Max–Plus Eigenvector Methods for Nonlinear H_∞ Problems: Error Analysis

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Abstract

The H_∞ problem for a nonlinear system is considered. The corresponding dynamic programming equation is a fully nonlinear, first-order, steady-state partial differential equation (PDE), possessing a term which is quadratic in the gradient. The solutions are typically nonsmooth, and further, there is non-uniqueness among the class of viscosity solutions. In the case where one tests a feedback control to see if it yields an H_∞ controller, or where either the controller or disturbance sufficiently dominate, the PDE is a Hamilton-Jacobi-Bellman equation. The computation of the solution of a nonlinear, steady-state, first-order PDE is typically quite difficult. In a companion paper, we developed an entirely new class of methods for the obtaining the "correct" solution of such PDEs. These methods are based on the linearity of the associated semi-group over the max-plus (or, in some cases, min-plus) algebra. In particular, solution of the PDE is reduced to solution of a max-plus (or min-plus) eigenvector problem for known unique eigenvalue 0 (the max-plus multiplicative identity). It was demonstrated that the eigenvector is unique, and that the power method converges to it. In the companion paper, the basic method was laid out without discussion of errors and convergence. In this paper, we both approach the error analysis for such an algorithm, and demonstrate convergence. The errors are due to both the truncation of the basis expansion and computation of the matrix whose eigenvector one computes.

Key words: Nonlinear H_∞ control, dynamic programming, numerical methods, partial differential equations, max-plus algebra.

1 Introduction

We consider the H_∞ problem for a nonlinear system. The corresponding dynamic programming equation (DPE) is a fully nonlinear, first-order, steady-state partial differential equation (PDE), possessing a term which is quadratic in the gradient (for background, see [1], [2], [18], [37] among

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many notable others). The solutions are typically nonsmooth, and further, there are multiple viscosity solutions – that is, one does not even have uniqueness among the class of viscosity solutions (cf. [32], [33]). The computation of the solution of a nonlinear, steady-state, first-order PDE is typically quite difficult, and possibly even more so in the presence of the non-uniqueness mentioned above. Some previous works in the general area of numerical methods for these problems are [3], [7], [8], [15], [16], [21], and the references therein. In the companion paper [24] to this article, the mathematical background and basic algorithm for a class of numerical methods for such PDEs was discussed. This class of methods employs the max–plus linearity of the associated semi-group. It is a completely new class of methods. The approach is appropriate for two classes of PDEs associated with nonlinear (infinite time-horizon) H_∞ problems. The first is the (design) case where one tests a feedback control to see if it yields an H_∞ controller; the corresponding PDE is a Hamilton-Jacobi-Bellman (HJB) equation. In the case where the "optimal" feedback control is being determined as well, the problem takes the form of a differential game, and the PDE is, in general, an Isaacs equation. However, if the controller sufficiently dominates the disturbance, the PDE is still an HJB equation, and this is the second case. There is a slight difference in that the second case makes use of the min-plus algebra rather than the max–plus algebra. In this paper, we will consider only the first case, so as to simplify the discussion. However, one can certainly generalize the discussion to the second case.

The recent history of this new class of methods stems from a study of the Robust/H_∞ filter ([31], [13], [11]; see also [5], [20], [6]), which has an associated time-dependent, fully nonlinear, first-order PDE. In [12], the linearity of the associated semi-group *over the max-plus algebra* was noted, and provided a key ingredient in the development of a numerical algorithm for this filter. This linearity had previously been noted in [23]. A second key ingredient (first noted to our knowledge in [12]) was the development of an appropriate basis for the solution space over the max-plus algebra, i.e. with the max–plus algebra replacing the standard underlying field. (See also [19], [22] for related work.) This reduced the problem of propagation of the solution of the PDE forward in time to max–plus matrix-vector multiplication – with dimension being that of the number of basis functions being used. A key point here is that only a finite number of basis functions are used, and so one needs to determine a bound on the induced errors.

Returning to the (design case) H_∞ problem, the associated steady–state PDE is solved to determine whether this is indeed an H_∞ controller with that disturbance attenuation level. (If there is a non-negative, locally bounded solution which is zero at the origin, then it is such an H_∞ controller; see for instance [1], [36].) The Hamiltonian is concave in the gradient variable. An example of such a PDE is

$$0 = H(x, \nabla W) = -\left[\frac{1}{2\gamma^2}(\nabla W)^T a(x)\nabla W + (f(x))^T \nabla W + l(x)\right]$$

where the notation will be described further below. There are typically multiple solutions of such PDEs – even when one normalizes by requiring $W(0) = 0$. In the linear-quadratic case, two of these solutions correspond to the stable and anti-stable manifolds associated with the Hamiltonian. The "correct" solution (i.e. the one corresponding to the available storage or value) was characterized in [36] as the smallest, non-negative viscosity solution which is zero at the origin. In [33], [32], for the class of problems considered here, a specific quadratic growth bound was given which isolated this correct solution as the unique, non-negative solution satisfying $0 \leq W(x) \leq C|x|^2$ for a specific C depending on the problem data.

The max–plus based methods make use of the fact that the solutions are actually fixed points of the associated semi-group, that is

$$W = S_\tau[W] \tag{1}$$

where S_τ is the semi-group with time-step $\tau > 0$. (See (10) for a definition of the semi-group.) In this case, one does not actually use the infinitesimal version of the semi-group (the PDE).

The max–plus algebra is a commutative semi-field over $\mathbf{R} \cup \{-\infty\}$ with the addition and multiplication given by

$$\begin{aligned} a \oplus b &= \max\{a, b\}, \\ a \otimes b &= a + b \end{aligned} \tag{2}$$

where the operations are defined for $-\infty$ in the obvious way. Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. Note that it is not a field since the additive inverses are missing. Roughly speaking, it can be extended to mimic the inclusion of additive inverses [4], but we do not need that here. Note that since 0 is the multiplicative identity, we can rewrite (1) as

$$0 \otimes W = S_\tau[W]. \tag{3}$$

In the companion paper [24] (and references therein), we showed that S_τ is linear over the max–plus algebra. With this in mind, one then thinks of W as an infinite-dimensional eigenvector (or eigenfunction) for S_τ corresponding to eigenvalue 0. If one approximates W by some finite-dimensional vector of coefficients in a max–plus basis expansion, then (3) can be re-cast as a finite-dimensional max–plus eigenvector equation (approximating the true solution). Thus, the nonlinear PDE problem is reduced to the solution of a (max–plus) linear eigenvector problem. In [24], an algorithm was generated under the assumption that the actual solution was spanned by a finite number of the basis functions. It also assumed exact computation of the finite–dimensional matrix which had the solution as the eigenvector. (Uniqueness of the eigenvalue and eigenvector were proven there.) In order to keep the paper at a reasonable size, further results regarding details of the numerical methods, convergence proofs and error bounds were delayed to the current paper (although one may note that [26], [27], [25] contain some of the main points).

Since, in reality, the value function would not have a *finite* max–plus expansion in any but the most unusual cases, we must consider the errors introduced by truncation of the expansion. In [17], the question was addressed in a broad sense. In [27], it was shown that as the number of basis functions increased, the approximation obtained by the algorithm converged to the true value function (assuming perfect computation of the matrix whose eigenvector one wants). We will now obtain some error estimates for the size of the errors introduced by this basis truncation. We also consider errors introduced by the approximation of the elements of the matrix corresponding to the H_∞ problem. Finally, these lead us to consider the relative rates at which the spacing between the basis functions and the improvement in the time–propagation errors in the matrix element computations must converge.

First we need to review some results from [24] and other earlier papers which will be needed here. This is done in Section 2. In Section 3, we obtain a bound on the size of the errors in the computation of the finite–dimensional matrix beyond which, one cannot guarantee that the method will produce an approximate solution. Then in Section 4, we consider the errors in the solution introduced by truncation of the basis functions. In Section 5, we consider the errors in the solution introduced by approximation of the elements of the finite–dimensional matrix. In Section 6, we combine these to determine the relative rates at which the spacing between the basis functions and the matrix element errors must go to zero together.

Portions of this paper have appeared previously in [29], [30] [27], [26], and [25], and the last two specifically discuss aspects of the convergence and error analysis.

2 Review of the Max–Plus Based Algorithm

In this section, the H_∞ problem class under consideration and accompanying assumptions are given. This is followed by a review of previous results regarding the max-plus based algorithm which are necessary for the error analysis to follow.

We will consider the infinite time–horizon H_∞ problem in the fixed–feedback case where the control is built into the choice of dynamics. Recall that the case of active control computation (i.e. the game case) is discussed in [24], [29] and [30]. Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{4}$$

where X is the state taking values in $\mathbf{R}^m$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0,\infty) \to \mathbf{R}^\kappa : \, w \in L_2[0,T] \;\; \forall T < \infty\}$, and σ is an $m \times \kappa$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are not necessary but are sufficient for the results to follow. No attempt has been made at this point to formulate tight assumptions. In particular, in order to provide some clear sketches of proofs, we will assume that all the functions f, σ and l (given below) are smooth, although that is not required for the results. We will assume that there exist $K_f, c \in (0,\infty)$ such that

$$\begin{aligned} &(x-y)^T(f(x)-f(y)) \le -c|x-y|^2 \quad \forall\, x,y \in \mathbf{R}^m \\ &f(0) = 0 \\ &|f_x(x)| \le K_f \quad \forall\, x \in \mathbf{R}^m \end{aligned} \tag{$A1$}$$

Note that this automatically implies the closed–loop stability criterion of H_∞ control. We assume that there exist $M, K_\sigma < \infty$ such that

$$\begin{aligned} &|\sigma(x)| \le M \quad \forall\, x \in \mathbf{R}^m \\ &|\sigma^{-1}(x)| \le M \quad \forall\, x \in \mathbf{R}^m \\ &|\sigma_x(x)| \le K_\sigma \quad \forall\, x \in \mathbf{R}^m \end{aligned} \tag{$A2$}$$

Here, we of course use σ^{-1} to indicate the Moore-Penrose pseudo-inverse, and it is implicit in the bound on $\sigma^{-1}(x)$ that σ is uniformly nondegenerate. Let $l(x)$ be the running cost (to which the L_2–norm disturbance penalty will be added). We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{aligned} &l_{xx}(x) \le \beta \quad \forall\, x \in \mathbf{R}^m \\ &0 \le l(x) \le \alpha|x|^2 \quad \forall\, x \in \mathbf{R}^m \end{aligned} \tag{$A3$}$$

where notation such as $l_{xx} \leq \beta$ will be used as a shorthand to indicate that the matrix $l_{xx} - \beta I$ is negative semi-definite. (There is a reason for allowing β to be greater than 2α, which one might notice; see [32].)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma \in (0, \infty)$ and a locally bounded available storage function (again, also referred to as the value function in the sequel), $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T < \infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2} |w(t)|^2 \, dt. \tag{5}$$

The corresponding DPE is

$$\begin{aligned} 0 &= -\sup_{w \in \mathbf{R}^\kappa} \left\{ [f(x) + \sigma(x) w]^T \nabla W + l(x) - \tfrac{\gamma^2}{2} |w|^2 \right\} \\ &= -\left[\tfrac{1}{2\gamma^2} (\nabla W)^T \sigma(x) \sigma^T(x) \nabla W + f^T(x) \nabla W + l(x) \right]. \end{aligned} \tag{6}$$

Since W itself does not appear in (6), one can always scale by an additive constant. It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \tag{$A4$}$$

We note that there are examples where $(A4)$ fails and the available storage is ∞. Then one has the following result. (See [32], Ths. 2.5 and 2.6, and [33], Th. 2.5, where the proofs also appear.)

Theorem 2.1 *There exists a unique continuous viscosity solution of (6) in the class*

$$0 \leq W(x) \leq c \frac{(\gamma - \delta)^2}{2M^2} |x|^2 \tag{7}$$

for sufficiently small $\delta > 0$. Further, this unique continuous viscosity solution is given by

$$W(x) = \lim_{T \to \infty} V(T, x) = \sup_{T < \infty} V(T, x) \tag{8}$$

where V is the value of the finite time horizon problem with dynamics (4) and payoff and value

$$\begin{aligned} &J(T, x, w) = \textstyle\int_0^T l(X(t)) - \frac{\gamma^2}{2} |w(t)|^2 \, dt \\ &V(T, x) = \sup_{w \in \mathcal{W}} J(T, x, w). \end{aligned} \tag{9}$$

Define the semi–group

$$S_\tau[W(\cdot)](x) = \sup_{w \in \mathcal{W}} \{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2} |w(t)|^2 \, dt + W(X(\tau)) \} \tag{10}$$

where X satisfies (4). The next result demonstrates that we may solve the problem by obtaining the fixed point of the semi-group. See [24], Th. 3.2 and the accompanying proof (or alternatively [30], Th. 3.2 and proof).

Theorem 2.2 *For any $\tau \in [0,\infty)$, W given by (8) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class (7).*

The following key result is proved in [24] (p. 1153) as well as in earlier references such as [30] (Th. 3.3) and [12] (pp.689–690). However, to the author's best knowledge, the first statement of the result is due to Maslov [23].

Theorem 2.3 *The solution operator, S_τ, is linear in the max–plus algebra.*

As noted in the introduction, the above linearity is a key to the development of the algorithms. A second key is the use of the space of semiconvex functions and a max–plus basis for the space. A function ϕ is **semiconvex** if for every $R < \infty$, there exists C_R such that $\widehat{\phi}(x) \doteq \phi(x) + (C_R/2)|x|^2$ is convex on the ball $B_R \doteq \{x \in \mathbf{R}^m : |x| < R\}$. The infimum over such C_R will be known as the semiconvexity constant for ϕ over B_R. We denote the space of semiconvex functions by $\mathcal{S}$. (The scalar C_R may sometimes be replaced by a symmetric, positive definite matrix where the condition becomes $\phi(x) + (1/2)x^T C_R x$ being convex; the case will be clear from the context.) Let $0 < R < \widehat{R}$, and suppose that ϕ is semiconvex over $B_{\widehat{R}}(0)$ with constant $C_{\widehat{R}}$. Then ϕ is Lipschitz over $B_R(0)$, with some constant L_R. See, for instance, [10] for a proof. Therefore any $\phi \in \mathcal{S}$ must be semiconvex and Lipschitz with some constants C_R and L_R over any ball $B_R(0)$. Consequently, we define the notation $\mathcal{S}^R_{C,L}$ to be the set of $\phi : \overline{B_R}(0) \to \mathbf{R}$ such that ϕ is semiconvex and Lipschitz over $\overline{B_R}(0)$ with constants C and L, respectively. For simplicity of notation, we will henceforth use the notation $\overline{B}_\rho$ for the closed ball $\overline{B_\rho}(0)$ for any $\rho \in (0,\infty)$. It is essential that the value, W, be semiconvex, and that is given by the next result. The proof appears on pp. 1154–1155 in [24].

Theorem 2.4 *W lies in $\mathcal{S}$; for any $R < \infty$, there exist $C_R, L_R < \infty$ such that $W \in \mathcal{S}^R_{C_R,L_R}$.*

We now turn to the max–plus basis over $\mathcal{S}^R_{C_R,L_R}$. The following theorem is a minor variant of the semiconvex duality result given in [12]. It is derived from convex duality [34], [35] in a straightforward manner. There is a change from [12] in that a scalar constant there is replaced by a symmetric matrix C such that $C - C_R I > 0$ where I is the (usual algebra) identity matrix. This replacement allows more freedom in the actual numerical implementation.

Theorem 2.5 *Let $\phi \in \mathcal{S}$. In particular, let $C_R, L_R \in (0,\infty)$ be the semi-convexity and Lipschitz constants, respectively, for ϕ over $\overline{B}_R$. Let C be a symmetric, positive definite matrix such that $C - C_R I > 0$. Let $D_R \geq R + |C^{-1}| L_R$ where $|C^{-1}|$ indicates the matrix norm of C^{-1}. (In particular, one may take $D_R = R + L_R/C_R$.) Then for all $x \in \overline{B}_R$,*

$$\phi(x) = \max_{\widetilde{x} \in \overline{B}_{D_R}} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) + a_{\widetilde{x}} \Big] = \max_{\widetilde{x} \in \mathbf{R}^m} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) + a_{\widetilde{x}} \Big] \tag{11}$$

where

$$a_{\widetilde{x}} = -\max_{x \in \overline{B}_R} \Big[-\frac{1}{2}(x-\widetilde{x})^T C (x-\widetilde{x}) - \phi(x) \Big]. \tag{12}$$

Corollary 2.6 *Let C_R, L_R, D_R be as in Theorem 2.5. Let $\phi \in \mathcal{S}^R_{C',L'}$ where in this case, C' may be a symmetric, positive definite matrix such that $C - C' > 0$, and $R + |C^{-1}|L' \leq D_R$. Then, (11), (12) hold.*

Remark 2.7 $R + |C^{-1}|L_R$ may be replaced by $|C^{-1}|\overline{L}_R$ where $\overline{L}_R$ is a Lipschitz constant for $\tilde{\phi}(x) \doteq \phi(x) + \frac{1}{2}x^T Cx$ over $\overline{B}_R$. Note that $\overline{L}_R \leq L_R + |C|R$.

Let $\phi \in \mathcal{S}^R_{C_R,L_R}$. Let $\{x_i\}$ be a countable, dense set over $\overline{B}_{D_R}$, and let symmetric $C - C_R I > 0$ where (again) $C_R > 0$ is a semiconvexity constant for ϕ over $\overline{B}_R$. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x - x_i)^T C(x - x_i) \quad \forall\, x \in \overline{B}_R$$

for each i. It may occasionally be handy to extend the domain beyond $\overline{B}_R$ by letting $\psi_i(x) = -\infty$ for $x \notin \overline{B}_R$. Then, using Theorem 2.5, one finds (see [12] pp. 695–698)

$$\phi(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \quad \forall x \in \overline{B}_R \quad \text{where} \quad a_i \doteq -\max_{x\in\overline{B}_R}[\psi_i(x) - \phi(x)]. \tag{13}$$

This is a countable max–plus basis expansion for ϕ. More generally, the set $\{\psi_i\}$ forms a max–plus basis for the space $\mathcal{S}^R_{C_R,L_R}$. We now have the following.

Theorem 2.8 *Given $R < \infty$, there exist semiconvexity and Lipschitz constants constant $C_R, L_R < \infty$ such that $W \in \mathcal{S}^R_{C_R,L_R}$. Let $C - C_R I > 0$ and $\{x_i\}$ be dense over $\overline{B}_{D_R}$, and define the basis $\{\psi_i\}$ as above. Then*

$$W(x) = \bigoplus_{i=1}^{\infty}[a_i \otimes \psi_i(x)] \quad \forall\, x \in \overline{B}_R \tag{14}$$

where

$$a_i \doteq -\max_{x\in\overline{B}_R}[\psi_i(x) - W(x)]. \tag{15}$$

For the remainder of the section, fix any $\tau \in (0, \infty)$. We also assume throughout this section that one may choose C such that $C - C_R I > 0$ and such that

$$S_\tau[\psi_i] \in \mathcal{S}^R_{C',L'} \text{ for all } i \tag{A5}$$

where $C - C' > 0$ and $R + |C^{-1}|L' \leq D_R$. This assures that each $S_\tau[\psi_i]$ has a max-plus basis expansion in terms of the basis $\{\psi_j\}$. We will not discuss this assumption in detail here, but simply note that we have verified that this assumption holds for the problems where we have used this max–plus method. We also note that this assumption will need to be replaced by a slightly stricter assumption ($A5'$) in Section 4 for the results there and beyond.

We now proceed to review the basics of the algorithm. In [24], the above theory was developed, and then, rather than proving convergence results for the algorithms, drastic assumptions were made so that the basic concept could be presented, while still keeping the paper to a reasonable length. In [24], [29], [30], it was simply assumed that there was a finite set of basis functions, $\{\psi_i\}_{i=1}^n$, such that W had a finite max–plus basis expansion over $\overline{B}_R$ in those functions, that is,

$$W(x) = \bigoplus_{i=1}^{n} a_i \otimes \psi_i, \tag{16}$$

and we let $a^T \doteq (a_1\ a_2 \cdots a_n)$, and $B_{j,i} = -\max_{x\in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$. Let B be the $n \times n$ matrix of elements $B_{j,i}$. Note that B actually depends on τ, but we suppress the dependence in the notation. We made the further drastic assumption that for each $j \in \{1, 2, \ldots, n\}$, $S_\tau[\psi_j]$ also had a finite basis expansion in the same set of basis functions, $\{\psi_i\}_{i=1}^n$, so that

$$S_\tau[\psi_j(x)] = \bigoplus_{i=1}^{n} B_{j,i} \otimes \psi_i(x) \tag{17}$$

for all $x \in \overline{B}_R$. Specifically, under (16), (17) one has (see, for instance, [24], Th. 5.1 or [30], Th. 5.1)

Theorem 2.9 *Suppose expansion (16) requires $a_i > -\infty$ for all i. $S_\tau[W] = W$ if and only if $a = B \otimes a$ where $B \otimes a$ represents max–plus matrix multiplication.*

Continuing with the review, suppose that one has computed B exactly. One must then compute the max–plus eigenvector. We should note that B has a unique max–plus eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [4]. By the above results, this eigenvalue must be zero. As discussed in [24], [29], [27], one can compute the max–plus eigenvector via the power method; this has the added benefit that the convergence analysis to follow is performed in an analogous way. In the power method, one computes an eigenvector, a by

$$a = \lim_{N\to\infty} B^N \otimes \vec{0}$$

where the power is to be understood in the max–plus sense and $\vec{0}$ is the zero vector. Throughout the paper, we let the $\{x_j\}$ be such that $x_1 = 0$, that is $\psi_1(x) = -\frac{1}{2}x^T Cx$. Since this is simply an approach to arrangement of the basis functions, we do not annotate it as an assumption. The fact that the power method works is encapsulated in the following series of three results which hold under (16), (17), and are proved in [24], pp. 1158–1160.

Lemma 2.10 $B_{1,1} = 0$. *Also, there exists $\delta > 0$ such that for all $j \neq 1$, $B_{j,j} \leq -\delta$.*

Theorem 2.11 *Let $N \in \{1, 2, \ldots, n\}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \leq k_i \leq n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $k_i = 1$ for all i. Then*

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \leq -\delta.$$

Recall that B has a unique max–plus eigenvalue, although possibly many max–plus eigenvectors corresponding to that eigenvalue [4], and that by the above results, this eigenvalue must be zero (ignoring errors due to approximation).

Theorem 2.12 $\lim_{N\to\infty} B^N \otimes 0$ *exists, converges in a finite number of steps, and satisfies* $e = B \otimes e$. *Further, this is the unique max–plus eigenvector up to a max–plus multiplicative constant.*

Thus, under the drastic assumptions above, one finds that the power method converges to the unique solution of the eigenvector problem (in a finite number of steps), and that this eigenvector is the finite set of coefficients in the max–plus basis expansion of the value function, W. The next sections will deal with the facts that we actually need to truncate infinite basis expansions, and that the computations of the elements of B are only approximate. Convergence results and error analysis will be performed. This will not only indicate that one can achieve arbitrarily good approximations to W via the above max–plus approach, but will also indicate the rate at which the distance between basis function centers should drop as the time-propagation errors in B drop so as to guarantee convergence. This is somewhat analogous to results for finite difference schemes which indicate the required relative rates at which the time and space step must go to zero for such problems.

For purposes of readability, we briefly *outline* the steps in the max-plus algorithm for (approximate) computation of W over a ball, $\overline{B}_R$.

1. Choose a set of max-plus basis functions of the form $\psi_i(x) = -\frac{1}{2}(x - x_i)^T C(x - x_i)$ where the x_i lie in $\overline{B}_{D_R}$. (In practice however, a rectangular grid has been used.) Choose a "time-step", τ.
2. Compute (approximately) elements of the matrix B given by

$$B_{j,i} = -\max_{x\in\overline{B}_R}(\psi_j(x) - S_\tau(\psi_i(x))).$$

   A reasonably efficient means of computing B is important, and a Runge-Kutta based approach is indicated in Section 5.1.
3. Compute the max-plus eigenvector of B corresponding to max–plus eigenvalue $\lambda = 0$ (i.e. the solution of $e = B \otimes e$). This is obtained from the max-plus power method $a_{k+1} = B \otimes a_k$. This converges exactly in a finite number of steps.
4. Construct the solution approximation from $\widehat{W}(x) \doteq \bigoplus_{i=1}^n a_i \otimes \psi_i(x)$ on $B_R(0)$.

3 Allowable Errors in Computation of B

In this section, we obtain a bound on the maximum allowable errors in the computation of B. If the errors are below this bound, then we can guarantee convergence of the power method to the unique eigenvector. In particular, the guaranteed convergence of the power method relies on Lemma 2.10 and Theorem 2.11 since these imply a certain structure to a directed graph associated with B (see [24], [29]). If there was a sequence $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \le k_i \le n$ for all i and $k_{N+1} = k_1$ such that one does not have $k_i = 1$ for all i, and such that

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \ge 0$$

then there would be no guarantee of convergence of the power method (nor the ensuing uniqueness result for that matter). In order to determine more exactly, the allowable errors in the computation of the elements of B, we first need to obtain a more exact expression for the δ that appears in Lemma 2.10 and Theorem 2.11, and this will appear in Theorems 3.4 and 3.6. That will be followed by results indicating the allowable error bounds. To begin, one needs the following lemma.

Lemma 3.1 *Let X satisfy (4) with initial state $X(0) = x \in \mathbf{R}^m$. Let $K, \tau \in (0, \infty)$, and let $w \in L_2[0, \tau]$. Suppose $\delta > 0$ sufficiently small so that*

$$\delta \leq KM^2/[c(1 - e^{-c\tau})] \tag{18}$$

where c, M are given in Assumptions $(A1)$,$(A2)$. Then

$$K|X(\tau) - x|^2 + \delta\|w\|^2_{L_2[0,\tau]} \geq \frac{\delta c}{8M^2}|x|^2(1 - e^{-c\tau})^4.$$

Remark 3.2 It may be of interest to note that the assumption on the size of δ does not seem necessary. At one point in the proof to follow, this assumption is used in order to eliminate a case which would lead to a more complex expression on the right-hand side in the result in the lemma statement. If some later technique benefited from not having such an assumption, the lemma proof could be revisited in order to eliminate it. However, at this point, that would seem to be a needless technicality.

Remark 3.3 It is perhaps also worth indicating the intuition behind the inequality obtained in Lemma 3.1. Essentially, it states that, due to the nature of the dynamics of the system, the only way that $|X(\tau) - x|^2$ can be kept small is through input disturbance energy $\|w\|^2$, and so their weighted sum is bounded from below. The dependence on $|x|$ on the right hand side is indicative of the fact that $|f(x)|$ goes to zero at the origin.

Proof. Note that by (4) and Assumptions $(A1)$ and $(A2)$,

$$\frac{d}{dt}|X|^2 \leq -2c|X|^2 + 2M|X||w| \tag{19}$$

$$\leq -c|X|^2 + \tfrac{M^2}{c}|w|^2. \tag{20}$$

Consequently, for any $t \in [0, \tau]$,

$$|X(t)|^2 \leq e^{-ct}|x|^2 + \tfrac{M^2}{c}\int_0^t |w(r)|^2\,dr$$

and so

$$\|w\|^2_{L_2(0,t)} \geq \tfrac{c}{M^2}\Big[|X(t)|^2 - |x|^2\Big] \quad \forall\, t \in [0, \tau]. \tag{21}$$

We may suppose

$$|X(t)| \leq \sqrt{1 + (1 - e^{-c\tau})^4/2}\,|x| \quad \forall\, t \in [0, \tau]. \tag{22}$$

Otherwise by (21) and the reverse of (22), there exists $t \in [0, \tau]$ such that

$$K|X(\tau) - x|^2 + \delta\|w\|^2_{L_2[0,\tau]} \geq \delta\|w\|^2_{L_2[0,t]} \geq \tfrac{\delta c}{2M^2}(1 - e^{-c\tau})^4|x|^2 \tag{23}$$

in which case one already has the desired result. Define $\overline{K} \doteq \sqrt{1+(1-e^{-c\tau})^4/2}$.

Recalling (19), and applying (22), one has

$$\frac{d}{dt}|X(t)|^2 \leq -2c|X(t)|^2 + 2M\overline{K}|x||w(t)|.$$

Solving this ODI for $|X(t)|^2$, and using the Hölder inequality, yields the bound

$$|X(\tau)|^2 \leq |x|^2 e^{-2c\tau} + \frac{M\overline{K}|x|\|w\|}{\sqrt{c}}(1-e^{-4c\tau})^{1/2}. \tag{24}$$

This implies

$$|X(\tau)| \leq |x|e^{-c\tau} + \frac{1}{c^{1/4}}\sqrt{M\overline{K}|x|\|w\|}(1-e^{-4c\tau})^{1/4}. \tag{25}$$

We consider two cases separately. First we consider the case where $|X(\tau)| \leq |x|$. Then, by (25)

$$|X(\tau)-x| \geq |x| - |X(\tau)| \geq |x|(1-e^{-c\tau}) - \frac{1}{c^{1/4}}\sqrt{M\overline{K}|x|\|w\|}(1-e^{-4c\tau})^{1/4}. \tag{26}$$

Now note that for general $a,b,c \in [0,\infty)$, $a+c \geq b$ implies

$$a^2 \geq \frac{b^2}{2} - c^2. \tag{27}$$

By (26) and (27) (and noting the non-negativity of the norm),

$$|X(\tau)-x|^2 \geq \max\left\{\tfrac{1}{2}|x|^2(1-e^{-c\tau})^2 - \frac{M\overline{K}}{\sqrt{c}}|x|\|w\|(1-e^{-4c\tau})^{1/2},\, 0\right\}$$

which implies

$$\begin{aligned} K|X(\tau)-x|^2 + \delta\|w\|^2 \geq \max\Big\{&\frac{K}{2}|x|^2(1-e^{-c\tau})^2 - \frac{KM\overline{K}}{\sqrt{c}}|x|\|w\|(1-e^{-4c\tau})^{1/2} + \delta\|w\|^2, \\ &\delta\|w\|^2\Big\}. \end{aligned} \tag{28}$$

The right hand side of (28) is a maximum of two convex quadratic functions of $\|w\|$. The second is monotonically increasing, while the first is positive at $\|w\| = 0$ and initially decreasing. This implies that there are two possibilities for the location of the minimum of the maximum of the two functions. If the minimum of the first function is to the left of the point where the two functions intersect, then the minimum occurs at the minimum of the first function; alternatively it occurs where the two functions intersect. The minimum of the first function occurs at $\|w\|_{min}$ (where we are abusing notation here, using the *min* subscript on the norm to indicate the value of $\|w\|$ at which the minimum occurs), and this is given by

$$\|w\|_{min} = \frac{KM\overline{K}|x|(1-e^{-4c\tau})^{1/2}}{2\sqrt{c}\delta}. \tag{29}$$

The point of intersection of the two functions occurs at

$$\|w\|_{int} = \frac{\sqrt{c}|x|(1-e^{-c\tau})^2}{2M\overline{K}(1-e^{-4c\tau})^{1/2}}. \tag{30}$$

The two points coincide when

$$\delta = \frac{KM^2\overline{K}^2(1-e^{-4c\tau})}{c(1-e^{-c\tau})^2} = \frac{KM^2[1+(1-e^{-c\tau})^4/2](1-e^{-4c\tau})}{c(1-e^{-c\tau})^2},$$

and $\|w\|_{int}$ occurs to the left of $\|w\|_{min}$ for δ less than this. It is easy to see that assumption (18) implies that δ is less than the value at which the points coincide, and consequently, the minimum of the right hand side of (28) occurs at $\|w\|_{int}$.

Using the value of the right hand side of (28) corresponding to $\|w\|_{int}$, we find that for any disturbance, w,

$$K|X(\tau)-x|^2 + \delta\|w\|^2 \geq \frac{\delta c|x|^2}{4M^2\overline{K}^2}\frac{(1-e^{-c\tau})^4}{(1-e^{-4c\tau})}$$

which, using definition of $\overline{K}$

$$= \frac{\delta c|x|^2}{4M^2}\frac{(1-e^{-c\tau})^4}{(1-e^{-4c\tau})[1+(1-e^{-c\tau})^4/2]} \geq \frac{\delta c|x|^2}{8M^2}(1-e^{-c\tau})^4. \qquad (31)$$

Now we turn to the second case,

$$|X(\tau)| > |x|. \qquad (32)$$

In this case, (32) and (25) yield

$$|x|e^{-c\tau} + \frac{1}{c^{1/4}}\sqrt{M\overline{K}|x|\|w\|}(1-e^{-4c\tau})^{1/4} > |x|. \qquad (33)$$

Upon rearrangement, (33) yields

$$\|w\| > \frac{\sqrt{c}|x|}{M\overline{K}}\frac{(1-e^{-c\tau})^2}{(1-e^{-4c\tau})^{1/2}}.$$

Consequently, using the definition of $\overline{K}$ and some simple manipulations,

$$\begin{aligned} K|X(\tau)-x|^2 + \delta\|w\|^2 &\geq \frac{\delta c|x|^2(1-e^{-c\tau})^4}{M^2(1-e^{-4c\tau})[1+(1-e^{-c\tau})^4/2]} \\ &\geq \frac{\delta c|x|^2}{2M^2}(1-e^{-c\tau})^4. \qquad (34) \end{aligned}$$

Combining (31) and (34) completes the proof. □

Now we turn to how Lemma 3.1 can be used to obtain a more detailed replacement for the δ that appears in 2.10 and Theorem 2.11. Fix $\tau > 0$. Let

$$\hat{\gamma}_0^2 \in \left(\frac{2M^2\alpha}{c^2}, \gamma^2\right), \qquad (35)$$

and in particular, let $\hat{\gamma}_0^2 = \gamma^2 - \delta$ where δ is sufficiently small so that

$$\delta < \gamma^2 - \frac{2M^2\alpha}{c^2}. \qquad (36)$$

Then all results of Section 2 for W hold with γ^2 replaced by $\hat{\gamma}_0^2$, and we denote the corresponding value by $W^{\hat{\gamma}_0}$. In particular, by Theorem 2.8, for any $R < \infty$ there exists semiconvexity constant $C_R^0 < \infty$ for $W^{\hat{\gamma}_0}$ over $\overline{B}_R$, and a Lipschitz constant, L_R^0. Note that the required constants satisfy $C_R^0 < C_R$ (see proof of Theorem 2.8 as given in [24]). If $L_R^0 > L_R$ sufficiently so that $R + |C^{-1}|L_R^0 > D_R$, we modify our basis to be dense over $\overline{B}_{D_R^0}$ where $D_R^0 \geq R + |C^{-1}|L_R^0$ (and redefine $D_R \doteq D_R^0$ in that case). Then, as before, the set $\{\psi_i\}$ forms a max–plus basis for the space of semiconvex functions over $\overline{B}_R$ with semiconvexity constant, C_R^0, i.e. $\mathcal{S}^R_{C_R^0, L_R^0}$.

For any j, let

$$\overline{x}_j \in \operatorname*{argmax}_{|x| \leq R} \{\psi_j(x) - W^{\hat{\gamma}_0}(x)\}. \tag{37}$$

Then for any $x \in \overline{B}_R$,

$$\psi_j(x) - \psi_j(\overline{x}_j) \leq W^{\hat{\gamma}_0}(x) - W^{\hat{\gamma}_0}(\overline{x}_j) - K_0|x - \overline{x}_j|^2 \tag{38}$$

where $K_0 > 0$ is the minimum eigenvalue of $C - C_R^0 I > 0$. Note that K_0 depends on $\hat{\gamma}_0$.

Theorem 3.4 *Let $\hat{\gamma}_0$ satisfy (35). Let $K = K_0$ satisfy (38) (where we may take $K_0 > 0$ to be the minimum eigenvalue of $C - C_R^0 I > 0$ if desired). Let $\delta > 0$ satisfy $\delta \leq \frac{\gamma^2}{2} - \frac{\hat{\gamma}_0^2}{2}$ and (18). Then, for any $j \neq 1$,*

$$B_{j,j} \leq \frac{-\delta c |\overline{x}_j|^2}{8M^2}(1 - e^{-c\tau})^4.$$

(Recall that by the choice of ψ_1 as the basis function centered at the origin, $B_{1,1} = 0$; see Lemma 2.10.)

Proof. Let K_0, τ, δ satisfy the assumptions (i.e. (18), (36), (38)). Then

$$S_\tau[\psi_j](\overline{x}_j) - \psi_j(\overline{x}_j) = \sup_{w \in L_2} \Big\{ \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\,dt + \psi_j(X(\tau)) - \psi_j(\overline{x}_j) \Big\} \tag{39}$$

where X satisfies (4) with $X(0) = \overline{x}_j$. Let $\varepsilon > 0$, and w^ε be ε–optimal . Then this implies

$$S_\tau[\psi_j](\overline{x}_j) - \psi_j(\overline{x}_j) \leq \int_0^\tau l(X^\varepsilon(t)) - \frac{\gamma^2}{2}|w^\varepsilon(t)|^2\,dt + \psi_j(X^\varepsilon(\tau)) - \psi_j(\overline{x}_j) + \varepsilon,$$

and by (38) and the definition of $\hat{\gamma}_0$

$$\begin{aligned}&\leq \int_0^\tau l(X^\varepsilon(t)) - \frac{\hat{\gamma}_0^2}{2}|w^\varepsilon(t)|^2 - \delta|w^\varepsilon(t)|^2\,dt + W^{\hat{\gamma}_0}(X^\varepsilon(\tau)) - W^{\hat{\gamma}_0}(\overline{x}_j)\\ &\quad -K_0|X^\varepsilon(\tau) - \overline{x}_j|^2 + \varepsilon\end{aligned}$$

and by Theorem 2.2 (for $W^{\hat{\gamma}_0}$),

$$\leq -\delta\|w^\varepsilon\|^2 - K_0|X^\varepsilon(\tau) - \overline{x}_j|^2 + \varepsilon.$$

Combining this with Lemma 3.1 yields

$$S_\tau[\psi_j](\overline{x}_j) - \psi_j(\overline{x}_j) \leq \frac{-\delta c|\overline{x}_j|^2}{8M^2}(1 - e^{-c\tau})^4 + \varepsilon.$$

Since this is true for all $\varepsilon > 0$, one has

$$S_\tau[\psi_j](\overline{x}_j) - \psi_j(\overline{x}_j) \leq \frac{-\delta c|\overline{x}_j|^2}{8M^2}(1-e^{-c\tau})^4. \tag{40}$$

But,

$$B_{j,j} = \min_{|x|\leq R}\{S_\tau[\psi_j](x) - \psi_j(x)\} \tag{41}$$

which by (40)

$$\leq \frac{-\delta c|\overline{x}_j|^2}{8M^2}(1-e^{-c\tau})^4. \quad \square$$

Remark 3.5 It is interesting to note that one may modify (41) as $B_{j,j} = \min_{x\in\mathbf{R}^m}\{S_\tau[\psi_j](x) - \psi_j(x)\}$ since one has $\psi_j(x) = -\infty$ for $x \notin \overline{B}_R$. One might also note that by the nondegeneracy of σ (Assumption $(A2)$), if any function $\phi > -\infty$ on $\overline{B}_R$, then $S_\tau[\phi] > -\infty$ on $\overline{B}_R$.

Theorem 3.6 *Let $\hat{\gamma}_0$ satisfy (35). Let K_0 be as in (38), and let $\delta > 0$ be given by*

$$\delta = \min\left\{\frac{K_0M^2}{c}, \frac{\gamma^2}{2} - \frac{\hat{\gamma}_0^2}{2}\right\} \tag{42}$$

(which is somewhat tighter than the requirement in the previous theorem). Let $N \in \mathcal{N}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \leq k_i \leq n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $k_i = 1$ for all i. Then

$$\sum_{i=1}^{N} B_{k_i,k_{i+1}} \leq -\max_{k_i}|\overline{x}_{k_i}|^2\frac{\delta c}{8M^2}(1-e^{-cN\tau})^4.$$

Proof. By Theorem 3.4, this is true for $N = 1$. We prove the case $N = 2$. The proof of the general case will then be obvious. First note the monotonicity of the semi–group in the sense that if $g_1(x) \leq g_2(x)$ for all x, then

$$S_\tau[g_1](x) \leq S_\tau[g_2](x) \quad \forall\, x \in \mathbf{R}^m. \tag{43}$$

Suppose either $i \neq 1$ or $j \neq 1$. By definition, $\psi_j(x) + B_{j,i} \leq S_\tau[\psi_i](x)$ for all $x \in \mathbf{R}^m$. Using (43) and the max–plus linearity of the semi–group yields

$$S_\tau[\psi_j](x) + B_{j,i} \leq S_{2\tau}[\psi_i](x) \quad \forall\, x$$

which implies in particular that

$$S_\tau[\psi_j](\overline{x}_i) + B_{j,i} \leq S_{2\tau}[\psi_i](\overline{x}_i). \tag{44}$$

Now, employing the same proof as that of Theorem 3.4, but with τ replaced by 2τ (noting that condition (18) is satisfied with 2τ replacing τ by our assumption (42)), one has as in (40)

$$S_{2\tau}[\psi_i](\overline{x}_i) - \psi_i(\overline{x}_i) \leq \frac{-\delta c|\overline{x}_i|^2}{8M^2}(1-e^{-2c\tau})^4. \tag{45}$$

Combining (44) and (45) yields

$$\Big[S_\tau[\psi_j](\overline{x}_i) - \psi_i(\overline{x}_i)\Big] + B_{j,i} \le \frac{-\delta c|\overline{x}_i|^2}{8M^2}(1-e^{-2c\tau})^4.$$

Using the definition of $B_{i,j}$, this implies

$$B_{i,j} + B_{j,i} \le \frac{-\delta c|\overline{x}_i|^2}{8M^2}(1-e^{-2c\tau})^4. \tag{46}$$

By symmetry, one also has

$$B_{i,j} + B_{j,i} \le \frac{-\delta c|\overline{x}_j|^2}{8M^2}(1-e^{-2c\tau})^4. \tag{47}$$

Combining (46) and (47) yields

$$B_{i,j} + B_{j,i} \le -\max\Big\{|\overline{x}_i|^2, |\overline{x}_j|^2\Big\}\frac{\delta c}{8M^2}(1-e^{-2c\tau})^4. \quad \square$$

The convergence of the power method (described in the previous section) relied on a certain structure of B ($B_{1,1} = 0$ and strictly negative loop sums as described in the assumptions of Theorem 2.11). Combining this with the above result on the size of loop sums, one can obtain a condition which guarantees convergence of the power method to a unique eigenvector corresponding to eigenvalue zero. This is given in the next theorem.

Theorem 3.7 *Let B be given by $B_{j,i} = -\max_{x\in\overline{B}_R}(\psi_j(x) - S_\tau[\psi_i](x))$ for all $i,j \le n$, and let $\widetilde{B}$ be an approximation of B with $\widetilde{B}_{1,1} = 0$ and such that there exists $\varepsilon > 0$ such that*

$$|\widetilde{B}_{i,j} - B_{i,j}| \le \max\{|\overline{x}_i|^2, |\overline{x}_j|^2\}\left(\frac{\delta c}{8M^2}\right)\frac{(1-e^{-c\tau})^4}{n^2} - \varepsilon \quad \forall\, i,j \text{ such that } (i,j) \ne (1,1) \tag{48}$$

where

$$\delta = \min\Big\{\frac{K_0 M^2}{c}, \frac{\gamma^2}{2} - \frac{\hat{\gamma}_0^2}{2}\Big\}. \tag{49}$$

Then the power method applied to $\widetilde{B}$ converges in a finite number of steps to the unique eigenvector $\widetilde{e}$ corresponding to eigenvalue zero, that is

$$\widetilde{e} = \widetilde{B} \otimes \widetilde{e}.$$

Proof. Let $N \in \mathcal{N}$, and consider a sequence of nodes $\{k_i\}_{i=1}^{N+1}$ with $k_1 = k_{N+1}$. We must show that if we are not in the case $k_i = 1$ for all i, then

$$\sum_{i=1}^{N} \widetilde{B}_{k_i,k_{i+1}} < 0.$$

Suppose $N > n^2$. Then any sequence $\{k_i\}_{i=1}^{N+1}$ with $k_1 = k_{N+1}$ must be composed of subloops of length no greater than n^2. Therefore, it is sufficient to prove the result for $N \le n^2$. Note that by the assumptions and Theorem 3.6,

$$\begin{aligned}\sum_{i=1}^{N}\widetilde{B}_{k_i,k_{i+1}} &\le \sum_{i=1}^{N} B_{k_i,k_{i+1}} + \sum_{i=1}^{N}|\widetilde{B}_{k_i,k_{i+1}} - B_{k_i,k_{i+1}}| \\ &\le -\max_{k_i}|\overline{x}_{k_i}|^2\frac{\delta c}{8M^2}(1-e^{-cN\tau})^4 + \max_{k_i}|\overline{x}_{k_i}|^2\frac{\delta c}{8M^2}(1-e^{-cN\tau})^4(N/n^2) - \varepsilon \\ &\le -\varepsilon.\end{aligned}$$

Then by the same proofs as for Theorem 2.12, the result follows. □

Theorem 3.7 will be useful later when we analyze the size of errors introduced by our computational approximation to the elements of B.

If the conditions of Theorem 3.7 are met, then one can ask what the size of the errors in the corresponding eigenvector are. Specifically, if eigenvector $\tilde{e}$ is computed using approximation $\widetilde{B}$, what is a bound on the size of the difference between e (the eigenvector of B) and $\tilde{e}$? The following theorem gives a rough, but easily obtained, bound.

Theorem 3.8 *Let B be given by $B_{i,j} = -\max_{x\in\overline{B}_R}(\psi_j(x) - S_\tau[\psi_i](x))$ for all $i,j \le n$, and let $\widetilde{B}$ be an approximation of B with $\widetilde{B}_{1,1} = 0$ and such that there exists $\varepsilon > 0$ such that*

$$|\widetilde{B}_{i,j} - B_{i,j}| \le \max\{|\overline{x}_i|^2, |\overline{x}_j|^2\}\left(\frac{\delta c}{8M^2}\right)\frac{(1-e^{-c\tau})^4}{n^\mu} - \varepsilon \quad \forall\, i,j \tag{50}$$

where $\mu \in \{2,3,4,...\}$ and δ is given by (49). Then the power method will yield the unique eigenvectors e and $\tilde{e}$ of B and $\widetilde{B}$ respectively, in finite numbers of steps, and

$$\|e - \tilde{e}\| \doteq \max_i |e_i - \tilde{e}_i| \le (D_R)^2\left(\frac{\delta c}{8M^2}\right)\frac{(1-e^{-c\tau})^4}{n^{\mu-2}} - \varepsilon.$$

Proof. By Theorem 3.7, one may use the power method to compute $\tilde{e}$, and so one has that for any $j \le n^2$,

$$\tilde{e}_j = [\widetilde{B}^{n^2}\otimes 0]_j = \max_{m\le n^2}[\widetilde{B}^m \otimes 0]_j = \max_{m\le n^2}\max_{\{k_l\}_{l=1}^m,\, k_1=j}\sum_{l=1}^{m}\widetilde{B}_{k_l,k_{l+1}}$$

where the exponents on $\widetilde{B}$ represent max–plus exponentiation and the bound $m \le n^2$ follows from the fact that under the assumption, the sum around any loop other than that of the trivial loop, $\widetilde{B}_{1,1} = 0$, are strictly negative. Therefore,

$$\tilde{e}_j \le \max_{m\le n^2}\max_{\{k_l\}_{l=1}^m,\, k_1=j}\left[\sum_{l=1}^{m}|\widetilde{B}_{k_l,k_{l+1}} - B_{k_l,k_{l+1}}| + \sum_{l=1}^{m}B_{k_l,k_{l+1}}\right]$$

which by the assumption (50) and the fact that e is the eigenvector of B,

$$\le (D_R)^2\left(\frac{\delta c}{8M^2}\right)\frac{(1-e^{-c\tau})^4}{n^{\mu-2}} - \varepsilon + e_j.$$

By a symmetrical argument, one obtains

$$|\tilde{e}_j - e_j| \le (D_R)^2\left(\frac{\delta c}{8M^2}\right)\frac{(1-e^{-c\tau})^4}{n^{\mu-2}} - \varepsilon. \;\square$$

We remark that by taking ε sufficiently small, and noting that $1-e^{-c\tau} \leq c\tau$ for nonnegative τ, Theorem 3.8 implies (under its assumptions)

$$\|e-\tilde{e}\| = \max_i |e_i - \tilde{e}_i| \leq (D_R)^2 \left(\frac{\delta c^5}{8M^2}\right) \frac{\tau^4}{n^{\mu-2}}. \tag{51}$$

Also note that aside from the case $i=j=1$ (recall $B_{1,1}=0$), one has

$$\min_{i\neq 1}\{|\overline{x}_i|^2\} \leq \max\{|\overline{x}_i|^2, |\overline{x}_j|^2\} \quad \forall\, i,j.$$

Using this, and choosing $\varepsilon > 0$ appropriately, one has the following theorem (where we note the condition on the errors in B is uniform but potentially significantly stricter). The proof is nearly identical to that for Theorem 3.8

Theorem 3.9 *Let B be as in Theorem 3.8, and let $\widetilde{B}$ be an approximation of B with $\widetilde{B}_{1,1} = 0$ and such that*

$$|\widetilde{B}_{i,j} - B_{i,j}| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\} \left(\frac{\delta c}{9M^2}\right) \frac{(1-e^{-c\tau})^4}{n^{\mu}} \quad \forall\, i,j \tag{52}$$

where $\mu \in \{2,3,4,...\}$ and δ is given by (49). Then the power method will yield the unique eigenvectors e and $\tilde{e}$ of B and $\widetilde{B}$ respectively, in finite numbers of steps, and

$$\|e-\tilde{e}\| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\} \left(\frac{\delta c}{9M^2}\right) \frac{(1-e^{-c\tau})^4}{n^{\mu-2}}.$$

A simpler variant on this result may be worth using. Note that for $\tau \in [0,1/c]$, one has $1-e^{-c\tau} \geq (c/2)\tau$. Then by a proof again nearly identical to that of Theorem 3.8, one has

Theorem 3.10 *Suppose $\tau \leq 1/c$. Let B be as in Theorem 3.8, and let $\widetilde{B}$ be an approximation of B with $\widetilde{B}_{1,1} = 0$ and such that*

$$|\widetilde{B}_{i,j} - B_{i,j}| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\} \left(\frac{\delta c^5}{9(16)M^2}\right) \frac{\tau^4}{n^{\mu}} \quad \forall\, i,j \tag{53}$$

where $\mu \in \{2,3,4,...\}$ and δ is given by (49). Then the power method will yield the unique eigenvectors e and $\tilde{e}$ of B and $\widetilde{B}$ respectively, in finite numbers of steps, and

$$\|e-\tilde{e}\| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\} \left(\frac{\delta c^5}{9(16)M^2}\right) \frac{\tau^4}{n^{\mu-2}}.$$

This variant is included since the simpler right hand sides might simplify analysis.

4 Convergence and Truncation Errors

In this section we consider the approximation due to using only a finite number of functions in the max–plus basis expansion. It will be shown that as the number of functions increases (in a reasonable way), the approximate solution obtained by the eigenvector computation of Section 2 converges from below to the value function, W. Error bounds will also be obtained.

4.1 Convergence

This subsection contains a quick proof that the errors due to truncation of the basis go to zero as the number of basis functions increases (in a reasonable way). No specific error bounds are obtained; those require the more tedious analysis of the next subsection.

Note that in this subsection, a slightly different notation for the indexing and numbers of basis functions in the sets of basis functions is used. This will make the proof simpler. *This alternate notation appears only in this subsection.* Specifically, let us have the sets of basis functions indexed by n, that is the *sets* are indexed by n. Let the cardinality of the n^{th} set be $\mathcal{I}^{(n)}$. For each n, let $\mathcal{X}^{(n)} \doteq \{x_i^{(n)}\}_{i=1}^{\mathcal{I}^{(n)}}$ and $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$. For instance, in the one-dimensional case, one might have $\mathcal{X}^{(1)} = \{0\}$, $\mathcal{X}^{(2)} = \{-1/2, 0, 1/2\}$, $\mathcal{X}^{(3)} = \{-3/4, -1/2, -1/4, 0, 1/4, 1/2, 3/4\}$, and so on. Further, we will let the basis functions be given by $\psi_i^{(n)} \doteq \frac{-1}{2}(x - x_i^{(n)})^T C(x - x_i^{(n)})$, and consider the sets of basis functions $\Psi^{(n)} \doteq \{\psi_i^{(n)} : i \in \mathcal{I}^{(n)}\}$. Then define the approximations to the semigroup operator, S_τ by

$$S_\tau^{(n)}[\phi](x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x) \tag{54}$$

where

$$a_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - S_\tau[\phi](x)\right]. \tag{55}$$

In other words, $S_\tau^{(n)}$ is the result of the application of the S_τ followed by the truncation due to a finite number of basis functions. More specifically, if one defines $\mathcal{T}^{(n)}[\phi](x) = \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x)$ with the $a_i^{(n)}$ given by (55), then $S_\tau^{(n)}[\phi] = \mathcal{T}^{(n)} \circ S_\tau[\phi]$. Also, let $\mathcal{Y}^{(n)} = \{\phi : \overline{B}_R(0) \to \mathbf{R} | \, \exists \{a_i^{(n)}\}$ such that $\phi(x) = \bigoplus_{i=1}^{\mathcal{I}^n} a_i^{(n)} \otimes \psi_i^{(n)}(x) \quad \forall x \in \overline{B}_R(0)\}$. Then note that for $\phi \in \mathcal{Y}^{(n)}$, one has

$$S_\tau^{(n)}[\phi](x) = \bigoplus_{i=1}^{\mathcal{I}^n} \left[\bigoplus_{j=1}^{\mathcal{I}^n} B_{i,j}^{(n)} \otimes a_j^{(n)}\right] \otimes \psi_i^{(n)}(x) \tag{56}$$

for where $B_{i,j}^{(n)}$ corresponds to $S_\tau^{(n)}[\phi]$.

Lastly, we use the notation ${S_\tau}^N$ to indicate repeated application of S_τ N times. (Of course, by the semigroup property, ${S_\tau}^N = S_{N\tau}$.) Correspondingly, we use the notation ${S_\tau^{(n)}}^N$ to indicate the application of $S_\tau^{(n)}$ N times.

Define $\phi_0(x) \equiv 0$ and

$$\phi_0^{(n)}(x) \doteq \bigoplus_{i=1}^{\mathcal{I}^n} {a^0}_i^{(n)} \otimes \psi_i^{(n)}(x), \qquad {a^0}_i^{(n)} \doteq -\max_x \left[\psi_i^{(n)}(x) - \phi_0(x)\right]. \tag{57}$$

It is well-known that (see [29], [32] among many others) that

$$\lim_{N\to\infty} {S_\tau}^N[\phi_0] = W. \tag{58}$$

Also, note that since $\mathcal{X}^{(n)} \subset \mathcal{X}^{(n+1)}$, one has

$${S_\tau^{(n)}}^N[\phi_0^{(n)}](x) \le {S_\tau}^{(n+1)^N}[\phi_0^{(n+1)}](x) \le {S_\tau}^N[\phi_0](x) \tag{59}$$

for all $x \in B_R$.

Note that by (57), and the definition of ϕ_0, the corresponding coefficients, $a^{0(n)}_i$, satisfy $a^{0(n)}_i = 0$ for all i. Combining this with Theorem 2.12 and (56), one finds that for each n, there exists $\overline{N}(n)$ such that

$$S_\tau^{(n)^N}[\phi_0^{(n)}] = S_\tau^{(n)^{\overline{N}(n)}}[\phi_0^{(n)}] \quad \forall\, N \geq \overline{N}(n). \tag{60}$$

Defining

$$W^{(n)^\infty} \doteq S_\tau^{(n)^{\overline{N}(n)}}[\phi_0^{(n)}], \tag{61}$$

we further find that the limit is the fixed point. That is,

$$S_\tau^{(n)}[W^{(n)^\infty}] = W^{(n)^\infty}. \tag{62}$$

Then, by (58), (59) and (61), we find that

$$W^{(n)^\infty} \text{ is monotonically increasing in } n \tag{63}$$

and

$$W^{(n)^\infty} \leq W. \tag{64}$$

Therefore, there exists $W^{\infty\infty} \leq W$ such that

$$W^{(n)^\infty} \uparrow W^{\infty\infty}, \tag{65}$$

and in fact, one can demonstrate equicontinuity of the $W^{(n)^\infty}$ on $\overline{B}_R$ given the assumptions (and consequently uniform convergence).

Under Assumption ($A5$), one can show (see for instance Lemma 4.3, although this is more specific than what is is needed, or Theorem 3.3 in [27]) that given $\varepsilon > 0$, there exists $n_\varepsilon < \infty$ such that

$$W^{(n)^\infty}(x) = S_\tau^{(n)}[W^{(n)^\infty}](x) \geq S_\tau[W^{(n)^\infty}](x) - \varepsilon$$

for all $x \in B_R$ for any $n \geq n_\varepsilon$. On the other hand, one always has

$$S_\tau^{(n)}[\phi] \leq S_\tau[\phi].$$

Combining these last two inequalities, one obtains

$$W^{(n)^\infty} = S_\tau^{(n)}[W^{(n)^\infty}] \leq S_\tau[W^{(n)^\infty}] \leq S_\tau^{(n)}[W^{(n)^\infty}] + \varepsilon = W^{(n)^\infty} + \varepsilon. \tag{66}$$

Combining this with (65), one finds

Theorem 4.1

$$W^{\infty\infty} = S_\tau[W^{\infty\infty}], \tag{67}$$

or in other words, $W^{\infty\infty}$ is a fixed point of S_τ.

Then, with some more work (see [27], Theorem 3.2), one obtains a convergence theorem.

Theorem 4.2

$$W^{\infty\infty}(x) = W(x) \quad \forall\, x \in \overline{B}_R.$$

4.2 Truncation Error Estimate

Theorem 4.2 demonstrates convergence of the algorithm to the value function as the basis function density increases. Here we outline one approach to obtaining specific error estimates. The estimates may be rather conservative due to the form of the truncation error bound used; this question will become more clear below. The main results are in Theorem 4.5 and Remark 4.6. Note that these are only the errors due to truncation to a finite number of basis functions; as noted above, analysis of the errors due to approximation of the entries in the B matrix is discussed further below.

Recall that we choose the basis functions throughout such that $x_1^{(n)} = 0$, or in other words, $\psi_1^{(n)}(x) = \frac{-1}{2}x^T Cx$ for all n. (Note that we return here to the notation where the (n) superscript corresponds to the number of basis functions - as opposed to the more complex notation with cardinality $\mathcal{I}^{(n)}$ which was used in the previous subsection only.) Also, we will use the notation

$$W_{N,\tau}^{(n)}(x) \doteq {S_\tau{}^{(n)}}^N[\phi_0^{(n)}](x)$$

and we reiterate that the N superscript indicates repeated application of the operator N times. Also, $\phi_0^{(n)}$ is the finite basis expansion of ϕ_0 (with n basis fucntions).

To specifically set C, we will replace Assumption $(A5)$ of Section 2 with the following. We assume throughout the remainder of the paper that one may choose matrix $C > 0$ and $\delta' \in (0,1)$ such that with $C' \doteq (1-\delta')C$

$$S_\tau[\psi_i] \in \mathcal{S}^R_{C',L'} \text{ for all } i \tag{$A5'$}$$

where $R + |C^{-1}|L' \le D_R$. Again, we do not discuss this assumption in detail, but simply note that we have verified that this assumption holds for the problems we have run. Also note that one could be more general, allowing C' to be a more general positive definite symmetric matrix such that $C - C' > 0$, but we will not include that here. Finally, it should be noted that δ' would depend on τ; as $\tau \downarrow 0$, one would need to take $\delta' \downarrow 0$. Since δ' will appear in the denominator of the error bound of the next lemma (as well as implicitly in the denominator of the fraction on the right–hand side of the error bound in Theorem 4.5), this implies that one does not want to take $\tau \downarrow 0$ as the means for reducing the errors. This will be discussed further in the next section.

The following lemma is a general result about the errors due to truncation when using the above max–plus basis expansion.

Lemma 4.3 *Let δ', C', L' be as in Assumption $(A5')$, and let $\phi \in \mathcal{S}_{R,C'}$ with $\phi(0) = 0$, ϕ differentiable at zero with $\nabla_x\phi(0) = 0$, and $-\frac{1}{2}x^T C'x \le \phi(x) \le \frac{1}{2}\widehat{\mathcal{M}}|x|^2$ for all x for some $\widehat{\mathcal{M}} < \infty$. Let $\{\psi_i\}_{i=1}^n$ consist of basis functions with matrix C, centers $\{x_i\} \subseteq \overline{B}_{D_R}$ such that $C - C'I > 0$, and let $\Delta \doteq \max_{x\in\overline{B}_{D_R}(0)} \min_i |x - x_i|$ Let*

$$\phi^\Delta(x) = \max_i[a_i + \psi_i(x)] \quad \forall\, x \in \overline{B}_R$$

where

$$a_i = -\max_{x\in\overline{B}_R}[\psi_i(x) - \phi(x)] \quad \forall\, i.$$

Then

$$0 \le \phi(x) - \phi^\Delta(x) \le \begin{cases} |C|\Big[2\widehat{\beta} + 1 + |C|/(\delta' C_R)\Big]|x|\Delta & \text{if } |x| \ge \Delta \\ \frac{1}{2}[\widehat{\mathcal{M}} + |C|]|x|\Delta & \text{otherwise} \end{cases}$$

where $\widehat{\beta}$ is specified in the proof.

Proof. Note that (see [12])

$$\phi(x) = \max_{\widetilde{x}\in\overline{B}_{D_R}} [a(\widetilde{x}) + \psi_{\widetilde{x}}(x)] \quad \forall\, x \in \overline{B}_R(0)$$

where

$$a(\widetilde{x}) = -\max_{x\in\overline{B}_R}[\psi_{\widetilde{x}}(x) - \phi(x)] \quad \forall\, \widetilde{x} \in \overline{B}_{D_R},$$

and

$$\psi_{\widetilde{x}}(x) \doteq -\tfrac{1}{2}(x-\widetilde{x})^T C(x-\widetilde{x}) \quad \forall\, x \in \overline{B}_R(0),\, \widetilde{x} \in \overline{B}_{D_R}.$$

It is obvious that $0 \le \phi(x) - \phi^{\Delta}(x)$, and so we prove the other bound.

Consider any $\overline{x} \in \overline{B}_R$. Then

$$\phi(\overline{x}) = a(\widetilde{x}) + \psi_{\widetilde{x}}(\overline{x}) \tag{68}$$

if and only if

$$C(\overline{x} - \widetilde{x}) \in -D_x^-\phi(\overline{x})$$

where

$$D_x^-\phi(x) = \left\{ p \in \mathbf{R}^m : \ \liminf_{|y-x|\to 0} \frac{\phi(y) - \phi(x) - (y-x)\cdot p}{|y-x|} \ge 0 \right\}.$$

We denote such an $\widetilde{x}$ corresponding to $\overline{x}$ (in (68)) as $\overline{\widetilde{x}}$. By the Lipschitz nature of ϕ, one can easily establish that

$$|\overline{\widetilde{x}} - \overline{x}| \le |C^{-1}|L'. \tag{69}$$

However, it will be desirable to have a bound where the right–hand side depends linearly on $|\overline{x}|$. (Actually, this may only be necessary for small $\overline{x}$, while (69) may be a smaller bound for large $\overline{x}$, but we will obtain it for general $\overline{x}$.) Noting that $\phi \ge -\frac{1}{2}x^T C' x \ge -\frac{1}{2}x^T C x$, one has

$$\tfrac{1}{2}(\overline{x} - \overline{\widetilde{x}})^T C(\overline{x} - \overline{\widetilde{x}}) \le a(\overline{\widetilde{x}}) + \tfrac{1}{2}\overline{x}^T C\overline{x}.$$

Also, since $a(\overline{\widetilde{x}}) + \psi_{\overline{\widetilde{x}}}(\cdot)$ touches ϕ from below at $\overline{x}$, one must have

$$\begin{aligned}
\tfrac{1}{2}(\overline{x} - \overline{\widetilde{x}})^T C(\overline{x} - \overline{\widetilde{x}}) - \tfrac{1}{2}(x - \overline{\widetilde{x}})^T C(x - \overline{\widetilde{x}}) &\le a(\overline{\widetilde{x}}) + \tfrac{1}{2}\overline{x}^T C\overline{x} - \tfrac{1}{2}(x - \overline{\widetilde{x}})^T C(x - \overline{\widetilde{x}}) \\
&\le \phi(x) + \tfrac{1}{2}\overline{x}^T C\overline{x} \le \tfrac{1}{2}\widehat{\mathcal{M}}|x|^2 + \tfrac{1}{2}\overline{x}^T C\overline{x}
\end{aligned}$$

for all $x \in \overline{B}_R$ where the last inequality is by assumption. Define

$$F(x) \doteq \tfrac{1}{2}(\overline{x} - \overline{\widetilde{x}})^T C(\overline{x} - \overline{\widetilde{x}}) - \tfrac{1}{2}(x - \overline{\widetilde{x}})^T C(x - \overline{\widetilde{x}}) - \tfrac{1}{2}\widehat{\mathcal{M}}|x|^2,$$

and we see that we require $F(x) \le \frac{1}{2}\overline{x}^T C\overline{x}$ for all $x \in \overline{B}_R$. Taking the derivative, we find the maximum of F at $\widehat{x}$ given by

$$\widehat{x} = (C + \widehat{\mathcal{M}}I)^{-1} C\overline{\widetilde{x}} \tag{70}$$

and so

$$\widehat{x} - \overline{\widetilde{x}} = -\widehat{\mathcal{M}}(C + \widehat{\mathcal{M}}I)^{-1}\overline{\widetilde{x}}. \tag{71}$$

(In the interests of readbility, we ignore the detail of the case where $\widehat{x} \notin B_R(0)$ here.) Therefore, $F(\widehat{x}) \le \frac{1}{2}\overline{x}^T C\overline{x}$ implies

$$\begin{aligned}(\overline{x}-\overline{\overline{x}})^T C(\overline{x}-\overline{\overline{x}}) &\le \overline{\overline{x}}^T \widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1}C(C+\widehat{\mathcal{M}}I)^{-1}\widehat{\mathcal{M}}\overline{\overline{x}} + \overline{\overline{x}}^T C(C+\widehat{\mathcal{M}}I)^{-1}\widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1}C\overline{\overline{x}} \\ &\quad + \overline{x}^T C\overline{x} \\ &= \widehat{\mathcal{M}}\overline{\overline{x}}^T \Big[\widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1}C(C+\widehat{\mathcal{M}}I)^{-1} + \widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1}\widehat{\mathcal{M}}I(C+\widehat{\mathcal{M}}I)^{-1} \\ &\qquad -\widehat{\mathcal{M}}^2(C+\widehat{\mathcal{M}}I)^{-2} + C(C+\widehat{\mathcal{M}}I)^{-2}C\Big]\overline{\overline{x}} + \overline{x}^T C\overline{x} \\ &= \widehat{\mathcal{M}}\overline{\overline{x}}^T \Big[\widehat{\mathcal{M}}C(C+\widehat{\mathcal{M}}I)^{-2} + C(C+\widehat{\mathcal{M}}I)^{-2}C\Big]\overline{\overline{x}} + \overline{x}^T C\overline{x} \\ &= \overline{\overline{x}}^T \widehat{\mathcal{M}}C(C+\widehat{\mathcal{M}}I)^{-1}\overline{\overline{x}} + \overline{x}^T C\overline{x}. \end{aligned} \tag{72}$$

Noting that C is positive definite symmetric, and writing it as $C = \sqrt{C}\sqrt{C}^T$ where $\sqrt{C} = S\sqrt{\Lambda}$ with S unitary and Λ the matrix of eigenvalues, one may rewrite the first term in the right–hand side of (72) as

$$\overline{\overline{x}}^T \widehat{\mathcal{M}}C(C+\widehat{\mathcal{M}}I)^{-1}\overline{\overline{x}} = \overline{\overline{x}}^T \widehat{\mathcal{M}}\tfrac{1}{2}\Big[C(C+\widehat{\mathcal{M}}I)^{-1} + (C+\widehat{\mathcal{M}}I)^{-1}C\Big]\overline{\overline{x}} = \overline{\overline{x}}^T \sqrt{C}Q\sqrt{C}^T\overline{\overline{x}}$$

where

$$Q \doteq \tfrac{1}{2}\widehat{\mathcal{M}}\Big[\sqrt{C}^T(C+\widehat{\mathcal{M}}I)^{-1}\sqrt{C}^{-T} + \sqrt{C}^{-1}(C+\widehat{\mathcal{M}}I)^{-1}\sqrt{C}\Big].$$

Making the change of variables $y = \sqrt{C}^T x$, (72) becomes

$$|\overline{y}-\overline{\overline{y}}|^2 \le \overline{\overline{y}}^T Q\overline{\overline{y}} + |\overline{y}|^2.$$

Noting that $\sqrt{C}^T(C+\widehat{\mathcal{M}}I)^{-1}\sqrt{C}^{-T}$ is a similarity transform of $(C+\widehat{\mathcal{M}}I)^{-1}$, one sees that the eigenvalues of Q are the eigenvalues of $\widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1}$. Now, since $(C+\widehat{\mathcal{M}}I)$ is positive definite,

$$(C+\widehat{\mathcal{M}}I) = \overline{S}\,\overline{\Lambda}\,\overline{S}^{-1}$$

with $\overline{\Lambda}$ the diagonal matrix of eigenvalues and $\overline{S}$ the unitary matrix of eigenvectors. Therefore, $\widehat{\mathcal{M}}(C+\widehat{\mathcal{M}}I)^{-1} = \overline{S}(\widehat{\mathcal{M}}\overline{\Lambda}^{-1})\overline{S}^{-1}$, and note that $\beta \doteq \max_i\{\widehat{\mathcal{M}}\overline{\lambda}_i^{-1}\} < 1$ where the $\overline{\lambda}_i$ are the diagonal elements of $\overline{\Lambda}$. Consequently,

$$|\overline{y}-\overline{\overline{y}}|^2 \le \beta|\overline{\overline{y}}|^2 + |\overline{y}|^2 \tag{73}$$

where $\beta \in (0,1)$. This implies

$$\begin{aligned}|\overline{\overline{y}}-\overline{y}|^2 &\le \beta|\overline{\overline{y}}-\overline{y}+\overline{y}|^2 + |\overline{y}|^2 \\ &= \beta\Big[|\overline{\overline{y}}-\overline{y}|^2 + |\overline{y}|^2 + 2(\overline{\overline{y}}-\overline{y})\cdot\overline{y}\Big] + |\overline{y}|^2 \\ &\le \beta|\overline{\overline{y}}-\overline{y}|^2 + (\beta+1)|\overline{y}|^2 + \beta\Big[\frac{(1-\beta)/2}{\beta}|\overline{\overline{y}}-\overline{y}|^2 + \frac{\beta}{(1-\beta)/2}|\overline{y}|^2\Big],\end{aligned}$$

which after some rearrangement, yields

$$|\overline{\overline{y}}-\overline{y}|^2 \le \frac{2(1+\beta^2)}{(1-\beta)^2}|\overline{y}|^2 \tag{74}$$

which implies

$$(\overline{x}-\overline{\overline{x}})^T C(\overline{x}-\overline{\overline{x}}) \le \left[\frac{2(1+\beta^2)}{(1-\beta)^2}\right]\overline{x}^T C\overline{x}.$$

Consequently, there exists $\widehat{\beta}<\infty$ (i.e. $\widehat{\beta}=[|\sqrt{C}|/C_R^0][\sqrt{2(1+\beta^2)}/(1-\beta)]$) such that

$$|\overline{x}-\overline{\overline{x}}| \le \widehat{\beta}|\overline{x}|. \tag{75}$$

Given $\overline{\overline{x}}$, let $\bar{i}\in \operatorname{argmin}_i |x_i-\overline{\overline{x}}|$, and note that

$$|x_{\bar{i}}-\overline{\overline{x}}| \le \Delta. \tag{76}$$

It is easy to see that

$$\begin{aligned}
|\psi_{\overline{\overline{x}}}(x)-\psi_{\bar{i}}(x)| &\le \tfrac{1}{2}|(x-\overline{\overline{x}})^T C(x-\overline{\overline{x}})-(x-\overline{\overline{x}})^T C(x-x_{\bar{i}})| \\
&\quad +\tfrac{1}{2}|(x-\overline{\overline{x}})^T C(x-x_{\bar{i}})-(x-x_{\bar{i}})^T C(x-x_{\bar{i}})| \\
&\le \tfrac{1}{2}|C|\Big[|\overline{\overline{x}}-x_{\bar{i}}||x-\overline{\overline{x}}|+|\overline{\overline{x}}-x_{\bar{i}}||x-x_{\bar{i}}|\Big] \\
&\le \tfrac{1}{2}|C|\Big[|\overline{\overline{x}}-x_{\bar{i}}||x-\overline{\overline{x}}|+|\overline{\overline{x}}-x_{\bar{i}}|(|x-\overline{\overline{x}}|+|\overline{\overline{x}}-x_{\bar{i}}|)\Big]
\end{aligned}$$

which by (76)

$$\le |C|\Big[|x-\overline{\overline{x}}|\Delta+\tfrac{1}{2}\Delta^2\Big]. \tag{77}$$

Combining (75) and (77), one finds

$$|\psi_{\overline{\overline{x}}}(\overline{x})-\psi_{\bar{i}}(\overline{x})| \le |C|\Big[\widehat{\beta}|\overline{x}|\Delta+\tfrac{1}{2}\Delta^2\Big]. \tag{78}$$

Now note that

$$\phi(\overline{x})-\phi^\Delta(\overline{x}) \le a(\overline{\overline{x}})+\psi_{\overline{\overline{x}}}(\overline{x})-[a_{\bar{i}}+\psi_{\bar{i}}(\overline{x})]$$

which by (78)

$$\le |C|\Big[\widehat{\beta}|\overline{x}|\Delta+\tfrac{1}{2}\Delta^2\Big]+a(\overline{\overline{x}})-a_{\bar{i}}. \tag{79}$$

We now deal with the last two terms in this bound. Let

$$\overline{x}_{\bar{i}} \doteq \operatorname*{argmax}_{x\in\overline{B}_R}[\psi_{\bar{i}}(x)-\phi(x)].$$

(Note that we will also skip the technical details of the additional case where $\overline{x}_{\bar{i}}$ lies on the boundary of $\overline{B}_R$.) Then,

$$-C(\overline{x}_{\bar{i}}-x_{\bar{i}}) \in D^-\phi(\overline{x}_{\bar{i}})$$

and

$$-C(\overline{x}-\overline{\overline{x}}) \in D^-\phi(\overline{x}).$$

By the semiconvexity, one has the general result that $p\in D^-\phi(x)$, $q\in D^-\phi(y)$ implies

$$(p-q)\cdot(x-y) \ge -(x-y)^T C'(x-y).$$

Consequently,

$$-(\overline{x}_{\overline{i}} - x_{\overline{i}} + \overline{\overline{x}} - \overline{x})^T C(\overline{x}_{\overline{i}} - \overline{x}) \geq -(\overline{x}_{\overline{i}} - \overline{x})^T C'(\overline{x}_{\overline{i}} - \overline{x}).$$

Recalling that $C' = (1-\delta')C$, we see that this implies

$$-(\overline{x}_{\overline{i}} - \overline{x})^T C(\overline{x}_{\overline{i}} - \overline{x}) + (1-\delta')(\overline{x}_{\overline{i}} - \overline{x})^T C(\overline{x}_{\overline{i}} - \overline{x}) \geq -|C|\,|\overline{x}_{\overline{i}} - \overline{x}|\,|x_{\overline{i}} - \overline{\overline{x}}| \geq -|C|\,|\overline{x}_{\overline{i}} - \overline{x}|\Delta,$$

or

$$\delta'(\overline{x}_{\overline{i}} - \overline{x})^T C(\overline{x}_{\overline{i}} - \overline{x}) \leq |C|\,|\overline{x}_{\overline{i}} - \overline{x}|\Delta.$$

Noting that $C - C_R I > 0$, this implies

$$|\overline{x}_{\overline{i}} - \overline{x}| \leq \frac{|C|}{\delta' C_R}\Delta. \tag{80}$$

Now,

$$\begin{aligned}\overline{\overline{a}} - a_{\overline{i}} &\leq \psi_{\overline{i}}(\overline{x}_{\overline{i}}) - \psi_{\overline{\overline{x}}}(\overline{x}_{\overline{i}}) \\ &= \psi_{\overline{i}}(\overline{x}) - \psi_{\overline{\overline{x}}}(\overline{x}) + [\psi_{\overline{i}}(\overline{x}_{\overline{i}}) - \psi_{\overline{\overline{x}}}(\overline{x}_{\overline{i}})] - [\psi_{\overline{i}}(\overline{x}) - \psi_{\overline{\overline{x}}}(\overline{x})]\end{aligned}$$

which, after cancellation,

$$\begin{aligned}&= \psi_{\overline{i}}(\overline{x}) - \psi_{\overline{\overline{x}}}(\overline{x}) - (\overline{x} - \overline{x}_{\overline{i}})C(x_{\overline{i}} - \overline{\overline{x}}) \\ &\leq |\psi_{\overline{i}}(\overline{x}) - \psi_{\overline{\overline{x}}}(\overline{x})| + |C|\Delta|\overline{x} - \overline{x}_{\overline{i}}|\end{aligned}$$

which by (78) and (80)

$$\leq |C|\Big[\widehat{\beta}|\overline{x}| + (\tfrac{1}{2} + |C|/(\delta' C_R))\Delta\Big]\Delta. \tag{81}$$

Combining (79) and (81) yields

$$\phi(\overline{x}) - \phi^{\Delta}(\overline{x}) \leq |C|\Big[2\widehat{\beta}|\overline{x}| + (1 + |C|/(\delta' C_R))\Delta\Big]\Delta. \tag{82}$$

Suppose $|\overline{x}| \geq \Delta$. Then, (82) implies

$$\phi(\overline{x}) - \phi^{\Delta}(\overline{x}) \leq |C|\Big[2\widehat{\beta} + 1 + |C|/(\delta' C_R)\Big]|\overline{x}|\Delta \tag{83}$$

which is the first case in right hand side of the assertion.

Lastly, suppose $|\overline{x}| < \Delta$. By assumption, there exists $\widehat{\mathcal{M}} < \infty$ such that $\phi(x) \leq \frac{1}{2}\widehat{\mathcal{M}}|x|^2$. Therefore,

$$\phi(\overline{x}) - \phi^{\Delta}(\overline{x}) \leq \tfrac{1}{2}(\widehat{\mathcal{M}} + |C|)|\overline{x}|^2 \leq \tfrac{1}{2}(\widehat{\mathcal{M}} + |C|)|\overline{x}|\Delta$$

which completes the proof. □

The above lemma is a general result about the errors due to truncation with the above max–plus basis expansion. In order to apply this to the problem at hand, one must consider the effect of repeated application of the truncated operator $S_\tau^{(n)}$. Note that $S_\tau^{(n)}$ may be written as the composition of S_τ and a truncation operator, $\mathcal{T}^{(n)}$ where we have

$$\mathcal{T}^{(n)}[\phi] = \phi^{\Delta}$$

in the notation of the previous lemma, where in particular, ϕ^Δ was given by

$$\phi^\Delta(x) = \max_i [a_i + \psi_i(x)] \quad \forall\, x \in \overline{B}_R$$

where

$$a_i = -\max_{x \in \overline{B}_R(0)} [\psi_i(x) - \phi(x)] \quad \forall\, i.$$

In other words, one has the following equivalence of notation

$$S_\tau^{(n)}[\phi] = \{\mathcal{T}^{(n)} \circ S_\tau\}[\phi] = \{S_\tau[\phi]\}^\Delta \tag{84}$$

which we shall use freely throughout.

We now proceed to consider how truncation errors accumulate. In order to simplify the analysis, we simply let

$$\mathcal{M}_{C'} \doteq \max\Big\{|C|[2\widehat{\beta} + 1 + |C|/(\delta' C_R)],\ \tfrac{1}{2}[\widehat{\mathcal{M}} + |C|]\Big\}.$$

Fix Δ. We suppose that we have n sufficiently large (with properly distributed basis function centers) so that

$$\max_{x \in \overline{B}_{D_R}} \min_i |x - x_i| \le \Delta.$$

Let ϕ_0 satisfy the conditions on ϕ in Lemma 4.3. (One can simply take $\phi_0 \equiv 0$.) Then, by Lemma 4.3,

$$\phi_0(x) - \mathcal{M}_{C'}|x|\Delta \le \phi_0^{(n)}(x) \le \phi_0(x) \quad \forall\, x \in \overline{B}_R(0). \tag{85}$$

Now, for any $x \in \overline{B}_R(0)$, let $w_x^{1,\overline{\varepsilon}}$ be $\overline{\varepsilon}/2$–optimal for $S_\tau[\phi_0](x)$, and let $X_x^{1,\overline{\varepsilon}}$ be the corresponding trajectory. Then,

$$\begin{aligned} 0 &\le S_\tau[\phi_0](x) - S_\tau[\phi_0^{(n)}](x) \\ &\le \phi_0(X_x^{1,\overline{\varepsilon}}(\tau)) - \phi_0^{(n)}(X_x^{1,\overline{\varepsilon}}(\tau)) + \frac{\overline{\varepsilon}}{2} \end{aligned}$$

which by (85)

$$\le \mathcal{M}_{C'}|X_x^{1,\overline{\varepsilon}}(\tau)|\Delta + \frac{\overline{\varepsilon}}{2}. \tag{86}$$

Proceeding along, one then finds

$$\begin{aligned} 0 &\le S_\tau[\phi_0](x) - S_\tau^{(n)}[\phi_0^{(n)}](x) \\ &= S_\tau[\phi_0](x) - S_\tau[\phi_0^{(n)}](x) + S_\tau[\phi_0^{(n)}](x) - S_\tau^{(n)}[\phi_0^{(n)}](x) \end{aligned}$$

which by Lemma 4.3, the fact that $S_\tau[\phi_0^{(n)}] \in \mathcal{S}^R_{C',L}$ (by Assumption $(A5')$), and (86)

$$\le \mathcal{M}_{C'}|X_x^{1,\overline{\varepsilon}}(\tau)|\Delta + \mathcal{M}_{C'}|x|\Delta + \frac{\overline{\varepsilon}}{2}. \tag{87}$$

Let us proceed one more step with this approach. For any $x \in \overline{B}_R(0)$, let $w_x^{2,\overline{\varepsilon}}$ be $\overline{\varepsilon}/4$–optimal for $S_\tau[S_\tau[\phi_0]](x)$ (that is $\overline{\varepsilon}/4$–optimal for problem S_τ with terminal cost $S_\tau[\phi_0]$), and let $X_x^{2,\overline{\varepsilon}}$ be the corresponding trajectory. Then, as before,

$$\begin{aligned} 0 &\le S_{2\tau}[\phi_0](x) - S_\tau[S_\tau^{(n)}\phi_0^{(n)}](x) \\ &= S_\tau[S_\tau[\phi_0]](x) - S_\tau[S_\tau^{(n)}[\phi_0^{(n)}]](x) \\ &\le S_\tau[\phi_0](X_x^{2,\overline{\varepsilon}}(\tau)) - S_\tau^{(n)}[\phi_0^{(n)}](X_x^{2,\overline{\varepsilon}}(\tau)) + \frac{\overline{\varepsilon}}{4}. \end{aligned} \tag{88}$$

Now let

$$w_2^{\overline{\varepsilon}}(t) \doteq \begin{cases} w_x^{2,\overline{\varepsilon}}(t) & \text{if } t \in [0,\tau] \\ w^{1,\overline{\varepsilon}}_{X_x^{2,\overline{\varepsilon}}(\tau)}(t-\tau) & \text{if } t \in (\tau, 2\tau], \end{cases}$$

and let $\overline{X}_x^{2,\overline{\varepsilon}}$ be the corresponding trajectory. Then combining (87) and (88), one has

$$\begin{aligned} 0 &\le S_{2\tau}[\phi_0](x) - S_\tau[S_\tau^{(n)}\phi_0^{(n)}](x) \\ &\le \mathcal{M}_{C'}|\overline{X}_x^{2,\overline{\varepsilon}}(2\tau)|\Delta + \mathcal{M}_{C'}|\overline{X}_x^{2,\overline{\varepsilon}}(\tau)|\Delta + \frac{\overline{\varepsilon}}{2} + \frac{\overline{\varepsilon}}{4}. \end{aligned} \tag{89}$$

Applying Lemma 4.3 again, but now using (89), one has

$$\begin{aligned} 0 &\le S_{2\tau}[\phi_0](x) - S_\tau^{(n)}[S_\tau^{(n)}[\phi_0^{(n)}]](x) \\ &= S_\tau[S_\tau[\phi_0]](x) - S_\tau[S_\tau^{(n)}[\phi_0^{(n)}]](x) + S_\tau[S_\tau^{(n)}[\phi_0^{(n)}]](x) - S_\tau^{(n)}[S_\tau^{(n)}[\phi_0^{(n)}]](x) \\ &\le \mathcal{M}_{C'}|\overline{X}_x^{2,\overline{\varepsilon}}(2\tau)|\Delta + \mathcal{M}_{C'}|\overline{X}_x^{2,\overline{\varepsilon}}(\tau)|\Delta + \mathcal{M}_{C'}|x|\Delta + \frac{\overline{\varepsilon}}{2} + \frac{\overline{\varepsilon}}{4} \\ &= \mathcal{M}_{C'}\Delta \sum_{i=0}^{2} |\overline{X}_x^{2,\overline{\varepsilon}}(i\tau)| + \sum_{i=1}^{2} \frac{\overline{\varepsilon}}{2^i}. \end{aligned} \tag{90}$$

It is then clear that, by induction, one obtains

Lemma 4.4

$$0 \le S_{N\tau}[\phi_0](x) - {S_\tau^{(n)}}^N[\phi_0](x) \le \mathcal{M}_{C'}\Delta \sum_{i=0}^{N} |\overline{X}_x^{N,\overline{\varepsilon}}(i\tau)| + \sum_{i=1}^{N} \frac{\overline{\varepsilon}}{2^i}. \tag{91}$$

where the construction of $\overline{\varepsilon}$–optimal $\overline{X}_x^{N,\overline{\varepsilon}}(\cdot)$ by induction follows in the obvious way as above.

Theorem 4.5 *Let $\{\psi_i\}_{i=1}^n$, C' and Δ be as in Lemma 4.3. Then, there exists $\overline{m}, \overline{\lambda} \in (0,\infty)$ such that*

$$0 \le W(x) - {W^{(n)}}^\infty(x) \le \mathcal{M}_{C'}\left(\frac{e^{\overline{m}}}{1-e^{-\overline{\lambda}\tau}}\right)|x|\Delta \quad \forall\, x \in \overline{B}_R(0).$$

Remark 4.6 By Theorem 2.12, there exists $N = N(n) < \infty$ such that

$${W^{(n)}}^\infty(x) = {W^{(n)}}^N(x) \quad \forall\, x \in \overline{B}_R(0),$$

and so Theorem 4.5 also implies

$$0 \le W(x) - {W^{(n)}}^N(x) \le \mathcal{M}_{C'}\left(\frac{e^{\overline{m}}}{1-e^{-\overline{\lambda}\tau}}\right)|x|\Delta \quad \forall\, x \in \overline{B}_R(0)$$

for $N \ge N(n)$.

Proof. Let $\overline{\varepsilon} \in (0,1)$. Fix ϕ_0 and x. For each $N < \infty$, construct $w_N^{\overline{\varepsilon}}(\cdot)$ as above along with the corresponding $\overline{X}_x^{N,\overline{\varepsilon}}$. Let $w_\infty^{\overline{\varepsilon}}(t) = w_N^{\overline{\varepsilon}}(t)$ if $t \in [0, N\tau]$, and similarly, $\overline{X}_x^{\infty,\overline{\varepsilon}}(t) = \overline{X}_x^{N,\overline{\varepsilon}}(t)$ if $t \in [0, N\tau]$. Then, by [32] (see also [33]), there exists $\widetilde{K} < \infty$ (independent of $\overline{\varepsilon} \in (0,1)$) such that

$$\|w_\infty^{\overline{\varepsilon}}\|_{L_2(0,N\tau)} \leq \widetilde{K}(1+|x|^2)$$

for all $N < \infty$. Consequently, using Assumptions $(A1)$ and $(A2)$, there exist $\overline{m}, \overline{\lambda} \in (0,\infty)$ such that

$$|\overline{X}_x^{\infty,\overline{\varepsilon}}(t)| \leq |x| e^{\overline{m}-\overline{\lambda}t} \quad \forall\, t \in [0,\infty). \tag{92}$$

Then, by Lemma 4.4 and (92),

$$\begin{aligned} 0 \leq S_{N\tau}[\phi_0](x) - S_\tau^{(n)^N}[\phi_0](x) &\leq \mathcal{M}_{C'} \Delta |x| e^{\overline{m}} \sum_{i=0}^{N} e^{-\overline{\lambda} i\tau} + \sum_{i=1}^{N} \frac{\overline{\varepsilon}}{2^i} \\ &\leq \mathcal{M}_{C'} \Delta |x| \frac{e^{\overline{m}}}{1-e^{-\overline{\lambda}\tau}} + \overline{\varepsilon}. \end{aligned}$$

Since this is true for all $N \in \mathcal{N}$, and $S_\tau^{(n)^N}[\phi_0](x) = S_\tau^{(n)^{N+1}}[\phi_0](x) = W^{(n)^\infty}(x)$ for all $N \geq N(n)$, one obtains the result by taking the limit as $N \to \infty$ and then as $\overline{\varepsilon} \downarrow 0$. □

Lastly, we note that for τ sufficiently small, where

$$\tau \leq 1/\overline{\lambda} \tag{93}$$

is sufficient (so that $\overline{\lambda}\tau/2 \leq (1-e^{-\overline{\lambda}\tau})$), one has

$$0 \leq W(x) - W^{(n)^\infty}(x) \leq \mathcal{M}_{C'} \Delta \Big(\frac{e^{\overline{m}}}{1-e^{-\overline{\lambda}\tau}} \Big) |x| \leq K_1 |x| (\Delta/\tau) \tag{94}$$

with

$$K_1 \doteq 2\mathcal{M}_{C'} e^{\overline{m}}/\overline{\lambda}. \tag{95}$$

5 Errors in the Approximation of B

In the previous section, we considered the errors due to truncation while assuming that B and consequently, the eigenvector, e were computed exactly. Of course, as discussed in Section 3, there is an allowable upper limit for errors in the elements of B, below which one can guarantee the convergence of the power method. The errors in B also translate into errors in the eigenvector and consequently the approximate solution as discussed in Sections 3 and 6. In this section, we consider a power series (in t) for $V(t,x) \doteq S_t[\psi_i](x)$ where we recall $B_{j,i} = -\max_{x \in \overline{B}_R(0)}[-\psi_j(x) - S_\tau[\psi_i](x)]$. With the power series for $V(t,x) = S_t[\psi_i](x)$ truncated at some level, $t^{n'-1}$ (for each i), we obtain a relationship between n', τ and basis function density which guarantees that the errors in B do not exceed the allowable bounds obtained in Section 3. In addition to the errors incurred by truncation of the power series, there may be errors in the computation of the terms in the series themselves. In subsection 5.1, one particular method for computing the power series terms to sufficient accuracy is given.

As noted above, one approach to the computation of B is a Taylor series (in t) approximation to $S_t[\psi_i](x)$. More specifically, letting $V(t,x) = S_t[\psi_i](x)$, so that V satisfies

$$\begin{aligned} V_t &= f \cdot \nabla V + l + \frac{1}{2\gamma^2} \nabla V^T \sigma \sigma^T \nabla V \\ V(0,x) &= \psi_i(x) \end{aligned} \tag{96}$$

one may approximate V as

$$V(t,x) = V_0(x) + V_1(x)t + \frac{1}{2} V_2(x) t^2 + \dots . \tag{97}$$

Here $V_0(x) = \psi_i(x)$ and V_1 is the right hand side of (96) with ψ_i replacing V. Specifically,

$$V_1(x) = f \psi_{ix} + l + \frac{1}{2\gamma^2} \psi_{ix} a \psi_{ix}$$

where $a = \sigma\sigma^T$ and we drop the gradient/vector notation for simplification here and below. The higher order terms are computed by differentiating (96) at $t = 0$. Of course this process requires some smoothness for V. The following is well–known, and so we only sketch a proof.

Theorem 5.1 *Given $R' < \infty$ and $n' \in \mathcal{N}$, there exists $\tau' > 0$ such that $V \in C^{n'}((0,\tau') \times B_{R'}(0))$.*

PROOF. The result for C^2 can be found, for instance, in [9] as well as many earlier works (see the references in [9] as well as [15]). In order to obtain continuity of higher derivatives, one simply differentiates (97), and applies the same technique. For example, the partial $V_{x_l}(t,x)$ satisfies

$$\begin{aligned} U_t &= \Big[f_{x_l} V_x + l_{x_l} + V_x a_{x_l} V_x \Big] + \Big[f + 2 V_x a \Big] U_x \\ U(0,x) &= \psi_{i x_l}(x). \end{aligned}$$

Note that τ' may depend on n'. □

Fix some $R', n' < \infty$. Let τ' be given by Theorem 5.1. We assume $\tau < \min\{\tau', 1, 1/c\}$ (where the motivation for the bounds of 1 and $1/c$ appear in (103) and (106) below) and $\widetilde{R} < R'$. Then we may approximate V over $(0,\tau) \times \overline{B}_{\widetilde{R}}(0)$ by

$$\widetilde{V}(t,x) = V_0(x) + V_1(x)t + V_2(x)\frac{t^2}{2} + \dots + V_{n'-1}(x) \frac{t^{n'-1}}{(n'-1)!}. \tag{98}$$

Letting

$$M_{R',n'} \doteq \max_{(t,x) \in [0,\tau] \times \overline{B}_{\widetilde{R}}(0)} |V_{t^{(n')}}(t,x)|,$$

one has

$$|V(t,x) - \widetilde{V}(t,x)| \le M_{R',n'} \frac{\tau^{n'}}{(n')!} \quad \forall\, (t,x) \in [0,\tau] \times \overline{B}_{\widetilde{R}}(0). \tag{99}$$

Now define the corresponding approximation to B by

$$\widetilde{B}_{j,i} = - \max_{x \in \overline{B}_{\widetilde{R}}(0)} \left\{ \psi_j(x) - \widetilde{V}(\tau, x) \right\}. \tag{100}$$

By (99) and (100), one has

$$|B_{j,i} - \widetilde{B}_{j,i}| \leq M_{R',n'} \frac{\tau^{n'}}{(n')!}. \tag{101}$$

Comparing (101) with Theorem 3.10, one finds that a sufficient condition for the convergence of the power method (using $\widetilde{B}$ computed from approximation $\widetilde{V}$) is that $\tau \leq 1/c$ and that for some $\mu \in \{2, 3, 4, \ldots\}$ ($\mu = 2$ is the weakest condition)

$$M_{R',n'} \frac{\tau^{n'}}{(n')!} \leq \left[\min_{i \neq 1} |\overline{x}_i|^2\right] \left(\frac{\delta c^5}{9(16)M^2}\right) \frac{\tau^4}{n^\mu}.$$

Note that the $\tau \leq 1/c$ condition can be removed by using Theorems 3.8 and 3.9 instead of 3.10.

Since computation of $\widetilde{B}_{j,i}$ requires the maximization operation, below we will introduce an approximation for $\widetilde{B}_{j,i}$, to be denoted by $\widehat{B}_{j,i}$ (where the maximum may only be computed approximately rather than exactly). Suppose further that

$$|\widetilde{B}_{j,i} - \widehat{B}_{j,i}| \leq M_{R',n'} \frac{\tau^{n'}}{(n')!}. \tag{102}$$

Then, by (102), (101) with Theorem 3.10, one finds that a sufficient condition for the convergence of the power method (using $\widehat{B}$) is that $\tau \leq 1/c$ and that for some $\mu \in \{2, 3, 4, \ldots\}$ ($\mu = 2$ is the weakest condition)

$$2M_{R',n'} \frac{\tau^{n'}}{(n')!} \leq \left[\min_{i \neq 1} |\overline{x}_i|^2\right] \left(\frac{\delta c^5}{9(16)M^2}\right) \frac{\tau^4}{n^\mu}, \tag{103}$$

and so a sufficient condition is

$$\tau^{n'-4} \leq \left[\min_{i \neq 1} |\overline{x}_i|^2\right] \left(\frac{\delta c^5 (n')!}{9(32)M^2 M_{R',n'}}\right) \frac{1}{n^\mu}. \tag{104}$$

Suppose a rectangular grid of evenly spaced basis function centers with N_D centerpoints per dimension, and recall that ψ_1 is centered at the origin which implies N_D is odd. (Perhaps it should be noted that this is conservative in that we are considering a rectangular grid encompassing $\overline{B}_{D_R}$ rather than just those basis functions centered in the sphere itself.) This implies $\min_{i \neq 1} |\overline{x}_i|^2 = 4D_R^2/(N_D - 1)^2$, and (104) becomes

$$\tau^{n'-4} \leq \left(\frac{D_R^2 \delta c^5 (n')!}{9(8)M^2 M_{R',n'}}\right) \left(\frac{1}{N_D}\right)^{m\mu} \left(\frac{1}{N_D - 1}\right)^2$$

which implies a sufficient condition is

$$\tau^{n'-4} \leq \left(\frac{D_R^2 \delta c^5 (n')!}{9(8)M^2 M_{R',n'}}\right) \left(\frac{1}{N_D}\right)^{m\mu+2} \doteq \widetilde{\mathcal{M}}_{R',n'} \left(\frac{1}{N_D}\right)^{m\mu+2} \tag{105}$$

where we recall that m is the dimension of the state space.

Therefore, if one fixes $\tau < \min\{1, 1/c\}$, then it is sufficient that

$$n' \geq 4 + \frac{\log \widetilde{\mathcal{M}}_{R',n'} + (m\mu + 2)\log(1/N_D)}{\log \tau}. \tag{106}$$

Alternatively, one may, without loss of generality, require $\widetilde{\mathcal{M}}_{R',n'} \geq 1$ in which case (noting that $\log \tau < 0$ since $\tau < 1$) (106) yields the sufficient condition

$$n' \geq 4 + \frac{(m\mu + 2)\log(1/N_D)}{\log \tau} \tag{107}$$

in which case the lower bound on n' scales like $\log(1/N_D)$. We remark, that this sufficient condition may be quite conservative.

5.1 A Method for Computing B

As noted above, one would not typically have a closed–form expression for the $B_{j,i}$ or even the $\widetilde{B}_{j,i}$ terms, and we denote the approximation of $\widetilde{B}$ by $\widehat{B}$. In this subsection, we indicate some specifics of a numerical method for the approximation. This is not essential to the paper, but we felt that it was useful to sketch an approximation technique so as to concretely indicate one approach to this subproblem.

The approach taken was to define

$$\widetilde{X}_{j,i}(t) \doteq \operatorname{argmax}\{\psi_j(x) - \widetilde{V}(t,x)\}$$

where $\widetilde{V}$ is given by (98) (i.e. the truncated power series expansion of $S_t[\psi_i](x)$), and then to propagate $\widetilde{X}_{j,i}$ as the solution of an ODE forward from $t = 0$ to τ via a Runge–Kutta method. One difficulty is that $\widetilde{X}_{j,i}(t)$ diverges as $t \downarrow 0$. In order to remedy this, and also remedy unbounded derivatives as $t \downarrow 0$, we replace $\psi_j(x)$ by $\psi^\tau_{j,i}(t,x)$ where

$$\psi^\tau_{j,i}(t,x) \doteq -\frac{1}{2}(x - \xi(t))^T[(C + \overline{\delta}(1 - t/\tau))I](x - \xi(t)) \tag{108}$$

where

$$\xi(t) \doteq x_i + (t/\tau)(x_j - x_i), \tag{109}$$

and $\overline{\delta} > 0$. Then one may define

$$\widetilde{X}^\tau_{j,i}(t) \doteq \operatorname*{argmax}_x\{\psi^\tau_{j,i}(t,x) - \widetilde{V}(t,x)\}, \tag{110}$$

and note that

$$\widetilde{X}^\tau_{j,i}(\tau) = \widetilde{X}_{j,i}(\tau) = \operatorname{argmax}\{\psi_j(x) - \widetilde{V}(t,x)\}.$$

Since $\widetilde{X}^\tau_{j,i}(t)$ is the argmax at each time $t \in [0,\tau]$, this implies

$$[\psi^\tau_{j,i}]_x(t, \widetilde{X}^\tau_{j,i}(t)) - \widetilde{V}_x(t, \widetilde{X}^\tau_{j,i}(t)) = 0$$

for all $t \in [0,\tau]$. Differentiating with respect to time, implies

$$\Big[[\psi^\tau_{j,i}]_{xx}(t, \widetilde{X}^\tau_{j,i}(t)) - \widetilde{V}_{xx}(t, \widetilde{X}^\tau_{j,i}(t))\Big]\dot{\widetilde{X}}^\tau_{j,i}(t) + \Big[]\psi^\tau_{j,i}]_{tx}(t, \widetilde{X}^\tau_{j,i}(t)) - \widetilde{V}_{tx}(t, \widetilde{X}^\tau_{j,i}(t))\Big] = 0,$$

or,

$$\dot{\widetilde{X}}^\tau_{j,i}(t) = \Big[[\psi^\tau_{j,i}]_{xx}(t, \widetilde{X}^\tau_{j,i}(t)) - \widetilde{V}_{xx}(t, \widetilde{X}^\tau_{j,i}(t))\Big]^{-1}\Big[[\psi^\tau_{j,i}]_{tx}(t, \widetilde{X}^\tau_{j,i}(t)) - \widetilde{V}_{tx}(t, \widetilde{X}^\tau_{j,i}(t))\Big]. \tag{111}$$

The initial state for (111) is

$$\widetilde{X}^{\tau}_{j,i}(0) = \operatorname*{argmax}_x\{\psi^{\tau}_{j,i}(0,x) - \widetilde{V}(0,x)\} = \operatorname*{argmax}_x\{-\tfrac{1}{2}(x-x_i)^T(C+\overline{\delta}I)(x-x_i) - \psi_i(x)\} = x_i.$$

Note that

$$\Big[[\psi^{\tau}_{j,i}]_{xx}(0,x) - \widetilde{V}_{xx}(0,x)\Big] = -[C+\overline{\delta}I] + C = -\overline{\delta}I \tag{112}$$

which is negative definite, and

$$\Big[[\psi^{\tau}_{j,i}]_{xx}(\tau,x) - \widetilde{V}_{xx}(\tau,x)\Big] = -C - \widetilde{V}_{xx}(\tau,x) \tag{113}$$

would be negative definite on $\overline{B}_R$ by Assumption $(A5')$ if approximation $\widetilde{V}(\tau,\cdot)$ were replaced by $S_\tau[\psi_i]$. Also,

$$\widetilde{X}^{\tau}_{j,i}(0) = x_i \in \overline{B}_{D_R}, \quad \text{and} \quad \widetilde{X}^{\tau}_{j,i}(\tau) \in \overline{B}_R \tag{114}$$

if approximation $\widetilde{V}(\tau,\cdot)$ is replaced by $S_\tau[\psi_i]$. This suggests the following assumption (which is only used for this approach to computing B). Suppose there exists $\hat{\delta} > 0$ such that

$$\begin{aligned} &\Big[[\psi^{\tau}_{j,i}]_{xx}(t,x) - \widetilde{V}_{xx}(t,x)\Big] + \hat{\delta}I < 0 \quad \forall\, |x| \le \hat{g}(t),\ \forall t \in [0,\tau] \\ \text{and} \quad & \\ &|\widetilde{X}^{\tau}_{j,i}(t)| \le \hat{g}(t) \quad \forall\, t \in [0,\tau] \end{aligned} \tag{115}$$

where $g : [0,\tau] \to \mathbf{R}$ is any function such that $\hat{g}(0) = D_R$, $\hat{g}(\tau) = R$ and $\hat{g}$ is monotonically decreasing. Note that, by (112)–(114), the conditions are satisfied at both endpoints ($t = 0$ and $t = \tau$) when $\widetilde{V}(\tau,\cdot)$ is replaced by $S_\tau[\psi_i]$. Consequently, this may not be significantly more restrictive than the general assumptions, and for the purposes of sketching this particular approach to computing B, let us assume (115). Note that this guarantees the existence of the inverse in (111), and further that $\widetilde{X}^{\tau}_{j,i}(\tau)$ is the unique maximizer in $\overline{B}_R$.

Analytical expressions for the right hand side of (111) can be obtained from (98) and (108). (These can be used to generate sufficient conditions that guarantee (115), but these are likely much too conservative.) Thus, one merely needs to propagate the n–dimensional ODE (111) forward to time τ. A Runge-Kutta method may be used for this, and the resulting approximate solution is denoted by $\widehat{X}^{\tau}_{j,i}$. The approximation of the elements of $\widehat{B}$ are then given by

$$\begin{aligned} \widehat{B}_{j,i} &= -\left\{\psi^{\tau}_{j,i}(\widehat{X}^{\tau}(\tau)) - \widetilde{V}(\tau,\widehat{X}^{\tau}(\tau))\right\} \\ &= -\left\{\psi_j(\widehat{X}^{\tau}(\tau)) - \widetilde{V}(\tau,\widehat{X}^{\tau}(\tau))\right\}. \end{aligned} \tag{116}$$

Note that

the number of steps in the Runge–Kutta algorithm must be controlled so that (102) is satisfied. (117)

6 Error Summary

The error analyses of the previous three sections will now be combined. In particular, the errors due to truncation and the errors in computation of B will be combined to produce overall error bounds (125), (126). A condition required for the algorithm to work (assuming one uses the power series of Section 5 for computation of B) is also obtained.

Theorems 3.7 to 3.10 provided sufficient conditions for the power method step to converge to the max–plus eigenvector. Employing the simplest condition (but also the strictest), that of Theorem 3.10, convergence of the power method with approximation $\widehat{B}$ to B is guaranteed if

$$|\widehat{B}_{i,j} - B_{i,j}| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\}\left(\frac{\delta c^5}{9(16)M^2}\right)\frac{\tau^4}{n^\mu} \quad \forall\, i,j \tag{118}$$

where $\mu \in \{2,3,4,...\}$ and δ is given by (49). Note that we are assuming $\tau \leq \min\{1, 1/c, \tau'\}$ as in Section 5 (as well as all assumptions including $(A5')$ and technical conditions (18), (36) which appear in Section 3). Then, Theorem 3.10 implies that a resulting error bound for the max–plus eigenvector given by

$$\|e - \hat{e}\| \doteq \max_i |e_i - \hat{e}_i| \leq \min_{i\neq 1}\{|\overline{x}_i|^2\}\left(\frac{\delta c^5}{9(16)M^2}\right)\frac{\tau^4}{n^{\mu-2}}. \tag{119}$$

where $\hat{e}$ corresponds to $\widehat{B}$. We remark that slightly different error estimates (under slightly different conditions) are also given in Theorems 3.8 and 3.9.

Suppose we adopt the notation $\widehat{W}(x) \doteq \bigoplus_{i=1}^n \hat{e}_i \otimes \psi_i(x)$ and $W^f(x) \doteq \bigoplus_{i=1}^n e_i \otimes \psi_i(x)$ so that W^f corresponds to the finite expansion with zero error in the computation/approximation of B. Then, by (119),

$$\begin{aligned}\|\widehat{W} - W^f\| \doteq \max_{|x|\leq R}|\widehat{W}(x) - W^f(x)| &\leq \min_{i\neq 1}\{|\overline{x}_i|^2\}\left(\frac{\delta c^5}{9(16)M^2}\right)\frac{\tau^4}{n^{\mu-2}} \\ &= \min_{i\neq 1}\{|\overline{x}_i|^2\}\left(\frac{\delta c^5\tau^4}{9(16)M^2}\right)\left(\frac{1}{N_D}\right)^{m(\mu-2)}\end{aligned} \tag{120}$$

where again, N_D is the number of centers of basis functions per dimension of the state space with a rectangular, evenly spaced grid of centers. It should be recalled that the basis functions are such that ψ_1 is centered at the origin ($\overline{x}_1 = 0$), and so N_D is odd. (Perhaps one should note that we are being sloppy here by using the number of basis functions corresponding to covering the entire rectangle which encloses the sphere $\overline{B}_{D_R}$, although only those with centers covering the sphere itself are required for the bound. Consequently, the above bound is conservative.) Also, with the evenly spaced basis function centers, (120) can be written as

$$\|\widehat{W} - W^f\| \leq \left(\frac{D_R^2\delta c^5\tau^4}{9(4)M^2}\right)\left(\frac{1}{N_D}\right)^{m(\mu-2)}\left(\frac{1}{N_D - 1}\right)^2 \tag{121}$$

Using the approach of Section 5, (118) is satisfied if

$$\tau^{n'-4} \leq \widetilde{\mathcal{M}}_{R',n'}\left(\frac{1}{N_D}\right)^{m\mu+2} \tag{122}$$

where $\widetilde{\mathcal{M}}_{R',n'}$ is given by (105) and n' is the number of terms (including zeroth order) in the Taylor series, **and** if (117) is satisfied.

This does not account for the truncation errors induced by using only a finite number of basis functions. Let W be the true value function (see Section 2). Then, by (94),

$$|W(x) - W^f(x)| \le K_1 \frac{|x|}{\tau}\Big(\frac{2D_R}{N_D-1}\Big) \quad \forall\, x \in \overline{B}_R \tag{123}$$

where K_1 is given by (95), $2D_R/(N_D-1) = \Delta$, and τ satisfies (93); τ now must satisfy

$$\tau \le \min\{1, 1/c, 1/\overline{\lambda}, \tau'\}. \tag{124}$$

The error bound (123) is not without drawbacks. In particular, τ appears in the denominator. However, it does not seem possible with the techniques of this paper to remove that term. This is the reason for concentrating in Section 5 on fixed τ with increasing n' as the means for reducing errors.

Combining (121) and (123), the total error bound (assuming convergence of the power method – for which (122) and (117) form a sufficient condition – and $\tau \le \min\{1, 1/c, 1/\overline{\lambda}, \tau'\}$) is given by

$$|W(x) - \widehat{W}(x)| \le \Big(\frac{D_R^2 \delta c^5 \tau^4}{9(4)M^2}\Big)\Big(\frac{1}{N_D}\Big)^{m(\mu-2)}\Big(\frac{1}{N_D-1}\Big)^2 + K_1\frac{|x|}{\tau}\Big(\frac{2D_R}{N_D-1}\Big)$$

which for $N_D \ge 3$

$$\le \Big(\frac{D_R^2 \delta c^5 \tau^4}{18M^2}\Big)\Big(\frac{1}{N_D}\Big)^{m(\mu-2)+2} + K_1\frac{|x|}{\tau}\Big(\frac{2D_R}{N_D}\Big). \tag{125}$$

Since the best error rate in the last term in $1/N_D$, we take $\mu = 2$, and find in that case

$$|W(x) - \widehat{W}(x)| \le \Big[\frac{D_R^2 \delta c^5 \tau^4}{18M^2} + 2K_1 D_R \frac{|x|}{\tau}\Big]\Big(\frac{1}{N_D}\Big). \tag{126}$$

That is, the total error goes down linearly in $(1/N_D)$. Note that this rate is constrained by the fact that the solutions are only viscosity solutions – which may have discontinuous first derivatives. It is conjectured that with smooth solutions, the rate would instead be $(1/N_D)^2$.

This assumes that conditions (122) and (117) are met as well as (124). Also, as in Section 5, one may prefer to write (122) as

$$n' \ge 4 + \frac{\log \widetilde{\mathcal{M}}_{R',n'} + (m\mu+2)\log(1/N_D)}{\log \tau},$$

or assuming without loss of generality that $\widetilde{\mathcal{M}}_{R',n'} \ge 1$, one has the less tight but clearer bound of

$$n' \ge 4 + \frac{(m\mu+2)\log(1/N_D)}{\log \tau}$$

in which case the lower bound on n' scales like $\log(1/N_D)$. From this, one sees for instance that doubling N_D would typically imply the addition of

$$\lceil\Big(\frac{(2m+2)\log(1/2)}{\log \tau}\Big) = \lceil\Big(\frac{(2m+2)\log 2}{\log(1/\tau)}\Big)$$

to n' where $\lceil(z)$ indicates the smallest integer greater than or equal to z. Again, this assumes the use of the Taylor series/Runge–Kutta approach of Section 5 toward the approximation of B. Alternate approaches may yield different conditions.

Remark 6.1 All error bounds are actually conceived as the errors which may be achieved with given computer effort. A key underlying assumption of this paper is that all the elements of B are computed. This requires substantial effort since the number of terms in B is the square of the number of basis functions. In practice, it has been observed that elements of B, $B_{i,j}$, corresponding to basis function pairs where $|x_i - x_j|$ is large generally do not contribute at all to the resulting eigenvector (recall that this is the max–plus algebra). By not computing these terms, one could greatly reduce the computations. This is a question for further study which lies beyond the bounds of the current paper.

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