The Dequantized Schrödinger Equation and a Complex-Valued Stationary-Action Diffusion Representation *

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Abstract

A stochastic representation for solution of the Schrödinger equation is obtained, utilizing complex-valued diffusion processes. The Maslov dequantization is employed, where the domain is complex-valued in the space variable. The notion of staticization is required to relate the Hamilton-Jacobi form of the dequantized Schrödinger equation to its stochastic control representation. A verification result is obtained, and existence is reduced to a real-valued space-variable case.

Key words. Schrödinger equation, stochastic control, Hamilton-Jacobi, stationary action, staticization, complex-valued diffusion.

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1 Introduction

We recall the Schrödinger initial value problem, given as

\begin{align}
0 &= i\hbar \psi_t(s, x) + \frac{\hbar^2}{2m} \Delta \psi(s, x) - \psi(s, x)V(x), \quad (s, x) \in \mathcal{D}, \\
\psi(0, x) &= \psi_0(x), \quad x \in \mathbb{R}^n,
\end{align}

where \(\hbar\) denotes Planck’s constant, \(m \in (0, \infty)\) denotes mass, initial condition \(\psi_0\) takes values in \(\mathbb{C}\), \(V\) denotes a known potential function, \(\Delta\) denotes the Laplacian with respect to the space (second) variable, \(\mathcal{D} \equiv (0, t) \times \mathbb{R}^n\), and subscript \(t\) will denote the derivative with respect to the time variable (the first argument of \(\psi\) here) regardless of the symbol being used for time in the argument list. We also let \(\mathcal{D} \equiv [0, t] \times \mathbb{R}^n\). We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf., [17]), which is \(S : \mathcal{D} \to \mathbb{C}\) given by \(\psi(s, x) = \exp\{\frac{i}{\hbar}S(s, x)\}\). Note that \(\psi_t = \frac{i}{\hbar} \psi S_t\),

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\[
\psi_x = \frac{i}{\hbar} \psi S_x \quad \text{and} \quad \Delta \psi = \frac{i}{\hbar} \psi \Delta S - \frac{1}{\hbar^2} \psi |S_x|^2 \quad \text{where for } y \in \mathbb{C}^n, \ |y|^2 = \sum_{j=1}^n y_j^2.
\]
(We remark that notation \(| \cdot |^2\) is not intended to indicate a squared norm; the range is complex.) We find that (1),(2) become
\[
0 = -S_t(s,x) + \frac{\hbar}{2m} \Delta S(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D},
\]
(3)
\[
S(0, x) = \phi(x), \quad x \in \mathbb{R}^n,
\]
(4)
where \(H : \mathbb{R}^n \times \mathbb{C}^n \to \mathbb{C}\) is the Hamiltonian given by
\[
H(x, p) = -\left(\frac{1}{2m} |p|^2 + V(x)\right) = \text{stat} \left\{ v \cdot p + \frac{m}{2} |v|^2 - V(x) \right\},
\]
(5)
and \(\text{stat}\) will be defined in the next section. We look for solutions in the space
\[
\mathcal{S} = \{ S : \overline{\mathcal{D}} \to \mathbb{C} | S \in C^{1,2}_p(\mathcal{D}) \cap C(\overline{\mathcal{D}}) \},
\]
(6)
where \(C^{1,2}_p\) denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound over the space variable. We will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the Hamilton-Jacobi partial differential equation (HJ PDE) problem given by
\[
0 = S_t(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D},
\]
(7)
\[
S(t,x) = \phi(x), \quad x \in \mathbb{R}^n.
\]
(8)

Working mainly with this last form, we will fix \(t \in (0, \infty)\), and allow \(s\) to vary in \([0, t]\).

Recall the semiclassical limit (cf. [1, 10, 11, 12]) where one views \(\hbar\) as a small parameter, and examines the limit as \(\hbar \downarrow 0\). This leads to an HJ PDE problem of the form
\[
0 = S_t(s,x) + \frac{\hbar}{2m} \Delta S(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D},
\]
(9)
\[
S(t,x) = \phi(x), \quad x \in \mathbb{R}^n.
\]
(10)

In recent work (cf. [2, 18, 19, 21]) it has been shown that stationary-action formulations of certain two-point boundary value problems (TPBVPs) for conservative dynamical systems also take the form (9),(10). This motivates the approach here, where we develop a stationary-action based representation for the solution of (7),(8) (and consequently (1),(2)). Due to the complex multiplier on the Laplacian, this representation is in terms of a stationary-action stochastic control problem with a complex-valued diffusion coefficient. It is important to note that the stationary-action diffusion process representation here is more closely related to the semiclassical limit analysis and to path-integral forms (cf. [3, 4]) than to another well-known area of research on diffusion-process representations, cf. [8, 14, 22, 23, 25]. In particular, those works discuss representations (and/or lack thereof) in the spirit of the Fokker-Planck equation, while here the representation generates \(S, \psi\), and consequently the position probability, through the stationary-action value of the stochastic process.

It is also important to note that the results are obtained here under strong assumptions. In particular, we assume the existence of a sufficiently smooth
potential function, defined over all of the complex space domain, which matches $V$ on $\mathbb{R}^n$. The assumptions allow for the inclusion of the case of the quantum harmonic oscillator, when one takes the potential to be a holomorphic quadratic over the complex domain. Potentials lacking such smoothness are beyond the scope here, and it is expected that such problems will be studied with the aid of viscosity-solution techniques in a later effort.

In Section 2, we recall the definitions necessary for staticization problems. In Section 3, the underlying space domain is extended from a space over the real field to a space over the complex field. This necessitates several other minor extensions, which are covered in the subsections. In particular, some classical existence and uniqueness results for stochastic differential equations (SDEs) are trivially extended to their complex-valued counterparts. In Section 4, the staticization-based stochastic-control value function representation for the dequantized Schrödinger equation is obtained. More specifically, a verification result is obtained demonstrating that if a solution of the HJ PDE over the “complexified” domain exists, then that solution has the indicated representation. Lastly, in Section 5, we indicate a result about existence of solutions of the HJ PDE over the complexified domain.

2 Stationarity definitions

Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. For this and other reasons, as in the definition of the Hamiltonian given in (5), we find it useful to develop additional notation and nomenclature. Specifically, in analogy with the language for minimization and maximization, we will refer to the search for stationary points as staticization, with these points being statica (in analogy with minima/maxima) and a single such point being labeled a staticum (in analogy with minimum/maximum). More specifically, we make the following definitions. Suppose $U$ is a generic normed vector space over $\mathbb{C}$ with $\mathcal{G} \subseteq U$, and suppose $F : \mathcal{G} \to \mathbb{C}$. We say $\bar{v} \in \argstat \{ F(v) \mid v \in \mathcal{G} \}$ if $\bar{v} \in \mathcal{G}$ and either

$$\limsup_{v \to \bar{v}, v \in \mathcal{G} \setminus \{ \bar{v} \}} \frac{|F(v) - F(\bar{v})|}{|v - \bar{v}|} = 0,$$

(11)
or there exists $\delta > 0$ such that $\mathcal{G} \cap B_\delta(\bar{v}) = \{ \bar{v} \}$ (where $B_\delta(\bar{v})$ denotes the ball of radius $\delta$ around $\bar{v}$). If argstat$\{ F(v) \mid v \in \mathcal{G} \} \neq \emptyset$, we define the possibly set-valued stat operation by

$$\text{stat}^s_{\bar{v} \in \mathcal{G}} F(v) \doteq \text{stat}^s \{ F(v) \mid v \in \mathcal{G} \} \doteq \{ F(\bar{v}) \mid \bar{v} \in \argstat \{ F(v) \mid v \in \mathcal{G} \} \}.$$

If argstat$\{ F(v) \mid v \in \mathcal{G} \} = \emptyset$, stat$^s_{v \in \mathcal{G}} F(v)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript $s$). In particular, if there exists $a \in \mathbb{C}$ such that stat$^s_{v \in \mathcal{G}} F(v) = \{ a \}$, then stat$^s_{v \in \mathcal{G}} F(v) \doteq a$; otherwise, stat$^s_{v \in \mathcal{G}} F(v)$ is undefined. At times, we may
abuse notation by writing \( \bar{v} = \text{argstat}\{F(v) | v \in \mathcal{G}\} \) in the event that the argstat is the single point \( \{\bar{v}\} \).

In the case where \( \mathcal{U} \) is a Hilbert space, and \( \mathcal{G} \subseteq \mathcal{U} \) is an open set, \( F : \mathcal{G} \to \mathbb{C} \) is Fréchet differentiable at \( \bar{v} \in \mathcal{G} \) with Riesz representation \( F_v(\bar{v}) \in \mathcal{U} \) if

\[
\lim_{w \to 0, \bar{v} + w \in \mathcal{G} \setminus \{\bar{v}\}} \frac{|F(\bar{v} + w) - F(\bar{v}) - \langle F_v(\bar{v}), w \rangle|}{|w|} = 0.
\]

The following is immediate from the above definitions.

**Lemma 1** Suppose \( \mathcal{U} \) is a Hilbert space, with open set \( \mathcal{G} \subseteq \mathcal{U} \) and \( \bar{v} \in \mathcal{G} \). Then, \( \bar{v} \in \text{argstat}\{F(y) | y \in \mathcal{G}\} \) if and only if \( F_v(\bar{v}) = 0 \).

For a discussion of staticization for general deterministic systems, we refer the reader to [18, 20].

### 3 Extensions to the complex domain

Various details of extensions to the complex domain must be considered prior to the development of the representation.

#### 3.1 Extended problem and assumptions

Although (1)–(2), (3)–(4) and (7)–(8) are typically given as HJ PDE problems over \( \overline{\mathcal{D}} \), we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to \( \mathbb{C}^n \), where \( \mathcal{V} : \mathbb{C}^n \to \mathbb{C} \), and we will abuse notation by employing the same symbol for the extended-domain functions. Let \( \mathcal{D}_C = (0, t) \times \mathbb{C}^n \) and \( \overline{\mathcal{D}}_C = [0, t] \times \mathbb{C}^n \), and define

\[
\mathcal{S}_C \doteq \{ S : \mathcal{D}_C \to \mathbb{C} : S \in C^{1,2}_p(\mathcal{D}_C) \cap C(\overline{\mathcal{D}}_C) \}.
\]

The extended-domain form of problem (7)–(8) is

\[
0 = \ddot{S}(s, x) + \frac{\hbar}{2m} \Delta S(s, x) + H(x, \dot{S}(s, x)), \quad (s, x) \in \mathcal{D}_C, \quad \left(14\right)
\]

\[
\dot{S}(t, x) = \phi(x), \quad x \in \mathbb{C}^n. \quad \left(15\right)
\]

Throughout Sections 3–4, we will assume:

(A.0) For all \( h \in (0, 1] \), there exists a solution, \( \bar{S} = \bar{S}^h \in \mathcal{S}_C \) to (14)–(15).

V, \( \phi : \mathbb{C}^n \to \mathbb{C} \) are entire, i.e., holomorphic on all of \( \mathbb{C}^n \). Further, there exists \( C_0 < \infty \) and \( q \in \mathbb{N} \) such that \( |V_{xx}(x)|, |\phi_{xx}(x)| < C_0(1 + |x|^{2q}) \) for all \( x \in \mathbb{C}^n \).

(A.1) There exists \( C_S < \infty \) such that \( |\bar{S}_x(r, x)| \leq C_S(1 + |x|) \) and \( |\bar{S}_{xx}(r, x)| \leq C_S(1 + |x|^{2q}) \) for all \( (r, x) \in \mathcal{D}_C \).

It may be helpful to make some remarks here. First, the potential for the quantum harmonic oscillator can be extended from a quadratic function on \( \mathbb{R}^n \).
We let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability triple, where \(\Omega\) denotes a sample space, \(\mathcal{F}\) denotes a \(\sigma\)-algebra on \(\Omega\), and \(\mathbb{P}\) denotes a probability measure on \((\Omega, \mathcal{F})\). Let \(\{\mathcal{F}_s \mid s \in [0, t]\}\) denote a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\) \((\mathcal{F}_{s_0} \subset \mathcal{F}_{s_1} \subset \mathcal{F}\) for all \(0 \leq s_0 < s_1 \leq t\), and let \(B\) denote an \(\mathcal{F}\)-adapted Brownian motion taking values in \(\mathbb{R}^n\). For \(s \in [0, t]\), let

\[
\mathcal{U}_s = \{ u : [s, t] \times \Omega \to \mathbb{C}^n \mid u \text{ is } \mathcal{F}_s \text{-adapted, right-continuous and such that } \mathbb{E} \int_s^t |u_r|^m \, dr < \infty \forall m \in \mathbb{N} \}. \tag{16}
\]

We supply \(\mathcal{U}_s\) with the norm

\[
\|u\|_{\mathcal{U}_s} = \max_{m \in [1, \bar{M}]} \left( \mathbb{E} \int_s^t |u_r|^m \, dr \right)^{1/m},
\]

where \(\bar{M} \geq 8q\).

We will be interested in diffusion processes given by

\[
\xi_r = \xi_r^{s,x} = x + \int_s^r u_r \, d\rho + \sqrt{\frac{h}{m} \frac{1+i}{\sqrt{2}}} \int_s^r dB_{\rho} \overset{\sim}{=} x + \int_s^r u_r \, d\rho + \sqrt{\frac{h}{m} \frac{1+i}{\sqrt{2}}} B_{r}^\Delta, \tag{17}
\]

where \(x \in \mathbb{C}^n\), \(s \in [0, t]\), \(u \in \mathcal{U}_s\), and \(B_r^\Delta = B_r - B_s\) for \(r \in [s, t]\). Note that because (1) and the dequantized versions thereof are generally defined over \((0, t) \times \mathbb{R}^n\), one could restrict the initial conditions to \(x \in \mathbb{R}^n\) rather than \(x \in \mathbb{C}^n\), although for \(r \in (s, t]\), one typically has \(\xi_r \in \mathbb{C}^n\). We will also be interested in the case where the control input is generated by a state-feedback. In particular, we will consider

\[
\tilde{\xi}_r = \tilde{\xi}_r^{s,x} = x + \int_s^r \left( \frac{1}{m} \right) \tilde{S}_x(\rho, \tilde{\xi}_r^{s,x}) \, d\rho + \sqrt{\frac{h}{m} \frac{1+i}{\sqrt{2}}} B_{r}^\Delta, \tag{18}
\]

where presuming for now existence and uniqueness of a solution, we may define the resulting \(u_r^{s,x} \in \mathcal{U}_s\) given by

\[
u_r^{s,x}(\omega) \overset{\sim}{=} \tilde{u}(r, \tilde{\xi}_r^{s,x}(\omega)) \overset{\sim}{=} \left( \frac{1}{m} \right) \tilde{S}_x(r, \tilde{\xi}_r^{s,x}(\omega)) \tag{19}
\]

for all \(r \in [s, t]\) and \(\omega \in \Omega\).
For $s \in [0, t]$, we let

$$X_s = \{ \xi : [s, t] \times \Omega \to \mathbb{C}^n \mid \xi \text{ is } \mathcal{F}_s\text{-adapted, right-continuous and such that} \}

$$

$$E\sup_{r \in [s, t]} |\xi_r|^m < \infty \forall m \in \mathbb{N}. \quad (20)$$

We supply $X_s$ with the norm

$$\|\xi\|_{X_s} \equiv \max_{m \in [1, M]} \left[ E\sup_{r \in [s, t]} |\xi_r|^m \right]^{1/m}.$$

It is natural to work with complex-valued state processes in this problem domain. However, in order to easily apply many of the existing results regarding existence, uniqueness and moments, we will find it handy to use a “vectorized” real-valued representation for the complex-valued state processes. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, $\mathbb{V}_0 : \mathbb{C} \to \mathbb{R}^2$, given by $\mathbb{V}_0(y + iz) = (y, z)^T$, where $y = \text{Re}(x)$ and $z = \text{Im}(x)$. This immediately yields the mapping $\mathbb{V}_0 : \mathbb{C}^n \to \mathbb{R}^{2n}$ given by $\mathbb{V}_0(y + iz) = (y^T, z^T)^T$, where component-wise, $(y_j, z_j)^T = \mathbb{V}_0(x_j)$ for all $j \in [1, n]$, where throughout, for integer $a \leq b$, we define $[a, b] = \{a, a + 1, \ldots, b\}$. Given control process, $u \in \mathcal{U}_s$, we define its vectorized analog by the isometric isomorphism, $\mathbb{V} : \mathcal{U}_s \to \mathcal{U}_s^\mathbb{R}$, where $\mathbb{V}(u)_r = \mathbb{V}(v + iw)_r = (v^T_r, w^T_r)^T$ and $(v^T_r, w^T_r)^T = \mathbb{V}(u_r)$ for all $r \in [s, t]$ and $\omega \in \Omega$, and where

$$\mathcal{U}_s^\mathbb{R} = \{(v, w) : [s, t] \times \Omega \to \mathbb{R}^{2n} \mid (v, w) \text{ is } \mathcal{F}_s\text{-adapted, right-continuous and such that} \}

$$

$$E\int_s^t |v_r|^m + |w_r|^m \, dr < \infty \forall m \in \mathbb{N}, \quad (21)$$

$$\|u\|_{\mathcal{U}_s^\mathbb{R}} \equiv \max_{m \in [1, M]} \left[ E\int_s^t |v_r|^m + |w_r|^m \, dr \right]^{1/m}. \quad (22)$$

Abusing notation, we also define the isometric isomorphism, $\mathbb{V} : X_s \to \mathcal{X}_s^\mathbb{R}$ by $\mathbb{V}(\xi)_r = \mathbb{V}(\eta + i\zeta)_r = (\eta^T_r, \zeta^T_r)^T$ for all $r \in [s, t]$ and $\omega \in \Omega$, where

$$\mathcal{X}_s^\mathbb{R} = \{(\eta, \zeta) : [s, t] \times \Omega \to \mathbb{R}^{2n} \mid (\eta, \zeta) \text{ is } \mathcal{F}_s\text{-adapted, right-continuous and such that} \}

$$

$$E\sup_{r \in [s, t]} |\eta_r|^m + |\zeta_r|^m < \infty \forall m \in \mathbb{N}, \quad (23)$$

$$\|\eta\|_{\mathcal{X}_s} \equiv \max_{m \in [1, M]} \left[ E\sup_{r \in [s, t]} |\eta_r|^m + |\zeta_r|^m \right]^{1/m}. \quad (24)$$

Under transformation by $\mathbb{V}$, (17) becomes

$$\begin{pmatrix} \eta_r \\ \zeta_r \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^t \begin{pmatrix} \nu_p \\ \ell_p \end{pmatrix} \, dp + \sqrt{\frac{1}{m} \frac{1}{\sqrt{2}}} \left( I_{n \times n} \right) B_r \Delta. \quad (25)$$

We may decompose $S \in \mathcal{S}_C$ as

$$(R(r, \mathbb{V}_0(x)), \bar{T}(r, \mathbb{V}_0(x)))^T = \mathbb{V}_0(\bar{S}(r, x)), \quad (26)$$
where \( \bar{R}, \bar{T} : \bar{\mathcal{D}}_2 \equiv [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R} \). We indicate some standard relations between derivative components. Partial derivatives in the space variable of \( \bar{S} \) are given by

\[
\bar{S}_{x_j}(r, x) = \bar{R}_{y_j}(r, y, z) - i\bar{R}_{z_j}(r, y, z) = \bar{T}_{x_j}(r, y, z) + i\bar{T}_{y_j}(r, y, z)
\]

for all \((r, x) = (r, y + iz) \in (0, t) \times \mathbb{C}^n \) and all \( j \in \{1, n\} \). That is, letting 
\[ a + ib = \bar{S}_{x_j}(r, x), \]
one has

\[
a = \bar{R}_{y_j} = T_{x_j}, \quad b = -\bar{R}_{z_j} = T_{y_j},
\]

evaluated at \((r, y, z)\) for all \( j \in \{1, n\} \). Further, letting 
\[ c + id = \bar{S}_{x_j, x_k}(r, x), \]
one has

\[
c = \bar{R}_{y_j, y_k} = -\bar{R}_{z_j, z_k} = T_{x_j, y_k} = -T_{x_j, y_k}, \]
\[
d = -\bar{R}_{y_j, z_k} = -\bar{R}_{z_j, y_k} = T_{x_j, y_k} = -T_{x_j, y_k},
\]

evaluated at \((r, y, z)\) for all \( j, k \in \{1, n\} \). Finally, letting 
\[ e + if = \bar{S}_{x_j, y_k, x_l}(r, x), \]
one has

\[
e = \bar{R}_{y_j, y_k, y_l} = -\bar{R}_{z_j, z_k, z_l} = -\bar{R}_{z_j, y_k, z_l} = -\bar{R}_{z_j, y_k, z_l}, \]
\[
f = -\bar{R}_{y_j, z_k, y_l} = \bar{R}_{z_j, z_k, z_l} = -\bar{R}_{z_j, y_k, y_l} = -\bar{R}_{z_j, y_k, y_l}, \]
\[
g = -\bar{R}_{z_j, y_k, z_l} = -T_{x_j, y_k, y_l} = -T_{x_j, y_k, y_l},
\]

evaluated at \((r, y, z)\) for all \( j, k, l \in \{1, n\} \).

One may also easily verify that for \( x = y + iz \in \mathbb{C}^n \),

\[
\mathcal{V}_{oo}(|x|^2) = \sum_{j=1}^{n} \left( \frac{y_j^2 - z_j^2}{2y_jz_j} \right).
\]

More specifically, with \( \bar{R}, \bar{T} \) given by (26) and \((y^T, z^T)^T = \mathcal{V}_0(x) \),

\[
\mathcal{V}_{oo}(|\bar{S}_x(r, x)|^2) = \left( \frac{\sum_{j=1}^{n} \bar{R}_{y_j}^2(r, y, z) - \bar{R}_{z_j}^2(r, y, z)}{\sum_{j=1}^{n} 2\bar{R}_{y_j}(r, y, z)\bar{R}_{z_j}(r, y, z)} \right)
\[
= \left( \frac{\sum_{j=1}^{n} \bar{T}_{y_j}^2(r, y, z) - \bar{T}_{z_j}^2(r, y, z)}{\sum_{j=1}^{n} 2\bar{T}_{y_j}(r, y, z)\bar{T}_{z_j}(r, y, z)} \right).
\]

Also, letting \((v^0, w^0) = \mathcal{V}_0(u^0) \) for all \( u^0 \in \mathbb{C}^n \), given \( s \in [0, t] \) and \( x \in \mathbb{C}^n \) with \((y^T, z^T)^T = \mathcal{V}_0(x) \),

\[
\mathcal{V}_0 \left( \arg\text{stat} \left[ \sum_{j=1}^{n} \bar{S}_{x_j}(r, x)u_j^0 + \frac{m}{2} |u_j^0|^2 \right] \right) = \mathcal{V}_0 \left( \frac{1}{m} \bar{S}_x(r, x) \right)
\]
\[
= \frac{1}{m} \left( \frac{-\bar{R}_y(r, y, z)}{\bar{R}_z(r, y, z)} \right) = \frac{1}{m} \left( \frac{-\bar{T}_y(r, y, z)}{-\bar{T}_y(r, y, z)} \right).
\]
Using the above, we see that under transformation \( V \), (18) becomes

\[
\begin{aligned}
\left( \eta^r, \zeta^r \right) &= \left( y, z \right) + \int_s^t \frac{1}{m} \left( \bar{R}_y (\rho, \eta^r, \zeta^r) \right) d\rho + \sqrt{\frac{n}{m}} \frac{1}{\sqrt{2}} \left( I_{n \times n} \right) B^\Delta_r \left( y, z \right) \left( y, z \right) + \int_s^r \sqrt{\frac{n}{m}} d\rho \left( I_{n \times n} \right) B^\Delta_r \left( y, z \right) + \sqrt{\frac{n}{m}} \frac{1}{\sqrt{2}} \left( I_{n \times n} \right) B^\Delta_r.
\end{aligned}
\]

Throughout, concerning both real and complex stochastic differential equations, typically given in integral form such as in (35), solution refers to a strong solution.

**Lemma 2** Let \( s \in [0, t) \), \( x \in \mathbb{C}^n \), \( u \in U_s \), \( (y, z) = V_0 (x) \) and \( (v, w) = V(u) \). There exists a unique solution, \( (\eta, \zeta) \in \mathcal{X}^v_s \), to (25), and a unique solution, \( (\eta^*, \zeta^*) \in \mathcal{X}^v_s \), to (35).

**Proof:** These are standard results. See [6], Section III.5 and Appendix D; [15], Lemma II.5.2 and Corollary II.5.12; and [16]. In the case of (35), one should note that by Assumption (A.2), \( \bar{R}_y \), \( \bar{R}_z \), \( \bar{T}_y \), \( \bar{T}_z \) grow at most linearly in \( y, z \).

The following is straightforward from the above definitions, cf. [24].

**Lemma 3** Let \( s \in [0, t) \), \( x \in \mathbb{C}^n \), \( u \in U_s \), \( (y^T, z^T)^T = V_0 (x) \) and \( (v, w) = V(u) \). \( \xi \in \mathcal{X} \) is a solution of (17) if and only if \( V(\xi) \in \mathcal{X}^v_s \), to (25). Similarly, \( \xi^* \in \mathcal{X} \) is a solution of (18) if and only if \( V(\xi^*) \in \mathcal{X}^v_s \) is a solution of (35).

Combining Lemmas 2 and 3, one has:

**Lemma 4** Let \( s \in [0, t) \), \( x \in \mathbb{C}^n \) and \( u \in U_s \). There exists a unique solution, \( \xi \in \mathcal{X} \), to (17), and a unique solution, \( \xi^* \in \mathcal{X} \), to (18).

### 3.3 A relationship among the solutions

Let \( \mathcal{D}_2 \doteq (0, t) \times \mathbb{R}^{2n} \) and \( \overline{\mathcal{D}}_2 \doteq [0, t] \times \mathbb{R}^{2n} \). By (27),

\[
|\bar{S}|^2_c = \sum_{j=1}^n \bar{R}_{y_j}^2 - \bar{R}_{z_j}^2 + 2i \bar{T}_{y_j} \bar{T}_{z_j},
\]

and by (29), (30)

\[
\Delta \bar{S} = \sum_{j=1}^n \bar{T}_{y_j, z_j} - i \bar{R}_{y_j, z_j}.
\]

Let

\[
(V^R(\mathcal{V}_0(x)), V^I(\mathcal{V}_0(x)))^T \doteq \mathcal{V}_{00}(V(x)),
\]

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\[(\phi^R(\mathcal{V}_0(x)), \phi^I(\mathcal{V}_0(x)))^T \doteq \mathcal{V}_{00}(\phi(x)), \quad (39)\]

for all \(x \in \mathbb{C}^n\). Substituting (36)–(38) into (14), we have

\[
0 = \hat{R}_t + i \hat{T}_t - \frac{\hbar}{2m} \sum_{j=1}^{n}(\hat{T}_{y_j,z_j} - \hat{R}_{y_j,z_j}) - \frac{1}{2m} \sum_{j=1}^{n} \left(\hat{R}_{y_j}^2 - \hat{R}_{z_j}^2 + 2i \hat{T}_{y_j} \hat{T}_{z_j}\right) - (V^R + iV^I)
\]
on \(\mathcal{D}_2\), which yields

\[
0 = \hat{R}_t - \frac{\hbar}{2m} \sum_{j=1}^{n} \hat{R}_{y_j,z_j} - \frac{1}{2m} \sum_{j=1}^{n} \left(\hat{R}_{y_j}^2 - \hat{R}_{z_j}^2\right) - V^R \quad \forall (s, y, z) \in \mathcal{D}_2, \quad (40)
\]

\[
0 = \hat{T}_t - \frac{\hbar}{2m} \sum_{j=1}^{n} \hat{T}_{y_j,z_j} - \frac{1}{2m} \sum_{j=1}^{n} \hat{T}_{y_j} \hat{T}_{z_j} - V^I \quad \forall (s, y, z) \in \mathcal{D}_2, \quad (41)
\]
on \(\mathcal{D}_2\), and, of course,

\[
\hat{R}(t, y, z) = \phi^R(y, z) \quad \forall (y, z) \in \mathbb{R}^{2n}, \quad (42)
\]

\[
\hat{T}(t, y, z) = \phi^I(y, z) \quad \forall (y, z) \in \mathbb{R}^{2n}. \quad (43)
\]

**Proposition 5** Let \(\hat{S} \in \mathcal{S}_C\) and \(\hat{R}, \hat{T}\) satisfy (26) for all \((r, x) \in \mathcal{D}_C\). If \(\hat{S}\) satisfies (14)–(15), then \(\hat{R}, \hat{T}\) satisfy (40)–(43). Alternatively, if \(\hat{R}, \hat{T} \in C_{p,2}^{1,2}(\mathcal{D}_2; \mathbb{R}) \cap C(\overline{\mathcal{D}_2}; \mathbb{R})\) satisfy (40)–(43), and \(\hat{S} \in \mathcal{S}_C\) is given by (26), then \(\hat{S}\) satisfies (14)–(15).

**Proof:** The first assertion follows by simple algebraic substitution, using (27)–(30). Now, suppose \(\hat{R}, \hat{T} \in C_{p,2}^{1,2}(\mathcal{D}_2; \mathbb{R}) \cap C(\overline{\mathcal{D}_2}; \mathbb{R})\) satisfy (40)–(43), and let \(\hat{S}\) be given by (26). We will show that \(\hat{R}, \hat{T}\) satisfy the Cauchy-Riemann relations, and hence that \(\hat{S} \in \mathcal{S}_C\). After that, one may again use simple algebraic substitutions to verify the final assertion.

Differentiating (40) with respect to \(y_k\), and (41) with respect to \(z_k\), yields

\[
\hat{R}_{t,y_k} = \frac{\hbar}{2m} \sum_{j=1}^{n} \hat{R}_{y_j,z_j,y_k} + \frac{1}{m} \sum_{j=1}^{n} \left(\hat{R}_{y_j} \hat{R}_{y_j,y_k} - \hat{R}_{z_j} \hat{R}_{z_j,y_k}\right) + V^R_{y_k}, \quad (44)
\]

\[
\hat{T}_{t,z_k} = \frac{\hbar}{2m} \sum_{j=1}^{n} \hat{T}_{y_j,z_j,z_k} + \frac{1}{m} \sum_{j=1}^{n} \left(\hat{T}_{y_j} \hat{T}_{y_j,z_k} + \hat{T}_{y_j} \hat{T}_{z_j,z_k}\right) + V^I_{z_k}, \quad (45)
\]

Applying (27)–(32) in (45), one finds

\[
\hat{T}_{t,z_k} = \frac{\hbar}{2m} \sum_{j=1}^{n} \hat{R}_{y_j,z_j,y_k} + \frac{1}{m} \sum_{j=1}^{n} \left(\hat{R}_{y_j} \hat{R}_{y_j,y_k} - \hat{R}_{z_j} \hat{R}_{z_j,y_k}\right) + V^R_{y_k}, \quad (46)
\]

which by (44),

\[
= \hat{R}_{t,y_k} \quad \forall (s, y, z) \in \mathcal{D}_2, \quad \forall k \in [1, n].
\]

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Also note that as $\phi$ is holomorphic,
\[
\bar{R}_{y_k}(t, y, z) = \phi_{y_k}^R(y, z) = \phi_{z_k}^I(y, z) = \bar{T}_{z_k}(t, y, z) \quad \forall y, z \in \mathbb{R}^n.
\]  
(47)

By the Fundamental Theorem of Calculus,
\[
\bar{T}_{z_k}(s, y, z) = \bar{T}_{z_k}(t, y, z) - \int_s^t \bar{T}_{z_k} (\sigma, y, z) d\sigma,
\]
which by (46),(47),
\[
\bar{R}_{y_k}(s, y, z) \quad \forall (s, y, z) \in D_2, \forall k \in ]1, n[.
\]  
(48)

Similarly, one obtains
\[
\bar{T}_{y_k}(s, y, z) = - \bar{R}_{z_k}(s, y, z) \quad \forall (s, y, z) \in D_2, \forall k \in ]1, n[.
\]  
(49)

By (48),(49), the Cauchy-Riemann conditions are satisfied. ■

4 The verification

We will obtain a verification result demonstrating that a solution of (7),(8) is the stationary value of the expectation of the action functional on process paths satisfying (17).

For $s \in (0, t)$ and $h \in (0, 1]$, we define payoff $J^h(s, \cdot, \cdot) : \mathbb{R}^n \times U_s \to \mathbb{C}$ by
\[
J^h(s, x, u) \doteq \mathbb{E}\left\{\int_s^t L(\xi_r, u_r) \, dr + \phi(\xi_t)\right\}
\]
\[
\doteq \mathbb{E}\left\{\int_s^t \left(\frac{m}{2} |u_r|^2 - V(\xi_r)\right) \, dr + \phi(\xi_t)\right\},
\]  
(50)

where $\xi$ satisfies (17) with input $u \in U_s$ and initial state $x \in \mathbb{R}^n$. The stationary value, $S^h : D \to \mathbb{C}$, is given by
\[
S^h(s, x) \doteq \text{stat}_{u \in U_s} J^h(s, x, u) \quad \forall (s, x) \in D.
\]  
(51)

We assume throughout Section 4 that
\[(A.3) \quad \text{argstat}_{u \in U_s} J^h(s, x, u) \text{ is single-valued for all } (s, x) \in D.
\]

This is the last assumption. We remark that one may want to weaken this assumption to uniqueness in some prespecified subset of $D$, but leave that additional complication to a later effort. The main result of the section is:

**Theorem 6** Let $h \in (0, 1]$. Suppose $\bar{S} \in \mathcal{S}_C \cap C^{1,4}(\mathcal{D}_C)$ satisfies (14)–(15), and there exists $\tilde{C}_S < \infty$ such that $|\bar{S}_{xxx}(r, x)|, |\bar{S}_{xrr}(r, x)|, |\bar{S}_{xxxx}(r, x)| \leq \tilde{C}_S (1 + |x|^{2q})$ for all $(s, x) \in \mathcal{D}_C$. Then, $\bar{S}(s, x) = S^h(s, x)$ for all $(s, x) \in \mathcal{D}_C$.

We remark that the representation is proved for general $h \in (0, 1]$ in anticipation of potential use in future semiclassical limit results. We begin with two lemmas.
Lemma 7 Let \( s \in [0,t], x \in \mathbb{C}^n, \ h \in (0,1] \) and \( u \in U_s \). Let \( \xi \in \mathcal{X}_s \) be given by (17). Suppose \( \bar{S} \in \mathcal{S}_C \) satisfies (14),(15). Let \( \bar{u}^* = \bar{u}^{*(s,x)}, \ \bar{\xi}^* = \bar{\xi}^{*(s,x)} \) be given by (18)-(19). Then,

\[
\begin{align*}
\bar{S}(s,x) &= \mathbb{E} \left\{ \int_s^t -\bar{S}_t(r,\xi_r) - \bar{S}^T_x(r,\xi_r)u_r - \frac{i}{2} \Delta \bar{S}(r,\xi_r) \, dr + \phi(\xi_t) \right\} \\
\text{and} \\
\bar{S}(s,x) &= \mathbb{E} \left\{ \int_s^t -\bar{S}_t(r,\bar{\xi}_r^*) - \bar{S}^T_x(r,\bar{\xi}_r^*)\bar{u}_r^* - \frac{i}{2} \Delta \bar{S}(r,\bar{\xi}_r^*) \, dr + \phi(\bar{\xi}_r^*) \right\}.
\end{align*}
\]

Proof: We prove only the second assertion; the proof of the first is similar and somewhat simpler. Fix \( h \in (0,1] \) and \( (s,x) \in \mathcal{D}_C \). Let \( \xi^* = \bar{\xi}^{*(s,x)} \) and \( \bar{u}^* = \bar{u}^{*(s,x)} \) be given by (18)-(19) with \( S \) replacing \( \bar{S} \). Let \( (y^T,z^T)^T = V_0(x), \ (\bar{\eta}^*,\bar{\xi}^*) = V(\bar{\xi}^*), \ (\bar{u}^*,\bar{u}^*) = V(\bar{u}^*), \) and \((\bar{R}(r,V_0(x)),\bar{T}(r,V_0(x))) = \bar{V}_0(\bar{S}(r,x)). \) Note that \((\bar{\eta}^*,\bar{\xi}^*)\) satisfy (35) with \( \bar{R},\bar{T} \) replacing \( (R,T) \). By Itô's formula,

\[
\mathbb{E}[\bar{R}(t,\bar{\eta}^*,\bar{\xi}^*)] = \bar{R}(s,y,z) + \mathbb{E} \left\{ \int_s^t \bar{R}_t(r,\bar{\eta}^*_r,\bar{\xi}^*_r) + \bar{R}^T_y(r,\bar{\eta}^*_r,\bar{\xi}^*_r)\bar{u}^*_r \\
+ \bar{R}^T_z(r,\bar{\eta}^*_r,\bar{\xi}^*_r)\bar{u}^*_r + \frac{h}{2} \sum_{j=1}^n \left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} \right] (r,\bar{\eta}^*_r,\bar{\xi}^*_r) \, dr \\
+ \frac{h}{2} \int_s^t \left[ \bar{R}_y(r,\bar{\eta}^*_r,\bar{\xi}^*_r) + \bar{R}_z(r,\bar{\eta}^*_r,\bar{\xi}^*_r) \right]^T dB_r \right\}, \tag{52}
\]

where, in the interests of space we let \([\bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j}] (r,\bar{\eta}^*_r,\bar{\xi}^*_r)\) denote \( \bar{R}_{y_j,y_j}(r,\bar{\eta}^*_r,\bar{\xi}^*_r) + 2\bar{R}_{y_j,z_j}(r,\bar{\eta}^*_r,\bar{\xi}^*_r) + \bar{R}_{z_j,z_j}(r,\bar{\eta}^*_r,\bar{\xi}^*_r) \), and use other similar notation where helpful throughout.

Now, by Assumption (A.2) and the definition of \( \bar{R},\bar{T} \), there exists \( \hat{C}_S < \infty \) such that

\[
|\bar{R}_y(r,y,z)|, |\bar{R}_z(r,y,z)| \leq \hat{C}_S(1 + |y| + |z|) \quad \forall (s,y,z) \in \mathcal{D}_2.
\]

Consequently, by Lemma 2 (noting that this implies \((\bar{\eta}^*,\bar{\xi}^*) \in \mathcal{X}_s^*)\), there exists \( M_1 = M_1(t-s,|x|) \) < \( \infty \) such that

\[
\mathbb{E} \int_s^t |\bar{R}_y(r,\bar{\eta}^*_r,\bar{\xi}^*_r)|^2 \, dr, \mathbb{E} \int_s^t |\bar{R}_z(r,\bar{\eta}^*_r,\bar{\xi}^*_r)|^2 \, dr < M_1. \tag{53}
\]

By (53) and standard results (cf. [5], Section V.3),

\[
\mathbb{E} \left\{ \int_s^t \left[ \bar{R}_y(r,\bar{\eta}^*_r,\bar{\xi}^*_r) + \bar{R}_z(r,\bar{\eta}^*_r,\bar{\xi}^*_r) \right] dB_r \right\} = 0. \tag{54}
\]

Combining (52) and (54) yields

\[
\bar{R}(s,y,z) = \mathbb{E} \left\{ \int_s^t -\bar{R}_t(r,\bar{\eta}^*_r,\bar{\xi}^*_r) - \bar{R}^T_y(r,\bar{\eta}^*_r,\bar{\xi}^*_r)\bar{u}^*_r - \bar{R}^T_z(r,\bar{\eta}^*_r,\bar{\xi}^*_r)\bar{w}^*_r \right\}
\]
\[-\frac{\hbar}{4m} \sum_{j=1}^{n} \left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} \right] (r, \eta^*_r, \zeta^*_r) \, dr + \bar{R}(t, \eta^*_r, \zeta^*_r) \right\} \right]. \tag{55}\]

Then, by (42) and (55),

\[
\bar{R}(s, y, z) = E \left\{ \int_{s}^{t} -\bar{T}(r, \eta^*_r, \zeta^*_r) - \bar{T}'(r, \eta^*_r, \zeta^*_r) \bar{v}^*_r - \bar{T}''(r, \eta^*_r, \zeta^*_r) \bar{w}^*_r
\right.
\]

\[
- \frac{\hbar}{4m} \sum_{j=1}^{n} \left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} \right] (r, \eta^*_r, \zeta^*_r) \, dr + \text{Re}(\phi(\zeta^*_r)) \right\}. \tag{56}\]

Similarly,

\[
\bar{T}(s, y, z) = E \left\{ \int_{s}^{t} -\bar{T}(r, \eta^*_r, \zeta^*_r) - \bar{T}'(r, \eta^*_r, \zeta^*_r) \bar{v}^*_r - \bar{T}''(r, \eta^*_r, \zeta^*_r) \bar{w}^*_r
\right.
\]

\[
- \frac{\hbar}{4m} \sum_{j=1}^{n} \left[ \bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j} \right] (r, \eta^*_r, \zeta^*_r) \, dr + \text{Im}(\phi(\zeta^*_r)) \right\}. \tag{57}\]

Now, applying $V_{00}^{-1}$ to (56),(57), one has

\[
\bar{S}(s, x) = \bar{R}(s, y, z) + i\bar{T}(s, y, z)
\]

\[
= E \left\{ \int_{s}^{t} -\bar{S}(r, \zeta^*_r) - \left[ \bar{R}(r, \eta^*_r, \zeta^*_r) + i\bar{T}(r, \eta^*_r, \zeta^*_r) \right]^T \bar{v}^*_r
\right.

\[
- \left[ \bar{T}(r, \eta^*_r, \zeta^*_r) - i\bar{R}(r, \eta^*_r, \zeta^*_r) \right]^T i\bar{w}^*_r
\]

\[
- \frac{\hbar}{4m} \sum_{j=1}^{n} \left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} + i(\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j}) \right] (r, \eta^*_r, \zeta^*_r) \, dr
\]

\[
+ \bar{S}(t, \zeta^*_r) \right\},
\]

and using (27), this is

\[
= E \left\{ \int_{s}^{t} -\bar{S}(r, \zeta^*_r) - \bar{S}'(r, \zeta^*_r) \bar{v}^*_r
\right.
\]

\[
- \frac{\hbar}{4m} \sum_{j=1}^{n} \left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} + i(\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j}) \right] (r, \eta^*_r, \zeta^*_r) \, dr
\]

\[
+ \bar{S}(t, \zeta^*_r) \right\}. \tag{58}\]

Also, by (29), (30),

\[
\left[ \bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} \right] (r, \eta^*_r, \zeta^*_r) = -2\text{Im}(\bar{S}(z_j, x_j)),
\]

\[
i[\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j}] (r, \eta^*_r, \zeta^*_r) = 2i\text{Re}(\bar{S}(z_j, x_j)),
\]
Applying these identities in (58) yields

\[ \bar{S}(s, x) = E\left\{ \int_s^t -\bar{S}_t(r, \bar{\xi}_r^*) - \bar{S}_x(r, \bar{\xi}_r^*) \bar{u}_r^* - \frac{ih}{2m} \Delta \bar{S}(r, \bar{\xi}_r^*) \, dr + \bar{S}(t, \bar{\xi}_t^*) \right\}. \]

\[ \text{Lemma 8} \]

Let \( h \in (0, 1) \), and suppose that \( \bar{S} \in \mathcal{S}_C \) satisfies (14)-(15). Then, \( \bar{S}(s, x) = J^h(s, x, \bar{u}^*(s, x)) \) for all \((s, x) \in \mathcal{D}_C\), where \( \bar{u}^*(s, x) \) is given by (18)-(19) with \( \bar{S} \) in place of \( S \).

\[ \text{Proof:} \]
Fix \( h \in (0, 1) \) and \((s, x) \in \mathcal{D}_C\). Let \( \bar{\xi}_r^* = \bar{\xi}^*(s, x, \bar{u}^*) \) be given by (18)-(19) with \( \bar{S} \) replacing \( S \). From the second assertion of Lemma 7, we have

\[ \bar{S}(s, x) = E\left\{ \int_S^t -\bar{S}_t(r, \bar{\xi}_r^*) - \bar{S}_x(r, \bar{\xi}_r^*) \bar{u}_r^* - \frac{ih}{2m} \Delta \bar{S}(r, \bar{\xi}_r^*) \, dr + \phi(\bar{\xi}_t^*) \right\}, \]

which by (19),

\[ = E\left\{ \int_s^t -\bar{S}_t(r, \bar{\xi}_r^*) + \frac{1}{2m} |\bar{S}_x(r, \bar{\xi}_r^*)|^2 - \frac{ih}{2m} \Delta \bar{S}(r, \bar{\xi}_r^*) \, dr + \phi(\bar{\xi}_t^*) \right\} \]

\[ = E\left\{ \int_s^t -\bar{S}_t(r, \bar{\xi}_r^*) + \frac{1}{2m} |\bar{S}_x(r, \bar{\xi}_r^*)|^2 + V(\bar{\xi}_r^*) - \frac{ih}{2m} \Delta \bar{S}(r, \bar{\xi}_r^*) \, dr + \phi(\bar{\xi}_t^*) \right\}, \]

which by (5) and (14),

\[ = E\left\{ \int_s^t \frac{m}{2} |\bar{S}_x(r, \bar{\xi}_r^*)|^2 - V(\bar{\xi}_r^*) \, dr + \phi(\bar{\xi}_t^*) \right\}, \]

which by the choice of \( \bar{u}^* = \bar{u}^*(s, x) \) and (50),

\[ = J^h(s, x, \bar{u}^*(s, x)). \]

\[ \text{Proof:} \quad [\text{proof of Theorem 6.}] \]
Fix \((s, x) \in \mathcal{D}_C\). Let \( L(x, v) = \frac{m}{2} |v|^2 - V(x) \) for all \( x, v \in \mathbb{C}^n \). For compactness of notation, let \( \bar{\xi}_r^* = \bar{\xi}^*(s, x, \bar{u}^*) \) and \( \bar{u}^* = \bar{u}^*(s, x) \).

By Lemma 8,

\[ \bar{S}(s, x) = E\left\{ \int_s^t L(\bar{\xi}_r^*, \bar{u}_r^*) \, dr + \phi(\bar{\xi}_t^*) \right\} = J^h(s, x, \bar{u}^*). \]

(59)

It remains to be shown the \( \bar{u}^* \) is the argmax over \( \mathcal{U}_c \) of \( J^h(s, x, \cdot) \).

Let \( u \in \mathcal{U}_c \) and \( \delta = u - \bar{u}^* \in \mathcal{U}_c \). Let \( \xi \in \mathcal{X}_u \) be the trajectory generated by \( u \), i.e., the solution of (17), and let \( \Delta = \xi - \bar{\xi}^* \in \mathcal{X}_u \), where we note that \( \Delta_t = \int_0^t \delta_{r_t} \, dp \) for all \( (r, \omega) \in [s, t] \times \Omega \). By (59),

\[ J^h(s, x, \bar{u}^*) = \bar{S}(s, x) = E\{\bar{S}(t, \xi_t)\} + [\bar{S}(s, x) - E\{\bar{S}(t, \xi_t)\}], \]

which by Lemma 7 and (15),
the assumed bounds on derivatives and the definition of
for all (r, ξ, x).

Now, by (5), (14),
\[ 0 = \tilde{S}_t(r, x) + \text{stat} \{ \tilde{S}_T^T(r, \xi^*) v + \frac{\Theta}{2} |v|^2 - V(x) \} + \frac{i \hbar}{2m} \Delta \tilde{S}(r, x). \]

Taking x = \xi^* in this, and using (27), (34) and the definition of \( \bar{u}^* \), we have
\[ 0 = \tilde{S}_t(r, \xi^*) + \tilde{S}_T^T(r, \xi^*) \bar{u}_r^* + \frac{\Theta}{2} |\bar{u}_x^*|^2 - V(\xi^*) + \frac{i \hbar}{2m} \Delta \tilde{S}(r, \xi^*) \]
for all (r, \omega) ∈ (s, t) × Ω. Combining (60) and (61), one has
\[
J^h(s, x, \bar{u}^*,(s,x)) = \mathbb{E}\left\{ \int_s^t L(\xi, u_r) \, dr + \phi(\xi_t) \right\} \\
+ \mathbb{E}\left\{ \int_s^t L(\xi^*_r, \bar{u}^*_r) - L(\xi, u_r) \right. \\
+ \tilde{S}_t(r, \xi^*_r) - \tilde{S}_t(r, \xi_r) \\
+ \tilde{S}_T^T(r, \xi^*_r) \bar{u}_r^* - \tilde{S}_T^T(r, \xi_r) u_r + \frac{i \hbar}{2m} [\Delta \tilde{S}(r, \xi^*_r) - \Delta \tilde{S}(r, \xi_r)] \, dr \left. \right\} \\
= J^h(s, x, u) + \mathbb{E}\left\{ \int_s^t L(\xi^*_r, \bar{u}^*_r) - L(\xi, u_r) \right. \\
+ \tilde{S}_t(r, \xi^*_r) - \tilde{S}_t(r, \xi_r) \\
+ \tilde{S}_T^T(r, \xi^*_r) \bar{u}_r^* - \tilde{S}_T^T(r, \xi_r) u_r + \frac{i \hbar}{2m} [\Delta \tilde{S}(r, \xi^*_r) - \Delta \tilde{S}(r, \xi_r)] \, dr \left. \right\}.
\]

This implies
\[
|J^h(s, x, \bar{u}^*,(s,x)) - J^h(s, x, u)| \\
\leq \mathbb{E}\left\{ \int_s^t |L(\xi^*_r, \bar{u}^*_r) - L(\xi, u_r) + \tilde{S}_t(r, \xi^*_r) - \tilde{S}_t(r, \xi_r) \\
+ \tilde{S}_T^T(r, \xi^*_r) \bar{u}_r^* - \tilde{S}_T^T(r, \xi_r) u_r + \frac{i \hbar}{2m} [\Delta \tilde{S}(r, \xi^*_r) - \Delta \tilde{S}(r, \xi_r)]| \, dr \right\} \\
\leq \mathbb{E}\left\{ \int_s^t |\Xi(\omega)| \, dr \right\}. \tag{62}
\]

Note that by Taylor’s Theorem, for x, \( \bar{x}, v, \bar{v} \in \mathbb{C}^n \) and r ∈ (s, t), and using the assumed bounds on derivatives and the definition of L,
\[
|L(x, v) - L(\bar{x}, \bar{v}) + \tilde{S}_t(r, x) - \tilde{S}_t(r, \bar{x}) + \tilde{S}_x(r, x)v - \tilde{S}_x(r, \bar{x})\bar{v} \\
+ \frac{i \hbar}{2m} [\Delta \tilde{S}(r, x) - \Delta \tilde{S}(r, \bar{x})]|
\]

14
for appropriate $K_1 = K_1(C_0, \hat{C}_S, h, m) < \infty$. This implies

$$|\Xi_r| \leq -V_x(\hat{\xi}_r^*) \Delta_r + \hat{S}_{xx}(r, \hat{\xi}_r^*) \Delta_r + \hat{S}_{xx}(r, \hat{\xi}_r^*) \Delta_r + [\hat{S}_x(r, \xi_r) u_r - \hat{S}_x(r, \xi_r) u_r] \Delta_r + \frac{i h}{2m} (\Delta \hat{S})_x(r, \hat{\xi}_r^*) \Delta_r + K_1 (1 + |\xi_r|^2 + |\hat{\xi}_r^*|^2) \Delta_r^2 + m |\delta_r|^2 \forall (r, \omega) \in (s, t) \times \Omega. \tag{64}$$

Now recall that

$$\hat{u}^* = \frac{1}{m} \hat{S}_x(r, \hat{\xi}_r^*) = \text{argstat}_{v \in \mathbb{C}} \left[ \frac{m}{2} |v|^2 + \hat{S}_x(r, \hat{\xi}_r^*) v \right],$$

and consequently, by Lemma 1,

$$m \hat{u}^* + \hat{S}_x(r, \hat{\xi}_r^*) = 0. \tag{65}$$

Substituting (65) into (64) yields

$$|\Xi_r| \leq -V_x(\hat{\xi}_r^*) \Delta_r + \hat{S}_{xx}(r, \hat{\xi}_r^*) \Delta_r + \hat{S}_{xx}(r, \hat{\xi}_r^*) \Delta_r + [\hat{S}_x(r, \xi_r) u_r - \hat{S}_x(r, \xi_r) u_r] \Delta_r + \frac{i h}{2m} (\Delta \hat{S})_x(r, \hat{\xi}_r^*) \Delta_r + K_1 (1 + |\xi_r|^2 + |\hat{\xi}_r^*|^2) \Delta_r^2 + m |\delta_r|^2 \forall (r, \omega) \in (s, t) \times \Omega. \tag{66}$$

Also, more generally, recalling definitions (19),

$$\hat{u}(r, x) = \frac{1}{m} \hat{S}_x(r, x) = \text{argstat}_{v \in \mathbb{C}} \left[ L(x, v) + \hat{S}_x(r, x) v \right] \forall (r, x) \in \mathcal{D}_\mathbb{C}. \tag{67}$$

Note that

$$-V_x(\hat{\xi}_r^*) + \hat{S}_{xx}(r, \hat{\xi}_r^*) + \hat{S}_{xx}(r, \hat{\xi}_r^*) \hat{u}_r^* + \frac{i h}{2m} (\Delta \hat{S})_x(r, \hat{\xi}_r^*)$$

$$= \frac{\partial}{\partial x} \left[ -V(x) + \hat{S}_l(r, x) + \hat{S}_x(r, x) v + \frac{i h}{2m} \hat{S}_{xx}(r, x) \right]_{x = \xi_r^*, v = \hat{u}(r, \xi_r^*)}$$

where the partial derivative notation indicates that the derivative is taken only over explicitly appearing arguments, and this is

$$= \frac{d}{dx} \left[ L(x, \hat{u}(r, x)) + \hat{S}_l(r, x) + \hat{S}_x(r, x) \hat{u}(r, x) + \frac{i h}{2m} \hat{S}_{xx}(r, x) \right]_{x = \xi_r^*}$$

$$- \frac{\partial}{\partial v} \left[ L(x, v) + \hat{S}_x(r, x) \right]_{x = \xi_r^*, v = \hat{u}(r, \xi_r^*)} \frac{d}{dx} \hat{u}(r, x)_{x = \xi_r^*},$$
which by (67),
\[
\frac{d}{dx} \left[ L(x, \tilde{u}(r, x)) + \bar{S}_t(r, x) + \bar{S}_x(r, x) \tilde{u}(r, x) + \frac{i h}{m} \bar{S}_{xx}(r, x) \right] \bigg|_{x = \xi^*_r,}
\]
which by (14) and (67),
\[
\frac{d}{dx}[0] = 0. \tag{68}
\]
Substituting (68) into (66), we have
\[
|\Xi| \leq \left| \bar{S}_x(r, \xi_r) u_r - \bar{S}_x(r, \xi^*_r) \bar{u}_r - \bar{S}_{xx}(r, \xi^*_r) \bar{u}_r \Delta_r - \bar{S}_x(r, \xi^*_r) \delta_r \right|
+ K_1 (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\Delta_r|^2 + m|\delta_r|^2 \quad \forall (r, \omega) \in (s, t) \times \Omega,
\]
which implies
\[
\begin{align*}
\mathbb{E} \int_s^t |\Xi| \, dr & \leq m\|\delta\|^2 \mathbb{E} + K_1 \mathbb{E} \int_s^t (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\Delta_r|^2 \, dr \\
& + \mathbb{E} \int_s^t \left| \bar{S}_x(r, \xi_r) u_r - \bar{S}_x(r, \xi^*_r) \bar{u}_r - \bar{S}_{xx}(r, \xi^*_r) \bar{u}_r \Delta_r - \bar{S}_x(r, \xi^*_r) \delta_r \right| \, dr. \tag{69}
\end{align*}
\]
Now,
\[
\begin{align*}
\mathbb{E} \int_s^t \left| \bar{S}_x(r, \xi_r) u_r - \bar{S}_x(r, \xi^*_r) \bar{u}_r - \bar{S}_{xx}(r, \xi^*_r) \bar{u}_r \Delta_r - \bar{S}_x(r, \xi^*_r) \delta_r \right| \, dr \\
= \mathbb{E} \int_s^t \left| \left[ \bar{S}_x(r, \xi_r) - \bar{S}_x(r, \xi^*_r) \right] \bar{u}_r - \bar{S}_{xx}(r, \xi^*_r) \bar{u}_r \Delta_r - \bar{S}_x(r, \xi^*_r) \delta_r \right| \, dr. \tag{70}
\end{align*}
\]
By Taylor’s Theorem and the assumptions,
\[
\left| \left[ \bar{S}_x(r, x) - \bar{S}_x(r, \bar{x}) \right] (v - \bar{v}) \right| \leq C_s (1 + |x|^{2q} + |\bar{x}|^{2q}) |x - \bar{x}| |v - \bar{v}|
\]
and
\[
\left| \bar{S}_x(r, x) - \bar{S}_x(r, \bar{x}) - \bar{S}_{xx}(r, \bar{x}) (x - \bar{x}) \right| \leq \frac{C_s}{2} (1 + |x|^{2q} + |\bar{x}|^{2q}) |x - \bar{x}|^2
\]
for all \( x, \bar{x}, v, \bar{v} \in \mathbb{C}^n \) and \( r \in (s, t) \). Applying these inequalities in (70), we have
\[
\begin{align*}
\mathbb{E} \int_s^t \left| \left[ \bar{S}_x(r, \xi_r) - \bar{S}_x(r, \xi^*_r) \right] \bar{u}_r - \bar{S}_{xx}(r, \xi^*_r) \bar{u}_r \Delta_r - \bar{S}_x(r, \xi^*_r) \delta_r \right| \, dr \\
\leq \mathbb{E} \int_s^t C_s (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\Delta_r| |\delta_r| \, dr \\
+ \mathbb{E} \int_s^t \frac{C_s}{2} (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\Delta_r|^2 |\bar{u}_r|^2 \, dr \\
\leq \frac{C_s}{2} \mathbb{E} \int_s^t (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\delta_r|^2 \, dr + \frac{C_s}{2} \mathbb{E} \int_s^t (1 + |\xi_r|^2q + |\xi^*_r|^2q) |\Delta_r|^2 \, dr.
\end{align*}
\]
\[ + \frac{C_2}{r} \mathbb{E} \int_s^t \left( 1 + |\xi_r'|^{2q} + |\xi_r^{*2q}| \right) |\bar{u}_r^*| |\Delta r|^2 \, dr. \quad (71) \]

Now, using Hölder’s inequality,

\[
\mathbb{E} \int_s^t \left( 1 + |\xi_r'|^{2q} + |\xi_r^{*2q}| \right) |\delta_r|^2 \, dr
\leq \left\{ 1 + \left[ 2 \mathbb{E} \int_s^t |\xi_r^*|^{4q} + |\xi_r^* + \Delta r|^{4q} \, dr \right]^{1/2} \right\} \|\delta\|_{\mathcal{L}_s}^2
\leq C_1 \left\{ 1 + \|\xi_r^*\|_{X_s}^{2q} + \left[ \mathbb{E} \int_s^t |\Delta r|^{4q} \, dr \right]^{1/2} \right\} \|\delta\|_{\mathcal{L}_s}^2, \quad (72) \]

for appropriate \( C_1 = C_1(q) < \infty \). Substituting (72) and (71) into (69) yields

\[
\mathbb{E} \int_s^t |\Xi_r| \, dr \leq \left\{ m + C_2 \mathbb{E} \left[ 1 + \|\xi_r^*\|_{X_s}^{2q} + \left[ \mathbb{E} \int_s^t |\Delta r|^{4q} \, dr \right]^{1/2} \right] \right\} \|\delta\|_{\mathcal{L}_s}^2
\]

\[
+ (K_1 + C_2) \mathbb{E} \int_s^t \left( 1 + |\xi_r'|^{2q} + |\xi_r^{*2q}| \right) |\Delta r|^2 \, dr
\]

\[
+ \frac{C_2}{r} \mathbb{E} \int_s^t \left( 1 + |\xi_r'|^{2q} + |\xi_r^{*2q}| \right) |\bar{u}_r^*| |\Delta r|^2 \, dr,
\]

which by the definition of \( \bar{u}^* \) and Assumption (A.2),

\[
\leq \left\{ m + C_2 \mathbb{E} \left[ 1 + \|\xi_r^*\|_{X_s}^{2q} + \left[ \mathbb{E} \int_s^t |\Delta r|^{4q} \, dr \right]^{1/2} \right] \right\} \|\delta\|_{\mathcal{L}_s}^2
\]

\[
+ C_2 \mathbb{E} \int_s^t \left( 1 + |\xi_r^*|^{2q} + |\xi_r^* + \Delta r|^{4q} \right) |\Delta r|^2 \, dr,
\]

for appropriate \( C_2 = C_2(C_0, C_S, \hat{C}_S, q) < \infty \), and this is

\[
\leq \left\{ m + C_2 \mathbb{E} \left[ 1 + \|\xi_r^*\|_{X_s}^{2q} + \left[ \mathbb{E} \int_s^t |\Delta r|^{4q} \, dr \right]^{1/2} \right] \right\} \|\delta\|_{\mathcal{L}_s}^2
\]

\[
+ C_3 \mathbb{E} \int_s^t \left( |\Delta r|^2 + |\xi_r^*|^{4q} |\Delta r|^2 + |\Delta r|^{4q+2} \right) \, dr, \quad (73) \]

for appropriate \( C_3 = C_3(C_0, C_S, \hat{C}_S, q) < \infty \).

Next one should note some simple estimates. First, using Hölder’s inequality,

\[
\mathbb{E} \int_s^t |\Delta r|^2 \, dr = \mathbb{E} \int_s^t \left( \int_s^r \delta_r \, d\rho \right)^2 \, dr \leq (t - s) \mathbb{E} \int_s^t |\delta_r|^2 \, d\rho \equiv (t - s) \|\delta\|_{\mathcal{L}_s}^2
\leq (t - s) \|\delta\|_{\mathcal{L}_s}^2. \quad (74) \]

Similarly, with Hölder’s inequality and some simple calculations,

\[
\mathbb{E} \int_s^t |\Delta r|^{4q+2} \, dr = \mathbb{E} \int_s^t \left( \int_s^r \delta_r \, d\rho \right)^{4q+2} \, dr \leq (t - s)^{4q+2} \mathbb{E} \int_s^t |\delta_r|^{4q+2} \, d\rho
\]

\[
\leq (t - s)^{4q+2} \|\delta\|_{\mathcal{L}_s}^{4q+2} \leq (t - s)^{4q+2} \|\delta\|_{\mathcal{L}_s}^{4q+2}, \quad (75) \]
and
\[
E \int_s^t |\Delta_r|^{4q} \, dr \leq (t-s)^{4q} ||\delta||_{L_4}^q.
\] (76)

Next,
\[
E \int_s^t |\tilde{\xi}_r|^{4q}|\Delta_r|^2 \, dr \leq \left[ E \int_s^t \left| \xi_s^{4q} \right| \, dr \right]^{1/2} \left[ E \int_s^t |\delta_r|^4 \, dr \right]^{1/2}
\leq (t-s) \left| \xi_s^{4q} \right| \left[ E \int_s^t |\delta_r|^4 \, dr \right]^{1/2}
\leq (t-s)^{5/3} \left| \xi_s^{4q} \right| ||\delta||_{L_4}^2.
\] (77)

Substituting (74)–(77) into (73), one finds
\[
E \int_s^t |\Xi_r| \, dr \leq \left\{ m + \frac{C_0 C_1}{2} \left[ 1 + \left| \xi_s^{4q} \right| + t^{2q} ||\delta||_{L_4}^{2q} \right] \right\} ||\delta||_{L_4}^2
+ C_3 \left| \left| \delta \right|_{L_4}^2 + t^{5/3} \left| \xi_s^{4q} \right| ||\delta||_{L_4}^2 + t^{4q+2} ||\delta||_{L_4}^{4q+2} \right]
\leq C_4 \left[ 1 + t^{4q+2} + (1 + t^{5/3}) \left| \xi_s^{4q} \right| \left[ ||\delta||_{L_4}^2 + ||\delta||_{L_4}^{4q+2} \right] \right],
\] (78)

for appropriate choice of \( C_4 = C_4(C_0, C_S, \hat{C}_S, q) < \infty \). Substituting (78) into (62) yields
\[
|J^h(s, x, \tilde{u}^{*,(s,x)}) - J^h(s, x, u)| \leq \overline{C} ||\delta||_{L_4}^2,
\]
for \( ||\delta||_{L_4} \leq 1 \) and appropriate choice of \( \overline{C} = \overline{C}(t, x, C_0, C_S, \hat{C}_S, q) < \infty \). By definition, this implies that \( \tilde{u}^* = \arg\max_{u \in U} J^h(s, x, u) \), where uniqueness of the argstat is guaranteed by Assumption (A.3).

It may be worth noting the following, which reflects the uniqueness implied by the above representation.

**Corollary 9** In addition to (A.0)–(A.3), assume the conditions of Theorem 6. There exists a unique solution \( \bar{S} \in S_C \) to (14), (15), where \( \bar{S} = S^h \). There also exists a solution, \( \hat{S} \in S \), to (7), (8), given by \( \hat{S}(r, y) = S^h(r, \nu^{-1}(y^T, 0^T)) \) for all \( (r, y) \in \overline{D} \). Lastly, any other solution in \( S \) to (7), (8) cannot be extended holomorphically to a solution of (14), (15) in \( S_C \).

**Proof:** The existence of \( \bar{S} \) follows from Assumption (A.0), and the uniqueness follows from Theorem 6. Let \( \bar{S} \) be as given in the corollary statement. Note that
\[
\bar{S}_l(r, y) = \bar{S}_l(r, y + i0) \quad \forall (r, y) \in \overline{D}.
\] (79)

Also, by (27), for \((r, y) \in \overline{D} \), and the fact that \( \bar{S} \) agrees with \( \hat{S} \) on \( \overline{D} \),
\[
|\bar{S}_x(r, y + i0)|^2 = \sum_{j=1}^n \left[ \bar{R}_{y_j}^2 - \bar{T}_{y_j}^2 + 2i \bar{R}_{y_j} \bar{T}_{y_j} \right] = \sum_{j=1}^n \left[ \bar{R}_{y_j}^2 - \bar{T}_{y_j}^2 + 2i \bar{R}_{y_j} \bar{T}_{y_j} \right]
\]
\[
\sum_{j=1}^{n} (\hat{S}_{yj}(r,y))^2 = |\hat{S}_y(r,y)|_c^2, \tag{80}
\]

where we take \((\hat{R}(r,V_0(x)),\hat{T}(r,V_0(x)))^T = \nu_{00}(S(r,x))\) for all \((r,x) \in \mathcal{D}_C\) and \((\hat{R}(r,y),\hat{T}(r,y))^T = \nu_{00}(\hat{S}(r,y))\) for all \((r,y) \in \mathcal{D}\). Similarly, using (29), (30),

\[
\Delta \hat{S}(r,y + i0) = \sum_{j=1}^{n} \hat{S}_{xj,xj}(r,y + i0)
\]

\[
= \sum_{j=1}^{n} [\hat{R}_{yj,yj}(r,y + i0) + i\hat{T}_{yj,yj}(r,y + i0)]
\]

\[
= \sum_{j=1}^{n} [\hat{R}_{yj,yj}(r,y + i0) + i\hat{T}_{yj,yj}(r,y + i0)] = \Delta \hat{S}(r,y) \tag{81}
\]

for all \((r,y) \in \mathcal{D}\). Then, by (14) and (79)–(81),

\[
\hat{S}_t(r,y + i0) + \frac{i\hbar}{2m} \Delta \hat{S}(r,y) + \hat{H}(y,\hat{S}_y(r,y)) = \hat{S}_t(r,y + i0) + \frac{i\hbar}{2m} \Delta \hat{S}(r,y + i0) + \hat{H}(y + i0,\hat{S}_x(r,y + i0)) = 0 \tag{82}
\]

for all \((r,y) \in \mathcal{D}\). That \(\hat{S}\) also satisfies the terminal condition is obvious, and we see that \(\hat{S}\) is a solution of (7),(8).

Regarding the last assertion, recall that if two holomorphic functions on \(\mathbb{C}^n\) agree on \(\{x = y + iz \in \mathbb{C}^n | z = 0\}\), then they agree on all of \(\mathbb{C}^n\), where this follows in the standard way from power series representations (cf. [9]). Noting the uniqueness of \(\hat{S} = \hat{S}^h\) yields the assertion. 

**Remark 10** The results concerning \(\hat{S}\) in Corollary 9 also extend to (1),(2) and (3),(4) in the obvious ways.

### 5 Existence

The results of Sections 3–4 were conditioned on an assumption of existence of a solution to (14),(15). However, for this class of systems, one can use simple complex-analysis equivalences to reduce the question of existence of a solution of the complex HJ PDE problem to that of existence of a solution of a real HJ PDE problem. The latter belongs to a class of problems for which there is a wide variety of existence results under varying assumptions, which are not repeated here. In this section, we drop the earlier assumptions.

**Theorem 11** Suppose \(V,\phi\) are holomorphic on \(\mathbb{C}^n\). Suppose also that \(R \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}_2})\) satisfies (40),(42). Then, there exists \(T \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}_2})\) satisfying (41),(43) such that for each \(s \in [0,t]\), \(T(s,\cdot,\cdot)\) is a harmonic conjugate of \(R(s,\cdot,\cdot)\). Further, letting \(S(s,x) = \nu_{00}^{-1}(R(s,V_0(x)),T(s,V_0(x)))\) for all \((s,x) \in \mathcal{D}_C\), \(S\) satisfies (14),(15).
Recalling that there is a free constant in the harmonic conjugate, for each \( s \in (0, t) \), we let

\[
T(s, 0, 0) = T(t, 0, 0) - \int_s^t -\frac{1}{m} \sum_{j=1}^n R_{y_j} (\rho, 0, 0) R_{z_j} (\rho, 0, 0) + \frac{\hbar}{2m} \sum_{j=1}^n R_{y_j, y_j} (\rho, 0, 0) + V'(0, 0) \, d\rho,
\]

which implies that for all \( s \in (0, t) \),

\[
T_t(s, 0, 0) = \left[ -\frac{1}{m} \sum_{j=1}^n R_{y_j} R_{z_j} + \frac{\hbar}{2m} \sum_{j=1}^n R_{y_j, y_j} + V' \right] (s, 0, 0), \tag{84}
\]

which by (27)–(30),

\[
= \left[ \frac{1}{m} \sum_{j=1}^n T_{y_j} T_{z_j} + \frac{\hbar}{2m} \sum_{j=1}^n T_{y_j, z_j} + V' \right] (s, 0, 0), \tag{85}
\]

which implies that (41) is satisfied at \((s, 0, 0)\) for all \( s \in (0, t) \).

By (27) and (44), for all \((s, y, z) \in D_2\) and all \( k \in [1, n] \),

\[
T_{t, z_k} = R_{t, y_k} = \frac{\hbar}{2m} \sum_{j=1}^n R_{y_j, z_j, y_k} + \frac{1}{m} \sum_{j=1}^n [R_{y_j} R_{y_j, y_k} - R_{z_j} R_{z_j, y_k}] + V_{y_k}, \tag{86}
\]

which by (27)–(32),

\[
= \frac{\hbar}{2m} \sum_{j=1}^n T_{y_j, z_j, z_k} + \frac{1}{m} \sum_{j=1}^n [T_{z_j} T_{y_j, z_k} + T_{y_j} T_{z_j, z_k}] + V'_{z_k}.
\]

For \( z_1 \in \mathbb{R} \), let \( \hat{z}^1(z_1) = (z_1, 0, 0 \ldots 0)^T \in \mathbb{R}^n \), and note that for \( s \in (0, t) \)

\[
T_t(s, 0, \hat{z}^1(z_1)) = T_t(s, 0, 0) + \int_0^{z_1} T_{t, z_1}(s, 0, \hat{z}^1(\zeta)) \, d\zeta, \tag{88}
\]

which by (87),

\[
= T_t(s, 0, 0) + \int_0^{z_1} \frac{d}{dz_1} \left\{ \left[ \frac{\hbar}{2m} \sum_{j=1}^n T_{y_j, z_j} + \frac{1}{m} \sum_{j=1}^n T_{y_j} T_{z_j} + V' \right] (s, 0, \hat{z}^1(\zeta)) \right\} \, d\zeta
\]
\[ T_t(s, 0, 0) = \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, 0, \hat{z}^1(z_1)) \]

\[ - \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, 0, 0), \]

which by \( (85) \),

\[ = \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, 0, \hat{z}^1(z_1)), \]

which implies that \( (41) \) is satisfied at \((s, 0, \hat{z}^1(z_1))\) for all \( s \in (0, t) \) and \( z_1 \in \mathbb{R} \). Proceeding from here similarly, first for \( z_2 \) and then \( z_3 \) and so on, yields finally

\[ T_t(s, 0, z) = \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, 0, z) \quad \forall s \in (0, t), \ z \in \mathbb{R}^n. \]

We now proceed to integrate along the \( y \)-components. First, for \( k \in [1, n] \), differentiating \( (40) \) with respect to \( z_k \), we have

\[ R_t, z_k = \frac{h}{2m} \sum_{j=1}^{n} R_{y_j^1, z_j, z_k} + \frac{1}{m} \sum_{j=1}^{n} [R_{y_j^1, y_j^1, z_k} - R_{z_j^1, z_j, z_k}] + V^R, \quad (89) \]

which by \((27)-(32)\),

\[ = -\frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j, y_k} - \frac{1}{m} \sum_{j=1}^{n} [T_{z_j^1, y_j^1, y_k} + T_{y_j^1, y_j^1, y_k}] - V^I \]

\[ = -\frac{d}{dy_k} \left\{ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right\}. \quad (90) \]

By \((27)\) and \((90)\), we have

\[ T_t, y_k = -R_t, z_k = -\frac{d}{dy_k} \left\{ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right\}. \quad (91) \]

For \( y_1 \in \mathbb{R} \), let \( \hat{y}^1(y_1) = (y_1, 0, 0 \ldots 0)^T \in \mathbb{R}^n \), and note that for \( s \in (0, t) \) and \( z \in \mathbb{R}^n \),

\[ T_t(s, \hat{y}^1(y_1), z) = T_t(s, 0, z) + \int_0^{y_1} T_t, y_1(s, \hat{y}^1(\eta), z) \ d\eta, \]

which by \( (91) \),

\[ = T_t(s, 0, z) + \int_0^{y_1} \frac{d}{dy_1} \left\{ \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, \hat{y}^1(\eta), z) \right\} \ d\eta \]

\[ = T_t(s, 0, z) + \left[ \frac{h}{2m} \sum_{j=1}^{n} T_{y_j^1, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j^1, z_j} + V^I \right](s, \hat{y}^1(y_1), z) \]

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\[ -\left[ \frac{\hbar}{2m} \sum_{j=1}^{n} T_{y_j, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{y_j} T_{z_j} + V}\right](s, 0, z), \]

which by (41),
\[ = \left[ \frac{\hbar}{2m} \sum_{j=1}^{n} T_{\dot{y}_j, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{\dot{y}_j} T_{z_j} + V\right](s, \dot{y}_1(s), z), \]

which implies that (41) is satisfied at \((s, \dot{y}_1(s), z)\) for all \(s \in (0, t)\), \(y_1 \in \mathbb{R}\) and \(z \in \mathbb{R}^n\). Proceeding from this along each component of \(y\), one finally obtains
\[ T_t(s, y, z) = \left[ \frac{\hbar}{2m} \sum_{j=1}^{n} T_{\dot{y}_j, z_j} + \frac{1}{m} \sum_{j=1}^{n} T_{\dot{y}_j} T_{z_j} + V\right](s, y, z) \quad \forall (s, y, z) \in D_2, \quad (92) \]

which is (41). By construction, \(T\) has the indicated smoothness. Lastly, by Proposition 5, one obtains the assertions concerning \(S\).

**Remark 12** As \(S\) obtained in Theorem 11 is holomorphic in \(x\) for each \(s \in (0, t)\) (and hence \(C^\infty\) in the space variable), noting that \(R, T\) are related to \(S\) by (26), one immediately sees that \(R, T\) are \(C^{1, \infty}(D_2) \cap C(D_2)\).

It should be remarked that the analogous result to Theorem 11, where one supposes existence of the solution to (41),(43) rather than (40),(42) is obtained in an analogous manner, and the redundant proof is omitted. The result is:

**Theorem 13** Suppose \(V, \dot{\phi}\) are holomorphic on \(\mathbb{C}^n\). Suppose also that \(T \in C^{1,3}(D_2) \cap C(\overline{D}_2)\) satisfies (41),(43). Then, there exists \(R \in C^{1,3}(D_2) \cap C(\overline{D}_2)\) satisfying (40),(42) such that for each \(s \in [0, t]\), \(T(s, \cdot, \cdot)\) is a harmonic conjugate of \(R(s, \cdot, \cdot)\). Further, letting \(S(s, x) = \mathcal{V}_{00}^{-1}(R(s, \mathcal{V}_0(x)), T(s, \mathcal{V}_0(x)))\) for all \((s, x) \in \overline{D}_C\), \(S\) satisfies (14),(15).

### References


