergence of the algorithm to the value function as the basis function density increased. Here we outline one approach to obtaining specific error estimates. The estimates may be rather conservative due to the form of the truncation error bound used; this question will become more clear below. Note that these are only the errors due to truncation to a finite number of basis functions; as noted above, analysis of the errors due to approximation of the entries in the B matrix is discussed further below.

Recall that we choose the basis functions throughout such that $x_1^{(n)} = 0$, or in other words, $\psi_1^{(n)}(x) = \frac{-c}{2}|x|^2$ for all n.

We will use the notation

$$W_{N,\tau}^{(n)}(x) \doteq {S_\tau}^{(n)N}[\phi_0^{(n)}](x)$$

where the N superscript indicates repeated application of the operator N times. We will not include the proof of the following which is essentially a statement about the nature of the truncation errors at each step.

Theorem 3.3 *Given $\varepsilon > 0$, there exists $n_\varepsilon < \infty$ such that for all $n \ge n_\varepsilon$, $N < \infty$ and $x \in B_R$,*

$$\begin{aligned} S_\tau[W_{N,\tau}^{(n)}(\cdot)](x) - \varepsilon|x| &\le {S_\tau}^{(n)}[W_{N,\tau}^{(n)}(\cdot)](x) \\ &\le S_\tau[W_{N,\tau}^{(n)}(\cdot)](x). \end{aligned}$$

Fix $\varepsilon > 0$ and choose $n > n_\varepsilon$ so that one also has $\phi_0(x) - \varepsilon|x| \le \phi_0^{(n)}(x) \le \phi_0(x)$.

Also fix $\overline{\varepsilon} > 0$, and let $w^{1,\overline{\varepsilon}}$ be $\overline{\varepsilon}/2$–optimal for $S_\tau[\phi_0]$, and let $X^{1,\overline{\varepsilon}}$ be the corresponding trajectory. Then,

$$0 \le S_\tau[\phi_0](x) - S_\tau[\phi_0^{(n)}](x) \le \varepsilon|X^{1,\overline{\varepsilon}}(\tau)| + \frac{\overline{\varepsilon}}{2}.$$

This implies

$$0 \le S_\tau[\phi_0](x) - {S_\tau}^{(n)}[\phi_0^{(n)}](x) \le \varepsilon\big(|x| + |X^{1,\overline{\varepsilon}}(\tau)|\big) + \frac{\overline{\varepsilon}}{2}.$$

Repeating this process (with $\overline{\varepsilon}/2^j$–optimal $w^{j,\overline{\varepsilon}}$ at the j^{th} step), one obtains the following result; we do not include the proof.

Theorem 3.4 *For any $x \in B_R$ and $N < \infty$,*

$$\begin{aligned} 0 &\le {S_\tau}^N[\phi_0](x) - {S_\tau}^{(n)N}[\phi_0^{(n)}](x) \\ &\le \varepsilon\Big[|x| + \sum_{j=1}^N |\overline{X}^{\overline{\varepsilon}}(j\tau)|\Big] + \overline{\varepsilon} \end{aligned}$$

where $\overline{X}^{\overline{\varepsilon}}$ is $\overline{\varepsilon}$–optimal for $S_{N\tau}[\phi_0]$.

From [14], we can also obtain that there exist $m, c \in (0, \infty)$ such that

$$|\overline{X}^{\overline{\varepsilon}}(t)| \le |x| e^{m - ct} \qquad \forall\, t \in [0, N\tau]. \tag{18}$$

Combining Theorem 3.4 and (18), one obtains

$$\begin{aligned} 0 &\le {S_\tau}^N[\phi_0](x) - {S_\tau}^{(n)N}[\phi_0^{(n)}](x) \\ &\le \varepsilon\Big(\frac{e^m}{1 - e^{-c\tau}}\Big)|x| + \overline{\varepsilon}. \end{aligned}$$

Noting that $\overline{\varepsilon} > 0$ was arbitrary, this then yields our bound on the errors introduced by truncation.

Theorem 3.5 *For $n \ge n_\varepsilon$,*

$$0 \le W(x) - W^{(n)\infty}(x) \le \varepsilon\Big(\frac{e^m}{1 - e^{-c\tau}}\Big)|x|$$

for all $x \in B_R$.

Consider the case where $\phi_0 = 0$. For simplicity, suppose the basis function centers are evenly distributed on a rectangular grid centered at the origin (still ordered so that x_1 is the origin), and contained within a ball of radius, R. (The even distribution across a square grid is almost certainly not optimal, but that issue is beyond this short paper.) Let there be $\overline{m}$ grid points per dimension, and so $\overline{m}^n$ grid points (basis functions) in total. Let the distance between grid points in each dimension be identical and denoted as $\overline{\Delta}$.

Note that it is easily shown that $\phi_0(x) - K\overline{\Delta}|x| \le \phi_0^{(n)}(x) \le \phi_0(x)$ where K depends on space dimension n and C. (Note here that one could obtain this same bound with basis function centers growing farter apart as one moves away from the origin.) Consequently, Theorem 3.5 implies

$$0 \le W(x) - W^{(n)\infty}(x) \le K\overline{\Delta}\Big(\frac{e^m}{1 - e^{-c\tau}}\Big)|x|$$

which for τ sufficiently small

$$\le K_1|x|(\overline{\Delta}/\tau)$$

where $K_1 = 2Ke^m/c$. Letting $\overline{\Delta} = \overline{K}\tau^2$ (for some $\overline{K} > 0$) yields

$$0 \le W(x) - W^{(n)\infty}(x) \le K_1|x|\tau. \tag{19}$$

More generally, letting $\overline{\Delta} = \overline{K}\tau^{\ell}$ (with $\ell > 1$) yields

$$0 \le W(x) - {W^{(n)}}^{\infty}(x) \le K_1|x|\tau^{\ell-1}. \qquad (20)$$

These last two inequalities are error bounds for the error due to truncation of the basis expansion, and we see that we have (at least with the above arguments) the requirement that $\overline{\Delta} = \overline{K}\tau^{\ell}$ for $\ell > 1$, and we will concentrate for simplicity on the case $\ell = 2$.

4 Approximation of B and Error Analysis

So far we have only considered the errors due to truncation while assuming that B and consequently, the eigenvector, e are computed exactly. Of course, the feasibility of the algorithm is dependent upon a feasible approximation algorithm for B. Here we begin a discussion of the errors in the approximation of B. This analysis has just begun. It is technical (at least by our methods), and so we do not include all steps.

One approach is a Taylor series (in t) approximation to $S_t[\psi_i](x)$. More specifically, letting $V(t,x) = S_t[\psi_i](x)$, so that V satisfies

$$\begin{aligned} V_t &= f \cdot \nabla V + l + \nabla V^T \sigma\sigma^T \nabla V \qquad (21)\\ V(0,x) &= \psi_i(x) \end{aligned}$$

one may approximate V as

$$V(t,x) = V_0(x) + V_1(x)t + \frac{1}{2}V_2(x)t^2 + \dots. \qquad (22)$$

Here $V_0(x) = \psi_i(x)$ and V_1 is the right hand side of (21) with ψ_i replacing V. The higher order terms are computed by differentiating (21), and we do not include them. Then

$$B_{j,i} = -\max_x \left\{ \psi_j(x) - [V_0 + V_1\tau + \frac{1}{2}V_2\tau^2 + \dots](x) \right\}. \qquad (23)$$

Note that the approximation of V via, say three terms, in the Taylor series at x out to time τ can be improved by reducing τ. However, the argmax moves off toward "∞" as $\tau \downarrow 0$, and one cannot show that the Taylor series approximation in τ converges. Consequently, one does not know that the approximation of $B_{j,i}$ improves as $\tau \downarrow 0$. The problem requires that the spacing between the basis function centers decreases as τ decreases. However, as noted in the previous section, we will be taking the spacing as $\overline{\Delta} = \tau^2$, and this will alleviate the problem. (In fact, $\overline{\Delta}$ decreasing linearly with τ would be sufficient.)

In order to see this more clearly, replace ψ_j in (23) by a time dependent variation that agrees with ψ_j at time τ. Let

$$\overline{\psi}_j(t,x) = -\frac{1}{2}(x-\xi(t))^T[C+\delta(1-t/\tau)I](x-\xi(t) \quad (24)$$

where $\xi(t) = x_i + (t/\tau)(x_j - x_i)$ and $\overline{\delta} > 0$.

Let $V(t,x)$ be as above (with ψ_i as initial condition). Let

$$\overline{X}^{\tau}(t) = \text{argmax}\left\{\overline{\psi}_j(t,x) - V(t,x)\right\}.$$

Note that

$$\overline{X}^{\tau}(\tau) = \text{argmax}\left\{\psi_j(x) - V(\tau,x)\right\}$$

which is the desired quantity. The replacement of $\psi_j(\cdot)$ by $\overline{\psi}_j(t,\cdot)$ prevents the argmax, $\overline{X}^{\tau}$, from "blowing up" at $t \downarrow 0$, that is $|\overline{X}^{\tau}|$ stays bounded as $t \downarrow 0$. In fact, $\overline{X}^{\tau}(t) \to x_i$ as $t \downarrow 0$. Further, the time derivative of $\overline{X}^{\tau}$ also remains bounded as $t \downarrow 0$. It is not difficult to show that $\overline{X}^{\tau}$ satisfies

$$\dot{\overline{X}}^{\tau} = (\overline{\psi}_{j_{xx}} - V_{xx})^{-1}[V_{tx} - \overline{\psi}_{j_{tx}}]. \qquad (25)$$

Although, for each particular τ, these trajectories are well-behaved, the requirement is to obtain uniform boundedness of $\overline{X}^{\tau}(tau)$ for small τ (which will again require the basis function discretization going to zero at the same time as τ).

We will make some additional assumptions which were not required for the above analysis, and repeat two (boundedness of f_x and l_{xx}) since they are particularly required here. We assume there exists $M_1 < \infty$ such that

$$|f_x| \le M,\ |f_{xx}| \le \frac{M_1}{1+|x|},\ |l_{xx}| \le M_1 \ \forall x \in \mathbf{R}^n \quad (H1)$$

and, for simplicity,

$$a = \sigma\sigma^T \text{ constant.} \qquad (H2)$$

Lastly, we will assume that there exists $\hat{\tau} > 0$ such that all derivatives of V higher than first order in x are uniformly bounded over $[0,\hat{\tau}] \times \overline{B}_R(0)$ for any $R < \infty$.

Note that

$$V_{xxt}(0,x) = -f_x x C(x-x_i) - 2f_x C + l_{xx} + \frac{1}{2}CaC.$$

Consequently, for C sufficiently large, there exists $\overline{\delta} > 0$ such that $V_{xxt}(0,x) > 2\overline{\delta}I$ for all x, and so there exists $\overline{\tau} \in (0,\hat{\tau}]$ such that for $t < \overline{\tau}$, $V_{xx}(t,x) \ge -C + \overline{\delta}tI$. This yields $(\overline{\psi}_{j_{xx}} - V_{xx}) \ge \min\{\delta, \overline{\delta}\tau\}I$ on $[0,\tau]$. To simplify the discussion here, let $\overline{\delta}\tau \le \delta$ so that

$$(\overline{\psi}_{j_{xx}} - V_{xx})^{-1} \le \frac{1}{\overline{\delta}\tau}I. \qquad (26)$$

Next, if one uses $\overline{\Delta} \le \overline{K}\tau^2$ required for (19) (actually, $\overline{\Delta} \le \overline{K}\tau$ will do), and one restricts consideration to pairs x_i, x_j such that $|x_i - x_j| \le K_2\overline{\Delta}$ (the sufficiency of using only this subset of B is not discussed here), one can obtain

$$|V_{tx}(t,\overline{X}^{\tau}) - \overline{\psi}_{j_{tx}}(t,\overline{X}^{\tau})| \le K_3|\overline{X}^{\tau}| \qquad (27)$$

for proper choice of $K_3 < \infty$. Combining (25), (26) and (27), we see that there exists $R < \infty$ such that

$$\text{argmax}\,\{\psi_j(x) - V(\tau, x)\} = |\overline{X}^{\tau}(\tau)| \leq R \qquad (28)$$

for all $\tau \in (0, \overline{\tau})$ where we remind the reader that $|x_i - x_j| \leq K_2\overline{\Delta} \leq \overline{K}\tau^2$. (There is some technical estimate here where one shows that for sufficiently small τ, this R may be the same that appears on the assumption of uniformly bounded derivatives of V above, which we do not discuss here.) Consequently, the Taylor series approximation (23) need only be considered on $B_R(0)$. Taking m terms in the Taylor expansion, and denoting the approximate B by $\widehat{B}^{(m)}$, one obtains an error bound of the form

$$|B_{j,i} - \widehat{B}^{(m)}_{j,i}| \leq K_{K_2,R}\tau^{m+1}.$$

Note that in order for the power method to work, and consequently for the results regarding the eigenvector and eigenvalue, one requires the result of Lemma 2.6 to also hold for the approximation $\widehat{B}^{(m)}$. Therefore, one has the contstraint

$$K_{K_2,R}\tau^{m+1} < M\overline{\Delta}^2\tau^3 \leq M\overline{K}\tau^5,$$

and so we require $m \geq 5$.

Further, using the representation of the eigenvector elements, e_j, as the maximum cost paths from node j to node 1, one sees that this could require $\overline{m}^n$ steps, i.e. traversing the entire graph to get to the origin node. (However, this seems highly unlikely, and in fact, one supposes something closer to order $\overline{m}$ steps, but we do not have analysis to support this claim.) Note that $\overline{m} \leq R_1/\overline{\Delta}$ for some R_1. Thus, the error in each component of the eigenvector e induced by the error in B could be bounded by

$$K_{K_2,R}R_2\tau^{m+1}\left(\frac{R_1}{\overline{\Delta}}\right)^n \leq K_4\tau^{m+1-2n}$$

for proper choice of K_4. Thus one would take $m+1 > 2n$ (or if one required only order $\overline{m}$ steps, this would be reduced to $m+1 > 2$).

In summary, choosing $\overline{\Delta} = \overline{K}\tau^2$, $m \geq 5$, the errors in B would be order τ^{m+1} with resulting errors in e of order (at worst) τ^{m+1-2n}. Also, recall that the additional errors due to basis truncation from the previous section were order τ.

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