A STOCHASTIC CONTROL VERIFICATION THEOREM FOR THE DEQUANTIZED SCHRÖDINGER EQUATION NOT REQUIRING A DURATION RESTRICTION *

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Abstract. A stochastic control representation for solution of the Schrödinger equation is obtained, utilizing complex-valued diffusion processes. The Maslov dequantization is employed, where the domain is complex-valued in the space variable. The notion of stationarity is utilized to relate the Hamilton-Jacobi form of the dequantized Schrödinger equation to its stochastic control representation. Convexity is not required, and consequently, there is no restriction on the duration of the problem. Additionally, existence is reduced to a real-valued domain case.

Key words. stochastic control, Schrödinger equation, Hamilton-Jacobi, stationary action, staticization, complex-valued diffusion.

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1. Introduction.

Diffusion representations have long been utilized in the study of Hamilton-Jacobi partial differential equations (HJ PDEs), cf. [6, 12, 16] among many others. The bulk of such results apply to real-valued HJ PDEs, that is, to HJ PDEs where the coefficients and solutions are real-valued. The Schrödinger equation is complex-valued, although generally defined over a real-valued space domain, which presents difficulties for the development of stochastic control representations. There is substantial existing work on the relation of stochastic processes to the Schrödinger equation, cf. [13, 18, 27, 28, 31]. The approach considered here is in the spirit of the Feynman path-integral interpretation [7, 8], where in particular, one looks at a certain action-based functional, S, where $\psi = \exp\{\frac{i}{\hbar} S\}$ and $\hbar$ denotes Planck's constant. One seeks a representation for S in the form of a value function for a stochastic control problem where the action functional is the payoff, cf. [2, 3, 7, 8, 11, 17, 21]. We note that this latter approach is also sometimes employed in analysis of semiclassical limits, cf. [1, 3, 11, 17].

An issue that arises in such approaches is that control has traditionally considered classical optimization (minimization or maximization) of some payoff. Implicit in that is an assumption that the payoff is real valued. In [4, 24, 26], the authors consider a least-action approach to obtaining fundamental solutions to two-point boundary value problems (TPBVPs) for conservative dynamical systems. However, that formulation, which was in terms of minimization of the action, induced duration limits on the problems which could be addressed, where those limits were also similar to duration limits present in existing results on the Schrödinger equation representation in terms of action, cf. [2, 3, 11]. We note that the duration limits are related to a loss of convexity of the payoff as the time horizon is extended. While in [4, 24, 26], the least-action principle was applied, the more generally applicable form is the stationary-action principle, which coincides with the least-action principle when the action functional is convex and coercive. Consequently and more recently, the notion of "staticization" was introduced for such TPBVPs, in which case one seeks a stationary point of the action over the space of control inputs. The extension to stationarity removes the restriction on problem duration. This yields a dynamic program which takes the form of an HJ PDE in the case of continuous-time/continuous-space processes, where these were studied in the context of deterministic dynamics in [22, 23, 25].

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As staticization seeks points where the derivative of a functional is zero, as opposed to optimization of the functional, it is easily extended to the case of complex-valued systems. The extension to stochastic dynamics is easily made as well. Also, as staticization does not require the imposition of duration limits on the problems, one can apply this new tool to the stochastic-control representation problem for the dequantized Schrödinger equation, and that is the topic considered herein.

In order to clarify the details in the above, we recall the Schrödinger initial value problem, given as

$$0 = i\hbar\psi_t(s,y) + \tfrac{\hbar^2}{2m}\Delta\psi(s,y) - \psi(s,y)V(y), \quad (s,y)\in\mathcal{D}, \tag{1.1}$$

$$\psi(0,y) = \psi_0(y), \quad y\in\mathbb{R}^n, \tag{1.2}$$

where $m\in(0,\infty)$ denotes mass, initial condition ψ_0 takes values in $\mathbb{C}$, V denotes a known potential function, Δ denotes the Laplacian with respect to the space (second) variable, $\mathcal{D}\doteq(0,t)\times\mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We also let $\overline{\mathcal{D}}\doteq(0,t]\times\mathbb{R}^n$. We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf. [20]), which as noted above, is $S:\overline{\mathcal{D}}\to\mathbb{C}$ given by $\psi(s,y)=\exp\{\frac{i}{\hbar}S(s,y)\}$. The Maslov dequantization is clearly similar to the logarithmic transform (cf. [9]), but with a modification induced through multiplication by an imaginary constant. Note that $\psi_t=\frac{i}{\hbar}\psi S_t$, $\psi_y=\frac{i}{\hbar}\psi S_y$ and $\Delta\psi=\frac{i}{\hbar}\psi\Delta S-\frac{1}{\hbar^2}\psi|S_y|_c^2$ where for $x\in\mathbb{C}^n$, $|x|_c^2\doteq\sum_{j=1}^n x_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We find that (1.1)–(1.2) become

$$0 = -S_t(s,y) + \tfrac{i\hbar}{2m}\Delta S(s,y) + H(y,S_y(s,y)), \quad (s,y)\in\mathcal{D}, \tag{1.3}$$

$$S(0,y) = \phi(y), \quad y\in\mathbb{R}^n, \tag{1.4}$$

where $H:\mathbb{R}^n\times\mathbb{C}^n\to\mathbb{C}$ is the Hamiltonian given by

$$H(y,p) = -\big[\tfrac{1}{2m}|p|_c^2 + V(y)\big] = \operatorname*{stat}_{u^0\in\mathbb{C}^n}\big\{(u^0)^T p + \tfrac{m}{2}|u^0|_c^2 - V(y)\big\}, \tag{1.5}$$

stat will be defined in the next section, and throughout, superscript T denotes transpose. We look for solutions in the space

$$\mathcal{S}\doteq\{S:\overline{\mathcal{D}}\to\mathbb{C}\,|\,S\in C_p^{1,2}(\mathcal{D})\cap C(\overline{\mathcal{D}})\}, \tag{1.6}$$

where $C_p^{1,2}$ denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound. We will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the Hamilton-Jacobi partial differential equation (HJ PDE) problem given by

$$0 = S_t(s,y) + \tfrac{i\hbar}{2m}\Delta S(s,y) + H(y,S_y(s,y)), \quad (s,y)\in\mathcal{D}, \tag{1.7}$$

$$S(t,y) = \phi(y), \quad y\in\mathbb{R}^n. \tag{1.8}$$

Working mainly with this last form, we will fix $t\in(0,\infty)$, and allow s to vary in $(0,t]$.

Recall that in semiclassical limit analysis, one views $\hbar$ as a small parameter, and examines the limit as $\hbar\downarrow 0$. Applying this in (1.7)–(1.8) yields an HJ PDE problem of the form

$$0 = S_t(s,y) + H(y,S_y(s,y)), \quad (s,y)\in\mathcal{D}, \tag{1.9}$$

$$S(t,y) = \phi(y), \quad y\in\mathbb{R}^n. \tag{1.10}$$

Recalling the above-noted recent work on least-action and stationary-action formulations of certain TPBVPs [4, 22, 24, 26, 23, 25], it was found that the associated HJ PDEs for such problems also take the form (1.9)–(1.10). This was the original motivation for the effort here, where we develop a stationary-action based representation for the solution of (1.7)–(1.8) (and consequently (1.1)–(1.2)). Due to the complex multiplier on the Laplacian, this representation is in terms of a stationary-action stochastic control problem with a complex-valued diffusion coefficient.

Again, a significant contribution of this effort is that the use of stationarity rather than optimization allows for the extension of the stochastic representation to arbitrary duration problems (Theorem 4.1). More specifically, we demonstrate that solutions of (1.7)–(1.8) are given by (4.2), where $J^{\hbar}$ and $\xi_{\cdot}$ are given by (4.1) and (3.6), respectively. Further, as this representation has a similar form to that of the stationary-action value for the limit system, but where the latter lacks the input diffusion term, one has the expectation that this will provide a new tool for the study of semiclassical limits. One should also note that the results are obtained here under strong assumptions. In particular, we assume the existence of a sufficiently smooth potential function, defined over all of the complex space domain, which matches V on $\mathbb{R}^n$. The assumptions allow for the inclusion of the case of the quantum harmonic oscillator, when one takes the potential to be a holomorphic quadratic over the complex domain. Potentials lacking such smoothness are beyond the scope here, and it is expected that such problems will be studied in a later effort.

In Section 2, we recall the definitions necessary for stationarity problems. In Section 3, the underlying space domain is extended from a space over the real field to a space over the complex field. This necessitates several other minor extensions, which are covered in the subsections. In particular, some classical existence and uniqueness results for stochastic differential equations (SDEs) are easily extended to their complex-valued counterparts. In Section 4, the main result of the paper, a stationarity-based stochastic-control value function representation for the dequantized Schrödinger equation, is obtained. More specifically, a verification result is obtained demonstrating that if a solution of the HJ PDE over the "complexified" domain exists, then that solution has the indicated representation. Lastly, in Section 5, we indicate a result about existence of solutions of the HJ PDE over the complexified domain.

2. Stationarity definitions. Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. For this and other reasons, as in the definition of the Hamiltonian given in (1.5), we find it useful to develop additional notation and nomenclature. Specifically, we will refer to the search for stationary points more succinctly as *staticization* (in analogy with minimization, and similar to that, based on the Latin "statica"). In particular, we make the following definitions. Suppose $(\mathcal{A}, |\cdot|)$ is a generic normed vector space over $\mathbb{C}$ with $\mathcal{G} \subseteq \mathcal{A}$, and suppose $F : \mathcal{G} \to \mathbb{C}$. We say $\bar{\alpha} \in \operatorname{argstat}\{F(\alpha) \,|\, \alpha \in \mathcal{G}\}$ if $\bar{\alpha} \in \mathcal{G}$ and either $\limsup_{\alpha \to \bar{\alpha}, \alpha \in \mathcal{G}\setminus\{\bar{\alpha}\}} |F(\alpha) - F(\bar{\alpha})|/|\alpha - \bar{\alpha}| = 0$, or there exists $\delta > 0$ such that $\mathcal{G} \cap B_\delta(\bar{\alpha}) = \{\bar{\alpha}\}$ (where $B_\delta(\bar{\alpha})$ denotes the ball of radius δ around $\bar{\alpha}$). If $\operatorname{argstat}\{F(\alpha) \,|\, \alpha \in \mathcal{G}\} \neq \emptyset$, we define the possibly set-valued stat^s operator by

$$\operatorname*{stat}^s_{\alpha \in \mathcal{G}} F(\alpha) \doteq \operatorname{stat}^s\{F(\alpha) \,|\, \alpha \in \mathcal{G}\} \doteq \big\{ F(\bar{\alpha}) \,\big|\, \bar{\alpha} \in \operatorname{argstat}\{F(\alpha) \,|\, \alpha \in \mathcal{G}\} \big\}.$$

If $\operatorname{argstat}\{F(\alpha) \,|\, \alpha \in \mathcal{G}\} = \emptyset$, $\operatorname{stat}^s_{\alpha \in \mathcal{G}} F(\alpha)$ is undefined. We will also be interested in a single-valued stat operation. In particular, if there exists $a \in \mathbb{C}$ such that

$\mathrm{stat}^s_{\alpha\in\mathcal{G}} F(\alpha) = \{a\}$, then $\mathrm{stat}_{\alpha\in\mathcal{G}} F(\alpha) \doteq a$; otherwise, $\mathrm{stat}_{\alpha\in\mathcal{G}} F(\alpha)$ is undefined. At times, we may abuse notation by writing $\bar\alpha = \mathrm{argstat}\{F(\alpha)\,|\,\alpha\in\mathcal{G}\}$ in the event that the argstat is the single point $\{\bar\alpha\}$. The following is immediate from the above definitions.

LEMMA 2.1. *Suppose $\mathcal{A}$ is a Hilbert space, with open set $\mathcal{G}\subseteq\mathcal{A}$, and that $F:\mathcal{G}\to\mathbb{C}$ is Fréchet differentiable at $\bar\alpha\in\mathcal{G}$ with Riesz representation $F_\alpha(\bar\alpha)\in\mathcal{A}$ Then, $\bar\alpha\in\mathrm{argstat}\{F(\alpha)\,|\,\alpha\in\mathcal{G}\}$ if and only if $F_\alpha(\bar\alpha)=0$.*

For further discussion, we refer the reader to [22, 25].

3. Extensions to the complex domain. Various details of extensions to the complex domain must be considered prior to the development of the representation.

3.1. Extended problem and assumptions. Although (1.1)–(1.2), (1.3)–(1.4) and (1.7)–(1.8) are typically given as HJ PDE problems over $\overline{\mathcal{D}}$, as in Doss et al. [1, 2, 3] we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to $\mathbb{C}^n$, i.e., $V:\mathbb{C}^n\to\mathbb{C}$, and we will abuse notation by employing the same symbol for the extended-domain functions. Throughout, for $k\in\mathbb{N}$, and $x\in\mathbb{C}^k$ or $x\in\mathbb{R}^k$, we let $|x|$ denote the Euclidean norm. Let $\mathcal{D}_\mathbb{C}\doteq(0,t)\times\mathbb{C}^n$ and $\overline{\mathcal{D}}_\mathbb{C}=(0,t]\times\mathbb{C}^n$, and define

$$\begin{aligned}\mathcal{S}_\mathbb{C}\doteq\{S:\overline{\mathcal{D}}_\mathbb{C}\to\mathbb{C}|\,&S \text{ is continuous on } \overline{\mathcal{D}}_\mathbb{C}, \text{ continuously differentiable in time on } \mathcal{D}_\mathbb{C},\\ &\text{and holomorphic on } \mathbb{C}^n \text{ for all } r\in(0,t]\,\}, \qquad (3.1)\\ \mathcal{S}^p_\mathbb{C}\doteq\{S\in\mathcal{S}_\mathbb{C}\,|\,&S \text{ satisfies a polynomial growth condition in space,}\\ &\text{uniformly on } (0,t]\,\}. \qquad (3.2)\end{aligned}$$

The extended-domain form of problem (1.7)–(1.8) is

$$0=\bar S_t(s,x)+\tfrac{i\hbar}{2m}\Delta\bar S(s,x)+H(x,\bar S_x(s,x)),\quad (s,x)\in\mathcal{D}_\mathbb{C}, \tag{3.3}$$
$$\bar S(t,x)=\phi(x),\quad x\in\mathbb{C}^n. \tag{3.4}$$

Throughout Sections 3–4, we will assume the following. In Section 5, existence will be discussed under weaker assumptions.

For each $\hbar\in(0,1]$, there exists a solution, $\bar S=\bar S^\hbar\in\mathcal{S}^p_\mathbb{C}$ to (3.3)–(3.4). $\quad (A.0)$

$V,\phi:\mathbb{C}^n\to\mathbb{C}$ are holomorphic on $\mathbb{C}^n$. Further, there exists $C_0<\infty$ and $q\in\mathbb{N}$ such that $|V_{xx}(x)|,|\phi_{xx}(x)|<C_0(1+|x|^{2q})$ for all $x\in\mathbb{C}^n$. $\quad (A.1)$

For each $\hbar\in(0,1]$, there exists $C_S=C_S^\hbar<\infty$ such that $|\bar S_x(r,x)|\le C_S(1+|x|)$ and $|\bar S_{xx}(r,x)|\le C_S(1+|x|^{2q})$ for all $(r,x)\in\mathcal{D}_\mathbb{C}$. $\quad (A.2)$

3.2. The underlying stochastic dynamics. We let $(\Omega,\mathcal{F},P)$ be a probability triple, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega,\mathcal{F})$. Let $\{\mathcal{F}_s\,|\,s\in[0,t]\}$ denote a filtration on $(\Omega,\mathcal{F},P)$, and let $B_\cdot$ denote an $\mathcal{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. For $s\in[0,t]$, let

$$\begin{aligned}\mathcal{U}_s\doteq\{u:\,[s,t]\times\Omega\to\mathbb{C}^n\,|\,&u \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous and such that}\\ &\mathbb{E}\textstyle\int_s^t|u_r|^m\,dr<\infty\ \forall m\in\mathbb{N}\,\}.\end{aligned} \tag{3.5}$$

We supply $\mathcal{U}_s$ with the norm $\|u\|_{\mathcal{U}_s}\doteq\max_{m\in]1,\bar M[}\left[\mathbb{E}\int_s^t|u_r|^m\,dr\right]^{1/m}$, where $\bar M\ge 8q$, and where throughout, for integer $a\le b$, we define $]a,b[=\{a,a+1,\dots b\}$. We will be interested in diffusion processes given by

$$\xi_r = \xi_r^{(s,x)} = x + \int_s^r u_\rho \, d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1+i}{\sqrt{2}} \int_s^r dB_\rho \doteq x + \int_s^r u_\rho \, d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1+i}{\sqrt{2}} \, B_r^\Delta, \quad (3.6)$$

where $x \in \mathbb{C}^n$, $s \in [0,t]$, $u \in \mathcal{U}_s$, and $B_r^\Delta \doteq B_r - B_s$ for $r \in [s,t]$. We will also be interested in the case where the control input is generated by a state-feedback. In particular, we will consider

$$\bar{\xi}_r^* = \bar{\xi}_r^{*,(s,x)} \doteq x + \int_s^r (\tfrac{-1}{m}) \bar{S}_x(\rho, \bar{\xi}_\rho^{*,(s,x)}) \, d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1+i}{\sqrt{2}} \, B_r^\Delta, \quad (3.7)$$

where presuming for now existence and uniqueness of a solution of (3.7), we may define the resulting $\bar{u}^{*,(s,x)} \in \mathcal{U}_s$ given by

$$\bar{u}_r^{*,(s,x)}(\omega) \doteq \hat{u}(r, \bar{\xi}^{*,(s,x)}(\omega)) \doteq (\tfrac{-1}{m}) \bar{S}_x(r, \bar{\xi}_r^{*,(s,x)}(\omega)) \quad \forall \, r \in [s,t], \; \omega \in \Omega. \quad (3.8)$$

For $s \in [0,t]$, we let

$$\mathcal{X}_s \doteq \{ \xi : [s,t] \times \Omega \to \mathbb{C}^n \,|\, \xi \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E} \sup_{r \in [s,t]} |\xi_r|^m < \infty \; \forall m \in \mathbb{N} \}. \quad (3.9)$$

We supply $\mathcal{X}_s$ with the norm $\|\xi\|_{\mathcal{X}_s} \doteq \max_{m \in \,]1, \bar{M}[} \left[\mathbb{E} \sup_{r \in [s,t]} |\xi_r|^m \right]^{1/m}$.

It is natural to work with complex-valued state processes in this problem domain. However, in order to easily apply many of the existing results regarding existence, uniqueness and moments, we will find it handy to use a "vectorized" real-valued representation for the complex-valued state processes. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, denoted here by $\mathcal{V}_{00} : \mathbb{C} \to \mathbb{R}^2$, with $\mathcal{V}_{00}(x) \doteq (y,z)^T$, where $y = \mathbf{Re}(x)$ and $z = \mathbf{Im}(x)$. This immediately yields the mapping $\mathcal{V}_0 : \mathbb{C}^n \to \mathbb{R}^{2n}$ given by $\mathcal{V}_0(y + iz) \doteq (y^T, z^T)^T$, where component-wise, $(y_j, z_j)^T = \mathcal{V}_{00}(x_j)$ for all $j \in]1, n[$. Also in the interests of a reduction of cumbersome notation, we will henceforth frequently abuse notation by writing (y,z) in place of $(y^T, z^T)^T$ when the meaning is clear.

Given control process, $u \in \mathcal{U}_s$, we define its vectorized analog by the isometric isomorphism, $\mathcal{V} : \mathcal{U}_s \to \mathcal{U}_s^v$, where $[\mathcal{V}(u)]_r \doteq (v_r^T, w_r^T)^T$ and $(v_r^T, w_r^T)^T = \mathcal{V}_0(u_r)$ for all $r \in [s,t]$ and $\omega \in \Omega$, and where

$$\mathcal{U}_s^v \doteq \{ (v,w) : [s,t] \times \Omega \to \mathbb{R}^{2n} \,|\, (v,w) \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E} \textstyle\int_s^t |v_r|^m + |w_r|^m \, dr < \infty \; \forall m \in \mathbb{N} \}, \quad (3.10)$$

$$\|u\|_{\mathcal{U}_s^v} \doteq \max_{m \in \,]1, \bar{M}[} \left[\mathbb{E} \textstyle\int_s^t |v_r|^m + |w_r|^m \, dr \right]^{1/m}. \quad (3.11)$$

Again abusing notation, we also define the isometric isomorphism, $\mathcal{V} : \mathcal{X}_s \to \mathcal{X}_s^v$ by $[\mathcal{V}(\xi)]_r \doteq [\mathcal{V}(\eta + i\zeta)]_r \doteq (\eta_r^T, \zeta_r^T)^T$ for all $r \in [s,t]$ and $\omega \in \Omega$, where

$$\mathcal{X}_s^v \doteq \{ (\eta,\zeta) : [s,t] \times \Omega \to \mathbb{R}^{2n} \,|\, (\eta,\zeta) \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E} \sup_{r \in [s,t]} [|\eta_r|^m + |\zeta_r|^m] < \infty \; \forall m \in \mathbb{N} \}, \quad (3.12)$$

$$\|(\eta,\zeta)\|_{\mathcal{X}_s^v} \doteq \max_{m \in \,]1, \bar{M}[} \left[\mathbb{E} \sup_{r \in [s,t]} (|\eta_r|^m + |\zeta_r|^m) \right]^{1/m}. \quad (3.13)$$

Under transformation by $\mathcal{V}$, (3.6) becomes

$$\begin{pmatrix} \eta_r \\ \zeta_r \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^r \begin{pmatrix} v_\rho \\ w_\rho \end{pmatrix} d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1}{\sqrt{2}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B_r^\Delta. \tag{3.14}$$

We may decompose $\bar{S} \in \mathcal{S}_{\mathbb{C}}$ as

$$(\bar{R}(r, \mathcal{V}_0(x)), \bar{T}(r, \mathcal{V}_0(x)))^T \doteq \mathcal{V}_{00}(\bar{S}(r,x)), \tag{3.15}$$

where $\bar{R}, \bar{T} : \overline{\mathcal{D}}_2 \doteq (0,t] \times \mathbb{R}^{2n} \to \mathbb{R}$, and we also let $\mathcal{D}_2 \doteq (0,t) \times \mathbb{R}^{2n}$. For later reference, it will be helpful to recall some standard relations between derivative components, which are induced by the Cauchy-Riemann equations. For all $(r,x) = (r, y+iz) \in (0,t) \times \mathbb{C}^n$ and all $j,k,\ell \in]1,n[$, and suppressing the arguments for reasons of space we have

$$\begin{aligned}
\mathbf{Re}[\bar{S}_{x_j,x_k}] &= \bar{R}_{y_j,y_k} = -\bar{R}_{z_j,z_k} = \bar{T}_{z_j,y_k} = \bar{T}_{y_j,z_k}, && (3.16)\\
\mathbf{Im}[\bar{S}_{x_j,x_k}] &= -\bar{R}_{y_j,z_k} = -\bar{R}_{z_j,y_k} = -\bar{T}_{z_j,z_k} = \bar{T}_{y_j,y_k}, && (3.17)\\
\mathbf{Re}[\bar{S}_{x_j,x_k,x_\ell}] &= \bar{R}_{y_j,y_k,y_\ell} = -\bar{R}_{y_j,z_k,z_\ell} = -\bar{R}_{z_j,z_k,y_\ell} = -\bar{R}_{z_j,y_k,z_\ell} \\
&= \bar{T}_{z_j,y_k,y_\ell} = -\bar{T}_{z_j,z_k,z_\ell} = \bar{T}_{y_j,z_k,y_\ell} = \bar{T}_{y_j,y_k,z_\ell}, && (3.18)\\
\mathbf{Im}[\bar{S}_{x_j,x_k,x_\ell}] &= -\bar{R}_{y_j,y_k,z_\ell} = -\bar{R}_{y_j,z_k,y_\ell} = \bar{R}_{z_j,z_k,z_\ell} = -\bar{R}_{z_j,y_k,y_\ell} \\
&= -\bar{T}_{z_j,y_k,z_\ell} = -\bar{T}_{z_j,z_k,y_\ell} = -\bar{T}_{y_j,z_k,z_\ell} = \bar{T}_{y_j,y_k,y_\ell}. && (3.19)
\end{aligned}$$

One may also easily verify that with $\bar{R}, \bar{T}$ given by (3.15) and $(y,z) = \mathcal{V}_0(x)$,

$$\begin{aligned}
\mathcal{V}_{00}(|\bar{S}_x(r,x)|_c^2) &= \begin{pmatrix} \sum_{j=1}^n \bar{R}_{y_j}^2(r,y,z) - \bar{R}_{z_j}^2(r,y,z) \\ \sum_{j=1}^n -2\bar{R}_{y_j}(r,y,z)\bar{R}_{z_j}(r,y,z) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \bar{T}_{z_j}^2(r,y,z) - \bar{T}_{y_j}^2(r,y,z) \\ \sum_{j=1}^n 2\bar{T}_{y_j}(r,y,z)\bar{T}_{z_j}(r,y,z) \end{pmatrix}. && (3.20)
\end{aligned}$$

Further, given $u^0 \in \mathbb{C}^n$, $s \in [0,t]$ and $x \in \mathbb{C}^n$ with $(y,z) \doteq \mathcal{V}_0(x)$,

$$\begin{aligned}
&\mathcal{V}_0\left(\operatorname*{argstat}_{u^0 \in \mathbb{C}^n} \Big[\sum_{j=1}^n \bar{S}_{x_j}(r,x) u_j^0 + \tfrac{m}{2} |u^0|_c^2 \Big] \right) = \mathcal{V}_0\left(\tfrac{-1}{m} \bar{S}_x(r,x) \right) \\
&= \tfrac{1}{m} \begin{pmatrix} -\bar{R}_y(r,y,z) \\ \bar{R}_z(r,y,z) \end{pmatrix} = \tfrac{1}{m} \begin{pmatrix} -\bar{T}_z(r,y,z) \\ -\bar{T}_y(r,y,z) \end{pmatrix}. && (3.21)
\end{aligned}$$

Using the above, we see that under transformation $\mathcal{V}$, (3.7) becomes

$$\begin{aligned}
\begin{pmatrix} \bar{\eta}_r^* \\ \bar{\zeta}_r^* \end{pmatrix} &= \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^r \tfrac{1}{m} \begin{pmatrix} -\bar{R}_y(\rho, \bar{\eta}_\rho^*, \bar{\zeta}_\rho^*) \\ \bar{R}_z(\rho, \bar{\eta}_\rho^*, \bar{\zeta}_\rho^*) \end{pmatrix} d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1}{\sqrt{2}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B_r^\Delta \\
&= \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^r \tfrac{-1}{m} \begin{pmatrix} \bar{T}_z(\rho, \bar{\eta}_\rho^*, \bar{\zeta}_\rho^*) \\ \bar{T}_y(\rho, \bar{\eta}_\rho^*, \bar{\zeta}_\rho^*) \end{pmatrix} d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1}{\sqrt{2}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B_r^\Delta. && (3.22)
\end{aligned}$$

Throughout, concerning both real and complex stochastic differential equations, typically given in integral form such as in (3.22), *solution* refers to a strong solution, unless specifically cited as a weak solution.

Lemma 3.1. *Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $u \in \mathcal{U}_s$, $(y,z) = \mathcal{V}_0(x)$ and $(v,w) = \mathcal{V}(u)$. There exists a unique solution, $(\eta,\zeta) \in \mathcal{X}_s^v$, to (3.14), and a unique solution, $(\bar{\eta}^*, \bar{\zeta}^*) \in \mathcal{X}_s^v$, to (3.22).*

Lemma 3.1 is easily obtained from well-known existing results. For completeness, a brief proof is given in Appendix 6.

The following is straightforward, cf. [29].

Lemma 3.2. *Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $u \in \mathcal{U}_s$, $(y,z) = \mathcal{V}_0(x)$ and $(v,w) = \mathcal{V}(u)$. $\xi \in \mathcal{X}_s$ is a solution of (3.6) if and only if $\mathcal{V}(\xi) \in \mathcal{X}_s^v$ is a solution of (3.14). Similarly, $\bar{\xi}^* \in \mathcal{X}_s$ is a solution of (3.7) if and only if $\mathcal{V}(\bar{\xi}^*) \in \mathcal{X}_s^v$ is a solution of (3.22).*

Combining Lemmas 3.1 and 3.2, one has:

Lemma 3.3. *Let $s \in [0,t)$, $x \in \mathbb{C}^n$ and $u \in \mathcal{U}_s$. There exists a unique solution, $\xi \in \mathcal{X}_s$, to (3.6), and a unique solution, $\bar{\xi}^* \in \mathcal{X}_s$, to (3.7).*

3.3. A relationship among the solutions. By the Cauchy-Riemann equations and (3.16)–(3.17),

$$|\bar{S}_x|_c^2 = \sum_{j=1}^n \bar{R}_{y_j}^2 - \bar{R}_{z_j}^2 + 2i\bar{T}_{y_j}\bar{T}_{z_j}, \quad \text{and} \quad \Delta\bar{S} = \sum_{j=1}^n \bar{T}_{y_j,z_j} - i\bar{R}_{y_j,z_j}. \tag{3.23}$$

Let

$$\big(V^R(\mathcal{V}_0(x)), V^I(\mathcal{V}_0(x))\big)^T \doteq \mathcal{V}_{00}(V(x)) \quad \text{and} \quad \big(\phi^R(\mathcal{V}_0(x)), \phi^I(\mathcal{V}_0(x))\big)^T \doteq \mathcal{V}_{00}(\phi(x)) \tag{3.24}$$

for all $x \in \mathbb{C}^n$. Substituting (3.23)–(3.24) into (3.3), and separating the real and imaginary parts, we have

$$0 = \bar{R}_t - \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{R}_{y_j,z_j} - \tfrac{1}{2m}\sum_{j=1}^n \big(\bar{R}_{y_j}^2 - \bar{R}_{z_j}^2\big) - V^R \quad \forall\, (s,y,z) \in \mathcal{D}_2, \tag{3.25}$$

$$0 = \bar{T}_t - \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{T}_{y_j,z_j} - \tfrac{1}{m}\sum_{j=1}^n \bar{T}_{y_j}\bar{T}_{z_j} - V^I \quad \forall\, (s,y,z) \in \mathcal{D}_2, \tag{3.26}$$

on $\mathcal{D}_2$, and, of course,

$$\bar{R}(t,y,z) = \phi^R(y,z) \quad \forall\, (y,z) \in \mathbb{R}^{2n}, \tag{3.27}$$

$$\bar{T}(t,y,z) = \phi^I(y,z) \quad \forall\, (y,z) \in \mathbb{R}^{2n}. \tag{3.28}$$

Proposition 3.4. *Let $\bar{S} \in \mathcal{S}_{\mathbb{C}}$ and $\bar{R},\bar{T}$ satisfy (3.15) for all $(r,x) \in \overline{\mathcal{D}}_{\mathbb{C}}$. If $\bar{S}$ satisfies (3.3)–(3.4), then $\bar{R},\bar{T}$ satisfy (3.25)–(3.28). Alternatively, if $\bar{R},\bar{T} \in C^{1,2}(\mathcal{D}_2;\mathbb{R}) \cap C(\overline{\mathcal{D}}_2;\mathbb{R})$ satisfy (3.25)–(3.28), and $\bar{S} \in \mathcal{S}_{\mathbb{C}}$ is given by (3.15), then $\bar{S}$ satisfies (3.3)–(3.4).*

In order to focus on the verification result of the next section, the proof of Proposition 3.4 is delayed to Appendix 6.

4. The verification. We will obtain a verification result demonstrating that a solution of (1.7)–(1.8) is the stationary value of the expectation of the action functional on process paths satisfying (3.6).

For $s \in (0,t)$ and $\hbar \in (0,1]$, we define payoff $J^\hbar(s,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{U}_s \to \mathbb{C}$ by

$$J^\hbar(s,x,u) \doteq \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 - V(\xi_r)\,dr + \phi(\xi_t)\right\}, \tag{4.1}$$

where ξ satisfies (3.6) with input $u \in \mathcal{U}_s$ and initial state $x \in \mathbb{R}^n$. The stationary value, $S^\hbar : \mathcal{D} \to \mathbb{C}$, is given by

$$S^\hbar(s,x) \doteq \operatorname*{stat}_{u \in \mathcal{U}_s} J^\hbar(s,x,u) \quad \forall\, (s,x) \in \mathcal{D}. \tag{4.2}$$

We assume throughout Section 4 that

$$\text{argstat}_{u \in \mathcal{U}_s} J^\hbar(s,x,u) \text{ is single-valued for all } (s,x) \in \mathcal{D}. \tag{$A.3$}$$

This is the last assumption. We remark that one may want to weaken this assumption to uniqueness in some prespecified subset of $\mathcal{D}$, but leave that additional complication to a later effort. The main result of the section is:

THEOREM 4.1. *Let $\hbar \in (0,1]$. Suppose $\bar S \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (3.3)–(3.4), and that there exists $\widehat{C}_S < \infty$ such that $|\bar S_{xxx}(r,x)|$, $|\bar S_{txx}(r,x)|$, $|\bar S_{xxxx}(r,x)| \le \widehat{C}_S(1+|x|^{2q})$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$. Then, $\bar S(s,x) = S^\hbar(s,x)$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$.*

We remark that the representation is proved for general $\hbar \in (0,1]$ in anticipation of possible use in semiclassical limit results. We begin with two lemmas.

LEMMA 4.2. *Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $\hbar \in (0,1]$ and $u \in \mathcal{U}_s$. Let $\xi \in \mathcal{X}_s$ be given by (3.6). Suppose $\bar S \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (3.3)–(3.4). Let $\bar u^* = \bar u^{*,(s,x)}$, $\bar\xi^* = \bar\xi^{*,(s,x)}$ be given by (3.7)-(3.8). Then,*

$$\begin{aligned}
\bar S(s,x) &= \mathbb{E}\bigg\{ \int_s^t -\bar S_t(r,\xi_r) - \bar S_x^T(r,\xi_r)u_r - \tfrac{i\hbar}{2m}\Delta \bar S(r,\xi_r)\,dr + \phi(\xi_t) \bigg\} \\
&= \mathbb{E}\bigg\{ \int_s^t -\bar S_t(r,\bar\xi^*_r) - \bar S_x^T(r,\bar\xi^*_r)\bar u^*_r - \tfrac{i\hbar}{2m}\Delta \bar S(r,\bar\xi^*_r)\,dr + \phi(\bar\xi^*_t) \bigg\}.
\end{aligned}$$

Proof. We prove only the second asserted representation; the proof of the first is similar and somewhat simpler. Fix $\hbar \in (0,1]$ and $(s,x) \in \overline{\mathcal{D}}_{\mathbb{C}}$. Let $\bar\xi^* = \bar\xi^{*,(s,x)}$ and $\bar u^* = \bar u^{*,(s,x)}$ be given by (3.7)-(3.8). Let $(y,z) = \mathcal{V}_0(x)$, $(\bar\eta^*,\bar\zeta^*) = \mathcal{V}(\bar\xi^*)$, $(\bar v^*,\bar w^*) = \mathcal{V}(\bar u^*)$, and $(\bar R(r,\mathcal{V}_0(x)), \bar T(r,\mathcal{V}_0(x))) = \mathcal{V}_{00}(\bar S(r,x))$ for all $(r,x) \in \overline{\mathcal{D}}_{\mathbb{C}}$. Note that $(\bar\eta^*,\bar\zeta^*)$ satisfy (3.22). By Itô's formula,

$$\begin{aligned}
\mathbb{E}\big[\bar R(t,\bar\eta^*_t,\bar\zeta^*_t)\big] &= \bar R(s,y,z) + \mathbb{E}\bigg\{ \int_s^t \bar R_t(r,\bar\eta^*_r,\bar\zeta^*_r) + \bar R_y^T(r,\bar\eta^*_r,\bar\zeta^*_r)\bar v^*_r \\
&\quad + \bar R_z^T(r,\bar\eta^*_r,\bar\zeta^*_r)\bar w^*_r + \tfrac{\hbar}{4m}\sum_{j=1}^n \big[\bar R_{y_j,y_j} + 2\bar R_{y_j,z_j} + \bar R_{z_j,z_j}\big](r,\bar\eta^*_r,\bar\zeta^*_r)\,dr \\
&\quad + \sqrt{\tfrac{\hbar}{2m}} \int_s^t \big[\bar R_y(r,\bar\eta^*_r,\bar\zeta^*_r) + \bar R_z(r,\bar\eta^*_r,\bar\zeta^*_r)\big]^T dB_r \bigg\},
\end{aligned} \tag{4.3}$$

where, in the interests of space we let $\big[\bar R_{y_j,y_j} + 2\bar R_{y_j,z_j} + \bar R_{z_j,z_j}\big](r,\bar\eta^*_r,\bar\zeta^*_r)$ denote $\bar R_{y_j,y_j}(r,\bar\eta^*_r,\bar\zeta^*_r) + 2\bar R_{y_j,z_j}(r,\bar\eta^*_r,\bar\zeta^*_r) + \bar R_{z_j,z_j}(r,\bar\eta^*_r,\bar\zeta^*_r)$, and use other similar notation where helpful throughout.

Now, by Assumption $(A.2)$ and the definition of $\bar R, \bar T$, there exists $\hat C_S < \infty$ such that

$$|\bar R_y(r,y,z)|, |\bar R_z(r,y,z)| \le \hat C_S(1+|y|+|z|) \quad \forall\,(s,y,z) \in \mathcal{D}_2.$$

Consequently, by Lemma 3.1 (noting that this implies $(\bar{\eta}^*, \bar{\zeta}^*) \in \mathcal{X}_s^v$), there exists $M_1 = M_1(t-s, |x|) < \infty$ such that

$$\mathbb{E}\int_s^t |\bar{R}_y(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)|^2 \, dr, \ \mathbb{E}\int_s^t |\bar{R}_z(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)|^2 \, dr < M_1. \tag{4.4}$$

By (4.4) and standard results (cf. [10], Section V.3),

$$\mathbb{E}\Big\{ \int_s^t \big[\bar{R}_y(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) + \bar{R}_z(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\big]^T \, dB_r \Big\} = 0. \tag{4.5}$$

Combining (4.3) and (4.5) yields

$$\begin{aligned} \bar{R}(s,y,z) = \mathbb{E}\Big\{ \int_s^t & -\bar{R}_t(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) - \bar{R}_y^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{v}_r^* - \bar{R}_z^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{w}_r^* \\ & - \tfrac{\hbar}{4m} \sum_{j=1}^n \big[\bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j}\big](r, \bar{\eta}_r^*, \bar{\zeta}_r^*) \, dr + \bar{R}(t, \bar{\eta}_t^*, \bar{\zeta}_t^*) \Big\}. \end{aligned} \tag{4.6}$$

Then, by (3.27) and (4.6),

$$\begin{aligned} \bar{R}(s,y,z) = \mathbb{E}\Big\{ \int_s^t & -\bar{R}_t(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) - \bar{R}_y^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{v}_r^* - \bar{R}_z^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{w}_r^* \\ & - \tfrac{\hbar}{4m} \sum_{j=1}^n \big[\bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j}\big](r, \bar{\eta}_r^*, \bar{\zeta}_r^*) \, dr + \mathbf{Re}(\phi(\bar{\xi}_t^*)) \Big\}. \end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned} \bar{T}(s,y,z) = \mathbb{E}\Big\{ \int_s^t & -\bar{T}_t(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) - \bar{T}_y^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{v}_r^* - \bar{T}_z^T(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)\bar{w}_r^* \\ & - \tfrac{\hbar}{4m} \sum_{j=1}^n \big[\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j}\big](r, \bar{\eta}_r^*, \bar{\zeta}_r^*) \, dr + \mathbf{Im}(\phi(\bar{\xi}_t^*)) \Big\}. \end{aligned} \tag{4.8}$$

Now, applying $\mathcal{V}_{00}^{-1}$ to (4.7),(4.8), one has

$$\begin{aligned} &\bar{S}(s,x) = \bar{R}(s,y,z) + i\bar{T}(s,y,z) \\ &= \mathbb{E}\Big\{ \int_s^t -\bar{S}_t(r, \bar{\xi}_r^*) - [\bar{R}_y(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) + i\bar{T}_y(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)]^T \bar{v}_r^* \\ &\qquad\qquad - [\bar{T}_z(r, \bar{\eta}_r^*, \bar{\zeta}_r^*) - i\bar{R}_z(r, \bar{\eta}_r^*, \bar{\zeta}_r^*)]^T i\bar{w}_r^* \\ &\quad - \tfrac{\hbar}{4m} \sum_{j=1}^n \big[\bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} + i(\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j})\big](r, \bar{\eta}_r^*, \bar{\zeta}_r^*) \, dr \\ &\quad + \bar{S}(t, \bar{\xi}_t^*) \Big\}, \end{aligned}$$

and using the Cauchy-Riemann equations, this is

$$\begin{aligned} &= \mathbb{E}\Big\{ \int_s^t -\bar{S}_t(r, \bar{\xi}_r^*) - \bar{S}_x^T(r, \bar{\xi}^*)\bar{u}_r^* \\ &- \tfrac{\hbar}{4m} \sum_{j=1}^n \big[\bar{R}_{y_j,y_j} + 2\bar{R}_{y_j,z_j} + \bar{R}_{z_j,z_j} + i(\bar{T}_{y_j,y_j} + 2\bar{T}_{y_j,z_j} + \bar{T}_{z_j,z_j})\big](r, \bar{\eta}_r^*, \bar{\zeta}_r^*) \, dr + \bar{S}(t, \bar{\xi}_t^*) \Big\}, \end{aligned} \tag{4.9}$$

which by (3.16), (3.17),

$$= \mathbb{E}\biggl\{\int_s^t -\bar{S}_t(r,\bar{\xi}^*_r) - \bar{S}_x^T(r,\bar{\xi}^*)\bar{u}^*_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}^*)\,dr + \bar{S}(t,\bar{\xi}^*_t)\biggr\}.$$

□

Lemma 4.3. *Let $\hbar \in (0,1]$, and suppose that $\bar{S} \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (3.3)–(3.4). Then, $\bar{S}(s,x) = J^\hbar(s,x,\bar{u}^{*,(s,x)})$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$, where $\bar{u}^{*,(s,x)}$ is given by (3.7)-(3.8) with $\bar{S}$ in place of S.*

Proof. Fix $\hbar \in (0,1]$ and $(s,x) \in \mathcal{D}_{\mathbb{C}}$. Let $\bar{\xi}^* = \bar{\xi}^{*,(s,x)}$ and $\bar{u}^* = \bar{u}^{*,(s,x)}$ be given by (3.7)-(3.8) with $\bar{S}$ replacing S. From the second assertion of Lemma 4.2, we have

$$\bar{S}(s,x) = \mathbb{E}\biggl\{\int_s^t -\bar{S}_t(r,\bar{\xi}^*_r) - \bar{S}_x^T(r,\bar{\xi}^*_r)\bar{u}^*_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}^*_r)\,dr + \phi(\bar{\xi}^*_t)\biggr\},$$

which by (3.8),

$$\begin{aligned}
&= \mathbb{E}\biggl\{\int_s^t -\bar{S}_t(r,\bar{\xi}^*_r) + \tfrac{1}{m}|\bar{S}_x(r,\bar{\xi}^*_r)|_c^2 - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}^*_r)\,dr + \phi(\bar{\xi}^*_t)\biggr\}\\
&= \mathbb{E}\biggl\{\int_s^t -\bar{S}_t(r,\bar{\xi}^*_r) + \tfrac{1}{2m}|\bar{S}_x(r,\bar{\xi}^*_r)|_c^2 + V(\bar{\xi}^*_r) - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}^*_r)\,dr\\
&\qquad + \int_s^t \tfrac{m}{2}|(\tfrac{-1}{m})\bar{S}_x(r,\bar{\xi}^*_r)|_c^2 - V(\bar{\xi}^*_r)\,dr + \phi(\bar{\xi}^*_t)\biggr\},
\end{aligned}$$

which by (1.5) and (3.3),

$$= \mathbb{E}\biggl\{\int_s^t \tfrac{m}{2}|(\tfrac{-1}{m})\bar{S}_x(r,\bar{\xi}^*_r)|_c^2 - V(\bar{\xi}^*_r)\,dr + \phi(\bar{\xi}^*_t)\biggr\},$$

which by the choice of $\bar{u}^* = \bar{u}^{*,(s,x)}$ and (4.1),

$$= J^\hbar(s,x,\bar{u}^{*,(s,x)}).$$

□

Proof. [proof of Theorem 4.1.] Fix $(s,x) \in \mathcal{D}_{\mathbb{C}}$. Let $L(x,u^0) \doteq \frac{m}{2}|u^0|_c^2 - V(x)$ for all $x, u^0 \in \mathbb{C}^n$. For compactness of notation, let $\bar{\xi}^* = \bar{\xi}^{*,(s,x)}$ and $\bar{u}^* = \bar{u}^{*,(s,x)}$. By Lemma 4.3,

$$\bar{S}(s,x) = \mathbb{E}\Bigl\{\int_s^t L(\bar{\xi}^*_r,\bar{u}^*_r)\,dr + \phi(\bar{\xi}^*_t)\Bigr\} = J^\hbar(s,x,\bar{u}^*). \tag{4.10}$$

It remains to be shown that $\bar{u}^*$ is the argstat over $\mathcal{U}_s$ of $J^\hbar(s,x,\cdot)$.

Let $u \in \mathcal{U}_s$ and $\delta \doteq u - \bar{u}^* \in \mathcal{U}_s$. Let $\xi \in \mathcal{X}_s$ be the trajectory generated by u, i.e., the solution of (3.6), and let $\Delta \doteq \xi - \bar{\xi}^* \in \mathcal{X}_s$, where we note that $\Delta_r = \int_s^r \delta_\rho\,d\rho$ for all $(r,\omega) \in [s,t]\times\Omega$. By (4.10),

$$J^\hbar(s,x,\bar{u}^*) = \bar{S}(s,x) = \mathbb{E}\{\bar{S}(t,\xi_t)\} + \bigl[\bar{S}(s,x) - \mathbb{E}\{\bar{S}(t,\xi_t)\}\bigr],$$

which by Lemma 4.2 and (3.4),

$$\begin{aligned}
&= \mathbb{E}\{\bar{S}(t,\xi_t)\} + \mathbb{E}\biggl\{\int_s^t -\bar{S}_t(r,\xi_r) - \bar{S}_x^T(r,\xi_r)u_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\xi_r)\,dr\biggr\}\\
&= \mathbb{E}\{\phi(\xi_t)\} + \mathbb{E}\biggl\{\int_s^t L(\xi_r,u_r)\,dr\biggr\}\\
&\quad + \mathbb{E}\biggl\{\int_s^t -L(\xi_r,u_r)\,dr - \bar{S}_t(r,\xi_r) - \bar{S}_x^T(r,\xi_r)u_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\xi_r)\,dr\biggr\}.
\end{aligned} \tag{4.11}$$

Now, by (1.5), (3.3),

$$0 = \bar{S}_t(r,x) + \operatorname*{stat}_{u^0\in\mathbb{C}^n}\left\{\bar{S}_x^T(r,x)u^0 + \tfrac{m}{2}|u^0|_c^2 - V(x)\right\} + \tfrac{i\hbar}{2m}\Delta\bar{S}(r,x).$$

Taking $x=\bar{\xi}_r^*$ in this, and using (3.8) and Lemma 2.1, we have

$$0 = \bar{S}_t(r,\bar{\xi}_r^*) + \bar{S}_x^T(r,\bar{\xi}^*)\bar{u}_r^* + \tfrac{m}{2}|\bar{u}_r^*|_c^2 - V(\bar{\xi}_r^*) + \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}_r^*) \tag{4.12}$$

for all $(r,\omega)\in(s,t)\times\Omega$. Combining (4.11), (4.12) and the definition of L from the top of the proof, one has

$$\begin{aligned}
J^\hbar(s,x,\bar{u}^*) &= \mathbb{E}\left\{\int_s^t L(\xi_r,u_r)\,dr + \phi(\xi_t)\right\}\\
&\quad + \mathbb{E}\left\{\int_s^t L(\bar{\xi}_r^*,\bar{u}_r^*) - L(\xi_r,u_r) + \bar{S}_t(r,\bar{\xi}_r^*) - \bar{S}_t(r,\xi_r)\right.\\
&\quad \left. + \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - \bar{S}_x^T(r,\xi_r)u_r + \tfrac{i\hbar}{2m}[\Delta\bar{S}(r,\bar{\xi}_r^*) - \Delta\bar{S}(r,\xi_r)]\,dr\right\}\\
&= J^\hbar(s,x,u) + \mathbb{E}\left\{\int_s^t L(\bar{\xi}_r^*,\bar{u}_r^*) - L(\xi_r,u_r) + \bar{S}_t(r,\bar{\xi}_r^*) - \bar{S}_t(r,\xi_r)\right.\\
&\quad \left. + \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - \bar{S}_x^T(r,\xi_r)u_r + \tfrac{i\hbar}{2m}[\Delta\bar{S}(r,\bar{\xi}_r^*) - \Delta\bar{S}(r,\xi_r)]\,dr\right\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|J^\hbar(s,x,\bar{u}^*) - J^\hbar(s,x,u)\right|\\
&\le \mathbb{E}\left\{\int_s^t \left|L(\bar{\xi}_r^*,\bar{u}_r^*) - L(\xi_r,u_r) + \bar{S}_t(r,\bar{\xi}_r^*) - \bar{S}_t(r,\xi_r)\right.\right. \qquad (4.13)\\
&\left.\left. + \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - \bar{S}_x^T(r,\xi_r)u_r + \tfrac{i\hbar}{2m}[\Delta\bar{S}(r,\bar{\xi}_r^*) - \Delta\bar{S}(r,\xi_r)]\right|dr\right\} \doteq \mathbb{E}\left\{\int_s^t \left|\Xi_r(\omega)\right|dr\right\}.
\end{aligned}$$

Note that by Taylor's Theorem, for $x,\bar{x},u^0,\bar{u}^0\in\mathbb{C}^n$ and $r\in(s,t)$, and using the assumed bounds on derivatives and the definition of L,

$$\begin{aligned}
&\Big|L(x,u^0) - L(\bar{x},\bar{u}^0) + \bar{S}_t(r,x) - \bar{S}_t(r,\bar{x}) + \bar{S}_x^T(r,x)u^0 - \bar{S}_x^T(r,\bar{x})\bar{u}^0\\
&+ \tfrac{i\hbar}{2m}[\Delta\bar{S}(r,x) - \Delta\bar{S}(r,\bar{x})]\Big|\\
&\le \Big| - V_x^T(\bar{x})(x-\bar{x}) + m(\bar{u}^0)^T(u^0-\bar{u}^0) + \bar{S}_{tx}^T(r,\bar{x})(x-\bar{x})\\
&\quad + [\bar{S}_{xx}(r,\bar{x})\bar{u}^0]^T(x-\bar{x}) + \bar{S}_x^T(r,\bar{x})(u^0-\bar{u}^0)\\
&\quad + \left\{\bar{S}_x^T(r,x)u^0 - \bar{S}_x^T(r,\bar{x})\bar{u}^0 - [\bar{S}_{xx}(r,\bar{x})\bar{u}^0]^T(x-\bar{x}) - \bar{S}_x^T(r,\bar{x})(u^0-\bar{u}^0)\right\}\\
&\quad + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x(r,\bar{x})(x-\bar{x})\Big| + K_1\big(1+|x|^{2q}+|\bar{x}|^{2q}\big)|x-\bar{x}|^2 + m|u^0-\bar{u}^0|^2, \quad (4.14)
\end{aligned}$$

for appropriate $K_1 = K_1(C_0,\widehat{C}_S,\hbar,m)<\infty$. This implies

$$\begin{aligned}
&\left|\Xi_r\right| \le \Big| - V_x^T(\bar{\xi}_r^*)\Delta_r + m(\bar{u}_r^*)^T\delta_r + \bar{S}_{tx}^T(r,\bar{\xi}_r^*)\Delta_r + [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r + \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\\
&+ \left\{\bar{S}_x^T(r,\xi_r)u_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\right\} \qquad (4.15)\\
&+ \tfrac{i\hbar}{2m}(\Delta\bar{S})_x^T(r,\bar{\xi}_r^*)\Delta_r\Big| + K_1\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2 + m|\delta_r|^2 \quad \forall\,(r,\omega)\in(s,t)\times\Omega.
\end{aligned}$$

Now recall that $\bar{u}^* = \frac{-1}{m}\bar{S}_x(r,\bar{\xi}_r^*) = \operatorname{argstat}_{u^0\in\mathbb{C}^n}\left[\frac{m}{2}|u^0|_c^2 + \bar{S}_x(r,\bar{\xi}_r^*)u^0\right]$, and consequently by Lemma 2.1,

$$m\bar{u}_r^* + \bar{S}_x(r,\bar{\xi}_r^*) = 0. \tag{4.16}$$

Substituting (4.16) into (4.15) yields

$$\begin{aligned}\big|\Xi_r\big| \le \Big| &- V_x^T(\bar{\xi}_r^*)\Delta_r + \bar{S}_{tx}^T(r,\bar{\xi}_r^*)\Delta_r + [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r + \big\{\bar{S}_x^T(r,\xi_r)u_r \\ &- \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\big\} + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x^T(r,\bar{\xi}_r^*)\Delta_r\Big| \\ &+ K_1\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2 + m|\delta_r|^2 \quad \forall\,(r,\omega)\in(s,t)\times\Omega. \end{aligned} \tag{4.17}$$

Also, more generally, recalling definitions (3.8),

$$\hat{u}(r,x) = \tfrac{-1}{m}\bar{S}_x(r,x) = \operatorname*{argstat}_{u^0\in\mathbb{C}^n}\left[L(x,u^0) + \bar{S}_x^T(r,x)u^0\right] \quad \forall\,(r,x)\in\mathcal{D}_{\mathbb{C}}. \tag{4.18}$$

Note that

$$\begin{aligned} &- V_x(\bar{\xi}_r^*) + \bar{S}_{tx}(r,\bar{\xi}^*) + \bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^* + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x(r,\bar{\xi}^*) \\ &= \frac{\partial}{\partial x}\Big[-V(x) + \bar{S}_t(r,x) + \bar{S}_x^T(r,x)u^0 + \tfrac{i\hbar}{2m}\bar{S}_{xx}(r,x)\Big]\Big|_{x=\bar{\xi}_r^*,\,u^0=\hat{u}(r,\bar{\xi}_r^*)}\end{aligned}$$

where the partial derivative notation indicates that the derivative is taken only over explicitly appearing arguments, and this is

$$\begin{aligned} &= \frac{d}{dx}\Big[L(x,\hat{u}(r,x)) + \bar{S}_t(r,x) + \bar{S}_x^T(r,x)\hat{u}(r,x) + \tfrac{i\hbar}{2m}\bar{S}_{xx}(r,x)\Big]\Big|_{x=\bar{\xi}_r^*} \\ &\quad - \frac{\partial}{\partial u^0}\Big[L(x,u^0) + \bar{S}_x^T(r,x)u^0\Big]\Big|_{x=\bar{\xi}_r^*,\,u^0=\hat{u}(r,\bar{\xi}_r^*)}\frac{d}{dx}\hat{u}(r,x)\Big|_{x=\bar{\xi}_r^*},\end{aligned}$$

which by (4.18),

$$= \frac{d}{dx}\Big[L(x,\hat{u}(r,x)) + \bar{S}_t(r,x) + \bar{S}_x^T(r,x)\hat{u}(r,x) + \tfrac{i\hbar}{2m}\bar{S}_{xx}(r,x)\Big]\Big|_{x=\bar{\xi}_r^*},$$

which by (3.3) and (4.18),

$$= \frac{d}{dx}\big[0\big] = 0. \tag{4.19}$$

Substituting (4.19) into (4.17), we have

$$\begin{aligned}\big|\Xi_r\big| \le &\Big|\bar{S}_x^T(r,\xi_r)u_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\Big| \\ &+ K_1\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2 + m|\delta_r|^2 \quad \forall\,(r,\omega)\in(s,t)\times\Omega,\end{aligned}$$

which implies

$$\begin{aligned}&\mathbb{E}\int_s^t \big|\Xi_r\big|\,dr \le m\|\delta\|_{\mathcal{U}_s}^2 + K_1\mathbb{E}\int_s^t \big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2\,dr \\ &+ \mathbb{E}\int_s^t \Big|\bar{S}_x^T(r,\xi_r)u_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\Big|\,dr.\end{aligned} \tag{4.20}$$

Now,

$$\mathbb{E}\int_s^t \left|\bar{S}_x^T(r,\xi_r)u_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\right| dr \tag{4.21}$$
$$= \mathbb{E}\int_s^t \left|\big[\bar{S}_x(r,\xi_r) - \bar{S}_x(r,\bar{\xi}_r^*)\big]^T\delta_r + \big[\bar{S}_x(r,\xi_r) - \bar{S}_x(r,\bar{\xi}_r^*) - \bar{S}_{xx}(r,\bar{\xi}_r^*)\Delta_r\big]^T\bar{u}_r^*\right| dr.$$

Also, by Taylor's Theorem and the assumptions,

$$\left|\big[\bar{S}_x(r,x) - \bar{S}_x(r,\bar{x})\big]^T(u^0 - \bar{u}^0)\right| \le C_s\big(1 + |x|^{2q} + |\bar{x}|^{2q}\big)|x-\bar{x}|\,|u^0-\bar{u}^0|$$

and

$$\left|\bar{S}_x(r,x) - \bar{S}_x(r,\bar{x}) - \bar{S}_{xx}(r,\bar{x})(x-\bar{x})\right| \le \tfrac{\widehat{C}_S}{2}\big(1 + |x|^{2q} + |\bar{x}|^{2q}\big)|x-\bar{x}|^2$$

for all $x, \bar{x}, u^0, \bar{u}^0 \in \mathbb{C}^n$ and $r \in (s,t)$. Applying these inequalities in (4.21), we have

$$\begin{aligned}
&\mathbb{E}\int_s^t \left|\bar{S}_x^T(r,\xi_r)u_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\bar{u}_r^* - [\bar{S}_{xx}(r,\bar{\xi}_r^*)\bar{u}_r^*]^T\Delta_r - \bar{S}_x^T(r,\bar{\xi}_r^*)\delta_r\right| dr \\
&\le \mathbb{E}\int_s^t C_s\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|\,|\delta_r|\,dr + \mathbb{E}\int_s^t \tfrac{\widehat{C}_s}{2}\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2\,|\bar{u}_r^*|\,dr \\
&\le \tfrac{C_S}{2}\,\mathbb{E}\int_s^t\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\delta_r|^2\,dr + \tfrac{C_S}{2}\mathbb{E}\int_s^t\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2\,dr \\
&\quad + \tfrac{\widehat{C}_S}{2}\mathbb{E}\int_s^t \big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\bar{u}_r^*|\,|\Delta_r|^2\,dr.
\end{aligned} \tag{4.22}$$

Now, using Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\int_s^t\big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\delta_r|^2\,dr \le \left\{1 + \left[2\mathbb{E}\int_s^t |\bar{\xi}_r^*|^{4q} + |\bar{\xi}_r^* + \Delta_r|^{4q}\,dr\right]^{1/2}\right\}\|\delta\|_{\mathcal{U}_s}^2 \\
&\le C_1\left\{1 + \|\bar{\xi}^*\|_{\mathcal{X}_s}^{2q} + \left[\mathbb{E}\int_s^t |\Delta_r|^{4q}\,dr\right]^{1/2}\right\}\|\delta\|_{\mathcal{U}_s}^2,
\end{aligned} \tag{4.23}$$

for appropriate $C_1 = C_1(q) < \infty$. Substituting (4.23) and (4.22) into (4.20) yields

$$\begin{aligned}
\mathbb{E}\int_s^t |\Xi_r|\,dr \le& \left\{m + \tfrac{C_1 C_S}{2}\left[1 + \|\bar{\xi}^*\|_{\mathcal{X}_s}^{2q} + \left[\mathbb{E}\int_s^t |\Delta_r|^{4q}\,dr\right]^{1/2}\right]\right\}\|\delta\|_{\mathcal{U}_s}^2 \\
&+ (K_1 + \tfrac{C_S}{2})\mathbb{E}\int_s^t \big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2\,dr \\
&+ \tfrac{\widehat{C}_S}{2}\mathbb{E}\int_s^t \big(1+|\xi_r|^{2q}+|\bar{\xi}_r^*|^{2q}\big)|\bar{u}_r^*|\,|\Delta_r|^2\,dr,
\end{aligned}$$

which by the definition of $\bar{u}^*$ and Assumption $(A.2)$,

$$\begin{aligned}
&\le \left\{m + \tfrac{C_1 C_S}{2}\left[1 + \|\bar{\xi}^*\|_{\mathcal{X}_s}^{2q} + \left[\mathbb{E}\int_s^t |\Delta_r|^{4q}\,dr\right]^{1/2}\right]\right\}\|\delta\|_{\mathcal{U}_s}^2 \\
&\quad + C_2\mathbb{E}\int_s^t \big(1+|\bar{\xi}_r^*|^{4q}+|\bar{\xi}_r^* + \Delta_r|^{4q}\big)|\Delta_r|^2\,dr,
\end{aligned}$$

for appropriate $C_2 = C_2(C_0, C_s, \widehat{C}_S, q) < \infty$, and this is

$$\leq \Big\{ m + \tfrac{C_1 C_S}{2} \Big[1 + \|\bar{\xi}^*\|_{\mathcal{X}_s}^{2q} + \Big[\mathbb{E} \int_s^t |\Delta_r|^{4q}\, dr \Big]^{1/2} \Big] \Big\} \|\delta\|_{\mathcal{U}_s}^2$$
$$+ C_3 \mathbb{E} \int_s^t \big(|\Delta_r|^2 + |\bar{\xi}_r^*|^{4q} |\Delta_r|^2 + |\Delta_r|^{4q+2} \big)\, dr, \tag{4.24}$$

for appropriate $C_3 = C_3(C_0, C_S, \widehat{C}_S, q) < \infty$.

Next one should note some simple estimates. First, using Hölder's inequality,

$$\mathbb{E} \int_s^t |\Delta_r|^2\, dr = \mathbb{E} \int_s^t \Big| \int_s^r \delta_\rho\, d\rho \Big|^2 dr \leq (t-s) \mathbb{E} \int_s^t |\delta_\rho|^2\, d\rho \doteq (t-s) \|\delta\|_2^2$$
$$\leq (t-s) \|\delta\|_{\mathcal{U}_s}^2, \tag{4.25}$$

where we let $\|\cdot\|_p$ denote the p-norm for $p \in [1, \infty)$. Similarly, with Hölder's inequality and some simple calculations,

$$\mathbb{E} \int_s^t |\Delta_r|^{4q+2}\, dr = \mathbb{E} \int_s^t \Big| \int_s^r \delta_\rho\, d\rho \Big|^{4q+2} dr \leq (t-s)^{4q+2} \mathbb{E} \int_s^t |\delta_\rho|^{4q+2}\, d\rho$$
$$= (t-s)^{4q+2} \|\delta\|_{4q+2}^{4q+2} \leq (t-s)^{4q+2} \|\delta\|_{\mathcal{U}_s}^{4q+2}, \tag{4.26}$$

and

$$\mathbb{E} \int_s^t |\Delta_r|^{4q}\, dr \leq (t-s)^{4q} \|\delta\|_{\mathcal{U}_s}^{4q}. \tag{4.27}$$

Next,

$$\mathbb{E} \int_s^t |\bar{\xi}_r^*|^{4q} |\Delta_r|^2\, dr \leq \Big[\mathbb{E} \int_s^t |\bar{\xi}_r^*|^{8q} \Big]^{1/2} \Big[\mathbb{E} \int_s^t \Big| \int_s^t \delta_\rho\, d\rho \Big|^4 dr \Big]^{1/2}$$
$$\leq (t-s)\, \|\bar{\xi}^*\|_{\mathcal{X}_s}^{4q} \Big[\mathbb{E} \Big| \int_s^t \delta_\rho\, d\rho \Big|^4 \Big]^{1/2} \leq (t-s)^{5/2}\, \|\bar{\xi}^*\|_{\mathcal{X}_s}^{4q} \Big[\mathbb{E} \int_s^t |\delta_\rho|^4\, d\rho \Big]^{1/2}$$
$$\leq (t-s)^{5/3}\, \|\bar{\xi}^*\|_{\mathcal{X}_s}^{4q}\, \|\delta\|_{\mathcal{U}_s}^2. \tag{4.28}$$

Substituting (4.25)–(4.28) into (4.24), one finds

$$\mathbb{E} \int_s^t \big| \Xi_r \big|\, dr \leq \Big\{ m + \tfrac{C_1 C_S}{2} \Big[1 + \|\bar{\xi}^*\|_{\mathcal{X}_s}^{2q} + t^{2q} \|\delta\|_{\mathcal{U}_s}^{2q} \Big] \Big\} \|\delta\|_{\mathcal{U}_s}^2$$
$$+ C_3 \Big[\|\delta\|_{\mathcal{U}_s}^2 + t^{5/3} \|\bar{\xi}^*\|_{\mathcal{X}_s}^{4q} \|\delta\|_{\mathcal{U}_s}^2 + t^{4q+2} \|\delta\|_{\mathcal{U}_s}^{4q+2} \Big]$$
$$\leq C_4 \Big[1 + t^{4q+2} + \big(1 + t^{5/3}\big) \|\bar{\xi}^*\|_{\mathcal{X}_s}^{4q} \Big] \Big[\|\delta\|_{\mathcal{U}_s}^2 + \|\delta\|_{\mathcal{U}_s}^{4q+2} \Big], \tag{4.29}$$

for appropriate choice of $C_4 = C_4(C_0, C_S, \widehat{C}_S, q) < \infty$. Substituting (4.29) into (4.13) yields

$$\big| J^\hbar(s, x, \bar{u}^*) - J^\hbar(s, x, u) \big| \leq \overline{\overline{C}} \|\delta\|_{\mathcal{U}_s}^2,$$

for $\|\delta\|_{\mathcal{U}_s} \leq 1$ and appropriate choice of $\overline{\overline{C}} = \overline{\overline{C}}(t, x, C_0, C_s, \widehat{C}_s, q) < \infty$. By definition, this implies that $\bar{u}^* = \operatorname{argstat}_{u \in \mathcal{U}_s} J^\hbar(s, x, u)$, where uniqueness of the argstat is guaranteed by Assumption $(A.3)$. □

It may be worth noting the following, which reflects the uniqueness implied by the above representation.

COROLLARY 4.4. *In addition to* $(A.0)$–$(A.3)$, *assume the conditions of Theorem 4.1. There exists a unique solution* $\tilde{S} \in \mathcal{S}^p_{\mathbb{C}}$ *to (3.3)–(3.4), where* $\tilde{S} = S^\hbar$. *There also exists a solution,* $\hat{S} \in \mathcal{S}$, *to (1.7)–(1.8), given by* $\hat{S}(r,y) = S^\hbar(r, \mathcal{V}_0^{-1}((y^T,0)^T))$ *for all* $(r,y) \in \overline{\mathcal{D}}$. *Lastly, any other solution in* $\mathcal{S}$ *to (1.7)–(1.8) cannot be extended holomorphically to a solution of (3.3)–(3.4) in* $\mathcal{S}^p_{\mathbb{C}}$.

Proof. The existence of $\tilde{S}$ follows from Assumption $(A.0)$, and the uniqueness follows from Theorem 4.1. Let $\hat{S}$ be as given in the corollary statement. Note that

$$\hat{S}_t(r,y) = \bar{S}_t(r, y+i0) \quad \forall\, (r,y) \in \mathcal{D}. \tag{4.30}$$

Also, by the Cauchy-Riemann equations, for $(r,y) \in \mathcal{D}$, and the fact that $\hat{S}$ agrees with $\bar{S}$ on $\overline{\mathcal{D}}$,

$$\begin{aligned} |\bar{S}_x(r,y+i0)|_c^2 &= \sum_{j=1}^n \left[\bar{R}^2_{y_j} - \bar{T}^2_{y_j} + 2i\bar{R}_{y_j}\bar{T}_{y_j}\right] = \sum_{j=1}^n \left[\hat{R}^2_{y_j} - \hat{T}^2_{y_j} + 2i\hat{R}_{y_j}\hat{T}_{y_j}\right] \\ &= \sum_{j=1}^n (\hat{S}_{y_j}(r,y))^2 = |\hat{S}_y(r,y)|_c^2, \end{aligned} \tag{4.31}$$

where we take $(\bar{R}(r,\mathcal{V}_0(x)), \bar{T}(r,\mathcal{V}_0(x)))^T = \mathcal{V}_{00}(\bar{S}(r,x))$ for all $(r,x) \in \mathcal{D}_{\mathbb{C}}$ and $(\hat{R}(r,y), \hat{T}(r,y))^T = \mathcal{V}_{00}(\hat{S}(r,y))$ for all $(r,y) \in \mathcal{D}$. Similarly, using (3.16), (3.17),

$$\begin{aligned} \Delta\bar{S}(r,y+i0) &= \sum_{j=1}^n \bar{S}_{x_j,x_j}(r,y+i0) = \sum_{j=1}^n \left[\bar{R}_{y_j,y_j}(r,y+i0) + i\bar{T}_{y_j,y_j}(r,y+i0)\right] \\ &= \sum_{j=1}^n \left[\hat{R}_{y_j,y_j}(r,y+i0) + i\hat{T}_{y_j,y_j}(r,y+i0)\right] = \Delta\hat{S}(r,y) \end{aligned} \tag{4.32}$$

for all $(r,y) \in \mathcal{D}$. Then, by (3.3) and (4.30)–(4.32),

$$\begin{aligned} &\hat{S}_t(r,y) + \tfrac{i\hbar}{2m}\Delta\hat{S}(r,y) + H(y, \hat{S}_y(r,y)) \\ &= \bar{S}_t(r,y+i0) + \tfrac{i\hbar}{2m}\Delta\bar{S}(r,y+i0) + H(y+i0, \bar{S}_x(r,y+i0)) = 0 \end{aligned} \tag{4.33}$$

for all $(r,y) \in \mathcal{D}$. That $\hat{S}$ also satisfies the terminal condition is obvious, and we see that $\hat{S}$ is a solution of (1.7)–(1.8).

Regarding the last assertion, recall that if two holomorphic functions on $\mathbb{C}^n$ agree on $\{x = y + iz \in \mathbb{C}^n \,|\, z = 0\}$, then they agree on all of $\mathbb{C}^n$ (cf. [14]). Noting the uniqueness of $\tilde{S} = S^\hbar$ yields the assertion. □

REMARK 4.5. *The results concerning* $\hat{S}$ *in Corollary 4.4 also extend to (1.1)–(1.2) and (1.3)–(1.4) in the obvious ways.*

5. Existence. The results of Sections 3–4 were conditioned on an assumption of existence of a solution to (3.3)–(3.4). However, for this class of systems, one can use simple complex-analysis equivalences to reduce the question of existence of a solution of the complex HJ PDE problem to that of existence of a solution of a real HJ PDE problem. In this section, we drop the earlier assumptions.

THEOREM 5.1. *Suppose* V, ϕ *are holomorphic on* $\mathbb{C}^n$. *Suppose also that* $R \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}}_2)$ *satisfies (3.25)–(3.27). Then, there exists* $T \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}}_2)$ *satisfying (3.26)–(3.28) such that for each* $s \in (0,t]$, $T(s,\cdot,\cdot)$ *is a harmonic conjugate* *of $R(s,\cdot,\cdot)$. Further, letting $S(s,x) = \mathcal{V}_{00}^{-1}(R(s,\mathcal{V}_0(x)),T(s,\mathcal{V}_0(x)))$ for all $(s,x) \in \overline{\mathcal{D}}_{\mathbb{C}}$, S satisfies (3.3)–(3.4).*

Proof. Let $T(t,\cdot,\cdot) = \phi^I(\cdot,\cdot)$. Then, by (3.27), $T(t,\cdot,\cdot)$ is a complex conjugate of $R(t,\cdot,\cdot)$, and satisfies (3.28). Now we begin the construction of T for $s \in (0,t)$. Recalling that there is a free constant in the harmonic conjugate, for each $s \in (0,t)$, we let

$$\begin{aligned} T(s,0,0) &\doteq T(t,0,0) - \int_s^t -\tfrac{1}{m}\sum_{j=1}^n R_{y_j}(\rho,0,0)R_{z_j}(\rho,0,0) \\ &\qquad\qquad + \tfrac{\hbar}{2m}\sum_{j=1}^n R_{y_j,y_j}(\rho,0,0) + V^I(0,0)\,d\rho \\ &\doteq T(t,0,0) - \int_s^t \Big[-\tfrac{1}{m}\sum_{j=1}^n R_{y_j}R_{z_j} + \tfrac{\hbar}{2m}\sum_{j=1}^n R_{y_j,y_j} + V^I\Big](\rho,0,0)\,d\rho, \end{aligned} \tag{5.1}$$

which implies that for all $s \in (0,t)$,

$$T_t(s,0,0) = \Big[-\tfrac{1}{m}\sum_{j=1}^n R_{y_j}R_{z_j} + \tfrac{\hbar}{2m}\sum_{j=1}^n R_{y_j,y_j} + V^I\Big](s,0,0), \tag{5.2}$$

which by the Cauchy-Riemann equations and (3.16)–(3.17),

$$= \Big[\tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + \tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + V^I\Big](s,0,0), \tag{5.3}$$

which implies that (3.26) is satisfied at $(s,0,0)$ for all $s \in (0,t)$.

By the Cauchy-Riemann equations and (6.2), for all $(s,y,z) \in \mathcal{D}_2$ and all $k \in]1,n[$,

$$T_{t,z_k} = R_{t,y_k} = \tfrac{\hbar}{2m}\sum_{j=1}^n R_{y_j,z_j,y_k} + \tfrac{1}{m}\sum_{j=1}^n [R_{y_j}R_{y_j,y_k} - R_{z_j}R_{z_j,y_k}] + V^R_{y_k}, \tag{5.4}$$

which by the Cauchy-Riemann equations and (3.16)–(3.19),

$$\begin{aligned} &= \tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j,z_k} + \tfrac{1}{m}\sum_{j=1}^n [T_{z_j}T_{y_j,z_k} + T_{y_j}T_{z_j,z_k}] + V^I_{z_k} \\ &= \frac{d}{dz_k}\Big\{\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big\}. \end{aligned} \tag{5.5}$$

For $z_1 \in \mathbb{R}$, let $\hat{z}^1(z_1) \doteq (z_1, 0, 0 \ldots 0)^T \in \mathbb{R}^n$, and note that for $s \in (0,t)$,

$$T_t(s,0,\hat{z}^1(z_1)) = T_t(s,0,0) + \int_0^{z_1} T_{t,z_1}(s,0,\hat{z}^1(\zeta))\,d\zeta, \tag{5.6}$$

which by (5.5),

$$= T_t(s,0,0) + \int_0^{z_1} \frac{d}{dz_1}\Big\{\Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,\hat{z}^1(\zeta))\Big\}\,d\zeta$$

$$= T_t(s,0,0) + \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,\hat{z}^1(z_1))$$

$$- \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,0),$$

which by (5.3),

$$= \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,\hat{z}^1(z_1)),$$

which implies that (3.26) is satisfied at $(s,0,\hat{z}^1(z_1))$ for all $s \in (0,t)$ and $z_1 \in \mathbb{R}$. Proceeding from here similarly, first for z_2 and then z_3 and so on, yields finally

$$T_t(s,0,z) = \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,z) \quad \forall\, s \in (0,t),\ z \in \mathbb{R}^n.$$

We now proceed to integrate along the y-components. First, for $k \in]1,n[$, differentiating (3.25) with respect to z_k, we have

$$R_{t,z_k} = \tfrac{\hbar}{2m}\sum_{j=1}^n R_{y_j,z_j,z_k} + \tfrac{1}{m}\sum_{j=1}^n [R_{y_j}R_{y_j,z_k} - R_{z_j}R_{z_j,z_k}] + V^R_{z_k}, \tag{5.7}$$

which by the Cauchy-Riemann equations and (3.16)–(3.19),

$$= -\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j,y_k} - \tfrac{1}{m}\sum_{j=1}^n [T_{z_j}T_{y_j,y_k} + T_{y_j}T_{z_j,y_k}] - V^I_{y_k}$$

$$= -\frac{d}{dy_k}\Big\{\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big\}. \tag{5.8}$$

By the Cauchy-Riemann equations and (5.8), we have

$$T_{t,y_k} = -R_{t,z_k} = \frac{d}{dy_k}\Big\{\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big\}. \tag{5.9}$$

For $y_1 \in \mathbb{R}$, let $\hat{y}^1(y_1) \doteq (y_1,0,0\ldots0)^T \in \mathbb{R}^n$, and note that for $s \in (0,t)$ and $z \in \mathbb{R}^n$,

$$T_t(s,\hat{y}^1(y_1),z) = T_t(s,0,z) + \int_0^{y_1} T_{t,y_1}(s,\hat{y}^1(\eta),z)\,d\eta,$$

which by (5.9),

$$= T_t(s,0,z) + \int_0^{y_1} \frac{d}{dy_1}\Big\{\Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,\hat{y}^1(\eta),z)\Big\}\,d\eta$$

$$= T_t(s,0,z) + \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,\hat{y}^1(y_1),z)$$

$$- \Big[\tfrac{\hbar}{2m}\sum_{j=1}^n T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^n T_{y_j}T_{z_j} + V^I\Big](s,0,z),$$

which by (3.26),

$$= \Big[\tfrac{\hbar}{2m}\sum_{j=1}^{n} T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^{n} T_{y_j}T_{z_j} + V^I\Big](s,\hat{y}^1(y_1),z),$$

which implies that (3.26) is satisfied at $(s,\hat{y}^1(y_1),z)$ for all $s \in (0,t)$, $y_1 \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Proceeding from this along each component of y, one finally obtains

$$T_t(s,y,z) = \Big[\tfrac{\hbar}{2m}\sum_{j=1}^{n} T_{y_j,z_j} + \tfrac{1}{m}\sum_{j=1}^{n} T_{y_j}T_{z_j} + V^I\Big](s,y,z) \quad \forall\,(s,y,z) \in \mathcal{D}_2, \tag{5.10}$$

which is (3.26). By construction, T has the indicated smoothness. Lastly, by Proposition 3.4, one obtains the assertions concerning S. □

REMARK 5.2. *As S obtained in Theorem 5.1 is holomorphic in x for each $s \in (0,t]$ (and hence C^∞ in the space variable), noting that R,T are related to S by (3.15), one immediately sees that R,T are $C^{1,\infty}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}}_2)$.*

The analogous result to Theorem 5.1, where one supposes existence of a solution to (3.26)–(3.28) rather than (3.25)–(3.27) is obtained similarly, and the redundant proof is omitted. The result is as follows.

THEOREM 5.3. *Suppose V,ϕ are holomorphic on $\mathbb{C}^n$. Suppose also that $T \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}}_2)$ satisfies (3.26)–(3.28). Then, there exists $R \in C^{1,3}(\mathcal{D}_2) \cap C(\overline{\mathcal{D}}_2)$ satisfying (3.25)–(3.27) such that for each $s \in (0,t]$, $T(s,\cdot,\cdot)$ is a harmonic conjugate of $R(s,\cdot,\cdot)$. Further, letting $S(s,x) = \mathcal{V}_{00}^{-1}(R(s,\mathcal{V}_0(x)),T(s,\mathcal{V}_0(x)))$ for all $(s,x) \in \overline{\mathcal{D}}_{\mathbb{C}}$, S satisfies (3.3)–(3.4).*

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6. Appendix A.

Proof of Lemma 3.1. The case of (3.14) is trivial, and we consider only (3.22). By Assumption $(A.2)$ and standard results (cf. [19] Theorem II.6.1; [5]), we have existence of a weak solution up to explosion time, $\tau = \tau(\omega) \doteq \inf\big\{r \in [s,t] \,\big|\, |(\bar\eta^*_r, \bar\zeta^*_r)| \notin \mathbb{R}^{2n}\big\}$. Let such a weak solution be denoted as probability space $(\hat\Omega, \hat{\mathcal{F}}, \hat P)$, filtration $\{\hat{\mathcal{F}}_r \,|\, r \in [s,t]\}$, $\hat{\mathcal{F}}_\cdot$-adapted Brownian motion $\hat B_\cdot$, and process $(\bar\eta^*_\cdot, \bar\zeta^*_\cdot)$. By $(A.2)$ and standard results (cf. [19] Lemma II.5.2 and Corollary II.5.12; [9] Section III.5 and Appendix D), there exists $C_1 = C_1(C_S, q) < \infty$ such that $\mathbb{E}\big\{\sup_{r\in[s,t]} |(\bar\eta^*_r, \bar\zeta^*_r)|^{2q}\big\} \le C_1(1 + |(y,z)|^{2q})$. Consequently, for almost every $\omega \in \hat\Omega$, there exists $C_2 = C_2(\omega, C_S, q) < \infty$ such that

$$|(\bar\eta^*_r, \bar\zeta^*_r)|^{2q} \le C_2[1 + |(y,z)|^{2q}] \quad \forall\, r \in [s,t], \tag{6.1}$$

which implies $\tau = t$ a.s.

Suppose there exist two weak solutions with the same probability space, filtration and filtration-adapted Brownian motion, but possibly different state processes, say $(\bar\eta^*_\cdot, \bar\zeta^*_\cdot)$ and $(\hat\eta^*_\cdot, \hat\zeta^*_\cdot)$. Using (3.22) and Assumption $(A.2)$, we find that there exists $C_3 = C_3(C_S, q) < \infty$ such that

$$\left|(\bar{\eta}^*_r, \bar{\zeta}^*_r) - (\hat{\eta}^*_r, \hat{\zeta}^*_r)\right| \le C_3 \int_s^r \left(1 + |\bar{\eta}^*_\rho|^{2q} + |\bar{\zeta}^*_\rho|^{2q} + |\hat{\eta}^*_\rho|^{2q} + |\hat{\zeta}^*_\rho|^{2q}\right) \cdot \left|(\bar{\eta}^*_\rho, \bar{\zeta}^*_\rho) - (\hat{\eta}^*_\rho, \hat{\zeta}^*_\rho)\right| d\rho,$$

and noting (6.1), we see that $\exists\; C_4 = C_4(\omega, C_S, q, |(y,z)|) < \infty$ such that this is

$$\le C_4 \int_s^r \left|(\bar{\eta}^*_\rho, \bar{\zeta}^*_\rho) - (\hat{\eta}^*_\rho, \hat{\zeta}^*_\rho)\right| d\rho.$$

Applying Gronwall's inequality to this, we find that $(\bar{\eta}^*_\cdot, \bar{\zeta}^*_\cdot) = (\hat{\eta}^*_\cdot, \hat{\zeta}^*_\cdot)$ a.s., i.e., that we have pathwise uniqueness. Then, by [15], Theorem IV.1.1, we see that there exists a unique strong solution. □

REMARK 6.1. *Alternatively, one may apply [30] Theorem V.38, to imply existence of a solution up to explosion time, τ, and then use (6.1) to imply that $\tau = t$ a.s. However, [30] does not make use of the substantial structure evident in (3.22), and with an eye to possible later efforts with less well-behaved drift terms but identical diffusion coefficients, we have used the slightly longer approach of the above proof.*

6.1. Proof of Proposition 3.4. The first assertion follows by simple algebraic substitution, using the Cauchy-Riemann equations and (3.16)–(3.17). Now, suppose $\bar{R}, \bar{T} \in C^{1,2}(\mathcal{D}_2; \mathbb{R}) \cap C(\overline{\mathcal{D}}_2; \mathbb{R})$ satisfy (3.25)–(3.28), and let $\bar{S}$ be given by (3.15). We will show that $\bar{R}, \bar{T}$ satisfy the Cauchy-Riemann relations, and hence that $\bar{S} \in \mathcal{S}_{\mathbb{C}}$. After that, one may again use simple algebraic substitutions to verify the final assertion.

Differentiating (3.25) with respect to y_k, and (3.26) with respect to z_k, yields

$$\bar{R}_{t,y_k} = \tfrac{\hbar}{2m} \sum_{j=1}^n \bar{R}_{y_j,z_j,y_k} + \tfrac{1}{m} \sum_{j=1}^n \left(\bar{R}_{y_j}\bar{R}_{y_j,y_k} - \bar{R}_{z_j}\bar{R}_{z_j,y_k}\right) + V^R_{y_k}, \tag{6.2}$$

$$\bar{T}_{t,z_k} = \tfrac{\hbar}{2m} \sum_{j=1}^n \bar{T}_{y_j,z_j,z_k} + \tfrac{1}{m} \sum_{j=1}^n \left(\bar{T}_{z_j}\bar{T}_{y_j,z_k} + \bar{T}_{y_j}\bar{T}_{z_j,z_k}\right) + V^I_{z_k}. \tag{6.3}$$

Applying the Cauchy-Riemann equations and (3.16)–(3.19) in (6.3), one finds

$$\bar{T}_{t,z_k} = \tfrac{\hbar}{2m} \sum_{j=1}^n \bar{R}_{y_j,z_j,y_k} + \tfrac{1}{m} \sum_{j=1}^n \left(\bar{R}_{y_j}\bar{R}_{y_j,y_k} - \bar{R}_{z_j}\bar{R}_{z_j,y_k}\right) + V^R_{y_k},$$

which by (6.2),

$$= \bar{R}_{t,y_k} \quad \forall\,(s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[\,. \tag{6.4}$$

Also note that as ϕ is holomorphic,

$$\bar{R}_{y_k}(t,y,z) = \phi^R_{y_k}(y,z) = \phi^I_{z_k}(y,z) = \bar{T}_{z_k}(t,y,z) \quad \forall\, y,z \in \mathbb{R}^n. \tag{6.5}$$

By the Fundamental Theorem of Calculus,

$$\bar{T}_{z_k}(s,y,z) = \bar{T}_{z_k}(t,y,z) - \int_s^t \bar{T}_{t,z_k}(\sigma,y,z)\, d\sigma,$$

which by (6.4),(6.5),

$$= \bar{R}_{y_k}(s,y,z) \quad \forall\,(s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[\,. \tag{6.6}$$

Similarly, one obtains

$$\bar{T}_{y_k}(s,y,z) = -\bar{R}_{z_k}(s,y,z) \quad \forall\,(s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[\,. \tag{6.7}$$

By (6.6),(6.7), the Cauchy-Riemann conditions are satisfied. □