AN IDEMPOTENT ALGORITHM FOR A CLASS OF NETWORK-DISRUPTION GAMES

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A game is considered where the communication network of the first player is explicitly modeled. The second player may induce delays in this network, while the first player may counteract such actions. Costs are modeled through expectations over idempotent probability measures. The idempotent probabilities are conditioned by observational data, the arrival of which may have been delayed along the communication network. This induces a game where the state space consists of the network delays. Even for small networks, the state-space dimension is high. Idempotent algebra-based methods are used to generate an algorithm not subject to the curse-of-dimensionality. An example is included.

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Classification: 15A80, 49L20, 90C35, 91A80, 14T05

1. INTRODUCTION

In recent years, algorithms based on idempotent algebras, most notably the max-plus algebra, have been demonstrated to be quite efficient for solution of classes of nonlinear control problems, [2, 9, 14, 15, 22]. Algebras such as the max-plus and min-plus semifields are the natural structures for the modeling of certain classes of network-traffic systems, cf. [10]. Most recently, it has also been seen that idempotent algebras are appropriate not only for solution of deterministic optimal control problems, but also for stochastic control problems and deterministic games, [18, 16, 13].

Here, we consider a game between two players, where we specifically model the flow of information along the communication network of the first player. The state will consist of the delays in information flow along this network. These delays will affect the ability of the first player to make optimal decisions regarding physical actions. An idempotent-algebra-based numerical method will be developed for solution of the game. The method will be in the class of idempotent curse-of-dimenmsionality-free algorithms. Note that dynamic programming methods are applicable to solution of a tremendous variety of problems in deterministic and stochastic control and games. However, they are subject to a computational cost which grows exponentially fast in the dimension of the state space – thus the famous "curse-of-dimensionality". The curse-of-dimenmsionality-free algorithms have costs which grow only at a cubic rate in space dimension, but are subject to other complexity growth problems which are attenuated by optimal idempotent projection, which is known to take the form of a pruning operation ([8] and the references therein). These methods have been demonstrated to have exceptionally low computational cost for high-dimensional nonlinear control problems ([9, 22] and the references therein).

This paper has a somewhat complex structure. In Section 2, the class of games of interest is described in more detail. In Section 3, some mathematical structures are recalled. The development of the game model begins in Section 4. This section is subdivided into several subsections. First, the model for controlled dynamics of delay on the network is presented in Subsection 4.1. In Subsection 4.2, the model of the running cost is developed. This requires that one first examine how a lack of information affects decisions, and ultimately, the resulting costs of actions in the physical domain. Throughout, we will distinguish between what will be called *physical actions* (e.g., movement of troops in the example game of Section 2) and the controls/dynamics in the network delay game. The running cost for the latter flows from the potentially suboptimal choice of physical actions, where we describe this as the value of information. Subsection 4.2 is a technical, but necessary precursor to study of the network-delay game. In Subsection 4.3, the payoff and value models for the network-delay game are given. Then, in Section 5, the algorithm for solution of the game is developed. As in related efforts [18, 16, 13], the algorithm is referred to as an "idempotent distributed dynamic program", where this indicates that an idempotent distributive property is used to convert the dynamic program into a curse-of-dimensionality-free form. In Section 6, computational efficiency and complexity bounds are considered in more detail. Lastly, in Section 7, an example application is discussed.

2. OVERALL GAME

In order to motivate the mathematical development to follow, it is necessary to describe the class of games to be studied. We find it helpful to use a specific example. Suppose there are two players, Blue and Red. We will be concerned chiefly with attack and defense of the Blue communication network, where these attacks will delay the flow of information along that network. The resulting costs will be determined by the effects of those delays on actions which will take place in a physical conflict between the players. We use a military example for illustrative purposes. Refer to Figures 1 and 2. Figure 1 depicts a Blue communication network. There are three subregions served by this network, and these are indicated by the circled areas. In those areas, the blue and red triangles denote physical Blue and Red entities, respectively. The blue crosses denote Blue observation assets. One may imagine that each of the subregions could correspond to a physical conflict such as that depicted in Figure 2. In the figure, one can see that the Blue observation assets may generate information on the activities of the physical Red entities, which may be helpful to the physical Blue entities in the subregion. The observational information must flow along some specified subnetwork of that indicated in Figure 1, with possible processing along the way, before the processed information and/or resulting commands can be delivered to the physical Blue entities in the subregion. We will use a max-plus stochastic model for the estimation of the physical Red state by Blue, where this is equivalent to a zero-sum game model while being a more useful form for

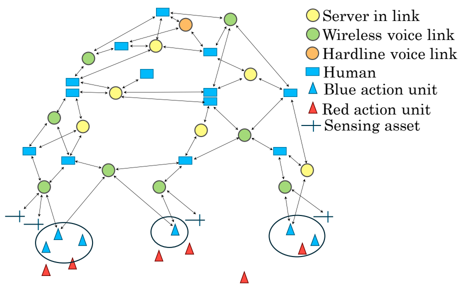

Fig. 1. Blue network with 3 physical regions.

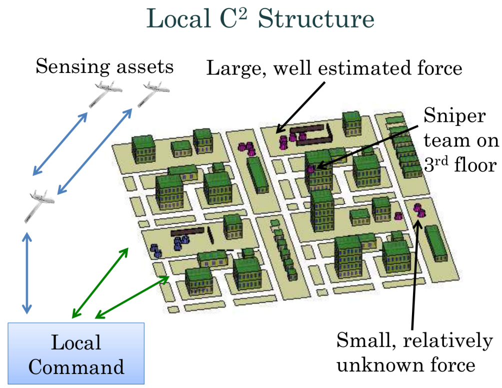

Fig. 2. Example physical region.

the construction of the computational method. This model will be used to generate the costs associated with the delay of information to the physical Blue entities. The larger game of interest here is that played on the space of delays along the Blue communication network, and the above costs will be used to define the payoffs in that game. *Note that we must first define the costs, themselves a result of a max-plus stochastic optimization problem, before they can be used to define the costs in the overall network-delay game.*

3. MATHEMATICAL PRELIMINARIES

Prior to development of the game model, we recall and define relevant mathematical objects. Specifically, we introduce the idempotent algebras, provide an overview of the max-plus probability structure, and introduce some standard results related to min-plus convex functions. These will prove useful in the development of our algorithm. Classical references on idempotent algebras (max-plus, min-plus or tropical, and min-max) include [3, 10, 11, 12, 14] among others.

3.1. Idempotent Algebras

The min-plus algebra is given by

$$a \oplus b \doteq \min\{a, b\}, \qquad a \otimes b \doteq a + b,$$

operating on $\mathbb{R}^+ \doteq \mathbb{R} \cup \{+\infty\}$. The additive and multiplicative identities are $\epsilon_\oplus = \infty$ and $\epsilon_\otimes = 0$, respectively. The max-plus algebra is given by

$$a \oplus^\vee b \doteq \max\{a, b\}, \qquad a \otimes^\vee b \doteq a + b,$$

operating on $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$. The additive and multiplicative identities are $\epsilon_{\oplus^\vee} = -\infty$ and $\epsilon_{\otimes^\vee} = 0$, respectively. The addition operator is idempotent in both the max-plus and the min-plus algebras. Namely, $a \oplus a = a$ and $a \oplus^\vee a = a$. We observe that both the max-plus and min-plus algebras define idempotent, commutative semifields.

In the max-min algebra, the operations are defined as

$$a \vee b \doteq \max\{a, b\}, \qquad a \wedge b \doteq \min\{a, b\},$$

operating on $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. The additive and multiplicative identities are $\epsilon_\vee = -\infty$ and $\epsilon_\wedge = \infty$, respectively. The max-min algebra defines an idempotent, commutative semiring [10]. Lastly, it will be helpful to also define the min-max algebra with these same operations, but with the $\wedge$ operation formally taking the role of addition and the $\vee$ operation formally taking the role of multiplication.

Regarding notation, we let $\bigoplus_{v\in\mathcal{V}}$ and $\bigotimes_{v\in\mathcal{V}}$ denote a min-plus sum and product over index set $\mathcal{V}$, with analogous notation for the other algebras. Also, in the interests of space when two objects are each indexed by the same variable that takes values in a finite set, we will use dot product notation as follows. For $a = \{a_v \,|\, v \in \mathcal{V}\}$ and $b = \{b_v \,|\, v \in \mathcal{V}\}$, we let $a \odot b \doteq \bigoplus_{v\in\mathcal{V}} a_v \otimes b_v$ and $a \odot^\vee b \doteq \bigoplus^\vee_{v\in\mathcal{V}} a_v \otimes b_v$. Lastly, when an object is indexed by a single variable taking values in a finite set, unless otherwise specified this may be understood to be a column vector.

Note that the min-plus and max-plus algebras are equivalent under a change of sign (i.e., $-[(-a) \oplus (-b)] = a \oplus^\vee b$ and $-[(-a) \otimes (-b)] = a \otimes^\vee b$), and similarly for the max-min and min-max algebras. We specifically note that the distributive property holds for idempotent algebras. For arbitrary finite index sets the following is easily obtained, cf. [13].

Lemma 3.1. Let $\mathcal{Y}$ and $\mathcal{V}$ be sets of finite cardinality. Given any $\Phi : \mathcal{Y} \times \mathcal{V} \to \mathbb{R}$, we have

$$\bigotimes_{y\in\mathcal{Y}} \bigoplus_{v\in\mathcal{V}} [\Phi(y,v)] = \bigoplus_{\{v.\}\in\hat{\mathcal{V}}} \bigotimes_{y\in\mathcal{Y}} [\Phi(y,v_y)], \quad \bigotimes^\vee_{y\in\mathcal{Y}} \bigoplus^\vee_{v\in\mathcal{V}} [\Phi(y,v)] = \bigoplus^\vee_{\{v.\}\in\hat{\mathcal{V}}} \bigotimes^\vee_{y\in\mathcal{Y}} [\Phi(y,v_y)],$$

$$\bigvee_{y\in\mathcal{Y}} \bigwedge_{v\in\mathcal{V}} [\Phi(y,v)] = \bigwedge_{\{v.\}\in\hat{\mathcal{V}}} \bigvee_{y\in\mathcal{Y}} [\Phi(y,v_y)] \text{ and } \bigwedge_{y\in\mathcal{Y}} \bigvee_{v\in\mathcal{V}} [\Phi(y,v)] = \bigvee_{\{v.\}\in\hat{\mathcal{V}}} \bigwedge_{y\in\mathcal{Y}} [\Phi(y,v_y)],$$

where $\hat{\mathcal{V}}$ denotes the set of sequences of elements of $\mathcal{V}$ indexed by $y \in \mathcal{Y}$.

We note that the subscript dot notation in $\{v.\} \in \hat{\mathcal{V}}$ used in the lemma is included as a reminder of the fact that each $v = v.$ is a function of a subscripted argument. It is also worth noting here that Φ may be instantiated as a matrix with $\#\mathcal{Y}$ rows and $\#\mathcal{V}$ columns, or vice-versa.

3.2. A Review of Max-plus Probability

One may define a probability measure with respect to the max-plus algebra [1, 7, 19, 21]. Let the pair $(\Omega, \mathcal{F})$ denote a sample space and associated sigma-algebra of sets on that space.

Definition 3.2. $p^{\oplus^\vee}$ is a max-plus probability measure on $(\Omega, \mathcal{F})$ if:

$$p^{\oplus^\vee}(E) \in [-\infty, 0] \;\; \forall E \in \mathcal{F}, \qquad p^{\oplus^\vee}(\Omega) = 0,$$

and for any countable collection of disjoint sets, $\{E_i\}$,

$$p^{\oplus^\vee}\Big(\bigcup E_i\Big) = \bigoplus_i{}^\vee p^{\oplus^\vee}(E_i).$$

We recall that $[-\infty, 0]$ in the max-plus algebra is analogous to $[0, 1]$ in the standard field, and consequently, max-plus probabilities take values between the max-plus additive and multiplicative identities, in analogy with standard-field probabilities.

Suppose $\mathcal{X}$ is a finite set, and let $X \doteq \#\mathcal{X}$. The max-plus probability simplex over $\mathcal{X}$ is

$$[S^{\oplus^\vee}]^X \doteq \Big\{ q \in [-\infty, 0]^X \Big| \bigoplus_{x\in\mathcal{X}}{}^\vee q_x = 0 \Big\},$$

where $[-\infty, 0]$ denotes $(-\infty, 0] \cup \{-\infty\}$ and the X superscript denotes outer product X times. Note that a vector consisting of the max-plus probabilities of the elements of $\mathcal{X}$ must lie in $[S^{\oplus^\vee}]^X$. For generic max-plus random variable, say $\widetilde{Z}$, taking values in $(\mathbb{R}^-)^X$, we define the expectation of $\widetilde{Z}$ as $\mathbf{E}^{\oplus^\vee}[\widetilde{Z}] \doteq \bigoplus^\vee_{x\in\mathcal{X}} \widetilde{Z}_x \otimes^\vee q_x$.

Remark 3.3. In standard-algebra probability, the probability of an event may be interpreted as the expected frequency of that event, although the interpretation is not required for the construction of the mathematics. In max-plus probability, one may interpret the probability as the additive-inverse of the relative cost to the opposing player. For example, the max-plus conditional probability of observation $y \in \mathcal{Y}$, given state $x \in \mathcal{X}$ may be interpreted as (the additive-inverse of) the relative cost to the opponent to cause us to observe y when the true state is x. This may be further interpreted as the cost of deception. These costs are taken to be relative in that given $x \in \mathcal{X}$, there exists $y_x \in \mathcal{Y}$ such that the conditional probability of y_x given x is $p^{\oplus^\vee}(y_x|x) = 0$ (the multiplicative identity). Similarly, the elements of max-plus Markov chain transition matrices may be viewed as costs in the game between ourselves and the opponent.

3.3. Min-Plus Convexity

For completeness of the presentation here, we recall some results from the theory of min-plus convexity [4, 8, 12, 20, 23, 24]. In order to proceed more quickly to the delay game model, where proofs are needed, they have been moved to the appendix.

Definition 3.4. A set $C \in (\mathbb{R}^+)^n$ is said to be min-plus convex if for all $x, y \in C$, $\alpha, \beta \in \mathbb{R}^+$, such that $\alpha \oplus \beta = 0$, we have $\alpha \otimes x \oplus \beta \otimes y \in C$.

Lemma 3.5. The intersection of a collection of min-plus convex sets is a min-plus convex set.

For $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$, the epigraph of f is $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^+ \,|\, y \geq f(x)\}$.

Definition 3.6. A function $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ is said to be min-plus convex if its epigraph is a min-plus convex set. We will let the set of min-plus convex functions over $\mathbb{R}^n$ be denoted by $\mathcal{S}(\mathbb{R}^n)$.

Definition 3.7. A function $f:\mathbb{R}^n\mapsto\overline{\mathbb{R}}$ is min-plus affine if it takes the form $f(x)=e\oplus b\odot x=e\oplus(b\odot x)$, with $e\in\overline{\mathbb{R}}$, $b\in\mathbb{R}^n$.

The following two results are also well-known (cf. [4, 12, 24]), and proofs are not included.

Proposition 3.8. A min-plus affine function is min-plus convex

Proposition 3.9. Let $\mathcal{Z}$ be a set which indexes a family $f_z(x)$ of min-plus convex functions. Then, $\bar{f}(x)\doteq\sup_{z\in\mathcal{Z}}f_z(x)$ is min-plus convex.

We employ the partial ordering on $\mathbb{R}^n$ given by

$$x\succeq y \text{ iff } x_i\geq y_i \text{ for each } i\doteq]1,\ldots,n[=\{1,2,\ldots,n\},$$

where throughout, we use the notation $]a,b[$ to denote $\{a,a+1,a+2,\ldots,b\}$ when $a\leq b$. Henceforth, we also let $\mathbb{R}^n$ be equipped with the norm $\|x\|_\infty\doteq\max_{i\in]1,n[}|x_i|$, and let $\mathcal{O}^n$ denote the closed first octant, i.e., $\mathcal{O}^n\doteq\{x\in\mathbb{R}^n|x\succeq 0\}$. Let

$$\mathcal{S}^1(\mathbb{R}^n)\doteq\{f:\mathbb{R}^n\mapsto\overline{\mathbb{R}}\ \mid\ 0\leq f(x+\delta)-f(x)\leq\|\delta\|_\infty,\forall x\in\mathbb{R}^n,\delta\succeq 0\}.$$

By a reversal of signs, the next two results follow immediately from their equivalent results for max-plus hypo-convex functions and the min-max algebra [8]. In particular, a function, f, is min-plus convex if and only if $-f$ is max-plus hypo-convex. Consequently, proofs are not included.

Theorem 3.10. $\mathcal{S}(\mathbb{R}^n)=\mathcal{S}^1(\mathbb{R}^n)$.

Theorem 3.11. There exist countable sets, $\mathcal{Z}$ and $\{b^z\in\mathbb{R}^n\,|\,z\in\mathcal{Z}\}$, such that for any $f\in\mathcal{S}(\mathbb{R}^n)$, there exists $\{e^z\in\overline{\mathbb{R}}\,|\,z\in\mathcal{Z}\}$ such that

$$f(x)=\bigvee_{z\in\mathcal{Z}}[e^z\oplus b^z\odot x]\doteq\sup_{z\in\mathcal{Z}}[e^z\oplus b^z\odot x]\quad\forall\, x\in\mathbb{R}^n. \tag{1}$$

Remark 3.12. Note that (1) can also be expressed in the form $f(x)=\bigvee_{z\in\mathcal{Z}}e^z\wedge(b^z\odot x)$. In this form, one can see that the min-plus linear functionals form a max-min basis for $\mathcal{S}(\mathbb{R}^n)$.

We can relax the Lipschitz requirement of Theorem 3.11 (implicit in the $\mathcal{S}^1(\mathbb{R}^n)$ representation for $\mathcal{S}(\mathbb{R}^n)$).

Definition 3.13. A function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ is generalized min-plus convex with coefficient $\hat{C}$, i.e., $f\in\mathcal{S}_{\hat{C}}(\mathbb{R}^n)$, if $\hat{C}\in\mathbb{R}^{n\times n}$ is positive-definite, symmetric and

$$0\leq f(x+\delta)-f(x)\leq\|\hat{C}\delta\|_\infty\quad\forall\, x\in\mathbb{R}^n,\ \delta\succeq 0.$$

We may now generalize Theorem 3.11 as follows. A proof is given in the appendix.

Corollary 3.14. There exist countable sets, $\mathcal{Z}$ and $\{b^z \in \mathbb{R}^n \,|\, z \in \mathcal{Z}\}$, such that given any $f \in \mathcal{S}_{\hat{C}}(\mathbb{R}^n)$, there exists $\{e^z \in \overline{\mathbb{R}} \,|\, z \in \mathcal{Z}\}$ such that

$$f(x) = \bigvee_{z\in\mathcal{Z}} \left[e^z \oplus b^z \odot (\hat{C}x)\right] \quad \forall\, x \in \mathbb{R}^n. \tag{2}$$

We will find it helpful to introduce the following definitions.

Definition 3.15. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be a finite-complexity min-plus convex function if it has representation (1), where $\mathcal{Z}$ has finite cardinality. A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ of the form of (1) but with domain $\mathcal{X} \subset \mathbb{R}^n$ where $\#\mathcal{X} < \infty$ is a finite-domain min-plus convex function.

A space is referred to as a max-min vector space [14] (otherwise referred to as a moduloid [3] or an idempotent semimodule [4, 12]) if the standard conditions as specified in [3, p. 108] are satisfied. Again, by reversal of signs, the next result follows exactly as the equivalent result for max-plus hypo-convex functions and the min-max algebra in [8].

Theorem 3.16. $\mathcal{S}(\mathbb{R}^n)$ is a max-min vector space.

The value of Theorem 3.16 is that it guarantees that max-min linear combinations of functions in $\mathcal{S}(\mathbb{R}^n)$ remain in $\mathcal{S}(\mathbb{R}^n)$. As the following is easily obtained, a proof is not included.

Theorem 3.17. Spaces of finite-complexity min-plus convex functions and spaces of finite-domain min-plus convex functions are max-min vector spaces.

The next result also follows from the equivalent result for max-plus hypo-convex functions, [8].

Lemma 3.18. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then, for any $\bar{x} \in \mathbb{R}^n$, $f(x) \geq f(\bar{x}) \oplus [(f(\bar{x}) \otimes (-\bar{x})) \odot x]$ for all $x \in \mathbb{R}^n$.

As would be expected, when the domain has finite cardinality, one should have a finite-complexity representation; this is guaranteed by the next result. A proof is included in the appendix.

Theorem 3.19. Let $f : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}^n$ and $\#\mathcal{X} < \infty$, and suppose $0 \leq f(x+\delta) - f(x) \leq \|\delta\|_\infty$ for all $x, x+\delta \in \mathcal{X}$ such that $\delta \succeq 0$. Then, it has representation (1) where $\#\mathcal{Z} \leq \#\mathcal{X}$.

The next corollary follows from Theorem 3.19 exactly as Corollary 3.14 followed from Theorem 3.11, and a proof is not included.

Corollary 3.20. Let $f : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X} \subset \mathbb{R}^n$ and $\#\mathcal{X} < \infty$, and suppose there exists positive-definite, symmetric $\hat{C}$ such that $0 \leq f(x+\delta) - f(x) \leq \|\hat{C}\delta\|_\infty$ for all $x, x+\delta \in \mathcal{X}$ such that $\delta \succeq 0$. Then, it has representation (2) where $\#\mathcal{Z} \leq \#\mathcal{X}$.

The results presented above will be critical to the algorithm we will obtain for solution of the delay game. In particular, the value function will lie in $\mathcal{S}(\mathbb{R}^n)$ for appropriate n, and the operations needed to propagate this value via an idempotent distributed dynamic program will involve taking max-min linear combinations of functions in $\mathcal{S}(\mathbb{R}^n)$.

4. PROBLEM DEFINITION

We now begin the mathematical construction of our network-disruption game problem. Recall that we are modeling delays on the Blue communication network, where Red would be expected to attempt to increase the delays, while Blue would attempt to alleviate such increases. The state of the game will be the set of delays of information at the nodes in the Blue network. We first develop the model for the dynamics of the delay state. The payoff for the game will be a max-plus sum over time (i.e., the worst-case-over-time) of the running cost. The motivation for this is that the physical Red entities in the example application of Section 2 could be expected to act at the worst-case time for Blue. The running cost will flow from the effects the delays have on the information-state available at the physical Blue entities, or Blue action nodes. The development of this running cost will be somewhat technical. We will be considering a zero-sum game, where Blue will be the minimizing player, and we will consider the upper value of the game, which corresponds to a worst-case analysis from the Blue perspective.

4.1. The Dynamics

We now introduce the framework for modeling the delays on the Blue network. We suppose that the Blue network will be defined as a finite undirected graph, $(\mathcal{G}, \mathcal{E})$, where $\mathcal{G}$ denotes the set of nodes, and $\mathcal{E}$ denotes the set of edges. Letting $G \doteq \#\mathcal{G}$, without loss of generality, we index the set of nodes as $\mathcal{G} =]1, G[\subset \mathbb{N}$. We index the set of edges, $\mathcal{E} \subseteq \mathcal{G} \times \mathcal{G}$, by $i \in]1, \#\mathcal{E}[$. Suppose the i^{th} edge connects nodes g and γ. We let $g_i^1 \doteq \min\{g, \gamma\}$ and $g_i^2 \doteq \max\{g, \gamma\}$. With this indexing scheme, we have $\mathcal{E} = \{(g_i^1, g_i^2) \subseteq \mathcal{G} \times \mathcal{G} \,|\, i \in]1, \#\mathcal{E}[\}$. We require $(g, g) \in \mathcal{E}$ for all $g \in \mathcal{G}$. The set of nodes will be decomposed as $\mathcal{G} = \mathcal{G}_s \cup \mathcal{G}_a \cup \mathcal{G}_c$ where $\mathcal{G}_s$ denotes the set of sensing nodes, $\mathcal{G}_a$ denotes the set of action nodes and $\mathcal{G}_c$ denotes the set of nodes at which communication and/or analysis and/or decision take place. Let $G_s = \#\mathcal{G}_s$, $G_a = \#\mathcal{G}_a$ and $G_c = \#\mathcal{G}_c$. We suppose that for each action node, say $\alpha \in \mathcal{G}_a$, there exists a set of relevant sensing nodes, $\hat{\mathcal{G}}_s(\alpha) \subseteq \mathcal{G}_s$ such that information from these sensing elements affects the min-plus probability distribution describing information relevant to action node α. For simplicity, we assume $\hat{\mathcal{G}}_s(\alpha_1) \cap \hat{\mathcal{G}}_s(\alpha_2) = \emptyset$ if $\alpha_1 \neq \alpha_2$.

At each time step, Red may act to increase the network delays, while Blue may act to counter this. We will use a fixed time-step model where, without loss of generality, we will let time be indexed by integer $k \in \{0, 1, 2, \dots\} \doteq \mathcal{I}_{\geq 0}$. For $k \in \mathcal{I}_{\geq 0}$, $g \in \mathcal{G}$ and $\sigma \in \mathcal{G}_s$, we let $d_{k,g}^{\sigma} \in \mathcal{I}_{\geq 0} \cup \{+\infty\}$ denote the delay in (possibly processed) information originating from sensor node σ at node g at time k. For each $g \in \mathcal{G}$, we let $\mathcal{N}_g \subseteq \mathcal{G}$ denote the set of neighboring nodes, that is, $\mathcal{N}_g \doteq \{\gamma \in \mathcal{G} \,|\, (g, \gamma) \in \mathcal{E} \text{ or } (\gamma, g) \in \mathcal{E}\}$. We suppose that under nominal operations, the delay in information as it passes from node γ to neighbor g increases by one time-step. As $\#\mathcal{N}_g$ may be greater than one, this would suggest a nominal delay dynamics model of the form

$$d_{k+1,g}^{\sigma} = \bigwedge_{\gamma \in \mathcal{N}_g} (d_{k,\gamma}^{\sigma} + 1). \tag{3}$$

However, we also expect that the dynamics may be affected by controls of the players. Let the Blue and Red control sets be $\mathcal{U}^b$ and $\mathcal{U}^r$, respectively, where $U^b \doteq \#\mathcal{U}^b < \infty$

and $U^r \doteq \#\mathcal{U}^r < \infty$. Control process values at time $k \in \mathcal{I}_{\geq 0}$ for Blue and Red will be denoted by $u_k^b \in \mathcal{U}^b$ and $u_k^r \in \mathcal{U}^r$, respectively. We suppose that the Red controls may include controls which completely block the flow of information on one or more links of the Blue network. There might also be Blue controls which can act to clear a backlog of information propagation along one or more links. The effect of any control pair, $(v^b, v^r) \in \mathcal{U}^b \times \mathcal{U}^r$, on the information-propagation process at node $g \in \mathcal{G} \setminus \mathcal{G}_s$ will be modeled through the use of a function, $f_g^p : \mathcal{U}^b \times \mathcal{U}^r \to]-1, \bar{J}[$, where $\bar{J}$ will denote the maximum possible delay-reduction in backlog that can occur in a single step at any node. We let

$$f_g^p(v^b, v^r) \doteq \begin{cases} -1 & \text{if node } g \text{ is blocked from receiving,} \\ 0 & \text{if node } g \text{ is receiving information at the nominal rate,} \\ J \in]1, \bar{J}[& \text{if information may flow into node } g \text{ sufficiently fast to} \\ & \text{reduce backlog by up to } J \text{ units per step.} \end{cases} \tag{4}$$

With the aid of this function, we let the delay dynamics at time k and node $g \in \mathcal{G} \setminus \mathcal{G}_s$ of information originating at sensor node $\sigma \in \mathcal{G}_s$ be given by $F_g^0 : \{d_{k,\gamma}^\sigma \,|\, \gamma \in \mathcal{N}_g\} \times \mathcal{U}^b \times \mathcal{U}^r \to \mathcal{I}_{\geq 0}$, where

$$d_{k+1,g}^\sigma = F^0(\{d_{k,\gamma}^\sigma \,|\, \gamma \in \mathcal{N}_g\}, u_k^b, u_k^r) \doteq \big[d_{k,g}^\sigma - f_g^p(u_k^b, u_k^r)\big] \vee \big[\bigwedge_{\gamma \in \mathcal{N}_g} d_{k,\gamma}^\sigma + 1\big]. \tag{5}$$

We include some clarifying remarks regarding the model of the dynamics. First, this is only one model, and other models maintaining this general form (roughly, taking a maximum of the minimum over a set of delays at nodes in $\mathcal{N}_g$ with an additional control-dependent amount) would fit the computational framework to appear below. Second, one may consider different control cases to obtain a heuristic sense of the dynamics. For example, if (u_k^b, u_k^r) induces "nominal flow" at g, then (5) should be equivalent to (3) there. Note that in this case, by (4), $f_g^p(u_k^b, u_k^r) = 0$, and (5) becomes

$$d_{k+1,g}^\sigma = d_{k,g}^\sigma \vee \big[\bigwedge_{\gamma \in \mathcal{N}_g} d_{k,\gamma}^\sigma + 1\big] = d_{k,g}^\sigma \vee \big[\bigwedge_{\gamma \in \mathcal{N}_g \setminus \{g\}} d_{k,\gamma}^\sigma + 1\big].$$

That is, the delay cannot decrease without backlog reduction control ($f_g^p(u_k^b, u_k^r) \in]1, \bar{J}[$), and the delay must also be at least the minimum over all neighbors (not including g itself) of these delays plus one. Similarly, in the case of the node g being blocked from receiving information, we have $f_g^p(u_k^b, u_k^r) = -1$, and (5) yields

$$d_{k+1,g}^\sigma = (d_{k,g}^\sigma + 1) \vee \big[\bigwedge_{\gamma \in \mathcal{N}_g} d_{k,\gamma}^\sigma + 1\big] \geq d_{k,g}^\sigma + 1,$$

and as $g \in \mathcal{N}_g$, we see $d_{k+1,g}^\sigma = d_{k,g}^\sigma + 1$.

The delay-state process will take values in $\mathcal{D} \doteq (I_{\geq 0})^{GG_s}$, and we let generic $\delta \in \mathcal{D}$ have components denoted as $\delta_g^\sigma \in I_{\geq 0}$ for $\sigma \in \mathcal{G}_s$, $g \in \mathcal{G}$. We let $d_{k,g} \doteq \{d_{k,g}^\sigma \,|\, \sigma \in \mathcal{G}_s\}$ and $D_k \doteq \{d_{k,g} \,|\, g \in \mathcal{G}\}$, where we note that D_k takes values in $\mathcal{D}$. That is, one may view elements of $\mathcal{D}$ as arrays indexed by $g \in \mathcal{G} \setminus \mathcal{G}_s$ and $\sigma \in \mathcal{G}_s$, where the (g, σ)-component of any D_k is $d^\sigma_{k,g}$. We let the collected dynamics of (5) be denoted by $\bar{F} : \mathcal{D}\times\mathcal{U}^b\times\mathcal{U}^r \to \mathcal{D}$, where given $D_k \in \mathcal{D}$, $u^b_k \in \mathcal{U}^b$ and $u^r_k \in \mathcal{U}^r$, and letting

$$D_{k+1} = \bar{F}(D_k, u^b_k, u^r_k), \tag{6}$$

the (g,σ)-component of D_{k+1} is $d^\sigma_{k+1,g}$.

4.2. The Running Cost

In order to determine the running cost, we must first analyze the subproblem faced by the physical Blue entities – the Blue action nodes.

4.2.1. The value of information

We suppose the *physical state* (as opposed to the network-delay state) can be decomposed according to domains partitioned by the action nodes. For each $\alpha \in \mathcal{G}_a$, let $\mathcal{X}^\alpha$ denote the finite set of possible physical states at action node, α, and let $X^\alpha \doteq \#\mathcal{X}^\alpha$. Without loss of generality, we let $\mathcal{X}^\alpha =]1, X^\alpha[$. The complete physical state will be denoted by $x \in \mathcal{X}$, where $\mathcal{X} \doteq \mathcal{X}^{\alpha_1} \times \mathcal{X}^{\alpha_2} \ldots \mathcal{X}^{\alpha_{G_a}}$, where we denote the elements of $\mathcal{G}_a$ as α_j for $j \in]1, G_a[$. We let $q^\alpha \in [S^{\oplus^\vee}]^{X^\alpha}$ denote the vector of probabilities of the possible states at action-note α. It is easily seen that for any $x \in \mathcal{X}$, $x = (x^1, x^2, \ldots x^{G_a})$, the max-plus probability of x is given by $q_x = \bigotimes^\vee_{\alpha\in\mathcal{G}_a} q^\alpha_{x^\alpha} \in [S^{\oplus^\vee}]^X$, where $X = \prod_{\alpha\in\mathcal{G}_a} X^\alpha$. Here, the max-plus probability, q^α, may be referred to as an information state at node α, as it describes the imperfectness of the information regarding the physical state there (see Remark 3.3). We must see how this translates into a cost for Blue.

Let $\mathcal{V}^\alpha$ denote the finite set of possible physical actions which the Blue action-node α may take. Let $\ell^\alpha : \mathcal{X}^\alpha \times \mathcal{V}^\alpha \to \mathbb{R}$, where for $x^\alpha \in \mathcal{X}^\alpha$ and $v^\alpha \in \mathcal{V}^\alpha$, $\ell^\alpha(x^\alpha, v^\alpha)$ denotes the cost to Blue for applying control action v^α while the true state is x^α. Then, given imperfect information described by $q^\alpha \in [S^{\oplus^\vee}]^{X^\alpha}$, the minimal expected cost to Blue at node α, assuming Blue applies a control action minimizing its cost, is

$$\hat{\psi}^\alpha(q^\alpha) = \bigwedge_{v^\alpha\in\mathcal{V}^\alpha} \Big\{ \mathbf{E}^{\oplus^\vee}\big[\ell^\alpha(\hat{X}^\alpha, v^\alpha)\big] \Big\} = \bigwedge_{v^\alpha\in\mathcal{V}^\alpha} \Big\{ \bigoplus^\vee_{x^\alpha\in\mathcal{X}^\alpha} \ell^\alpha(x^\alpha, v^\alpha) \otimes^\vee q^\alpha_x \Big\}, \tag{7}$$

where $\hat{X}^\alpha$ denotes a random variable distributed according to q^α. For $v^\alpha \in \mathcal{V}^\alpha$, let $L^\alpha(v^\alpha)$ denote the vector of length X^α with elements $\ell^\alpha(x^\alpha, v^\alpha)$, that is, $[L^\alpha(v^\alpha)]_{x^\alpha} \doteq \ell^\alpha(x^\alpha, v^\alpha)$ for all $x^\alpha \in \mathcal{X}^\alpha$. Then we may rewrite (7) as

$$\hat{\psi}^\alpha(q^\alpha) = \bigwedge_{v^\alpha\in\mathcal{V}^\alpha} \Big\{ L^\alpha(v^\alpha) \odot^\vee q^\alpha \Big\}, \tag{8}$$

where $\odot^\vee$ denotes the max-plus dot product. We see that $\hat{\psi}(q^\alpha)$ is the max-plus expected cost to Blue at action node α given information state q^α. In order to see how this translates into a cost for delay of information along the Blue network, we must examine how a max-plus partially-observed Markov chain propagates.

4.2.2. Max-plus conditional probability propagation

We model the dynamics of the physical Red entities using a max-plus Markov chain (see also [16]). For heuristic purposes, note that in the example depicted in Figure 2, the Red actions might simply be movement from one location to another, where each state, $x^\alpha \in \mathcal{X}^\alpha$, might denote a specific possible configuration of Red-entity positions.

We let $q_k^\alpha \in [S^{\oplus^\vee}]^{X^\alpha}$ denote the observation-conditioned max-plus probability distribution for the physical state at action node α and time k, where $[q_k^\alpha]_{x^\alpha}$ denotes the x^α component of q_k^α. We suppose that in the absence of observations, the distribution propagates as a max-plus Markov chain. That is,

$$q_{k+1}^\alpha = (\mathbb{P}^\alpha)^T \otimes^\vee q_k^\alpha,$$

for each $\alpha \in \mathcal{G}_a$, where $\mathbb{P}^\alpha$ is the max-plus probability transition matrix corresponding to node $\alpha \in \mathcal{G}_a$, and throughout, we use $\otimes^\vee$ to denote max-plus matrix and matrix-vector multiplication as well as scalar max-plus multiplication. Note $\mathbb{P}^\alpha_{\zeta^\alpha,x^\alpha} \in [-\infty, 0]$ for all $\zeta^\alpha, x^\alpha \in \mathcal{X}^\alpha$ and $\bigoplus^\vee_{x^\alpha \in \mathcal{X}^\alpha} \mathbb{P}^\alpha_{\zeta^\alpha,x^\alpha} = 0$. That is, each transition probability lies between that additive and multiplicative inverses, and each rows sum is the multiplicative inverse.

We now consider the introduction of observation updates. We suppose q_k^α denotes the *a priori* distribution at node α and time k, and let $\hat{q}_k^\alpha$ denote the *a posteriori*. Suppose that Blue obtains observation of x^α, $y \in \mathcal{Y}^\alpha$, (which we recall may be at least partially controlled by Red). We assume that the set of possible observations at each action-node location is finite, i.e., $\#\mathcal{Y}^\alpha \in \mathbb{N}$ for all $\alpha \in \mathcal{G}_a$. Let the max-plus probability that sensor $\sigma \in \hat{\mathcal{G}}_s(\alpha)$ observes $y \in \mathcal{Y}^\alpha$ given true state $x^\alpha \in \mathcal{X}^\alpha$ be denoted by $p^{\oplus^\vee}(y|x^\alpha;\sigma) \in [-\infty, 0]$. Recall that we may interpret a max-plus probability as the additive inverse of a cost to Red (recall Remark 3.3), or equivalently, as a negative cost for minimizing-player, Blue. Then, the max-plus probability (equivalently, cost) for any true state $x^\alpha \in \mathcal{X}^\alpha$ would be $[\tilde{q}_k^\alpha]_{x^\alpha} = p^{\oplus^\vee}(y|x^\alpha;\sigma) \otimes^\vee [q_k^\alpha]_{x^\alpha}$. However, as we are concerned only with relative costs, we normalize so that the max-plus sum over $x^\alpha \in \mathcal{X}^\alpha$ is zero. The normalized cost/max-plus probability is

$$[\hat{q}_k^\alpha]_{x^\alpha} = p^{\oplus^\vee}(y|x^\alpha;\sigma) \otimes^\vee [q_k^\alpha]_{x^\alpha} \oslash^\vee \Big\{ \bigoplus_{\zeta^\alpha \in \mathcal{X}^\alpha}^\vee \big[p^{\oplus^\vee}(y|\zeta^\alpha;\sigma) \otimes^\vee [q_k^\alpha]_{\zeta^\alpha} \big] \Big\}, \tag{9}$$

where $\oslash^\vee$ indicates max-plus division (standard-sense subtraction). Note that (9) is the max-plus equivalent of Bayes' rule. We may interpret the change from $[q_k^\alpha]_{x^\alpha}$ to $[\hat{q}_k^\alpha]_{x^\alpha}$, as the additive inverse of the minimal relative cost to Red for modification of the observation process to yield y given true state x^α. Next, for any $y \in \mathcal{Y}^\alpha$, let $C_\sigma^{\alpha,y}$ be the $X^\alpha \times X^\alpha$ max-plus diagonal matrix with diagonal elements $p^{\oplus^\vee}(y|x^\alpha;\sigma)$ (where we note that a square matrix is max-plus diagonal if all off-diagonal elements are $-\infty$). Also let $R_\sigma^{\alpha,y}$ be the X^α-length vector with elements $p^{\oplus^\vee}(y|x^\alpha;\sigma)$. Written in vector form, update (9) takes the form

$$\hat{q}_k^\alpha = \mathcal{B}_\sigma^y[q_k^\alpha] \doteq [C_\sigma^{\alpha,y} \otimes^\vee q_k^\alpha] \oslash^\vee [(R_\sigma^{\alpha,y}) \odot^\vee q_k^\alpha]. \tag{10}$$

As the presentation is already dense, we will assume that every sensor observes at every time step. Recalling that the set of sensors allocated to action node $\alpha \in \mathcal{G}_a$ is

$\hat{\mathcal{G}}_s(\alpha)$, the full one-step update corresponding to $\alpha \in \mathcal{G}_a$ is

$$q_{k+1}^\alpha = (\mathbb{P}^\alpha)^T \otimes^\vee \Bigg(\prod_{\sigma \in \hat{\mathcal{G}}_s(\alpha)} \mathcal{B}_\sigma^{y_{k,\sigma}} \Bigg) [q_k^\alpha]. \tag{11}$$

4.2.3. Constructing the cost of delay

We now examine how the actual running cost will depend on the delays along the network, where this running cost for the delay game will flow from the physical-action costs described above. The observation-conditioned max-plus probability distribution update of (11) was developed under the assumption that all observations made at time k were actually available for processing at time k. Refer to Section 2, and in particular, to Figure 1. The observation of x^α obtained by $\sigma \in \hat{\mathcal{G}}_s(\alpha)$ may need to proceed through some specified subset of the Blue network, with processing possibly occurring along the way, before the resulting updated conditional probability and/or command decisions can be applied by Blue action node α. We need to determine how the resulting cost to Blue will depend on the delay-state of the network.

We will let $\hat{G}_s(\alpha) = \#\hat{\mathcal{G}}_s(\alpha)$ and $\vec{y}_k \in [\mathcal{Y}^\alpha]^{\hat{G}_s(\alpha)}$ be the vector of observations at time k with components $y_{k,\sigma}$ for $\sigma \in \hat{\mathcal{G}}_s(\alpha)$. The total expected payoff at time k and action node α of an action at time $k+1$ given current distribution q_k^α then becomes

$$\hat{\psi}_1^\alpha(q_k^\alpha, \hat{\mathcal{G}}_s(\alpha)) \doteq \mathbf{E}^{\oplus^\vee}_{\vec{y}_k \in [\mathcal{Y}^\alpha]^{\hat{G}_s(\alpha)}} \hat{\psi}^\alpha \Big([\mathbb{P}^\alpha]^T \otimes^\vee \Big(\prod_{\sigma \in \hat{\mathcal{G}}_s(\alpha)} \mathcal{B}_\sigma^{y_{k,\sigma}} \Big) [q_k^\alpha] \Big), \tag{12}$$

where $\hat{\psi}^\alpha$ is given in (8), and the subscript on the max-plus expectation indicates that it is taken over the observations of the Red state at action node α and time k. For notational simplicity, assume for the moment that there is only one sensor. Substituting (8), (10) and (11) into (12) yields, after a bit of work,

$$\begin{aligned} \hat{\psi}_1^\alpha(q_k^\alpha, \hat{\mathcal{G}}_s(\alpha)) &= \bigoplus^\vee_{y_k \in \mathcal{Y}^\alpha} \bigwedge_{v^\alpha \in \mathcal{V}^\alpha} \Big\{ [L^\alpha(v^\alpha)]^T \otimes^\vee [\mathbb{P}^\alpha]^T \otimes^\vee C_\sigma^{\alpha, y_k} \otimes^\vee q_k^\alpha \Big\} \\ &= \bigoplus^\vee_{y_k \in \mathcal{Y}^\alpha} \bigwedge_{v^\alpha \in \mathcal{V}^\alpha} \Big\{ \Big(C_\sigma^{\alpha, y_k} \otimes^\vee \mathbb{P}^\alpha \otimes^\vee L^\alpha(v^\alpha) \Big) \odot^\vee q_k^\alpha \Big\}. \end{aligned}$$

Now, returning to the case of multiple sensor nodes, and continuing to assume the same observation set for each, this becomes

$$\hat{\psi}_1^\alpha(q_k^\alpha, \hat{\mathcal{G}}_s(\alpha)) = \bigoplus^\vee_{\vec{y}_k \in [\mathcal{Y}^\alpha]^{\hat{G}_s(\alpha)}} \bigwedge_{v^\alpha \in \mathcal{V}^\alpha} \Big\{ \Big[\Big(\bigotimes^\vee_{\sigma \in \hat{\mathcal{G}}_s(\alpha)} C_\sigma^{\alpha, y_{k,\sigma}} \Big) \otimes^\vee \mathbb{P}^\alpha \otimes^\vee L^\alpha(v^\alpha) \Big] \odot^\vee q^\alpha \Big\}, \tag{13}$$

where for simplicity, we henceforth take $\mathcal{Y}^\alpha = \mathcal{Y}$ for all $\alpha \in \mathcal{G}_a$.

We now expand this to take into account that some of the observations may have been delayed, which will require some rather technical notation. Let $\delta \in \mathcal{D}$ denote a generic delay state. If information from node $\sigma \in \hat{\mathcal{G}}_s(\alpha)$ is delayed at $\alpha \in \mathcal{G}_a$ by some time, δ_α^σ, then the observation updates in (13) for time-steps more recent than δ_α^σ steps back will not take place. Corresponding to $\delta \in \mathcal{D}$, the maximum delay is

$d^*(\delta) = \bigvee_{\alpha\in\mathcal{G}_a}\bigvee_{\sigma\in\hat{\mathcal{G}}_s(\alpha)}\delta^\sigma_\alpha$. Let $j^*(\delta) \doteq -d^*(\delta)$. (Recall $\hat{\mathcal{G}}_s(\alpha_1)\cap\hat{\mathcal{G}}_s(\alpha_2)=\emptyset$ if $\alpha_1\neq\alpha_2$.) Then, for each $j\in\,]j^*(\delta),0[$, and each $\alpha\in\mathcal{G}_a$, we see that the set of sensors from which observations for $-j$ steps back are available is $\tilde{\mathcal{G}}_{s,j}(\alpha,\delta)\doteq\{\sigma\in\hat{\mathcal{G}}_s(\alpha)\,|\,j\le-\delta^\sigma_\alpha\}$, and let $\tilde{G}_{s,j}(\alpha,\delta)\doteq\#\tilde{\mathcal{G}}_{s,j}(\alpha,\delta)$. Also, given $\alpha\in\mathcal{G}_a$, $\delta\in\mathcal{D}$ and an observation-history length $-\bar{k}\ge 0$, let

$$\hat{\mathcal{Y}}(\alpha,\delta,\bar{k})\doteq\mathcal{Y}^{\tilde{G}_{s,\bar{k}}(\alpha,\delta)}\times\mathcal{Y}^{\tilde{G}_{s,\bar{k}+1}(\alpha,\delta)}\times\cdots\mathcal{Y}^{\tilde{G}_{s,0}(\alpha,\delta)},$$

with elements $\vec{\vec{y}}=\{\vec{y}_j\,|\,j\in\,]\bar{k},0[\}=\{y_{j,\sigma}\,|\,\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta),\ j\in\,]\bar{k},0[\}$.

We suppose that the information state has some initial value, $\bar{q}^\alpha_0$. Proceeding as in (13) but now with information processing over time-step range $]\bar{k},0[$, we see that the max-plus expected cost of delay state δ is given by

$$\breve{\psi}^\alpha(\delta)=\breve{\psi}^\alpha(\delta;\bar{q}^\alpha_0,\bar{k})=\bigvee_{\vec{\vec{y}}\in\hat{\mathcal{Y}}(\alpha,\delta,\bar{k})}\ \bigwedge_{v^\alpha\in\mathcal{V}^\alpha}\Bigg\{\Bigg[\Bigg(\bigotimes^{\vee}_{j\in\,]\bar{k},0[}\Bigg[\Bigg(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta)}C^{\alpha,y_{j,\sigma}}_\sigma\Bigg)\otimes^\vee\mathbb{P}^\alpha\Bigg]\Bigg)\otimes^\vee L^\alpha(v^\alpha)\Bigg]\odot^\vee\bar{q}^\alpha_0\Bigg\},\tag{14}$$

where we note that as the matrices in the product over j are not necessarily diagonal, we must indicate order, and to limit space we use the following notation. For integers $a\le b$ and generic sequence of matrices, M_i, we let

$$\bigotimes^{\vee}_{j\in]a,b[}M_i\doteq M_a\otimes^\vee M_{a+1}\otimes^\vee\dots M_b\ \text{ and }\ \bigotimes^{\vee}_{j\in]a,b['}M_i\doteq M_b\otimes^\vee M_{b-1}\otimes^\vee\dots M_a.$$

Also, the left-most side of (14) is included to indicate that we will be most interested here in the dependence on the delay state, δ, rather than on the initial max-plus probability distribution, $\bar{q}^\alpha_0$. As an aside, it is interesting to note that by modifying the choice of symbols for the idempotent operations, (14) also takes the form

$$\breve{\psi}^\alpha(\delta)=\breve{\psi}^\alpha(\delta;\bar{q}^\alpha_0,\bar{k})=\bigvee_{\vec{\vec{y}}\in\hat{\mathcal{Y}}(\alpha,\delta,\bar{k})}\ \bigoplus_{v^\alpha\in\mathcal{V}^\alpha}\Bigg\{\Bigg[\Bigg(\bigotimes_{j\in\,]\bar{k},0[}\Bigg[\Bigg(\bigotimes_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta)}C^{\alpha,y_{j,\sigma}}_\sigma\Bigg)\otimes\mathbb{P}^\alpha\Bigg]\Bigg)\otimes L^\alpha(v^\alpha)\Bigg]\odot\bar{q}^\alpha_0\Bigg\},$$

in which case, one sees that this is a min-plus convex function of $\bar{q}^\alpha_0$. Lastly, we take the total cost over all $\alpha\in\mathcal{G}_a$ to be

$$\bar{\psi}(\delta)=\bar{\psi}(\delta;\bar{q}_0,\bar{k})\doteq\bigvee_{\alpha\in\mathcal{G}_a}\breve{\psi}^\alpha(\delta;\bar{q}^\alpha_0,\bar{k}),\tag{15}$$

where $[\bar{q}_0]_{x^1,x^2,\dots x^{G_a}}=\bigotimes^\vee_{\alpha\in\mathcal{G}_a}[\bar{q}^\alpha_0]_{x^\alpha}$ for all $(x^1,x^2,\dots x^{G_a})\in[\mathcal{X}^\alpha]^{G_a}$. Model (15) suggests Red would act at, and only at, the worst-case location for Blue. A more complex cost could also be considered, possibly with additional technical difficulties, which is beyond the scope of this already-technical effort.

4.3. The Payoff and Value

So far, we have defined the game dynamics and the running cost. We now proceed to define the payoff and the value. Fix a time-horizon, $K < \infty$. As we will be taking a dynamic-programming perspective, we consider games with any initial time, $k_0 \in]0, K[$. The sets of control sequences for each player over $]k_0, K-1[$ are $\widetilde{\mathcal{U}}^b_{k_0} \doteq \left\{ \{u^b_k\}_{k \in]k_0, K-1[} \,\middle|\, u^b_k \in \mathcal{U}^b \,\forall k \right\}$ and $\widetilde{\mathcal{U}}^r_{k_0} \doteq \left\{ \{u^r_k\}_{k \in]k_0, K-1[} \,\middle|\, u^r_k \in \mathcal{U}^r \,\forall k \right\}$. We are interested in the upper value, and specifically the upper value under an assumption of nonanticipative strategies (cf. [5]). The set of nonanticipative strategies for Red given initial time k_0 will be denoted as $\mathcal{R}_{k_0} \doteq \left\{ \rho : \widetilde{\mathcal{U}}^b_{k_0} \to \widetilde{\mathcal{U}}^r_{k_0} \,|\, \text{nonanticipative} \right\}$. For use below, we recall the definition of a nonanticipative strategy:

Definition 4.1. A map $\rho : \widetilde{\mathcal{U}}^b_{k_0} \to \widetilde{\mathcal{U}}^r_{k_0}$ is nonanticipative if, for any $k \in]k_0, K-1[$, and any control strategies $u^1, u^2 \in \widetilde{\mathcal{U}}^b_{k_0}$ such that $u^1_i = u^2_i$ for all $i \in]k_0, k[$, one has $\rho_i[u^1] = \rho_i[u^2]$ for all $i \in]k_0, k[$.

For $k_0 \in]0, K[$, $\delta \in \mathcal{D}$, $u^b \in \widetilde{\mathcal{U}}^b_{k_0}$ and $u^r \in \widetilde{\mathcal{U}}^b_{k_0}$, the game payoff will be given by

$$\bar{J}^K(k_0, \delta, u^b_\cdot, u^r_\cdot) = \bar{J}^K(k_0, \delta, u^b_\cdot, u^r_\cdot; \bar{q}_0) \doteq \bigvee_{k \in]k_0, K[} \bar{\psi}(D_k; \bar{q}_0, -k), \tag{16}$$

where $D_\cdot$ satisfies dynamics (6) with initial condition $D_{k_0} = \delta$. The upper value is then given by

$$\overline{W}^K(k_0, \delta) = \overline{W}^K(k_0, \delta; \bar{q}_0) \doteq \bigvee_{\rho \in \mathcal{R}_{k_0}} \bigwedge_{u^b \in \widetilde{\mathcal{U}}^b_{k_0}} \bar{J}^K(k_0, \delta, u^b_\cdot, \rho[u^b_\cdot]; \bar{q}_0) \quad \forall\, k_0 \in]0, K[,\ \delta \in \mathcal{D}. \tag{17}$$

5. DEVELOPMENT OF THE IDEMPOTENT ALGORITHM

5.1. The Form of the Running Cost

We will demonstrate that the running cost is a min-plus convex function of delay. For simplicity of presentation here, we let $\mathcal{G}_a = \{\alpha\}$ so that $\bar{\psi}(\delta) = \bar{\psi}(\delta; \bar{q}_0, \bar{k}) = \breve{\psi}^\alpha(\delta; \bar{q}^\alpha_0, \bar{k})$.

As we will be mainly interested below in the dependence of $\bar{\psi}$ on δ, we will sometimes suppress the dependence of $\bar{\psi}$ on $\bar{q}_0$ and $\bar{k}$ in the notation.

Theorem 5.1. $\breve{\psi}^\alpha(\cdot) = \breve{\psi}^\alpha(\cdot; \bar{q}^\alpha_0, \bar{k}) \in \mathcal{S}_{\hat{C}}(\mathbb{R}^{\hat{G}_s(\alpha)})$ for some positive definite, diagonal $\hat{C} \in \mathbb{R}^{\hat{G}_s(\alpha) \times \hat{G}_s(\alpha)}$ (where we recall $\mathcal{S}_{\hat{C}}(\mathbb{R}^{\hat{G}_s(\alpha)})$ is specified in definition 3.13).

Proof. Recall the expression for $\breve{\psi}^\alpha(\delta; \bar{q}^\alpha_0, \bar{k})$ given in (14). Suppose $0 \preceq \delta_1 \preceq \delta_2$. Then, $\tilde{\mathcal{G}}_{s,j}(\alpha, \delta_2) \subseteq \tilde{\mathcal{G}}_{s,j}(\alpha, \delta_1)$ for all $j \in]\bar{k}, 0[$, where we recall that $\tilde{\mathcal{G}}_{s,j}(\alpha, \delta)$ was defined below (13). Let $\vec{y}^{\,1,*} \in \hat{\mathcal{Y}}(\alpha, \delta_1, \bar{k})$ achieve the maximum in (14) for $\delta = \delta_1$. Let $\vec{y}^{\,2,*}$ be the element of $\hat{\mathcal{Y}}(\alpha, \delta_2, \bar{k})$ such that

$$y^{2,*}_{j,\sigma} = y^{1,*}_{j,\sigma} \quad \forall\, \sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha, \delta_2),\ j \in]\bar{k}, 0[\,.$$

Also let

$$v^{\alpha,*,2} \in \operatorname*{argmin}_{v^\alpha \in \mathcal{V}^\alpha} \left\{ \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y^{2,*}_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^\alpha) \right] \odot^\vee \bar{q}_0^\alpha \right\}.$$

Then,

$$\begin{aligned}
\breve{\psi}^\alpha(\delta_1; \bar{q}_0^\alpha, \bar{k}) &\le \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)} C_\sigma^{\alpha, y^{1,*}_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^{\alpha,*,2}) \right] \odot^\vee \bar{q}_0^\alpha \\
&= \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)\setminus\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y^{1,*}_{j,\sigma}} \right) \right. \right. \right. \\
&\qquad \left. \left. \left. \otimes^\vee \left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y^{2,*}_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^{\alpha,*,2}) \right] \odot^\vee \bar{q}_0^\alpha. \qquad (18)
\end{aligned}$$

Recall that the $C_\sigma^{\alpha,y}$ matrices are max-plus diagonal, and that their diagonal elements, being max-plus probabilities, are nonpositive. Then, recalling that $a \otimes^\vee b = a + b$, (18) implies

$$\breve{\psi}^\alpha(\delta_1; \bar{q}_0^\alpha, \bar{k}) \le \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y^{2,*}_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^{\alpha,*,2}) \right] \odot^\vee \bar{q}_0^\alpha,$$

which by the choice of $v^{\alpha,*,2}$,

$$\begin{aligned}
&= \bigwedge_{v^\alpha \in \mathcal{V}^\alpha} \left\{ \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y^{2,*}_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^\alpha) \right] \right. \\
&\qquad \left. \odot^\vee \bar{q}_0^\alpha \right\} \\
&\le \bigvee_{\vec{y} \in \hat{\mathcal{Y}}(\alpha,\delta_2,\bar{k})} \bigwedge_{v^\alpha \in \mathcal{V}^\alpha} \left\{ \left[\left(\bigotimes^\vee_{j\in\,]\bar{k},0[} \left[\left(\bigotimes^\vee_{\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha, y_{j,\sigma}} \right) \otimes^\vee \mathbb{P}^\alpha \right] \right) \otimes^\vee L^\alpha(v^\alpha) \right] \right. \\
&\qquad \left. \odot^\vee \bar{q}_0^\alpha \right\} \\
&= \breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}),
\end{aligned}$$

and we have $0 \le \breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}) - \breve{\psi}^\alpha(\delta_1; \bar{q}_0^\alpha, \bar{k})$.

It remains to prove that there exists positive definite, symmetric $\hat{C} \in \mathbb{R}^{\hat{G}_s(\alpha)\times\hat{G}_s(\alpha)}$ such that $\breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}) - \breve{\psi}^\alpha(\delta_1; \bar{q}_0^\alpha, \bar{k}) \le \|\hat{C}(\delta_2 - \delta_1)\|_\infty$. Let $\vec{y}^{\,2,*} \in \hat{\mathcal{Y}}(\alpha, \delta_2, \bar{k})$ achieve the maximum in (14) for $\delta = \delta_2$. Let $\vec{y}^{\,1,*}$ be the element of $\hat{\mathcal{Y}}(\alpha, \delta_1, \bar{k})$ such that

$$y^{1,*}_{j,\sigma} \begin{cases} = y^{2,*}_{j,\sigma} & \text{if } \sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2),\ j \in\,]\bar{k},0[\\ \in \operatorname{argmax}_{y\in\mathcal{Y}} \{\min_{x^\alpha \in \mathcal{X}^\alpha} p^{\oplus^\vee}(y \,|\, x^\alpha)\} & \text{if } \sigma \notin \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2),\ j \in\,]\bar{k},0[\,. \end{cases}$$

Let

$$\hat{c} \doteq \max_{j\in\,]\bar{k},0[}\ \max_{\sigma \notin \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)\setminus\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)}\ \max_{y_{j,\sigma}\in\mathcal{Y}}\ \min_{x^\alpha\in\mathcal{X}^\alpha} p^{\oplus^\vee}(y_{j,\sigma} \,|\, x^\alpha).$$

In this case, let

$$v^{\alpha,*,1} \in \operatorname*{argmin}_{v^\alpha \in \mathcal{V}^\alpha} \Big\{ \Big[\Big(\bigotimes^{\vee}_{j\in\,]\bar{k},0[} \Big[\Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)} C_\sigma^{\alpha,y^{1,*}_{j,\sigma}} \Big) \otimes^\vee \mathbb{P}^\alpha \Big] \Big) \otimes^\vee L^\alpha(v^\alpha) \Big] \odot^\vee \bar{q}^\alpha_0 \Big\}.$$

Then,

$$\breve{\psi}^\alpha(\delta_1; \bar{q}^\alpha_0, \bar{k}) \geq \Big[\Big(\bigotimes^{\vee}_{j\in\,]\bar{k},0[} \Big[\Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)} C_\sigma^{\alpha,y^{1,*}_{j,\sigma}} \Big) \otimes^\vee \mathbb{P}^\alpha \Big] \Big) \otimes^\vee L^\alpha(v^{\alpha,*,1}) \Big] \odot^\vee \bar{q}^\alpha_0,$$

and by the definition of max-plus matrix-vector multiplication, for all $x^\alpha, \zeta^\alpha \in \mathcal{X}^\alpha$, this is

$$\geq \ell^\alpha(\zeta^\alpha, v^{\alpha,*,1}) + [\bar{q}^\alpha_0]_{x^\alpha} + \Big[\bigotimes^{\vee}_{j\in\,]\bar{k},0['} \Big[(\mathbb{P}^\alpha)^T \otimes^\vee \Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)} C_\sigma^{\alpha,y^{1,*}_{j,\sigma}} \Big) \Big] \Big]_{\zeta^\alpha, x^\alpha},$$

where we recall the $C^{\alpha,y}_\sigma$ matrices are max-plus diagonal, and this is

$$\begin{aligned} = \ell^\alpha(\zeta^\alpha, v^{\alpha,*,1}) + [\bar{q}^\alpha_0]_{x^\alpha} + \Big[\bigotimes^{\vee}_{j\in\,]\bar{k},0['} \Big[(\mathbb{P}^\alpha)^T \otimes^\vee \Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1)\setminus\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha,y^{1,*}_{j,\sigma}} \Big) \\ \otimes^\vee \Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha,y^{2,*}_{j,\sigma}} \Big) \Big] \Big]_{\zeta^\alpha, x^\alpha}. \end{aligned} \tag{19}$$

Now, by the choice of $y^{1,*}_{j,\sigma}$, the diagonal elements of $C_\sigma^{y^{1,*}_{j,\sigma}} \geq \hat{c}$ for $\sigma \in \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_1) \setminus \tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)$. Consequently, (19) implies

$$\begin{aligned} \breve{\psi}^\alpha(\delta_1; \bar{q}^\alpha_0, \bar{k}) \geq & \ell^\alpha(\zeta^\alpha, v^{\alpha,*,1}) + [\bar{q}^\alpha_0]_{x^\alpha} + \Big[\bigotimes^{\vee}_{j\in\,]\bar{k},0['} \Big[(\mathbb{P}^\alpha)^T \otimes^\vee \Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha,y^{2,*}_{j,\sigma}} \Big) \Big] \Big]_{\zeta^\alpha, x^\alpha} \\ & + \hat{c} \sum_{j\in]\bar{k},0[} [\tilde{G}_{s,j}(\alpha,\delta_1) - \tilde{G}_{s,j}(\alpha,\delta_2)]. \end{aligned}$$

As this is true for all $x^\alpha, \zeta^\alpha \in \mathcal{X}^\alpha$, we have

$$\begin{aligned} \breve{\psi}^\alpha(\delta_1; \bar{q}^\alpha_0, \bar{k}) \geq & \bigoplus^{\vee}_{x^\alpha,\zeta^\alpha\in\mathcal{X}^\alpha} \Big\{ \ell^\alpha(\zeta^\alpha, v^{\alpha,*,1}) + [\bar{q}^\alpha_0]_{x^\alpha} + \Big[\bigotimes^{\vee}_{j\in\,]\bar{k},0['} \Big[(\mathbb{P}^\alpha)^T \otimes^\vee \\ & \Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha,y^{2,*}_{j,\sigma}} \Big) \Big] \Big]_{\zeta^\alpha, x^\alpha} \Big\} + \hat{c} \sum_{j\in]\bar{k},0[} [\tilde{G}_{s,j}(\alpha,\delta_1) - \tilde{G}_{s,j}(\alpha,\delta_2)] \\ = & \Big[\Big(\bigotimes^{\vee}_{j\in\,]\bar{k},0[} \Big[\Big(\bigotimes^{\vee}_{\sigma\in\tilde{\mathcal{G}}_{s,j}(\alpha,\delta_2)} C_\sigma^{\alpha,y^{2,*}_{j,\sigma}} \Big) \otimes^\vee \mathbb{P}^\alpha \Big] \Big) \otimes^\vee L^\alpha(v^{\alpha,*,1}) \Big] \odot^\vee \bar{q}^\alpha_0 \\ & + \hat{c} \sum_{j\in]\bar{k},0[} [\tilde{G}_{s,j}(\alpha,\delta_1) - \tilde{G}_{s,j}(\alpha,\delta_2)] \end{aligned}$$

which by the choice of $\vec{y}^{2,*}$,

$$
\begin{aligned}
&\geq \breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}) + \hat{c} \sum_{j \in]\bar{k},0[} [\tilde{G}_{s,j}(\alpha, \delta_1) - \tilde{G}_{s,j}(\alpha, \delta_2)] \\
&= \breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}) + \hat{c} \sum_{\sigma \in \hat{\mathcal{G}}_s(\alpha)} \big([\delta_2]_\sigma - [\delta_1]_\sigma\big)
\end{aligned}
$$

and recalling that $\hat{c} \leq 0$, this is

$$
\geq \breve{\psi}^\alpha(\delta_2; \bar{q}_0^\alpha, \bar{k}) + \hat{c}\hat{G}_s(\alpha)\|\delta_2 - \delta_1\|_\infty,
$$

which yields the result. $\square$

Remark 5.2. Henceforth, for simplicity of presentation, we will assume that the $\bar{\psi}$ may be normalized such that the components of δ are integer-valued, that $\bar{\psi}(\delta)$ is integer-valued, and that $\bar{\psi}$ is Lipschitz with constant one.

Theorem 5.3. The cost of delay, $\bar{\psi}$, can be expressed as a finite-domain min-plus convex function. That is, there exists index set, $\mathcal{Z}$, and corresponding sets of coefficients, such that

$$
\bar{\psi}(\delta) = \bigvee_{z \in \mathcal{Z}} \Big[e^z \oplus b^z \odot \delta \Big] \quad \forall\, \delta \in \mathcal{D}, \tag{20}
$$

where we recall that the $\odot$ product is given by $b^z \odot \delta \doteq \bigoplus_{(g,\sigma) \in \mathcal{G} \times \mathcal{G}_s} b^z_{g,\sigma} \otimes \delta^\sigma_g$.

P r o o f. This follows directly from Theorems 5.1 and 3.19. $\square$

We remark here that $\bar{\psi}$ depends only on the components of δ, δ^σ_g, such that $g \in \mathcal{G}_a$ and $\sigma \in \hat{\mathcal{G}}_s(g)$. In particular, the components of $b^z_{g,\sigma}$ such that either $g \notin \mathcal{G}_a$ or such that $g \in \mathcal{G}_a$ but $\sigma \notin \hat{\mathcal{G}}_s(g)$ are $-\infty$.

The value of Theorem 5.3 is that the asserted form will allow us to obtain an idempotent distributed dynamic program for reduced-complexity computation.

5.2. Idempotent Distributed Dynamic Program

We first obtain the general dynamic program for the game (16)–(17). After that we move to the idempotent distributed dynamic program (IDDP) for solution of the game.

Theorem 5.4. Value $\overline{W}^K$ is the unique solution of the dynamic program

$$
W^K(K, \delta) = \bar{\psi}(\delta) \quad \forall\, \delta \in \mathcal{D}, \tag{21}
$$

$$
W^K(k, \delta) = \bigwedge_{v^b \in \mathcal{U}^b} \bigvee_{v^r \in \mathcal{U}^r} \Big[\bar{\psi}(\delta) \vee W^K(k+1, D_{k+1}) \Big] \quad \forall k \in]k_0, K-1[,\ \delta \in \mathcal{D}, \tag{22}
$$

where $D_{k+1} = \bar{F}(\delta, v^b, v^r)$.

P r o o f. The proof of the above theorem is similar to existing results in the area of dynamic programming for max-plus control [7, 18], and consequently, we only sketch it. It is sufficient to show that a solution to (22) must be identical to $\overline{W}^K$. Suppose $W^K(\hat{k}+1,\cdot) = \overline{W}^K(\hat{k}+1,\cdot)$, which is true for $\hat{k}+1 = K$ by (21). Then, by (22),

$$W^K(\hat{k},\delta) = \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \Big[\bar{\psi}(\delta) \vee \overline{W}^K(k+1, D_{\hat{k}+1})\Big],$$

where $D_{\hat{k}+1} = \bar{F}(\delta, v^b, v^r)$, and by the definition of value, (17), this is

$$= \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \bigvee_{\rho\in\mathcal{R}_{\hat{k}+1}} \bigwedge_{u^b\in\widetilde{\mathcal{U}}^b_{\hat{k}+1}} \Big[\bigvee_{k\in\,]\hat{k},K[} \bar{\psi}(D_k)\Big]$$

$$= \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{(v^r,\rho)\in\mathcal{U}^r\times\mathcal{R}_{\hat{k}+1}} \bigwedge_{u^b\in\widetilde{\mathcal{U}}^b_{\hat{k}+1}} \Big[\bigvee_{k\in\,]\hat{k},K[} \bar{\psi}(D_k)\Big], \tag{23}$$

where $D_\cdot$ satisfies (6) with $D_{\hat{k}} = \delta$. Applying the min-max distributive property, Lemma 3.1, to (23), we have

$$W^K(\hat{k},\delta) = \bigvee_{\{(v^r,\rho)_{v^b}\}_{v_b\in\mathcal{U}^b}\in(\mathcal{U}^r\times\mathcal{R}_{\hat{k}+1})^{U^b}} \bigwedge_{v^b\in\mathcal{U}^b} \bigwedge_{u^b\in\widetilde{\mathcal{U}}^b_{\hat{k}+1}} \Big[\bigvee_{k\in\,]\hat{k},K[} \bar{\psi}(D_k)\Big]$$

$$= \bigvee_{\{(v^r,\rho)_{v^b}\}_{v_b\in\mathcal{U}^b}\in(\mathcal{U}^r\times\mathcal{R}_{\hat{k}+1})^{U^b}} \bigwedge_{(v^b,u^b)\in\mathcal{U}^b\times\widetilde{\mathcal{U}}^b_{\hat{k}+1}} \Big[\bigvee_{k\in\,]\hat{k},K[} \bar{\psi}(D_k)\Big],$$

where $(\mathcal{U}^r\times\mathcal{R}_{\hat{k}+1})^{U^b}$ denotes the outer product of $\mathcal{U}^r\times\mathcal{R}_{\hat{k}+1}$, U^b times, and one then shows that this is

$$= \bigvee_{\rho\in\mathcal{R}_{\hat{k}}} \bigwedge_{u^b\in\widetilde{\mathcal{U}}^b_{\hat{k}}} \Big[\bigvee_{k\in\,]\hat{k},K[} \bar{\psi}(D_k)\Big] = \bigvee_{\rho\in\mathcal{R}_{\hat{k}}} \bigwedge_{u^b\in\widetilde{\mathcal{U}}^b_{\hat{k}}} \bar{J}^K(\hat{k},\delta,u^b,\rho[u^b]) = \overline{W}^K(\hat{k},\delta),$$

and we do not include the technical details. □

We can now obtain the IDDP (cf. [18]) for $\overline{W}^K$ from this dynamic program. The IDDP will be helpful in computation as it is a finite-complexity computation utilizing idempotent algebraic operations. We assume that one has precomputed $\bar{\psi}(\delta)$ for all $\delta \in (]0,\bar{k}[)^{GG_s}$, where we recall that (14) for computation of $\bar{\psi}(\delta)$ requires only idempotent matrix-vector operations.

Theorem 5.5. For $k\in]0,K-1[$, there exist $Z_k\in\mathbb{N}$, $\mathcal{Z}_k =]1,Z_k[$ and $e^z_k\in\mathbb{R}$ and $b^z_k\in\mathbb{R}^{GG_s}$ for all $z\in\mathcal{Z}_k$, such that

$$W^K(k,\delta) = \bigvee_{z\in\mathcal{Z}_k} \Big[e^z_k \oplus b^z_k \odot \delta \Big] \quad \forall\, \delta\in\mathcal{D},$$

where $b^z_k\odot\delta \doteq \bigoplus_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s} [b^z_k]_{g,\sigma}\otimes\delta^\sigma_g$. Further, $\{e^z_k \,|\, z\in\mathcal{Z}_k\}$ and $\{b^z_k \,|\, z\in\mathcal{Z}_k\}$ may be computed via idempotent algebraic operations.

Proof. We will proceed inductively, backwards in time. We have by Theorem 5.3 and Theorem 5.4 that $W^K(K,\delta) = \bigvee_{z\in\mathcal{Z}_K} \Big[e^z_K \oplus b^z_K \odot \delta \Big]$ for all $\delta\in\mathcal{D}$. Now, let $k\in]0,K-1[$.

We have by Theorem 5.4 that,

$$W^K(k,\delta) = \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \Big[\bar{\psi}(\delta) \vee W^K(k+1, D_{k+1})\Big],$$

where $D_{k+1} = \bar{F}(\delta, v^b, v^r)$. Making the induction assumption that the result holds for $W^K(k+1,\cdot)$, we have

$$\begin{aligned} W^K(k,\delta) &= \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \left\{\bar{\psi}(\delta) \vee \bigvee_{z_{k+1}\in\mathcal{Z}_{k+1}} \left[e_{k+1}^{z_{k+1}} \oplus b_{k+1}^{z_{k+1}} \odot D_{k+1}\right]\right\} \\ &= \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \left\{\bar{\psi}(\delta) \vee \bigvee_{z_{k+1}\in\mathcal{Z}_{k+1}} \left[e_{k+1}^{z_{k+1}} \oplus \bigoplus_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s}^{\vee} b_{k+1,g}^{z_{k+1},\sigma} \otimes d_{k+1,g}^{\sigma}\right]\right\}. \quad (24) \end{aligned}$$

Substituting delay dynamics (5) into (24) with $d_k = \delta$ yields,

$$\begin{aligned} W^K(k,\delta) = \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \Bigg\{\bar{\psi}(\delta)\vee \bigvee_{z_{k+1}\in\mathcal{Z}_{k+1}} \Bigg[e_{k+1}^{z_{k+1}} \oplus \bigoplus_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s}^{\vee} b_{k+1,g}^{z_{k+1},\sigma} \quad (25) \\ \otimes \Big([\delta_g^\sigma - f_g^p(v^b,v^r)] \vee [\bigwedge_{\gamma\in\mathcal{N}_g\setminus\{g\}} \delta_\gamma^\sigma + 1]\Big)\Bigg]\Bigg\}. \end{aligned}$$

Defining for $\gamma \in \mathcal{G}$,

$$\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},0} = \begin{cases} b_{k+1,g}^{z_{k+1},\sigma} - f_g^p(v^b,v^r) & \text{if } \gamma = g \\ +\infty & \text{otherwise,} \end{cases} \quad (26)$$

$$\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},1} = \begin{cases} b_{k+1,g}^{z_{k+1},\sigma} + 1 & \text{if } \gamma \in \mathcal{N}_g\setminus\{g\} \\ +\infty & \text{otherwise,} \end{cases} \quad (27)$$

and observing that $\bar{\psi}(\delta)$ is independent of v^b and v^r allows us to express (25) as

$$\begin{aligned} W^K(k,\delta) = \bar{\psi}(\delta) \vee \Bigg\{\bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{v^r\in\mathcal{U}^r} \bigvee_{z_{k+1}\in\mathcal{Z}_{k+1}} \Bigg[e_{k+1}^{z_{k+1}} \oplus \bigwedge_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s} \\ \bigvee_{i\in\{0,1\}} \bigwedge_{\gamma\in\mathcal{G}} [\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},i} \otimes \delta_\gamma^\sigma]\Bigg]\Bigg\}, \end{aligned} \quad (28)$$

and by the min-max distributive property, this becomes,

$$\begin{aligned} = \bar{\psi}(\delta) \vee \Bigg\{ \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{(v^r,z_{k+1})\in\mathcal{U}^r\times\mathcal{Z}_{k+1}} \Bigg[e_{k+1}^{z_{k+1}} \oplus \bigwedge_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s} \quad (29) \\ \bigvee_{i\in\{0,1\}} \bigwedge_{\gamma\in\mathcal{G}} [\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},i} \otimes \delta_\gamma^\sigma]\Bigg]\Bigg\}. \end{aligned}$$

Applying the max-min distributive property again, we find

$$W^K(k,\delta) = \bar\psi(\delta) \vee \Bigg\{ \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{(v^r,z_{k+1})\in\mathcal{U}^r\times\mathcal{Z}_{k+1}} \Bigg[e_{k+1}^{z_{k+1}} \oplus \bigvee_{\bar i.\in\bar{\mathcal{I}}} \bigwedge_{(\gamma,\sigma)\in\mathcal{G}\times\mathcal{G}_s} \bigwedge_{g\in\mathcal{G}} \big[\hat b_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},\bar i_{(g,\sigma)}} \otimes \delta_\gamma^\sigma\big]\Bigg]\Bigg\}, \tag{30}$$

where $\bar{\mathcal{I}} \doteq \{\bar i_{(g,\sigma)}\}_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s}$, that is, $\bar{\mathcal{I}}$ is the set of sequences of elements of $\{0,1\}$ indexed by $(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s$, and in particular, $\bar i_{(g,\sigma)} \in \{0,1\}$ for each $(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s$. This is

$$= \bar\psi(\delta) \vee \Bigg\{ \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{(v^r,z_{k+1})\in\mathcal{U}^r\times\mathcal{Z}_{k+1}} \Bigg[e_{k+1}^{z_{k+1}} \oplus \bigvee_{\bar i\in\bar{\mathcal{I}}} \bigwedge_{(\gamma,\sigma)\in\mathcal{G}\times\mathcal{G}_s} \big[\tilde b_{k,\gamma,\sigma}^{v^b,v^r,z_{k+1},\bar i_{(\cdot,\sigma)}} \otimes \delta_\gamma^\sigma\big]\Bigg]\Bigg\},$$

where $\tilde b_{k,\gamma,\sigma}^{v^b,v^r,z_{k+1},\bar i_{(\cdot,\sigma)}} \doteq \bigwedge_{g\in\mathcal{G}} \hat b_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},\bar i_{(g,\sigma)}}$, which is

$$= \bar\psi(\delta) \vee \Bigg\{ \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{(v^r,z_{k+1},\bar i)\in\mathcal{U}^r\times\mathcal{Z}_{k+1}\times\bar{\mathcal{I}}} \Big[e_{k+1}^{z_{k+1}} \oplus \bar b_k^{v^b,v^r,z_{k+1},\bar i} \odot \delta \Big]\Bigg\}, \tag{31}$$

where $\bar b_{k,\gamma,\sigma}^{v^b,v^r,z_{k+1},\bar i} \doteq \tilde b_{k,\gamma,\sigma}^{v^b,v^r,z_{k+1},\bar i_{(\cdot,\sigma)}}$, and we are using the $\odot$ notation as indicated in the theorem statement. Letting $\hat{\mathcal{Z}}_{k+1}$ be an indexing of $\mathcal{U}^r\times\mathcal{Z}_{k+1}\times\bar{\mathcal{I}}$, with $e_{k+1}^{\hat z}$, $\breve b_k^{\hat z}$ defined appropriately for each $\hat z\in\hat{\mathcal{Z}}_{k+1}$, this is equivalently,

$$W^K(k,\delta) = \bar\psi(\delta) \vee \Bigg\{ \bigwedge_{v^b\in\mathcal{U}^b} \bigvee_{\hat z\in\hat{\mathcal{Z}}_{k+1}} \Big[e_{k+1}^{\hat z} \oplus \breve b_k^{\hat z} \odot \delta \Big]\Bigg\},$$

and applying the max-min distributive property again, this is

$$= \bar\psi(\delta) \vee \Bigg\{ \bigvee_{\{\hat z_{v^b}\}_{v^b\in\mathcal{U}^b}\in(\hat{\mathcal{Z}}_{k+1})^{U^b}} \bigwedge_{v^b\in\mathcal{U}^b} \Big[\breve e_{k+1}^{\hat z_{v^b}} \oplus \breve b_k^{\hat z_{v^b}} \odot \delta \Big]\Bigg\}$$

$$= \bar\psi(\delta) \vee \Bigg\{ \bigvee_{\{\hat z_{v^b}\}_{v^b\in\mathcal{U}^b}\in(\hat{\mathcal{Z}}_{k+1})^{U^b}} \Big[\bar{\breve e}_{k+1}^{\hat z.} \oplus \bar{\breve b}_k^{\hat z.} \odot \delta \Big]\Bigg\}, \tag{32}$$

where $\bar{\breve e}_{k+1}^{\hat z.} \doteq \bigwedge_{v^b\in\mathcal{U}^b} \breve e_{k+1}^{\hat z_{v^b}}$ and $\bar{\breve b}_{k,\gamma,\sigma}^{\hat z.} \doteq \bigwedge_{v^b\in\mathcal{U}^b} \breve b_{k,\gamma,\sigma}^{\hat z_{v^b}}$ for all $(\gamma,\sigma)\in\mathcal{G}\times\mathcal{G}_s$, and (32) has the asserted form. □

6. COMPUTATIONAL COMPLEXITY

Dynamic programming suffers from a curse-of-dimensionality, where the computational cost grows exponentially with state-space dimension. The algorithm implied by Theorem 5.5 involves only idempotent-algebraic matrix-vector operations, and does not require propagating values on a grid over state space. This is typical for this class of methods [13, 14, 15, 16, 18]. However, these methods suffer from an apparent "curse-of-complexity", where the computational cost grows at a high exponential rate as one propagates backward in time. The curse-of-complexity is evident in (32), where one sees an extreme complexity growth via the increasing cardinality of the index set. This is typically attenuated by projection onto the optimal idempotent subspace of a specified dimension at each step, cf. [8]. Here however, experimentation has demonstrated that the overwhelming number of functionals in (32) contribute nowhere to $W^K(k,\cdot)$, and it is senseless to compute them merely to throw them away later. Instead, one may use Theorem 5.5 to demonstrate the finite-complexity min-plus convex function form of $W^K(k,\cdot)$, while performing the computations without full application of the distributive property, as for example at the level of either (29) or (31). In the next section, we indicate techniques for computation of the essential min-plus affine functions in (32) with less wasted operations. We also demonstrate some initial results indicating potential explanations as to why one should not typically expect the extreme complexity growth of (32). However, fuller analysis of complexity bounds appears rather technical and beyond the scope of this effort.

6.1. Efficient Computation

As the results in this section are generic, we work on $\mathbb{R}^n$ rather than $\mathcal{D}$, and we let $\mathcal{N} \doteq]1,n[$. Also, in order to conserve space, we use the abbreviation, FCMPCF, for "finite-complexity min-plus convex function". We will find that the notion of *crux* will be quite helpful in understanding the structure of FCMPCFs.

Definition 6.1. Consider the FCMPCF given by

$$f(d) \doteq \bigvee_{j\in\mathcal{J}} h^j(d) \doteq \bigvee_{j\in\mathcal{J}} \left[e^j \oplus b^j \odot d\right] \quad \forall\, d \in \mathbb{R}^n, \tag{33}$$

where $\mathcal{J}$ denotes an arbitrary finite index set. For each h^j, the crux value is e^j, the crux location is $c^j = e^j \otimes (-b^j)$, and the crux is the pair (c^j, e^j). The coefficient set is $\{(b^j, e^j) \,|\, j \in \mathcal{J}\}$, and the crux set $\{(c^j, e^j) \,|\, j \in \mathcal{J}\}$.

We note that the crux is the unique point where the $n+1$ hyperplane sections that form the graph of h^j intersect. Note also that, given a crux set $\{(c^j, e^j) \,|\, j \in \mathcal{J}\}$, one can obtain the coefficient set elements from $b^j = e^j \otimes (-c^j)$ for all $j \in \mathcal{J}$, and hence the FCMPCF as well. In particular, this relationship establishes one-to-one correspondences between the FCMPCF in form (33), the coefficient set and the crux set.

Definition 6.2. Consider the FCMPCF of (33). We say a component affine functional, $h^j(\cdot)$, is strictly active if it is the unique maximizing function at some point.

Remark 6.3. A component affine function which is not strictly active can be removed from the representation without loss of accuracy. That is, if $h^{\hat{j}}$ is not strictly active in (33), then $f(\cdot) = \bigvee_{j\in\mathcal{J}\setminus\{\hat{j}\}} h^j(\cdot)$.

Proposition 6.4. A min-plus affine functional is strictly active in (33) if and only if it is strictly active at its crux location.

Proof. Sufficiency is obvious, and so we only prove necessity. Suppose affine function $h^{\bar{j}}$ is strictly active. Then, there exists $\hat{d} \in \mathbb{R}^n$ such that

$$h^{\bar{j}}(\hat{d}) = e^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > e^j \oplus b^j \odot \hat{d} = h^j(\hat{d}) \quad \forall\, j \in \mathcal{J} \setminus \{\bar{j}\}. \tag{34}$$

Fix any $j \neq \bar{j}$. Suppose $e^j \leq b^j \odot \hat{d}$. Then, by (34), $e^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > e^j$. This implies

$$h^{\bar{j}}(c^{\bar{j}}) = e^{\bar{j}} > e^j \geq e^j \oplus b^j \odot c^{\bar{j}} = h^j(c^{\bar{j}}),$$

where (c^j, e^j) and $(c^{\bar{j}}, e^{\bar{j}})$ are the cruxes of h^j and $h^{\bar{j}}$, respectively. Now instead, suppose $e^j > b^j \odot \hat{d}$. Then, by (34), $e^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > b^j \odot \hat{d}$, which implies $b^{\bar{j}} \odot \hat{d} > b^j \odot \hat{d}$. This implies that there exists $\tilde{i} \in \mathcal{N}$, such that $b^{\bar{j}} \odot \hat{d} > b^j_{\tilde{i}} \otimes \hat{d}_{\tilde{i}}$, which implies $b^{\bar{j}}_{\tilde{i}} \otimes \hat{d}_{\tilde{i}} > b^j_{\tilde{i}} \otimes \hat{d}_{\tilde{i}}$, and hence $b^{\bar{j}}_{\tilde{i}} > b^j_{\tilde{i}}$. Consequently, $\bigoplus_{i\in\mathcal{N}} b^j_i - b^{\bar{j}}_i < 0$. This implies $h^{\bar{j}}(c^{\bar{j}}) = e^{\bar{j}} > \bigoplus_{i\in\mathcal{N}} b^j_i + e^{\bar{j}} - b^{\bar{j}}_i = \bigoplus_{i\in\mathcal{N}} b^j_i + c^{\bar{j}} = b^j \odot c^{\bar{j}} \geq h^j(c^{\bar{j}})$. □

Definition 6.5. The right-hand side of (33) is a minimal realization if for any $\hat{j} \in \mathcal{J}$, there exists $\hat{d} \in \mathbb{R}^n$ such that $\bigvee_{j\in\mathcal{J}\setminus\{\hat{j}\}} h^j(\hat{d}) < \bigvee_{j\in\mathcal{J}} h^j(\hat{d})$. In that case, $\{(b^j, e^j) \,|\, j \in \mathcal{J}\}$ is a minimal coefficient set, and $\{(c^j, e^j) \,|\, j \in \mathcal{J}\}$ is a minimal crux set.

Remark 6.6. By Proposition 6.4, an FCMPCF given by (33) is a minimal realization if for each $j \in \mathcal{J}$, $h^{\hat{j}}(c^{\hat{j}}) > h^j(c^{\hat{j}})$ for all $j \neq \hat{j}$.

Proposition 6.4 also suggests an efficient algorithm for reduction of a coefficient set for an FCMPCF to a minimal coefficient set.

An algorithm for reduction to a minimal coefficient set.

1. Suppose we are given $\{(b^j, e^j) \,|\, j \in \mathcal{J}\}$ and $\{(c^j, e^j) \,|\, j \in \mathcal{J}\}$, where $\mathcal{J} =]1, J[$. Set $\hat{j} = 1$.
2. Compute $h^j(c^{\hat{j}}) = e^j \oplus (b^j \odot c^{\hat{j}})$ for all $j \in \mathcal{J} \setminus \{\hat{j}\}$.
3. If there exists $j \in \mathcal{J} \setminus \{\hat{j}\}$ such that $e^{\hat{j}} \leq h^j(c^{\hat{j}})$, set $\mathcal{J}$ to be $\mathcal{J} \setminus \{\hat{j}\}$.
4. If all $j \in \mathcal{J}$ have been examined, we are done. Otherwise, let $\hat{j}$ be the next index in $\mathcal{J}$, and return to step 2.

We will require an efficient algorithm for computation of an FCMPCF given in form

$$f(d) \doteq \bigwedge_{k\in\mathcal{K}} \Big[\bigvee_{j\in\mathcal{J}^k} e^{k,j} \oplus b^{k,j} \odot d \Big]. \tag{35}$$

Experimentation has demonstrated that this is substantially more efficiently performed through a serial repetition of pointwise minima of pairs of FCMPCFs. Consider a generic case of two FCMPCFs with minimal coefficient sets given by $\mathcal{A}^1 \doteq \{(b^{1,j}, e^{1,j}) \,|\, j \in \mathcal{J}^1\}$ and $\mathcal{A}^2 \doteq \{(b^{2,j}, e^{2,j}) \,|\, j \in \mathcal{J}^2\}$, where $\mathcal{J}^1$ and $\mathcal{J}^2$ denote arbitrary finite index sets, with corresponding min-plus affine functionals $h^{1,j}$ and $h^{2,j}$. Then, by the distributive property,

$$\bigwedge_{k\in\{1,2\}} \Big[\bigvee_{j\in\mathcal{J}^k} e^{k,j} \oplus b^{k,j} \odot d \Big] = \bigvee_{(j_1,j_2)\in\mathcal{J}^1\times\mathcal{J}^2} \tilde{e}^{j_1,j_2} \oplus \tilde{b}^{j_1,j_2} \odot d,$$

where $\tilde{e}^{j_1,j_2} = e^{1,j_1} \oplus e^{2,j_2}$ and $\tilde{b}^{j_1,j_2} = b^{1,j_1} \oplus b^{2,j_2}$, and this is

$$\doteq \bigvee_{\breve{j}\in\breve{\mathcal{J}}} \breve{e}^{\breve{j}} \oplus \breve{b}^{\breve{j}} \odot d, \tag{36}$$

where $\breve{\mathcal{J}} =]1, \breve{J}[=]1, (\#\mathcal{J}^1)(\#\mathcal{J}^2)[$ is an indexing of $\mathcal{J}^1 \times \mathcal{J}^2$, and we let $\breve{\mathcal{A}} \doteq \{(\breve{b}^{\breve{j}}, \breve{e}^{\breve{j}}) \,|\, \breve{j} \in \breve{\mathcal{J}}\}$ denote the reindexed coefficient set $\{(\tilde{b}^{j_1,j_2}, \tilde{e}^{j_1,j_2}) \,|\, (j_1, j_2) \in \mathcal{J}^1 \times \mathcal{J}^2\}$.

An algorithm for efficient computation of (35).

1. Suppose we are given $\{(b^{k,j}, e^{k,j}) \,|\, j \in \mathcal{J}^k,\ k \in \mathcal{K}\}$, where $\mathcal{K} =]1, K[$ and $\mathcal{J}^k =]1, J^k[$ for all $k \in \mathcal{K}$. Let $\mathcal{B}^o \doteq \{(b^{1,j}, e^{1,j}) \,|\, j \in \mathcal{J}^1\}$, and set $\hat{k} = 2$.
2. Let $\mathcal{B}^n \doteq \{(b^{\hat{k},j}, e^{\hat{k},j}) \,|\, j \in \mathcal{J}^{\hat{k}}\}$, Create $\widetilde{\mathcal{B}}^o$ from (36) with $\mathcal{B}^o$, $\mathcal{B}^n$ and $\widetilde{\mathcal{B}}^o$ replacing $\mathcal{A}^1$, $\mathcal{A}^2$ and $\breve{\mathcal{A}}$ there, respectively.
3. Prune $\widetilde{\mathcal{B}}^o$ using the above "algorithm for reduction to a minimal coefficient set", and label the result $\mathcal{B}^o$.
4. If $\hat{k} = K$, we are done; $f(d) = \bigvee_{\breve{j}\in\breve{\mathcal{J}}} e^{\breve{j}} \oplus b^{\breve{j}} \odot d$. Otherwise, set $\hat{k}$ to be $\hat{k} + 1$, and return to step 2.

Remark 6.7. We have not included an algorithm for computing the pointwise maximum of a set of FCMPCFs, as this is rather straight-forward. Suppose h^1 and h^2 are two FCMPCFs with coefficient sets $\mathcal{A}^1 = \{(b^{1,j}, e^{1,j}) \,|\, j \in \mathcal{J}^1\}$ and $\mathcal{A}^2 = \{(b^{2,j}, e^{2,j}) \,|\, j \in \mathcal{J}^2\}$. Then, as no new cruxes are generated by the maximization operation, a resulting set of cruxes for $h^1 \vee h^2$ is the union of the corresponding sets of cruxes. Consequently, a coefficient set for $h^1 \vee h^2$ is $\mathcal{A}^1 \cup \mathcal{A}^2$, and one reduces this to a minimal coefficient set by application of the "algorithm for reduction to a minimal coefficient set"

As suggested above, we remark that application of the above algorithms for the IDDP iteration in form (29) or (31) is generally far more efficient than using (32). In the latter approach one applies the "Algorithm for reduction to a minimal coefficient set" only to prune the extremely large index set obtained there.

6.2. Some Computational Complexity Bounds

Although the possible computational complexity growth with backward iteration of the IDDP appears extreme, the actual complexity growth has been quite low for all examples so far tested. This phenomenon is evident in the example included below. Before proceeding to the example, we indicate some computational complexity growth bounds for pointwise minima of pairs of FCMPCFs, which as indicated above, is a key step in the IDDP propagation. Specifically, we consider complexity bounds for the computation of the coefficient set obtained from

$$\begin{aligned} f(d) &\doteq h^1(d) \wedge h^2(d) \doteq \Big[\bigvee_{j_1 \in \mathcal{J}^1} e^{1,j_1} \oplus b^{1,j_1} \odot d \Big] \wedge \Big[\bigvee_{j_2 \in \mathcal{J}^2} e^{2,j_2} \oplus b^{2,j_2} \odot d \Big] \\ &= \bigvee_{(j_1,j_2) \in \mathcal{J}^1 \times \mathcal{J}^2} \bar{e}^{j_1,j_2} \oplus \bar{b}^{j_1,j_2} \odot d, \end{aligned}$$

where the forms for h^1 and h^2 are minimal realizations, $\mathcal{J}^1 =]1, J^1[$, $\mathcal{J}^2 =]1, J^2[$, $\bar{e}^{j_1,j_2} = e^{1,j_1} \oplus e^{2,j_2}$ and $\bar{b}^{j_1,j_2} = b^{1,j_1} \oplus b^{2,j_2}$ for all $j_1 \in \mathcal{J}^1$, $j_2 \in \mathcal{J}^2$. The minimal coefficient sets for the original FCMPCFs are $\mathcal{A}^1 \doteq \{(b^{1,j}, e^{1,j}) \,|\, j \in \mathcal{J}^1\}$ and $\mathcal{A}^2 \doteq \{(b^{2,j}, e^{2,j}) \,|\, j \in \mathcal{J}^2\}$, and we seek bounds on the size of the *minimal* coefficient subset of $\{(\bar{b}^{j_1,j_2}, \bar{e}^{j_1,j_2}) \,|\, j_1 \in \mathcal{J}^1,\, j_2 \in \mathcal{J}^2\}$. We let $\mathcal{C}^1 \doteq \{(c^{1,j}, e^{1,j}) \,|\, j \in \mathcal{J}^1\}$ and $\mathcal{C}^2 \doteq \{(c^{2,j}, e^{2,j}) \,|\, j \in \mathcal{J}^2\}$ be the corresponding minimal crux sets. Also, let $\bar{\mathcal{J}} \subseteq \mathcal{J}^1 \times \mathcal{J}^2$ be an index set corresponding to a minimal realization of f, that is, let $\bar{\mathcal{A}} \doteq \{(\bar{b}^{j_1,j_2}, \bar{e}^{j_1,j_2}) \,|\, (j_1, j_2) \in \bar{\mathcal{J}}\}$ be a minimal coefficient set for f, and let $\bar{\mathcal{C}} \doteq \{(\bar{c}^{j_1,j_2}, \bar{e}^{j_1,j_2}) \,|\, (j_1, j_2) \in \bar{\mathcal{J}}\}$ be the corresponding minimal crux set.

Lemma 6.8. Suppose (b, e) and $(\hat{b}, \hat{e})$ are in the minimal coefficient set for an FCMPCF, and that $b \preceq \hat{b}$. Then $e > \hat{e}$.

P r o o f. This is obvious, as otherwise, $e \oplus b \odot d \le \hat{e} \oplus \hat{b} \odot d$ for all $d \in \mathbb{R}^n$. □

The next two results indicate conditions where the existence of one element in a minimal coefficient set, $\bar{\mathcal{A}}$, precludes inclusion of other possible elements. First, note that either $\bar{e}^{\bar{j}_1,\bar{j}_2} = e^{1,\bar{j}_1}$ or $\bar{e}^{\bar{j}_1,\bar{j}_2} = e^{2,\bar{j}_2}$.

Theorem 6.9. Suppose $(\bar{b}^{\bar{j}_1,\bar{j}_2}, \bar{e}^{\bar{j}_1,\bar{j}_2}) \in \bar{\mathcal{A}}$ and $\bar{e}^{\bar{j}_1,\bar{j}_2} = e^{1,\bar{j}_1}$. Suppose $(b^{2,\bar{j}_3}, e^{2,\bar{j}_3}) \in \mathcal{A}_2$, $\bar{j}_3 \neq \bar{j}_2$, where $b^{2,\bar{j}_3} \preceq b^{2,\bar{j}_2}$. Let $(\hat{b}, \hat{e}) \doteq (\bar{b}^{\bar{j}_1,\bar{j}_3}, \bar{e}^{\bar{j}_1,\bar{j}_3}) = (b^{1,\bar{j}_1} \oplus b^{2,\bar{j}_3}, e^{1,\bar{j}_1} \oplus e^{2,\bar{j}_3})$. Then, $(\hat{b}, \hat{e}) \notin \bar{\mathcal{A}}$.

P r o o f. As $(b^{2,\bar{j}_2}, e^{2,\bar{j}_2}), (b^{2,\bar{j}_3}, e^{2,\bar{j}_3}) \in \mathcal{A}_2$, and $b^{2,\bar{j}_3} \preceq b^{2,\bar{j}_2}$, by Lemma 6.8, $e^{2,\bar{j}_3} > e^{2,\bar{j}_2} \ge \bar{e}^{\bar{j}_1,\bar{j}_2} = e^{1,\bar{j}_1}$, and consequently,

$$\hat{e} = e^{1,\bar{j}_1} = \bar{e}^{\bar{j}_1,\bar{j}_2}. \tag{37}$$

Also, letting $(\hat{c}, \hat{e})$ denote the crux corresponding to $(\hat{b}, \hat{e})$,

$$\hat{c} = \hat{e} \otimes (-\hat{b}) = \hat{e} \otimes [-(b^{1,\bar{j}_1} \oplus b^{2,\bar{j}_3})] \ge \hat{e} \otimes [-(b^{1,\bar{j}_1} \oplus b^{2,\bar{j}_2})] = \hat{e} \otimes (-\bar{b}^{\bar{j}_1,\bar{j}_2}),$$

which by (37),

$$= \bar{e}^{\bar{j}_1,\bar{j}_2} \otimes (-\bar{b}^{\bar{j}_1,\bar{j}_2}) = \bar{c}^{\bar{j}_1,\bar{j}_2}. \tag{38}$$

By (38) and the monotonicity of f (implied by Theorem 3.10),

$$f(\hat{c}) \ge f(\bar{c}^{\bar{j}_1,\bar{j}_2}) = e^{1,\bar{j}_1} = \hat{e} = \hat{e} \oplus \hat{b} \odot \hat{c},$$

which implies that $\hat{e} \oplus \hat{b} \odot d$ is not strictly active in f at its crux, and consequently, $(\hat{b}, \hat{e}) \notin \bar{\mathcal{A}}$. □

Theorem 6.10. Suppose $(\bar{b}^{\bar{j}_1,\bar{j}_2}, \bar{e}^{\bar{j}_1,\bar{j}_2}) \in \bar{\mathcal{A}}$ and $\bar{e}^{\bar{j}_1,\bar{j}_2} = e^{1,\bar{j}_1}$. Suppose $(b^{1,\bar{j}_3}, e^{1,\bar{j}_3}) \in \mathcal{A}_1$, $\bar{j}_3 \neq \bar{j}_1$, and that $e^{1,\bar{j}_3} \leq e^{1,\bar{j}_1}$. Let $\mathcal{I} \doteq \{i \in \mathcal{N} \,|\, b_i^{1,\bar{j}_1} \geq b_i^{2,\bar{j}_2}\}$ and $\hat{\mathcal{I}} \doteq \{i \in \mathcal{N} \,|\, b_i^{1,\bar{j}_1} < b_i^{1,\bar{j}_3}\}$, and suppose $\hat{\mathcal{I}} \subseteq \mathcal{I}$. Let $(\hat{b}, \hat{e}) \doteq (\bar{b}^{\bar{j}_3,\bar{j}_2}, \bar{e}^{\bar{j}_3,\bar{j}_2}) = (b^{1,\bar{j}_3} \oplus b^{2,\bar{j}_2}, e^{1,\bar{j}_3} \oplus e^{2,\bar{j}_2})$. Then, $(\hat{b}, \hat{e}) \notin \bar{\mathcal{A}}$.

Proof. Because $\hat{\mathcal{I}} \subseteq \mathcal{I}$, for $i \in \hat{\mathcal{I}}$,

$$\hat{b}_i = b_i^{1,\bar{j}_3} \oplus b_i^{2,\bar{j}_2} = b_i^{2,\bar{j}_2} = \bar{b}_i^{\bar{j}_1,\bar{j}_2}. \tag{39}$$

On the other hand, for $i \notin \hat{\mathcal{I}}$.

$$\hat{b}_i = b_i^{1,\bar{j}_3} \oplus b_i^{2,\bar{j}_2} \leq b_i^{1,\bar{j}_1} \oplus b_i^{2,\bar{j}_2} = \bar{b}_i^{\bar{j}_1,\bar{j}_2}. \tag{40}$$

Combining (39) and (40), yields

$$\hat{b} \preceq \bar{b}^{\bar{j}_1,\bar{j}_2}. \tag{41}$$

Also, using our assumptions, $\hat{e} = e^{1,\bar{j}_3} \oplus e^{2,\bar{j}_2} \leq e^{1,\bar{j}_1} \oplus e^{2,\bar{j}_2} = \bar{e}^{\bar{j}_1,\bar{j}_2}$. Combining this with (41), and applying Lemma 6.8, $(\hat{b}, \hat{e}) \notin \bar{\mathcal{A}}$. □

Theorems 6.9 and 6.10 provide partial motivation for the observed, unexpectedly reasonable growth of complexity under the minimum operation on FCMPCFs. However, analysis in support of more complete complexity growth bounds appears to be quite technical (specifically in the main case of $n > 1$), and consequently, such is beyond the scope of this effort.

7. EXAMPLE

We consider a delay game played over a network with multiple sensor nodes. Figure 3 depicts the network in relation to physical space. The corresponding network graph is given in Figure 4. The network contains three sensor nodes (nodes 1,10,13) with corresponding action nodes (nodes 8,7,6). The remaining nodes are general communication, analysis and/or decision nodes. The network mimics a battlefield where information is conveyed from the sensors to commanders and analysts, and where processed information and commands are then propagated to the action nodes. We consider the network delay-game played over $K = 3$ time steps where $U^b = U^r = 3$. The control function $f_g^p(u_k^b, u_k^r)$ has the form:

$$f_g^p(u_k^b, u_k^r) = \begin{cases} 1 & \text{if } (g, u_k^b, u_k^r) \in \{(16,1,2),(17,2,1),(15,3,2)\}, \\ \text{-1} & \text{if } (g, u_k^b, u_k^r) \in \{(16,1,1),(17,2,2),(15,1,3)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the value is defined over the very-high-dimensional space, $\mathcal{D}$, and consequently, the solution is difficult to visualize. In order to provide some intuitive sense of the solution propagation, we plot the value over a two-dimensional affine subspace at three times. This is depicted in Figures 5, 6 and 7.

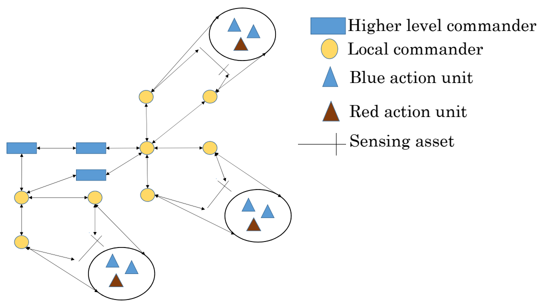

Fig. 3. Physical network.

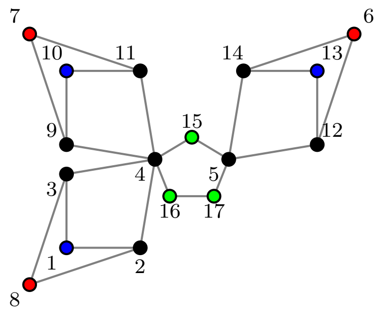

Fig. 4. Labelled graphical network.

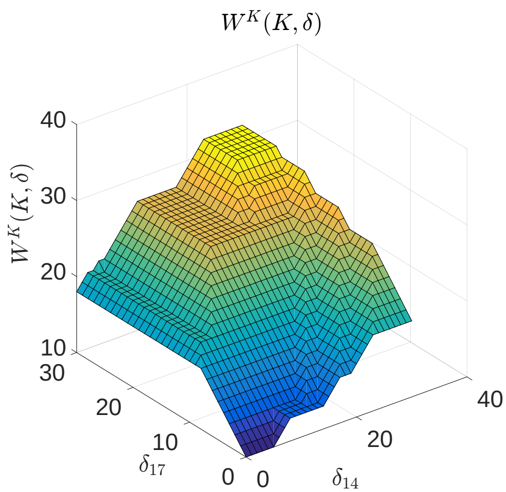

Fig. 5. $W^K(K, \delta)$.

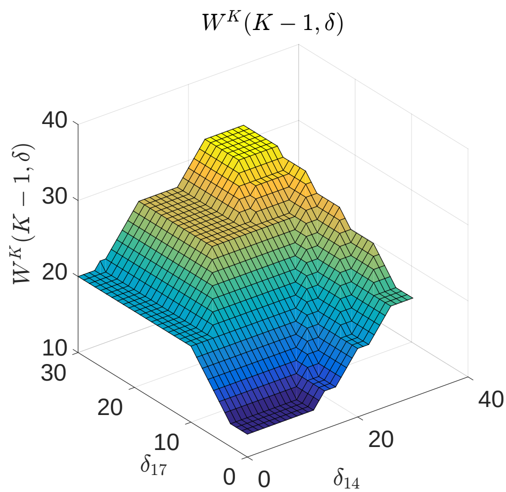

Fig. 6. $W^K(K-1, \delta)$.

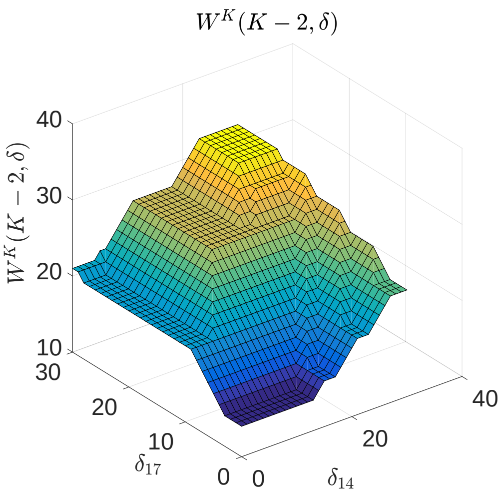

Fig. 7. $W^K(K,\delta)$.

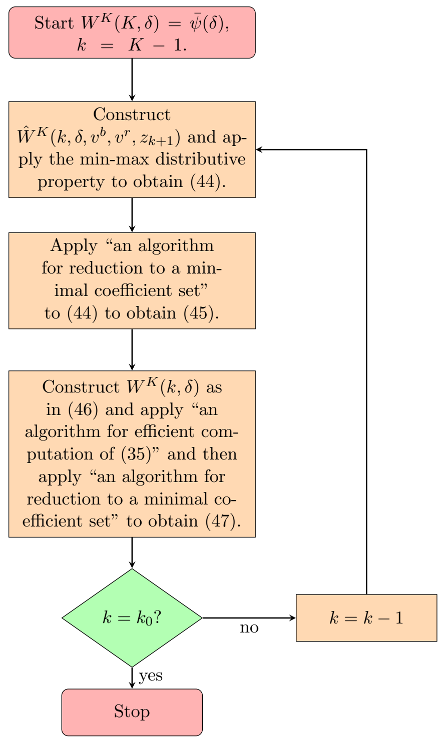

Fig. 8. Computation of solution.

8. COMMENTS ON COMPUTATION

The main component of the computation is the backward propagation of the value function, W^K, via the IDDP, and here we include some comments on the implementation of that algorithm. It may be helpful to refer to Figure 8, which provides a high-level flow chart of this backward propagation. Suppose one has constructed a terminal payoff function, $\bar{\psi}$, in the form of an FCMPCF as in (20). Without loss of generality, we have the value at the terminal time given, as in (20)–(21), by

$$W^K(K,\delta) = \bar{\psi}(\delta) = \bigvee_{z_K \in \mathcal{Z}_K} \left[e_K^{z_K} \oplus b_K^{z_K} \odot \delta \right] \quad \forall\, \delta \in \mathcal{D},$$

More generally, suppose we have

$$W^K(k+1,\delta) = \bigvee_{z_{k+1} \in \mathcal{Z}_{k+1}} \left[e_{k+1}^{z_{k+1}} \oplus b_{k+1}^{z_{k+1}} \odot \delta \right] \quad \forall\, \delta \in \mathcal{D},$$

which is certainly true at time-step $k+1=K$. One then employs (29) from the proof of Theorem 5.5 to obtain

$$W^K(k,\delta)=\bar{\psi}(\delta)\vee\bigwedge_{v^b\in\mathcal{U}^b}\ \bigvee_{(v^r,z_{k+1})\in\mathcal{U}^r\times\mathcal{Z}_{k+1}}\left[\hat{W}^K(k,\delta,v^b,v^r,z_{k+1})\right]\quad\forall\,\delta\in\mathcal{D},\tag{42}$$

where each

$$\hat{W}^K(k,\delta,v^b,v^r,z_{k+1})=e_{k+1}^{z_{k+1}}\oplus\bigwedge_{(g,\sigma)\in\mathcal{G}\times\mathcal{G}_s}\ \bigvee_{i\in\{0,1\}}\ \bigwedge_{\gamma\in\mathcal{G}}\left[\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},i}\otimes\delta_\gamma^\sigma\right],\tag{43}$$

and the values of the $\hat{b}_{k,g,\sigma,\gamma}^{v^b,v^r,z_{k+1},i}$ coefficients are given in (26),(27). Continuing as in the proof of Theorem 5.5 one applies the min-max distributive property to (43) to obtain

$$\hat{W}^K(k,\delta,v^b,v^r,z_{k+1})=\bigvee_{\bar{i}\in\bar{\mathcal{I}}}\left[e_{k+1}^{z_{k+1}}\oplus\bar{b}_k^{v^b,v^r,z_{k+1},\bar{i}}\odot\delta\right]\quad\forall\,\delta\in\mathcal{D},\tag{44}$$

where $\bar{b}_k^{v^b,v^r,z_{k+1},\bar{i}}$ and $\bar{\mathcal{I}}$ are given above (31) in the proof of Theorem 5.5. Observing that for each $k\in]0,K-1[$, $v^b\in\mathcal{U}^b$, $v^r\in\mathcal{U}^r$ and $z_{k+1}\in\mathcal{Z}_{k+1}$, (44) is an FCMPCF, one applies "an algorithm for reduction to a minimal coefficient set" to obtain

$$\hat{W}^K(k,\delta,v^b,v^r,z_{k+1})=\bigvee_{j\in\mathcal{J}^{(v^b,v^r,z_{k+1})}}\left[\check{e}_{k+1}^{z_{k+1}}\oplus\check{b}_k^{v^b,v^r,z_{k+1},j}\odot\delta\right]\quad\forall\,\delta\in\mathcal{D},\tag{45}$$

where $\mathcal{J}^{(v^b,v^r,z_{k+1})}\doteq]1,J^{(v^b,v^r,z_{k+1})}[$ and $J^{(v^b,v^r,z_{k+1})}$ represents the complexity of the $\hat{W}^K(k,\delta,v^b,v^r,z_{k+1})$ after reduction to a minimal coefficient set. Substituting this into (42) yields

$$W^K(k,\delta)=\bigwedge_{v^b\in\mathcal{U}^b}\left[\bar{\psi}(\delta)\vee\bigvee_{v^r,z_{k+1}\in\mathcal{U}^r\times\mathcal{Z}_{k+1}}\ \bigvee_{j\in\mathcal{J}^{(v^b,v^r,z_{k+1})}}\left(\check{e}_{k+1}^{z_{k+1}}\oplus\check{b}_k^{v^b,v^r,z_{k+1},j}\odot\delta\right)\right]\tag{46}$$

for all $\delta\in\mathcal{D}$. We observe that the right-hand side of (46) is in the form of (35). Applying "an algorithm for efficient computation of (35)", will result in an FCMPCF representation that is equivalent to (46). Another application of "an algorithm for reduction to a minimal coefficient set" will compute the minimal coefficient set of that FCMPCF, resulting in

$$W^K(k,\delta)=\bigvee_{z\in\mathcal{Z}_k}\left[e_k^z\oplus b_k^z\odot\delta\right]\quad\forall\,\delta\in\mathcal{D},\tag{47}$$

where $\mathcal{Z}_k=]1,Z_k[$, and Z_k is the minimal complexity of $W^K(k,\cdot)$. This is in the same form as $W^K(k+1,\cdot)$. If $k=k_0$, then the computation is finished; otherwise one repeats the procedure detailed here with $k\to k-1$.

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A. SOME DELAYED PROOFS

Proof. *[Proof of Corollary 3.14.]* Let $\hat{f}:\mathbb{R}^n\to\mathbb{R}$ be given by $\hat{f}(x)\doteq f(\hat{C}^{-1}x)$ for all $x\in\mathbb{R}^n$. Then,

$$|\hat{f}(x)-\hat{f}(y)| = |f(\hat{C}^{-1}x)-f(\hat{C}^{-1}y)| \le \|\hat{C}(\hat{C}^{-1}x-\hat{C}^{-1}y)\|_\infty = \|x-y\|_\infty.$$

Noting that $\hat{f}$ also maintains the monotonicity of f, $\hat{f}$ satisfies the conditions of Theorem 3.11, and consequently, there exist countable sets, $\hat{\mathcal{Z}}$ and $\{\hat{b}^z\in\mathbb{R}^n\,|\,z\in\mathcal{Z}\}$, such that for any such $\hat{f}$, there exist $\{\hat{e}^z\in\overline{\mathbb{R}}\,|\,z\in\mathcal{Z}\}$ such that $\hat{f}(x)=\bigvee_{z\in\hat{\mathcal{Z}}}\left[\hat{e}^z\oplus\hat{b}^z\odot x\right]$ for all $x\in\mathbb{R}^n$. This implies $f(x)=\bigvee_{z\in\hat{\mathcal{Z}}}\left[\hat{e}^z\oplus\hat{b}^z\odot(\hat{C}x)\right]$. □

Proof. *[Proof of Theorem 3.19.]* For $\bar{x}\in\mathcal{X}$, let $e^{\bar{x}}\doteq f(\bar{x})$ and $b^{\bar{x}}=e^{\bar{x}}\otimes(-\bar{x})$. Let $h^{\bar{x}}(x)\doteq e^{\bar{x}}\oplus b^{\bar{x}}\odot x$ for all $x\in\mathcal{X}$. Then, $\bigvee_{\bar{x}\in\mathcal{X}}h^{\bar{x}}(x)\ge h^x(x)=f(x)$ for all $x\in\mathcal{X}$. On the other hand, by Lemma 3.18, $h^{\bar{x}}(x)\le f(x)$ for all $x,\bar{x}\in\mathcal{X}$, and consequently, $\bigvee_{\bar{x}\in\mathcal{X}}h^{\bar{x}}(x)\le f(x)$ for all $x\in\mathcal{X}$. □

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