A Stationary-Action Control Representation for the Dequantized Schrödinger Equation

William M. McEneaney

***Abstract*— A stochastic control representation for solution of the Schrödinger equation is obtained, utilizing complex-valued diffusion processes. The Maslov dequantization is employed, where the domain is complex-valued in the space variable. The control problem is defined in terms of stationarity rather than optimality. The notion of stationarity is utilized to relate the Hamilton-Jacobi form of the dequantized Schrödinger equation to its stochastic control representation. Convexity of the action functional is not required, and there is no restriction on the duration of the problem.**

I. Introduction

Diffusion representations have long been utilized in the study of Hamilton-Jacobi partial differential equations (HJ PDEs). The bulk of such results apply to real-valued HJ PDEs, that is, to HJ PDEs where the coefficients and solutions are real-valued. The Schrödinger equation is complex-valued, although generally defined over a real-valued space domain, which presents difficulties for the development of stochastic control representations. There is substantial existing work on the relation of stochastic processes to the Schrödinger equation, cf. [11], [15], [24], [25], [28]. The approach considered here is in the spirit of the Feynman path-integral interpretation [6], [7], where in particular, one looks at a certain action-based functional, S, where $\psi = \exp\{\frac{i}{\hbar}S\}$ and $\hbar$ denotes Planck's constant. One seeks a representation for S in the form of a value function for a stochastic control problem where the action functional is the payoff, cf. [2], [3], [6], [7], [10], [14], [19]. We note that this latter approach is also sometimes employed in analysis of semiclassical limits, cf. [1], [3], [10], [14].

An issue that arises in such approaches is that control has traditionally considered classical optimization (minimization or maximization) of some payoff. Implicit in that is an assumption that the payoff is real valued. In [4], [22], the authors consider a least-action approach to obtaining fundamental solutions to two-point boundary value problems (TPBVPs) for conservative dynamical systems. However, that formulation induced duration limits on the problems which could be addressed, where those limits were similar to duration limits present in existing results on the Schrödinger equation representation in terms of action, cf. [2], [3], [10]. We note that the duration limits are related to a loss of convexity of the payoff as the time horizon is extended. The least-action principle is a special case of the more generally applicable stationary-action principle, where the latter is equivalent to the former when the action functional is convex and coercive. Consequently and more recently, the notion of "staticization" was introduced for such TPBVPs, in which case one seeks a stationary point of the action over the space of control inputs. The extension to stationarity removes the restriction on problem duration. This yields a dynamic program which takes the form of an HJ PDE in the case of continuous-time/continuous-space processes, where these were studied in the context of deterministic dynamics in [21], [20], [23].

As staticization seeks points where the derivative of a functional is zero, as opposed to optimization of the functional, it is easily extended to the case of complex-valued systems. The extension to stochastic dynamics is easily made as well. Also, as staticization does not require the imposition of duration limits on the problems, one can apply this new tool to the stochastic-control representation problem for the dequantized Schrödinger equation, and that is the topic considered herein.

In order to clarify the details in the above, we recall the Schrödinger initial value problem, given as

$$0 = i\hbar\psi_t(s,x) + \frac{\hbar^2}{2m}\Delta\psi(s,x) - \psi(s,x)V(x),\ (s,x) \in \mathcal{D}, \tag{1}$$

$$\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $m \in (0,\infty)$ denotes mass, initial condition ψ_0 takes values in $\mathbb{C}$, V denotes a known potential function, Δ denotes the Laplacian with respect to the space (second) variable, $\mathcal{D} \doteq (0,t) \times \mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We also let $\overline{\mathcal{D}} \doteq (0,t] \times \mathbb{R}^n$. We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf., [18]), which as noted above, is $S : \overline{\mathcal{D}} \to \mathbb{C}$ given by $\psi(s,x) = \exp\{\frac{i}{\hbar}S(s,x)\}$. Note that $\psi_t = \frac{i}{\hbar}\psi S_t$, $\psi_x = \frac{i}{\hbar}\psi S_x$ and $\Delta\psi = \frac{i}{\hbar}\psi\Delta S - \frac{1}{\hbar^2}\psi|S_x|_c^2$ where for $y \in \mathbb{C}^n$, $|y|_c^2 \doteq y^T y = \sum_{j=1}^n y_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We find that (1)–(2) become

$$0 = -S_t(s,x) + \frac{i\hbar}{2m}\Delta S(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D}, \tag{3}$$

$$S(0,x) = \phi(x), \quad x \in \mathbb{R}^n, \tag{4}$$

where $H : \mathbb{R}^n \times \mathbb{C}^n \to \mathbb{C}$ is the Hamiltonian given by

$$\begin{aligned} H(x,p) &= -\left[\tfrac{1}{2m}|p|_c^2 + V(x)\right] \\ &= \operatorname*{stat}_{v \in \mathbb{C}^n} \left\{v^T p + \tfrac{m}{2}|v|_c^2 - V(x)\right\}, \end{aligned} \tag{5}$$

Research partially supported by grants from AFOSR and NSF.

University of California San Diego, La Jolla, CA 92093-0411, USA. wmceneaney@ucsd.edu

and stat will be defined in the next section. We look for solutions in the space

$$\mathcal{S} \doteq \{S : \overline{\mathcal{D}} \to \mathbb{C} \,|\, S \in C_p^{1,2}(\mathcal{D}) \cap C(\overline{\mathcal{D}})\}, \tag{6}$$

where $C_p^{1,2}$ denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound. We will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the HJ PDE problem given by

$$0 = S_t(s,x) + \tfrac{i\hbar}{2m}\Delta S(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D}, \tag{7}$$

$$S(t,x) = \phi(x), \quad x \in \mathbb{R}^n. \tag{8}$$

Working mainly with this last form, we will fix $t \in (0,\infty)$, and allow s to vary in $(0,t]$.

Recall that in semiclassical limit analysis, one views $\hbar$ as a small parameter, and examines the limit as $\hbar \downarrow 0$. Applying this in (7)–(8) yields an HJ PDE problem of the form

$$0 = S_t(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D}, \tag{9}$$

$$S(t,x) = \phi(x), \quad x \in \mathbb{R}^n. \tag{10}$$

Recalling the above-noted recent work on least-action and stationary-action formulations of certain TPBVPs [4], [21], [22], [20], [23], it was found that the associated HJ PDEs for such problems also take the form (9)–(10). This was the original motivation for the effort here, where we develop a stationary-action based representation for the solution of (7)–(8) (and consequently (1)–(2)). Due to the complex multiplier on the Laplacian, this representation is in terms of a stationary-action stochastic control problem with a complex-valued diffusion coefficient.

The main contribution of this effort is that the use of stationarity rather than optimization allows for the extension of the stationary-action stochastic control representation to arbitrary duration problems (Theorem 6). More specifically, we demonstrate that solutions of (7)–(8) are given by (48), where $J^\hbar$ and $\xi_\cdot$ are given by (47) and (16), respectively. Further, as this representation has a similar form to that of the stationary-action value for the limit system, but where the latter lacks the input diffusion term, one has the expectation that this will provide a new tool for the study of semiclassical limits.

In Section II, we recall the definitions necessary for stationarity problems. In Section III, the underlying space domain is extended from a space over the real field to a space over the complex field. This necessitates several other minor extensions, which are covered in the subsections. In particular, some classical existence and uniqueness results for stochastic differential equations (SDEs) are easily extended to their complex-valued counterparts. In Section IV, the main result of the paper, a stationarity-based stochastic-control value function representation for the dequantized Schrödinger equation, is obtained. More specifically, a verification result is obtained demonstrating that if a solution of the HJ PDE over the "complexified" domain exists, then that solution has the indicated representation.

II. Stationarity definitions

Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. For this and other reasons, as in the definition of the Hamiltonian given in (5), we find it useful to develop additional notation and nomenclature. Specifically, we will refer to the search for stationary points more succinctly as *staticization* (in analogy with minimization, and similarly to that, based on the Latin "statica"). In particular, we make the following definitions. Suppose $(\mathcal{Y}, |\cdot|)$ is a generic normed vector space over $\mathbb{C}$ with $\mathcal{G} \subseteq \mathcal{Y}$, and suppose $F : \mathcal{G} \to \mathbb{C}$. We say $\bar{y} \in \mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\}$ if $\bar{y} \in \mathcal{G}$ and either $\limsup_{y\to\bar{y}, y\in\mathcal{G}\setminus\{\bar{y}\}} |F(y) - F(\bar{y})|/|y-\bar{y}| = 0$, or there exists $\delta > 0$ such that $\mathcal{G} \cap B_\delta(\bar{y}) = \{\bar{y}\}$ (where $B_\delta(\bar{y})$ denotes the ball of radius δ around $\bar{y}$). If $\mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\} \neq \emptyset$, we define the possibly set-valued stat^s operator by

$$\begin{aligned}\operatorname*{stat^s}_{y\in\mathcal{G}} F(y) &\doteq \mathrm{stat}^s\{F(y) \,|\, y \in \mathcal{G}\} \\ &\doteq \big\{F(\bar{y}) \,\big|\, \bar{y} \in \mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\}\big\}.\end{aligned}$$

If $\mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\} = \emptyset$, $\mathrm{stat}^s_{y\in\mathcal{G}} F(y)$ is undefined. We will also be interested in a single-valued stat operation. In particular, if there exists $a \in \mathbb{C}$ such that $\mathrm{stat}^s_{y\in\mathcal{G}} F(y) = \{a\}$, then $\mathrm{stat}_{y\in\mathcal{G}} F(y) \doteq a$; otherwise, $\mathrm{stat}_{y\in\mathcal{G}} F(y)$ is undefined. At times, we may abuse notation by writing $\bar{y} = \mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\}$ in the event that the $\mathrm{argstat}$ is the single point $\{\bar{y}\}$. The following is immediate from the above definitions.

Lemma 1: Suppose $\mathcal{Y}$ is a Hilbert space, with open set $\mathcal{G} \subseteq \mathcal{Y}$, and that $F : \mathcal{G} \to \mathbb{C}$ is Fréchet differentiable at $\bar{y} \in \mathcal{G}$ with Riesz representation $F_y(\bar{y}) \in \mathcal{Y}$ Then, $\bar{y} \in \mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}\}$ if and only if $F_y(\bar{y}) = 0$.

For further discussion, we refer the reader to [21], [23].

III. Extensions to the complex domain

Various details of extensions to the complex domain must be considered prior to the development of the representation.

A. Extended problem and assumptions

Although (1)–(2), (3)–(4) and (7)–(8) are typically given as HJ PDE problems over $\overline{\mathcal{D}}$, as in Doss et al. [1], [2], [3] we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to $\mathbb{C}^n$, i.e., $V : \mathbb{C}^n \to \mathbb{C}$, and we will abuse notation by employing the same symbol for the extended-domain functions. Throughout, for $k \in \mathbb{N}$, and $x \in \mathbb{C}^k$ or $x \in \mathbb{R}^k$, we let $|x|$ denote the Euclidean norm. Let $\mathcal{D}_\mathbb{C} \doteq (0,t) \times \mathbb{C}^n$ and $\overline{\mathcal{D}}_\mathbb{C} = (0,t] \times \mathbb{C}^n$, and define

$$\begin{aligned}\mathcal{S}_\mathbb{C} \doteq \{S : \overline{\mathcal{D}}_\mathbb{C} \to \mathbb{C} |\, &S \text{ is continuous on } \overline{\mathcal{D}}_\mathbb{C}, \text{ continuously} \\ &\text{differentiable in time on } \mathcal{D}_\mathbb{C}, \text{ and} \\ &\text{holomorphic on } \mathbb{C}^n \text{ for all } r \in (0,t]\,\},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{S}^p_\mathbb{C} \doteq \{S \in \mathcal{S}_\mathbb{C} \,|\, &S \text{ satisfies a polynomial growth condition} \\ &\text{in space, uniformly on } (0,t]\,\}.\end{aligned} \tag{12}$$

The extended-domain form of problem (7)–(8) is

$$0=\bar{S}_t(s,x)+\tfrac{i\hbar}{2m}\Delta\bar{S}(s,x)+H(x,\bar{S}_x(s,x)),\ (s,x)\in\mathcal{D}_{\mathbb{C}}, \tag{13}$$

$$\bar{S}(t,x)=\phi(x),\quad x\in\mathbb{C}^n. \tag{14}$$

Throughout, we assume the following.

For each $\hbar\in(0,1]$, there exists a solution, $\bar{S}=\bar{S}^\hbar\in\mathcal{S}^p_{\mathbb{C}}$ to (13)–(14). $(A.0)$

$V,\phi:\mathbb{C}^n\to\mathbb{C}$ are holomorphic on $\mathbb{C}^n$. Further, there exists $C_0<\infty$ and $q\in\mathbb{N}$ such that $|V_{xx}(x)|,|\phi_{xx}(x)|<C_0(1+|x|^{2q})$ for all $x\in\mathbb{C}^n$. $(A.1)$

For each $\hbar\in(0,1]$, there exists $C_S=C_S^\hbar<\infty$ such that $|\bar{S}_x(r,x)|\le C_S(1+|x|)$ and $|\bar{S}_{xx}(r,x)|\le C_S(1+|x|^{2q})$ for all $(r,x)\in\mathcal{D}_{\mathbb{C}}$. $(A.2)$

B. The underlying stochastic dynamics

We let $(\Omega,\mathcal{F},P)$ be a probability triple, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega,\mathcal{F})$. Let $\{\mathcal{F}_s\,|\,s\in[0,t]\}$ denote a filtration on $(\Omega,\mathcal{F},P)$, and let $B.$ denote an $\mathcal{F}.$-adapted Brownian motion taking values in $\mathbb{R}^n$. For $s\in[0,t]$, let

$$\mathcal{U}_s\doteq\{u:[s,t]\times\Omega\to\mathbb{C}^n\,|u \text{ is } \mathcal{F}.\text{-adapted, right-cts,}\ \mathbb{E}\textstyle\int_s^t|u_r|^m\,dr<\infty\ \forall m\in\mathbb{N}\}. \tag{15}$$

We supply $\mathcal{U}_s$ with the norm $\|u\|_{\mathcal{U}_s}\doteq\max_{m\in]1,\bar{M}[}\left[\mathbb{E}\int_s^t|u_r|^m\,dr\right]^{1/m}$, where $\bar{M}\ge 8q$. We will be interested in diffusion processes given by

$$\xi_r=\xi_r^{(s,x)}=x+\int_s^r u_\rho\,d\rho+\sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\,B_r^\Delta, \tag{16}$$

where $x\in\mathbb{C}^n$, $s\in[0,t]$, $u\in\mathcal{U}_s$, and $B_r^\Delta\doteq B_r-B_s$ for $r\in[s,t]$. We will also be interested in the case where the control input is generated by a state-feedback. In particular, we will consider

$$\bar{\xi}_r^*=\bar{\xi}_r^{*,(s,x)}\doteq x+\int_s^r(\tfrac{-1}{m})\bar{S}_x(\rho,\bar{\xi}_\rho^{*,(s,x)})\,d\rho+\sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\,B_r^\Delta, \tag{17}$$

where presuming for now existence and uniqueness of a solution of (17), we may define the resulting $u^{*,(s,x)}\in\mathcal{U}_s$ given by

$$u_r^{*,(s,x)}(\omega)\doteq\hat{u}(r,\bar{\xi}^{*,(s,x)}(\omega))\doteq(\tfrac{-1}{m})\bar{S}_x(r,\bar{\xi}_r^{*,(s,x)}(\omega)) \tag{18}$$

for all $r\in[s,t]$ and $\omega\in\Omega$. For $s\in[0,t]$, we let

$$\mathcal{X}_s\doteq\{\xi:[s,t]\times\Omega\to\mathbb{C}^n\,|\,\xi \text{ is } \mathcal{F}.\text{-adapted, right-cts,}\ \mathbb{E}\sup_{r\in[s,t]}|\xi_r|^m<\infty\ \forall m\in\mathbb{N}\}. \tag{19}$$

We supply $\mathcal{X}_s$ with the norm $\|\xi\|_{\mathcal{X}_s}\doteq\max_{m\in]1,\bar{M}[}\left[\mathbb{E}\sup_{r\in[s,t]}|\xi_r|^m\right]^{1/m}$.

It is natural to work with complex-valued state processes in this problem domain. However, in order to easily apply many of the existing results regarding existence, uniqueness and moments, we will find it handy to use a "vectorized" real-valued representation for the complex-valued state processes. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, denoted here by $\mathcal{V}_{00}:\mathbb{C}\to\mathbb{R}^2$, with $\mathcal{V}_{00}(x)\doteq(y,z)^T$, where $y=\mathbf{Re}(x)$ and $z=\mathbf{Im}(x)$. This immediately yields the mapping $\mathcal{V}_0:\mathbb{C}^n\to\mathbb{R}^{2n}$ given by $\mathcal{V}_0(y+iz)\doteq(y^T,z^T)^T$, where component-wise, $(y_j,z_j)^T=\mathcal{V}_{00}(x_j)$ for all $j\in]1,n[$, where throughout, for integer $a\le b$, we define $]a,b[=\{a,a+1,\dots b\}$. Also in the interests of a reduction of cumbersome notation, we will henceforth frequently abuse notation by writing (y,z) in place of $(y^T,z^T)^T$ when the meaning is clear.

Given control process, $u\in\mathcal{U}_s$, we define its vectorized analog by the isometric isomorphism, $\mathcal{V}:\mathcal{U}_s\to\mathcal{U}_s^v$, where $[\mathcal{V}(u)]_r\doteq(v_r^T,w_r^T)^T$ and $(v_r^T,w_r^T)^T=\mathcal{V}_0(u_r)$ for all $r\in[s,t]$ and $\omega\in\Omega$, and where

$$\mathcal{U}_s^v\doteq\{(v,w):[s,t]\times\Omega\to\mathbb{R}^{2n}\,|\,(v,w)\text{ is }\mathcal{F}.\text{-adapted, right-cts, }\mathbb{E}\textstyle\int_s^t|v_r|^m+|w_r|^m\,dr<\infty\ \forall m\in\mathbb{N}\}, \tag{20}$$

$$\|u\|_{\mathcal{U}_s^v}\doteq\max_{m\in]1,\bar{M}[}\left[\mathbb{E}\textstyle\int_s^t|v_r|^m+|w_r|^m\,dr\right]^{1/m}. \tag{21}$$

Again abusing notation, we also define the isometric isomorphism, $\mathcal{V}:\mathcal{X}_s\to\mathcal{X}_s^v$ by $[\mathcal{V}(\xi)]_r\doteq[\mathcal{V}(\eta+i\zeta)]_r\doteq(\eta_r^T,\zeta_r^T)^T$ for all $r\in[s,t]$ and $\omega\in\Omega$, where

$$\mathcal{X}_s^v\doteq\{(\eta,\zeta):[s,t]\times\Omega\to\mathbb{R}^{2n}\,|\,(\eta,\zeta)\text{ is }\mathcal{F}.\text{-adapted, right-cts, }\mathbb{E}\sup_{r\in[s,t]}[|\eta_r|^m+|\zeta_r|^m]<\infty\ \forall m\in\mathbb{N}\}, \tag{22}$$

$$\|(\eta,\zeta)\|_{\mathcal{X}_s^v}\doteq\max_{m\in]1,\bar{M}[}\left[\mathbb{E}\sup_{r\in[s,t]}(|\eta_r|^m+|\zeta_r|^m)\right]^{1/m}. \tag{23}$$

Under transformation by $\mathcal{V}$, (16) becomes

$$\begin{pmatrix}\eta_r\\ \zeta_r\end{pmatrix}=\begin{pmatrix}y\\ z\end{pmatrix}+\int_s^r\begin{pmatrix}v_\rho\\ w_\rho\end{pmatrix}d\rho+\sqrt{\tfrac{\hbar}{m}}\tfrac{1}{\sqrt{2}}\begin{pmatrix}I_{n\times n}\\ I_{n\times n}\end{pmatrix}B_r^\Delta. \tag{24}$$

We may decompose $\bar{S}\in\mathcal{S}_{\mathbb{C}}$ as

$$(\bar{R}(r,\mathcal{V}_0(x)),\bar{T}(r,\mathcal{V}_0(x)))^T\doteq\mathcal{V}_{00}(\bar{S}(r,x)), \tag{25}$$

where $\bar{R},\bar{T}:\overline{\mathcal{D}}_2\doteq(0,t]\times\mathbb{R}^{2n}\to\mathbb{R}$, and we also let $\mathcal{D}_2\doteq(0,t)\times\mathbb{R}^{2n}$. For later reference, it will be helpful to recall some standard relations between derivative components, which are induced by the Cauchy-Riemann equations. For all $(r,x)=(r,y+iz)\in(0,t)\times\mathbb{C}^n$ and all $j,k,\ell\in]1,n[$, and suppressing the arguments for reasons of space we have

$$\mathbf{Re}[\bar{S}_{x_j,x_k}]=\bar{R}_{y_j,y_k}=-\bar{R}_{z_j,z_k}=\bar{T}_{z_j,y_k}=\bar{T}_{y_j,z_k}, \tag{26}$$

$$\mathbf{Im}[\bar{S}_{x_j,x_k}]=-\bar{R}_{y_j,z_k}=-\bar{R}_{z_j,y_k}=-\bar{T}_{z_j,z_k}=\bar{T}_{y_j,y_k}, \tag{27}$$

$$\begin{aligned}\mathbf{Re}[\bar{S}_{x_j,x_k,x_\ell}]&=\bar{R}_{y_j,y_k,y_\ell}=-\bar{R}_{y_j,z_k,z_\ell}=-\bar{R}_{z_j,z_k,y_\ell}\\&=-\bar{R}_{z_j,y_k,z_\ell}=\bar{T}_{z_j,y_k,y_\ell}=-\bar{T}_{z_j,z_k,z_\ell}\\&=\bar{T}_{y_j,z_k,y_\ell}=\bar{T}_{y_j,y_k,z_\ell},\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{Im}[\bar{S}_{x_j,x_k,x_\ell}]&=-\bar{R}_{y_j,y_k,z_\ell}=-\bar{R}_{y_j,z_k,y_\ell}=\bar{R}_{z_j,z_k,z_\ell}\\&=-\bar{R}_{z_j,y_k,y_\ell}=-\bar{T}_{z_j,y_k,z_\ell}=-\bar{T}_{z_j,z_k,y_\ell}\\&=-\bar{T}_{y_j,z_k,z_\ell}=\bar{T}_{y_j,y_k,y_\ell}.\end{aligned} \tag{29}$$

One may also easily verify that with $\bar{R},\bar{T}$ given by (25) and $(y^T,z^T)^T=\mathcal{V}_0(x)$,

$$\mathcal{V}_{00}(|\bar{S}_x(r,x)|_c^2)=\begin{pmatrix}\sum_{j=1}^n\bar{R}_{y_j}^2(r,y,z)-\bar{R}_{z_j}^2(r,y,z)\\ \sum_{j=1}^n-2\bar{R}_{y_j}(r,y,z)\bar{R}_{z_j}(r,y,z)\end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^n \bar{T}_{z_j}^2(r,y,z) - \bar{T}_{y_j}^2(r,y,z) \\ \sum_{j=1}^n 2\bar{T}_{y_j}(r,y,z)\bar{T}_{z_j}(r,y,z) \end{pmatrix}. \tag{30}$$

Further, given $u^0 \in \mathbb{C}^n$, $s \in [0,t]$ and $x \in \mathbb{C}^n$ with $(y^T, z^T)^T \doteq \mathcal{V}_0(x)$,

$$\mathcal{V}_0\left(\operatorname*{argstat}_{u^0\in\mathbb{C}^n} \Big[\sum_{j=1}^n \bar{S}_{x_j}(r,x)u_j^0 + \tfrac{m}{2}|u^0|_c^2 \Big] \right)$$
$$= \tfrac{1}{m} \begin{pmatrix} -\bar{R}_y(r,y,z) \\ \bar{R}_z(r,y,z) \end{pmatrix} = \tfrac{1}{m} \begin{pmatrix} -\bar{T}_z(r,y,z) \\ -\bar{T}_y(r,y,z) \end{pmatrix}. \tag{31}$$

Using the above, we see that under transformation $\mathcal{V}$, (17) becomes

$$\begin{aligned} \begin{pmatrix} \eta_r^* \\ \zeta_r^* \end{pmatrix} &= \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^r \tfrac{1}{m} \begin{pmatrix} -\bar{R}_y(\rho,\eta_\rho^*,\zeta_\rho^*) \\ \bar{R}_z(\rho,\eta_\rho^*,\zeta_\rho^*) \end{pmatrix} d\rho \\ &\quad + \sqrt{\tfrac{\hbar}{m}}\tfrac{1}{\sqrt{2}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B_r^\Delta \\ &= \begin{pmatrix} y \\ z \end{pmatrix} + \int_s^r \tfrac{-1}{m} \begin{pmatrix} \bar{T}_z(\rho,\eta_\rho^*,\zeta_\rho^*) \\ \bar{T}_y(\rho,\eta_\rho^*,\zeta_\rho^*) \end{pmatrix} d\rho \\ &\quad + \sqrt{\tfrac{\hbar}{m}}\tfrac{1}{\sqrt{2}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B_r^\Delta. \end{aligned} \tag{32}$$

Throughout, concerning both real and complex stochastic differential equations, typically given in integral form such as in (32), *solution* refers to a strong solution, unless specifically cited as a weak solution.

Lemma 2: Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $u \in \mathcal{U}_s$, $(y,z) = \mathcal{V}_0(x)$ and $(v,w) = \mathcal{V}(u)$. There exists a unique solution, $(\eta,\zeta) \in \mathcal{X}_s^v$, to (24), and a unique solution, $(\eta^*,\zeta^*) \in \mathcal{X}_s^v$, to (32).

Lemma 2 is easily obtained from minor extensions of well-known existing results (specifically, [16, Theorem II.6.1] ; and [13, Theorem IV.1.1] or [27, Theorem V.38)]), and in the interests of space, we do not include the proof. The following is also straightforward, cf. [26].

Lemma 3: Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $u \in \mathcal{U}_s$, $(y^T,z^T)^T = \mathcal{V}_0(x)$ and $(v,w) = \mathcal{V}(u)$. $\xi \in \mathcal{X}_s$ is a solution of (16) if and only if $\mathcal{V}(\xi) \in \mathcal{X}_s^v$ is a solution of (24). Similarly, $\bar{\xi}^* \in \mathcal{X}_s$ is a solution of (17) if and only if $\mathcal{V}(\bar{\xi}^*) \in \mathcal{X}_s^v$ is a solution of (32).

Combining Lemmas 2 and 3, one has:

Lemma 4: Let $s \in [0,t)$, $x \in \mathbb{C}^n$ and $u \in \mathcal{U}_s$. There exists a unique solution, $\xi \in \mathcal{X}_s$, to (16), and a unique solution, $\bar{\xi}^* \in \mathcal{X}_s$, to (17).

C. A relationship among the solutions

By the Cauchy-Riemann equations and (26)–(27),

$$|\bar{S}_x|_c^2 = \sum_{j=1}^n \bar{R}_{y_j}^2 - \bar{R}_{z_j}^2 + 2i\bar{T}_{y_j}\bar{T}_{z_j}, \tag{33}$$

$$\Delta\bar{S} = \sum_{j=1}^n \bar{T}_{y_j,z_j} - i\bar{R}_{y_j,z_j}. \tag{34}$$

Let

$$\big(V^R(\mathcal{V}_0(x)), V^I(\mathcal{V}_0(x))\big)^T \doteq \mathcal{V}_{00}(V(x)), \tag{35}$$

$$\big(\phi^R(\mathcal{V}_0(x)), \phi^I(\mathcal{V}_0(x))\big)^T \doteq \mathcal{V}_{00}(\phi(x)), \tag{36}$$

for all $x \in \mathbb{C}^n$. Substituting (33)–(36) into (13), and separating the real and imaginary parts, we have

$$0 = \bar{R}_t - \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{R}_{y_j,z_j} - \tfrac{1}{2m}\sum_{j=1}^n \big(\bar{R}_{y_j}^2 - \bar{R}_{z_j}^2\big) - V^R, \tag{37}$$

$$0 = \bar{T}_t - \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{T}_{y_j,z_j} - \tfrac{1}{m}\sum_{j=1}^n \bar{T}_{y_j}\bar{T}_{z_j} - V^I, \tag{38}$$

on $\mathcal{D}_2$, and, of course,

$$\bar{R}(t,y,z) = \phi^R(y,z) \quad \forall\, (y,z) \in \mathbb{R}^{2n}, \tag{39}$$

$$\bar{T}(t,y,z) = \phi^I(y,z) \quad \forall\, (y,z) \in \mathbb{R}^{2n}. \tag{40}$$

Proposition 5: Let $\bar{S} \in \mathcal{S}_\mathbb{C}$ and $\bar{R}, \bar{T}$ satisfy (25) for all $(r,x) \in \overline{\mathcal{D}}_\mathbb{C}$. If $\bar{S}$ satisfies (13)–(14), then $\bar{R}, \bar{T}$ satisfy (37)–(40). Alternatively, if $\bar{R}, \bar{T} \in C^{1,2}(\mathcal{D}_2;\mathbb{R}) \cap C(\overline{\mathcal{D}}_2;\mathbb{R})$ satisfy (37)–(40), and $\bar{S} \in \mathcal{S}_\mathbb{C}$ is given by (25), then $\bar{S}$ satisfies (13)–(14).

Proof: The first assertion follows by simple algebraic substitution, using the Cauchy-Riemann equations and (26)–(27). Now, suppose $\bar{R}, \bar{T} \in C^{1,2}(\mathcal{D}_2;\mathbb{R}) \cap C(\overline{\mathcal{D}}_2;\mathbb{R})$ satisfy (37)–(40), and let $\bar{S}$ be given by (25). We will show that $\bar{R}, \bar{T}$ satisfy the Cauchy-Riemann relations, and hence that $\bar{S} \in \mathcal{S}_\mathbb{C}$. After that, one may again use simple algebraic substitutions to verify the final assertion.

Differentiating (37) with respect to y_k, and (38) with respect to z_k, yields

$$\begin{aligned} \bar{R}_{t,y_k} &= \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{R}_{y_j,z_j,y_k} + \tfrac{1}{m}\sum_{j=1}^n \big(\bar{R}_{y_j}\bar{R}_{y_j,y_k} - \bar{R}_{z_j}\bar{R}_{z_j,y_k}\big) \\ &\quad + V_{y_k}^R, \end{aligned} \tag{41}$$

$$\begin{aligned} \bar{T}_{t,z_k} &= \tfrac{\hbar}{2m}\sum_{j=1}^n \bar{T}_{y_j,z_j,z_k} + \tfrac{1}{m}\sum_{j=1}^n \big(\bar{T}_{z_j}\bar{T}_{y_j,z_k} + \bar{T}_{y_j}\bar{T}_{z_j,z_k}\big) \\ &\quad + V_{z_k}^I. \end{aligned} \tag{42}$$

Applying the Cauchy-Riemann equations and (26)–(29) in (41) and (42), with a little work one finds

$$\bar{T}_{t,z_k} = \bar{R}_{t,y_k} \quad \forall\, (s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[\,. \tag{43}$$

Also note that as ϕ is holomorphic,

$$\bar{R}_{y_k}(t,y,z) = \phi_{y_k}^R(y,z) = \phi_{z_k}^I(y,z) = \bar{T}_{z_k}(t,y,z) \tag{44}$$

for all $y,z \in \mathbb{R}^n$. By the Fundamental Theorem of Calculus,

$$\bar{T}_{z_k}(s,y,z) = \bar{T}_{z_k}(t,y,z) - \int_s^t \bar{T}_{t,z_k}(\sigma,y,z)\,d\sigma,$$

which by (43),(44),

$$= \bar{R}_{y_k}(s,y,z) \quad \forall\, (s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[. \tag{45}$$

Similarly, one obtains

$$\bar{T}_{y_k}(s,y,z) = -\bar{R}_{z_k}(s,y,z) \quad \forall\, (s,y,z) \in \mathcal{D}_2,\ \forall k \in]1,n[. \tag{46}$$

By (45),(46), the Cauchy-Riemann conditions are satisfied. ■

IV. The verification

We will obtain a verification result demonstrating that a solution of (7)–(8) is the stationary value of the expectation of the action functional on process paths satisfying (16).

For $s \in (0,t)$ and $\hbar \in (0,1]$, we define payoff $J^\hbar(s,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{U}_s \to \mathbb{C}$ by

$$J^\hbar(s,x,u) \doteq \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 - V(\xi_r)\,dr + \phi(\xi_t) \right\}, \quad (47)$$

where ξ satisfies (16) with input $u \in \mathcal{U}_s$ and initial state $x \in \mathbb{R}^n$. The stationary value, $S^\hbar : \mathcal{D} \to \mathbb{C}$, is given by

$$S^\hbar(s,x) \doteq \operatorname*{stat}_{u\in\mathcal{U}_s} J^\hbar(s,x,u) \quad \forall\,(s,x) \in \mathcal{D}. \quad (48)$$

We assume throughout Section IV that

$\operatorname{argstat}_{u\in\mathcal{U}_s} J^\hbar(s,x,u)$ is single-valued for all $(s,x) \in \mathcal{D}$. $\quad (A.3)$

This is the last assumption. We remark that one may want to weaken this assumption to uniqueness in some prespecified subset of $\mathcal{D}$, but leave that additional complication to a later effort. The main result of the section is:

Theorem 6: Let $\hbar \in (0,1]$. Suppose $\bar{S} \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (13)–(14), and there exists $\widehat{C}_S < \infty$ such that $|\bar{S}_{xxx}(r,x)|, |\bar{S}_{txx}(r,x)|, |\bar{S}_{xxxx}(r,x)| \le \widehat{C}_S(1+|x|^{2q})$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$. Then, $\bar{S}(s,x) = S^\hbar(s,x)$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$.

The proof of Theorem 6 follows a somewhat similar path as that in the stationary-action dynamic programming equation results of [21], [23]. However, the stochastic and complex-valued aspects of the problem at hand introduce substantial difficulties not present in those results. We begin with two lemmas. The proofs are technical but relatively straightforward, and are not included.

Lemma 7: Let $s \in [0,t)$, $x \in \mathbb{C}^n$, $\hbar \in (0,1]$ and $u \in \mathcal{U}_s$. Let $\xi \in \mathcal{X}_s$ be given by (16). Suppose $\bar{S} \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (13)–(14). Let $\bar{u}^* = \bar{u}^{*,(s,x)}$, $\bar{\xi}^* = \bar{\xi}^{*,(s,x)}$ be given by (17)-(18). Then,

$$\bar{S}(s,x) = \mathbb{E}\bigg\{ \int_s^t -\bar{S}_t(r,\xi_r) - \bar{S}_x^T(r,\xi_r)u_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\xi_r)\,dr + \phi(\xi_t) \bigg\}$$

and

$$\bar{S}(s,x) = \mathbb{E}\bigg\{ \int_s^t -\bar{S}_t(r,\bar{\xi}^*_r) - \bar{S}_x^T(r,\bar{\xi}^*_r)\bar{u}^*_r - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\bar{\xi}^*_r)\,dr + \phi(\bar{\xi}^*_t) \bigg\}.$$

Lemma 8: Let $\hbar \in (0,1]$, and suppose that $\bar{S} \in \mathcal{S}^p_{\mathbb{C}}$ satisfies (13)–(14). Then, $\bar{S}(s,x) = J^\hbar(s,x,\bar{u}^{*,(s,x)})$ for all $(s,x) \in \mathcal{D}_{\mathbb{C}}$, where $\bar{u}^{*,(s,x)}$ is given by (17)-(18) with $\bar{S}$ in place of S.

Note that Lemma 7 only implies that the solution of HJ PDE problem (13)–(14) satisfies a specific complex-valued version of Itô's formula. After that, Lemma 8 shows that the solution of (13)–(14) is the payoff under the control that is asserted to be the stationary control. It remains to show that this control does indeed achieve an $\operatorname{argstat}$ of the payoff. That last step is accomplished in the remaining proof of Theorem 6. Only an outline of the long, rather technical, proof is included.

Proof: [outline of the proof of Theorem 6.] Fix $(s,x) \in \mathcal{D}_{\mathbb{C}}$. Let $L(x,v) \doteq \frac{m}{2}|v|_c^2 - V(x)$ for all $x,v \in \mathbb{C}^n$. For compactness of nontation, let $\bar{\xi}^* = \bar{\xi}^{*,(s,x)}$ and $\bar{u}^* = \bar{u}^{*,(s,x)}$. By Lemma 8,

$$\bar{S}(s,x) = \mathbb{E}\Big\{ \int_s^t L(\bar{\xi}^*_r,\bar{u}^*_r)\,dr + \phi(\bar{\xi}^*_t) \Big\} = J^\hbar(s,x,\bar{u}^*). \quad (49)$$

It remains to be shown the $\bar{u}^*$ is the $\operatorname{argstat}$ over $\mathcal{U}_s$ of $J^\hbar(s,x,\cdot)$.

Let $u \in \mathcal{U}_s$ and $\delta \doteq u - \bar{u}^* \in \mathcal{U}_s$. Let $\xi \in \mathcal{X}_s$ be the trajectory generated by u, i.e., the solution of (16), and let $\Delta \doteq \xi - \bar{\xi}^* \in \mathcal{X}_s$, where we note that $\Delta_r = \int_s^r \delta_\rho\,d\rho$ for all $(r,\omega) \in [s,t] \times \Omega$. By (49),

$$\begin{aligned} J^\hbar(s,x,\bar{u}^*) &= \bar{S}(s,x) = \mathbb{E}\{\bar{S}(t,\xi_t)\} \\ &\quad + \big[\bar{S}(s,x) - \mathbb{E}\{\bar{S}(t,\xi_t)\}\big], \end{aligned}$$

which by Lemma 7 and (14),

$$\begin{aligned} &= \mathbb{E}\{\phi(\xi_t)\} + \mathbb{E}\Big\{ \int_s^t L(\xi_r,u_r)\,dr \Big\} \\ &\quad + \mathbb{E}\bigg\{ \int_s^t -L(\xi_r,u_r)\,dr - \bar{S}_t(r,\xi_r) - \bar{S}_x^T(r,\xi_r)u_r \\ &\qquad - \tfrac{i\hbar}{2m}\Delta\bar{S}(r,\xi_r)\,dr \bigg\}. \end{aligned} \quad (50)$$

Using (13) and the Cauchy-Riemann equations, (50) yields

$$\begin{aligned} &\big|J^\hbar(s,x,\bar{u}^{*,(s,x)}) - J^\hbar(s,x,u)\big| \\ &\le \mathbb{E}\bigg\{ \int_s^t \Big| L(\bar{\xi}^*_r,\bar{u}^*_r) - L(\xi_r,u_r) + \bar{S}_t(r,\bar{\xi}^*_r) - \bar{S}_t(r,\xi_r) \\ &\qquad + \bar{S}_x^T(r,\bar{\xi}^*_r)\bar{u}^*_r - \bar{S}_x^T(r,\xi_r)u_r \\ &\qquad + \tfrac{i\hbar}{2m}[\Delta\bar{S}(r,\bar{\xi}^*_r) - \Delta\bar{S}(r,\xi_r)]\Big|\,dr \bigg\} \\ &\doteq \mathbb{E}\Big\{ \int_s^t \big|\Xi_r(\omega)\big|\,dr \Big\}. \end{aligned} \quad (51)$$

It remains to show that

$$\big|J^\hbar(s,x,\bar{u}^{*,(s,x)}) - J^\hbar(s,x,u)\big| \le \overline{\overline{C}}\|\delta\|^2_{\mathcal{U}_s},$$

for an appropriate $\overline{\overline{C}} < \infty$ on a sufficiently small ball.

Using Taylor's theorem and the assumed bounds on derivatives, one finds

$$\begin{aligned} \big|\Xi_r\big| &\le \Big| - V_x(\bar{\xi}^*_r)\Delta_r + m\bar{u}^*_r\delta_r + \bar{S}_{tx}(r,\bar{\xi}^*_r)\Delta_r \\ &\quad + \bar{S}_{xx}(r,\bar{\xi}^*_r)\bar{u}^*_r\Delta_r + \bar{S}_x(r,\bar{\xi}^*_r)\delta_r + \big[\bar{S}_x(r,\xi_r)u_r \\ &\quad - \bar{S}_x(r,\bar{\xi}^*_r)\bar{u}^*_r - \bar{S}_{xx}(r,\bar{\xi}^*_r)\bar{u}^*_r\Delta_r - \bar{S}_x(r,\bar{\xi}^*_r)\delta_r\big] \\ &\quad + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x(r,\bar{\xi}^*_r)\Delta_r \Big| + K_1\big(1+|\xi_r|^{2q}+|\bar{\xi}^*_r|^{2q}\big)|\Delta_r|^2 \\ &\quad + m|\delta_r|^2 \quad \forall\,(r,\omega) \in (s,t)\times\Omega, \end{aligned} \quad (52)$$

for appropriate $K_1 = K_1(C_0, \widehat{C}_S, \hbar, m) < \infty$.

Next, using Lemma 1, one shows

$$m\bar{u}_r^* + \bar{S}_x(r, \bar{\xi}^*) = 0. \tag{53}$$

Next, note that

$$\begin{aligned} &- V_x(\bar{\xi}_r^*) + \bar{S}_{tx}(r, \bar{\xi}^*) + \bar{S}_{xx}(r, \bar{\xi}_r^*)\bar{u}_r^* + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x(r, \bar{\xi}^*) \\ &= \frac{\partial}{\partial x}\Big[- V(x) + \bar{S}_t(r, x) + \bar{S}_x(r, x)v \\ &\qquad + \tfrac{i\hbar}{2m}\bar{S}_{xx}(r, x)\Big]\Big|_{x=\bar{\xi}_x^*,\, v=\hat{u}(r,\bar{\xi}_r^*)} \end{aligned} \tag{54}$$

where the partial derivative notation indicates that the derivative is taken only over explicitly appearing arguments. Recalling definitions (18), one has

$$\hat{\hat{u}}(r, x) = \tfrac{-1}{m}\bar{S}_x(r, x) = \operatorname*{argstat}_{v\in\mathbb{C}^n} \big[L(x, v) + \bar{S}_x(r, x)v\big], \tag{55}$$

for all $(r, x) \in \mathcal{D}_{\mathbb{C}}$. Working with (54), and applying (55), one finds

$$\begin{aligned} &- V_x(\bar{\xi}_r^*) + \bar{S}_{tx}(r, \bar{\xi}^*) + \bar{S}_{xx}(r, \bar{\xi}_r^*)\bar{u}_r^* + \tfrac{i\hbar}{2m}(\Delta\bar{S})_x(r, \bar{\xi}^*) \\ &= 0. \end{aligned} \tag{56}$$

Substituting (53) and (56) into (52), we have

$$\begin{aligned} \big|\Xi_r\big| \le &\Big|\bar{S}_x(r, \xi_r)u_r - \bar{S}_x(r, \bar{\xi}_r^*)\bar{u}_r^* - \bar{S}_{xx}(r, \bar{\xi}_r^*)\bar{u}_r^*\Delta_r \\ &- \bar{S}_x(r, \bar{\xi}_r^*)\delta_r\Big| + K_1\big(1 + |\xi_r|^{2q} + |\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2 \\ &+ m|\delta_r|^2 \quad \forall\, (r, \omega) \in (s, t) \times \Omega, \end{aligned}$$

which implies

$$\begin{aligned} &\big|J^\hbar(s, x, \bar{u}^{*,(s,x)}) - J^\hbar(s, x, u)\big| = \mathbb{E}\int_s^t \big|\Xi_r\big|\, dr \\ &\le m\|\delta\|_{\mathcal{U}_s}^2 + K_1\mathbb{E}\int_s^t \big(1 + |\xi_r|^{2q} + |\bar{\xi}_r^*|^{2q}\big)|\Delta_r|^2\, dr \\ &+ \mathbb{E}\int_s^t \Big|\bar{S}_x(r, \xi_r)u_r - \bar{S}_x(r, \bar{\xi}_r^*)\bar{u}_r^* - \bar{S}_{xx}(r, \bar{\xi}_r^*)\bar{u}_r^*\Delta_r \\ &- \bar{S}_x(r, \bar{\xi}_r^*)\delta_r\Big|\, dr. \end{aligned}$$

Then, working with the assumptions, and using Taylor's theorem, the definition of $\bar{u}_r^*$ and various Hölder inequalities, one eventually obtains

$$\big|J^\hbar(s, x, \bar{u}^{*,(s,x)}) - J^\hbar(s, x, u)\big| \le \overline{\overline{C}}\|\delta\|_{\mathcal{U}_s}^2,$$

for $\|\delta\|_{\mathcal{U}_s} \le 1$ and appropriate choice of $\overline{\overline{C}} = \overline{\overline{C}}(t, x, C_0, C_s, \widehat{C}_s, q) < \infty$. By definition, this implies that $\bar{u}^* = \mathrm{argstat}_{u\in\mathcal{U}_s} J^\hbar(s, x, u)$, where uniqueness of the argstat is guaranteed by Assumption $(A.3)$. ■

It may be worth noting the following, which reflects the uniqueness implied by the above representation.

Corollary 9: In addition to $(A.0)$–$(A.3)$, assume the conditions of Theorem 6. There exists a unique solution $\tilde{S} \in \mathcal{S}_{\mathbb{C}}^p$ to (13)–(14), where $\tilde{S} = S^\hbar$. There also exists a solution, $\hat{S} \in \mathcal{S}$, to (7)–(8), given by $\hat{S}(r, y) = S^\hbar(r, \mathcal{V}_0^{-1}((y^T, 0)^T))$ for all $(r, y) \in \overline{\mathcal{D}}$. Lastly, any other solution in $\mathcal{S}$ to (7)–(8) cannot be extended holomorphically to a solution of (13)–(14) in $\mathcal{S}_{\mathbb{C}}^p$.

Remark 10: The results concerning $\hat{S}$ in Corollary 9 also extend to (1)–(2) and (3)–(4) in the obvious ways.

REFERENCES

[1] R. Azencott and H. Doss, "L'équation de Schrödinger quand $\hbar$ tend vers zéro: une approche probabiliste", Stochastic Aspects of Classical and Quantum Systems, Lecture Notes in Math., 1109 (1985), 1–17.

[2] H. Doss, "On a probabilistic approach to the Schrödinger equation with a time-dependent potential", J. Functional Analysis, 260 (2011), 1824–1835.

[3] H. Doss, "Sur une resolution stochastique de l'equation de Schrödinger à coefficients analytiques", Comm. Math, Phys., 73 (1980), 247–264.

[4] P.M. Dower and W.M. McEneaney, "A fundamental solution for an infinite dimensional two-point boundary value problem via the principle of stationary action", Proc. 2013 Australian Control Conf., 270–275.

[5] S. Fang and T. Zhang, "A study of a class of stochastic differential equations with non-Lipschitzian coefficients", Prob. Theory and Related Fields, 132 (2005), 356–390.

[6] R.P. Feynman, "Space-time approach to non-relativistic quantum mechanics", Rev. of Mod. Phys., 20 (1948), 367–387.

[7] R.P. Feynman, *The Feynman Lectures on Physics, Vol. 2,* Basic Books, (1964) 19-1–19-14.

[8] W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions, Second Ed.,* Springer, New York, 2006.

[9] W.H. Fleming and R.W. Rishel, *Deterministic and Stochastic Optimal Control,* Springer, New York, 1982.

[10] W.H. Fleming, "Stochastic calculus of variations and mechanics", J. Optim. Theory and Applics., 41 (1983), 55–74.

[11] H. Grabert, P. Hänggi and P. Talkner, "Is quantum mechanics equivalent to a classical stochastic process?", Phys. Rev. A, 19 (1979), 2440–2445.

[12] K. Gürlebeck, K. Habetha and W. Spröβig, *Holomorphic Functions in the Plane and n-dimensional Space,* Birkhäuser, Basel, 2008.

[13] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes,* Sec. Ed., North-Holland, 1989.

[14] V.N. Kolokoltsov, *Semiclassical Analysis for Diffusions and Stochastic Processes,* Springer Lec. Notes in Math., 1724, Springer, Berlin, 2000.

[15] A.J. Krener, "Reciprocal diffusions in flat space", Prob. Theory and Related Fields, 107 (1997), 243–281.

[16] N.V. Krylov, *Controlled Diffusion Processes,* Springer, New York, 1980.

[17] Y.-J. Lee and K.-G. Yen, "Analysis of complex Brownian motion", Comm. Stochastic Analysis, 2 (2008), 97–107.

[18] G.L. Litvinov, "The Maslov dequantization, idempotent and tropical mathematics: A brief introduction", J. Math. Sciences, 140 (2007), 426–444.

[19] V.P. Maslov. "A new superposition principle for optimization problem", Russian Math. Surveys [translation of Uspekhi Mat. Nauk], 42 (1987), 39–48.

[20] W.M. McEneaney and P.M. Dower, "Static duality and a stationary-action application", J. Diffl. Eqs. (submitted).

[21] W.M. McEneaney and P.M. Dower, "Staticization, its dynamic program and solution propagation", Automatica (to appear).

[22] W.M. McEneaney and P.M. Dower, "The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems", SIAM J. Control and Optim., 53 (2015), 2898–2933.

[23] W.M. McEneaney and P.M. Dower, "Staticization and associated Hamilton-Jacobi and Riccati equations", Proc. SIAM Conf. on Control and Its Applics., (2015), 376–383.

[24] M. Nagasawa, *Schrödinger Equations and Diffusion Theory,* Birkhäuser, Basel, 1993.

[25] E. Nelson, "Derivation of the Schrödinger equation from Newtonian mechanics", Phys. Rev., 150 (1966), 1079–1085.

[26] C. Perret, "The Stability of Numerical Simulations of Complex Stochastic Differential Equations", ETH, Zurich, 2010.

[27] P.E. Protter, *Stochastic Integration and Differential Equations,* Sec. Ed., Springer-Verlag, Berlin, 2005.

[28] J.C. Zambrini, "Probability in quantum mechanics according to E. Schrödinger", Physica B+C, 151 (1988), 327–331.