

Idempotent Method for Continuous-Time Stochastic Control and Complexity Attenuation[★]

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Abstract: We consider a min-plus based numerical method for solution of finite time-horizon control of nonlinear diffusion processes. The approach belongs to the class of curse-of-dimensionality-free methods. The min-plus distributive property is required. The price to pay is a very heavy curse-of-complexity. These methods perform well due to the complexity-attenuation step. This projects the solution down onto a near-optimal min-plus subspace. Keywords: stochastic, nonlinear control, numerical methods.

1. INTRODUCTION

It is now well-known that many classes of deterministic control problems may be solved by max-plus or min-plus numerical methods. These methods include max-plus basis-expansion approaches [1], [5], [7], as well as the more recently developed curse-of-dimensionality-free methods [7], [10]. It has recently been discovered that idempotent methods are applicable to stochastic control and games. The methods are related to the above curse-of-dimensionality-free methods for deterministic control. In particular, a min-plus based method was developed for stochastic control problems [9].

The first such methods for stochastic control were developed only for discrete-time problems. Here, we will remove the severe restriction to discrete-time problems. This extension requires overcoming significant technical hurdles. We will first define a parameterized set of operators, approximating the dynamic programming operator. We obtain the solutions to the problem of backward propagation by repeated application of the approximating operators. Using techniques from the theory of viscosity solutions, we show that the solutions converge to the viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) associated with the original problem.

The problem is now reduced to backward propagation by these approximating operators. The min-plus distributive property is employed. A generalization of this distributive property, applicable to continuum versions will be obtained. This will allow interchange of expectation over normal random variables (and other random variables with range in \mathbb{R}^m) with infimum operators. At each time-step,

the solution will be represented as an infimum over a set of quadratic forms. Use of the min-plus distributive property will allow us to maintain that solution form as one propagates backward in time. Backward propagation is reduced to simple standard-sense linear algebraic operations for the coefficients in the representation. We also demonstrate that the assumptions on the representation which allow one to propagate backward one step are inherited by the representation at the next step. The difficulty with the approach is an extreme curse-of-complexity, wherein the number of terms in the min-plus expansion grows very rapidly as one propagates. The complexity growth will be attenuated via projection onto a lower dimensional min-plus subspace at each time step. That is, at each step, one desires to project onto the optimal subspace relative to the solution approximation. Importantly, the subspace is not set a priori. Using some tools from convex analysis and minimax theory, we show the optimal projection is achieved by pruning the current set of quadratic forms.

2. DEFINITION AND DYNAMIC PROGRAM

We begin by defining the specific class of problems which will be addressed here. Let the dynamics take the form

$$d\xi_s = f^{\mu_s}(\xi_s, u_s) ds + \sigma^{\mu_s}(\xi_s, u_s) dB_s, \quad \xi_t = x \in \mathbb{R}^n, \quad (1)$$

where $f^m(x, u)$ is measurable, with more assumptions on it to follow. The u_s and μ_s will be control inputs taking values in $U \subset \mathbb{R}^p$ and $\mathcal{M} =]1, M[= \{1, 2, \dots, M\}$, respectively. In practice, we often find it useful to allow both a continuum-valued control component and a finite set-valued component, where the latter is used to allow approximation of more general nonlinear Hamiltonians, c.f. [7] for motivation. Also, $\{B, \mathcal{F}\}$ is an l -dimensional Brownian motion on the probability space (Ω, \mathcal{F}, P) , where \mathcal{F}_0 contains all the P -negligible elements of \mathcal{F} and $\sigma^m(x, u)$ is an $n \times l$ matrix-valued diffusion coefficient. We will be

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examining a finite time-horizon formulation, with terminal time, T , and will take initial time $t \in [0, T]$.

The payoff (to be minimized) will be

$$J(t, x, u, \mu) \doteq \mathbf{E} \left\{ \int_t^T l^{\mu_s}(\xi_s, u_s) ds + \Psi(\xi_T) \right\} \quad (2)$$

where

$$\Psi(x) \doteq \inf_{z_T \in Z'_T} \{g_T(x, z_T)\}, \quad (3)$$

where $l^m(x, u)$ and $g_T(x, z)$ are measurable, and $(Z'_T, d_{Z'_T})$ is a separable metric space. The value function is

$$V(t, x) = \inf_{u \in \mathcal{U}_t, \mu \in \tilde{\mathcal{M}}_t} J(t, x, u, \mu), \quad (4)$$

where \mathcal{U}_t (*resp.* $\tilde{\mathcal{M}}_t$) is the set of \mathcal{F}_t -progressively measurable controls, taking values in U (*resp.* \mathcal{M}) such that there exists a strong solution to (1).

We will assume that the given data in the dynamics and payoff satisfy the following conditions:

- (A1) U is a closed subset of \mathbb{R}^p .
(A2) There exist $L_f, L_\sigma, K_f, K_\sigma > 0$ such that for any $x, x' \in \mathbb{R}^n, u, u' \in U, m, m' \in \mathcal{M}$,

$$|f^m(x, u) - f^{m'}(x', u')| \leq L_f(|x - x'| + |u - u'| + |m - m'|),$$

$$\|\sigma^m(x, u) - \sigma^{m'}(x', u')\| \leq L_\sigma(|x - x'| + |u - u'| + |m - m'|),$$

$$|f^m(x, u)| \leq K_f(1 + |x| + |u|), \quad \|\sigma^m(x, u)\| \leq K_\sigma.$$

- (A3) There exist $L_l, K_l > 0$ and $\nu > 0$ such that for any $x, x' \in \mathbb{R}^n, u, u' \in U, m, m' \in \mathcal{M}$

$$|l^m(x, u) - l^{m'}(x', u')| \leq L_l(1 + |x| + |x'| + |u| + |u'|) \cdot (|x - x'| + |u - u'| + |m - m'|),$$

$$-K_l + \frac{\nu}{2}|u|^2 \leq l^m(x, u) \leq K_l(1 + |x|^2 + |u|^2).$$

- (A4) There exist $L_\Psi, K_\Psi > 0$ such that for any $x, x' \in \mathbb{R}^n$,

$$|\Psi(x) - \Psi(x')| \leq L_\Psi(1 + |x| + |x'|)|x - x'|,$$

$$-K_\Psi \leq \Psi(x) \leq K_\Psi(1 + |x|^2).$$

At a formal level, one expects that $V(t, x)$ will satisfy the following Hamilton-Jacobi-Bellman partial differential equation (HJB PDE):

$$\frac{\partial V}{\partial t}(t, x) + \mathcal{H}(x, D_x V(t, x), D_x^2 V(t, x)) = 0, \quad (5)$$

$$V(T, x) = \Psi(x), \quad x \in \mathbb{R}^n, \quad (6)$$

$$\begin{aligned} & \mathcal{H}(x, D_x V(t, x), D_x^2 V(t, x)) \\ &= \inf_{u \in U} \min_{m \in \mathcal{M}} \left\{ \frac{1}{2} \text{tr}(\sigma^m(x, u) \sigma^m(x, u)^T D_x^2 V(t, x)) \right. \\ & \quad \left. + f^m(x, u) \cdot D_x V(t, x) + l^m(x, u) \right\}. \end{aligned} \quad (7)$$

Under restrictive conditions, it is known that $V(t, x)$ can be characterized as the unique viscosity solution of (5)/(6). However, for more general conditions including assumptions (A1)–(A4), if $V(t, x)$ is associated with the set of control processes satisfying $\mathbf{E} \left[\int_t^T |u_s|^2 ds \right] < \infty$, it

is shown in [4] that $V(t, x)$ is the unique viscosity solution of (5)/(6) satisfying the quadratic growth condition

$$|V(t, x)| \leq K(1 + |x|^2), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \quad (8)$$

See [4, Theorem 3.1 and Remark 3.1]. We need to additionally show convergence of discrete-time approximations to the viscosity solution. We approximate the viscosity solution of (5)/(6) by discrete-time stochastic control problems. To define a discrete-time stochastic control value, we introduce a family of parameterized operators $\{F_{t,s}\}_{t < s}$ acting on $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$:

$$F_{t,s}\phi(x) = \inf_{u \in U, m \in \mathcal{M}} \{l^m(x, u)(s - t) + \mathbf{E}[\phi(x + f^m(x, u)(s - t) + \sigma^m(x, u)(B_s - B_t))]\}. \quad (9)$$

Let $\pi_N = \{t_0 = 0 < t_1 < \dots < t_N = T\}$ be a partition of $[0, T]$ with the step size $\Delta_N = t_{i+1} - t_i = T/N$ ($i = 0, 1, \dots, N - 1$). We define a discrete-time value function $V^N(t, x)$ ($(t, x) \in [0, T] \times \mathbb{R}^n$) associated with π_N recursively backward in time:

$$V^N(t, x) = \begin{cases} \Psi(x), & t = T, \\ F_{t, t_{i+1}} V^N(t_{i+1}, \cdot)(x), & t_i \leq t < t_{i+1}, \end{cases} \quad (10)$$

where $F_{t, t_{i+1}} V^N(t_{i+1}, \cdot)(x)$ is $F_{t, t_{i+1}} \phi(x)$ with $\phi(\cdot) = V^N(t_{i+1}, \cdot)$. We obtain the following theorem regarding approximation of the viscosity solution.

Theorem 2.1. Under (A1)–(A4), $V^N(t, x)$ converges to a continuous viscosity solution $V(t, x)$ of (5) with (6) as $N \rightarrow \infty$ uniformly on each compact set of $[0, T] \times \mathbb{R}^n$. Indeed, $V(t, x)$ is the unique viscosity solution satisfying the quadratic growth condition: for some $K > 0$,

$$|V(t, x)| \leq K(1 + |x|^2), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \quad (11)$$

3. MIN-PLUS DISTRIBUTIVE PROPERTY

We will use an infinite version of the min-plus distributive property to move a certain infimum from inside an expectation operator to outside. It will be familiar to control and game theorists who often work with notions of non-anticipative mappings and strategies. One version of such appeared in [9]. However, the assumptions in that result are too restrictive for the class of problems we are considering. Instead, we generalize that result to the following, where the proof appears in [6].

Theorem 3.1. Let (Z, d_Z) be a separable metric space and (W, d_W) be a separable Banach space with Borel sets \mathcal{B}^W . Let p be a finite measure on (W, \mathcal{B}^W) , and let $\bar{D} \doteq p(W)$. Let $h: W \times Z \rightarrow \mathbb{R}$ be Borel measurable. Suppose there exists $\bar{z} \in Z$ such that

$$\int_W h(w, \bar{z}) dP(w) < \infty \quad (12)$$

and suppose for given $\varepsilon > 0$, there exists $R < \infty$ such that

$$\int_{(\bar{B}_R(0))^c} \inf_{z \in Z} h(w, z) dP(w) \geq -\varepsilon. \quad (13)$$

Also, suppose that given $\varepsilon > 0$ and $R < \infty$, there exists $\delta > 0$ such that $|h(w, z) - h(\bar{w}, z)| < \varepsilon$ for all $z \in Z$ and all $w, \bar{w} \in \bar{B}_R(0)$ such that $d_W(w, \bar{w}) < \delta$. Lastly, we suppose that either Z is countable or $h(w, z)$ is continuous on z for each $w \in W$ (where of course, the

former supposition can be embedded within the latter, but that is less illuminating). Then,

$$\int_W \inf_{z \in Z} h(w, z) dP(w) = \inf_{\tilde{z} \in \tilde{Z}} \int_W h(w, \tilde{z}(w)) dP(w),$$

where $\tilde{Z} \doteq \{\tilde{z} : W \rightarrow Z \mid \text{Borel measurable}\}$.

4. DISTRIBUTED DYNAMIC PROGRAMMING

We will use the above infinite-version of the min-plus distributive property in conjunction with the dynamic programming principle of Section 2. This will yield what we refer to as an idempotent distributed dynamic programming principle (IDDPP).

Recall our discrete-time value function, $V^N(t_k, x)$ given by (10) for $t_k \in \pi_N$ and $x \in \mathbb{R}^n$. Suppose that at time, t_{k+1} , one has representation

$$V^N(t_{k+1}, x) = \inf_{z \in Z_{k+1}} g_{k+1}^N(x, z), \quad (14)$$

where $(Z_{k+1}, d_{Z_{k+1}})$ is a separable metric space. Recalling form (3), and letting $g_N^N(x, z) = g_T(x, z)$ and $Z_N = Z'_T$, we see that $V^N(t_N, x) = V^N(T, x) = \Psi(x)$ has this form. Then the dynamic program of (9), (10) with $\Delta = T/N$ becomes

$$\begin{aligned} V^N(t_k, x) &= \inf_{u \in U} \min_{m \in \mathcal{M}} \left\{ l^m(x, u) \Delta \right. \\ &\quad \left. + \mathbf{E} \left[\inf_{z \in Z_{k+1}} g_{k+1}^N(x + f^m(x, u) \Delta + \sigma^m(x, u) w, z) \right] \right\} \\ &= \inf_{u \in U} \min_{m \in \mathcal{M}} \int_W \inf_{z \in Z_{k+1}} \left[l^m(x, u) \Delta \right. \\ &\quad \left. + g_{k+1}^N(x + f^\Delta(x, u, m, w), z) \right] dP_\Delta(w), \end{aligned} \quad (15)$$

where $f^\Delta(x, u, m, w) = f^m(x, u) \Delta + \sigma^m(x, u) w$, P_Δ is the measure corresponding to a Gaussian random variable over \mathbb{R}^l with mean zero and covariance ΔI , and $W = \mathbb{R}^l$.

We outline the approach to be followed. Specifically, we will use the min-plus distributive property of Theorem 3.1 to move the infimum over Z_{k+1} outside the integral. Letting

$$\tilde{Z}_{k+1} \doteq \{\tilde{z}_{k+1} : W \rightarrow Z_{k+1} \mid \text{Borel measurable}\},$$

we will have

$$\begin{aligned} V^N(t_k, x) &= \inf_{u \in U} \min_{m \in \mathcal{M}} \inf_{\tilde{z}_{k+1} \in \tilde{Z}_{k+1}} \int_W l^m(x, u) \Delta \\ &\quad + g_{k+1}^N(x + f^\Delta(x, u, m, w), \tilde{z}_{k+1}(w)) dP_\Delta(w). \end{aligned} \quad (16)$$

Since we will suppose that Z_k is bounded and U is possibly unbounded, it is convenient to handle the infimum over u separately in (16). From (16), we can take

$$V^N(t_k, x) = \inf_{z \in Z_k} g_k^N(x, z), \quad (17)$$

where $Z_k = \mathcal{M} \times \tilde{Z}_{k+1}$ and for $x \in \mathbb{R}^n$ and $z = (m, \tilde{z}_{k+1}) \in Z_k$,

$$\begin{aligned} g_k^N(x, z) &= \inf_{u \in U} \int_W l^m(x, u) \Delta \\ &\quad + g_{k+1}^N(x + f^\Delta(x, u, m, w), \tilde{z}_{k+1}(w)) dP_\Delta(w). \end{aligned} \quad (18)$$

Consequently, the general form of (14) will be inherited from $V^N(t_{k+1}, \cdot)$ to $V^N(t_k, \cdot)$, and one can propagate

backward in this manner indefinitely. This is what we referred to above as the IDDPP. We note that (16) itself can be used as an IDDPP if U is bounded. With significant effort, one obtains our main IDDPP result.

Theorem 4.1. In addition to (A1)–(A3), suppose that $(Z'_T, d_{Z'_T})$ and $g_T : \mathbb{R}^n \times Z'_T \rightarrow \mathbb{R}$ satisfy the following:

(i-T) $(Z'_T, d_{Z'_T})$ is a bounded and closed subset of a separable Banach space \mathcal{X}'_T where metric $d_{Z'_T}$ is induced by the norm of \mathcal{X}'_T .

(ii-T) There exist $K, L > 0$ such that for any $x, x' \in \mathbb{R}^n$, $z \in Z'_T$,

$$\begin{aligned} -K &\leq g_T(x, z) \leq K(1 + |x|^2), \\ |g_T(x, z) - g_T(x', z)| &\leq L(1 + |x| + |x'|)|x - x'|. \end{aligned}$$

(iii-T) There exists $\hat{L} > 0$ such that for any $z, z' \in Z'_T$, $x \in \mathbb{R}^n$,

$$|g_T(x, z) - g_T(x, z')| \leq \hat{L}(1 + |x|^2) d_{Z'_T}(z, z').$$

Letting $Z_N = Z'_T$ and $g_N^N(x, z) = g_T(x, z)$, (17) with (18) holds for all $k \in]0, N[$.

5. FULLY QUADRATIC FORMS

The above theory was somewhat general in form. With a small modification, one could include u in the index set, letting $Z_k = U \times \mathcal{M} \times \tilde{Z}_{k+1}$ rather than $Z_k = \mathcal{M} \times \tilde{Z}_{k+1}$. However, we believe the most computationally useful form will occur when the problem is quadratic in u with $U = \mathbb{R}^p$. Consequently, we now consider such a special problem class, where in particular, we use the fact that the quadratic nature of the control-dependence allows one to analytically obtain the g_k^N for each $z \in Z_k$.

To be precise, we now suppose

$$f^m(x, u) = A^m x + B^m u + \lambda_d^m, \quad \sigma^m(x, u) = \sigma^m, \quad (19)$$

$$\begin{aligned} l^m(x, u) &= \frac{1}{2} x^T \bar{H}_1^m x + x^T \bar{H}_2^m u + \frac{1}{2} u^T \bar{H}_3^m u + \bar{\lambda}_1^m \cdot x \\ &\quad + \bar{\lambda}_2^m \cdot u + \bar{c}^m, \quad U = \mathbb{R}^p, \end{aligned} \quad (20)$$

$$g_T(x, z) = \frac{1}{2} x^T Q_T(z) x + b_T^T(z) x + c_T(z). \quad (21)$$

We do not use the earlier assumptions. Instead, we replace them with assumptions on the problem data in the above form which will yield a special case of the earlier assumptions. We assume:

(AQ.1) $(Z'_T, d_{Z'_T})$ is a bounded and closed subset of a separable Banach space \mathcal{X}'_T where metric $d_{Z'_T}$ is induced by norm $\|\cdot\|_{\mathcal{X}'_T}$ of \mathcal{X}'_T .

(AQ.2) There exists $D_1 < \infty$ such that $|Q_T(z)|, |b_T(z)|, |c_T(z)| \leq D_1$ for all $z \in Z'_T$.

(AQ.3) There exists $L < \infty$ such that

$$\begin{aligned} \max \{ |Q_T(z) - Q_T(\bar{z})|, |b_T(z) - b_T(\bar{z})|, |c_T(z) - c_T(\bar{z})| \} \\ \leq L d_{Z'_T}(z, \bar{z}) \quad \forall z, \bar{z} \in Z'_T. \end{aligned}$$

(AQ.4) $Q_T(z) \geq 0$ for all $z \in Z'_T$.

$$(AQ.5) \quad \overline{H}_3^m > 0 \quad \text{and} \quad \begin{pmatrix} \overline{H}_1^m & \overline{H}_2^m \\ \overline{H}_2^{mT} & \overline{H}_3^m \end{pmatrix} \geq 0 \quad \forall m \in \mathcal{M}.$$

Note that (AQ.1)–(AQ.5), along with the specific form of (19)–(21) immediately imply that (A1)–(A4) and (i-T)–(iii-T) hold. Recall (18), and suppose that for some $k \in]0, N-1[$,

$$V_{k+1}^N(x) \doteq V^N(t_{k+1}, x) = \inf_{z \in Z_{k+1}} g_{k+1}^N(x, z), \quad (22)$$

where

$$g_{k+1}^N(x, z) = \frac{1}{2} x^T Q_{k+1}(z) x + b_{k+1}^T(z) x + c_{k+1}(z). \quad (23)$$

Note that $g_N^N(x, z) = g_T(x, z)$ is of this form.

After some computations, we arrive at the following, where we again do not include the details of the algebra.

$$\begin{aligned} g_k^N(x, z) &= \frac{1}{2} x^T Q_k(\tilde{z}, m) x + b_k^T(\tilde{z}, m) x + c_k(\tilde{z}, m) \\ &= \frac{1}{2} x^T Q_k(z) x + b_k^T(z) x + c_k(z) \end{aligned} \quad (24)$$

for appropriate Q_k, b_k, c_k , where $z = (m, \tilde{z}) \in Z_k$. Using this form, one can verify that conditions (AQ.1)–(AQ.4) are inherited. We do not include the details. This yields the following minor, but computationally useful, variant of main Theorem 4.1.

Theorem 5.1. Consider the special dynamics and cost of (19)–(21). Suppose (AQ.1)–(AQ.5) hold. Let $g_N^N(x, z) = g_T(x, z)$ and $Z_N = Z_T'$. Then, for all $k \in]0, N-1[$

$$V_k^N(x) = V^N(t_k, x) = \inf_{z \in Z_k} g_k^N(x, z),$$

where $g_k^N(x, z)$ is given by (24) with $Z_k = \tilde{Z}_{k+1} \times \mathcal{M}$, for appropriate Q_k, b_k, c_k .

Due to space limitations, we do not include the computations which yield Q_k, b_k, c_k from $Q_{k+1}, b_{k+1}, c_{k+1}$.

6. COMPLEXITY REDUCTION

The key to this class of methods lies in the repeated projection of the solution down onto a low-dimensional (min-plus) subspace. Importantly, the subspace is chosen at each step so as to minimize the error induced by this projection. One sees from Theorem 5.1 that after one step of the IDDPP, the set Z_k will have the cardinality of the continuum even in the case where Z_{k+1} is finite. Consequently, the projection down to a finite-dimensional subspace is a critical step. For the class of problems where the solution may be well-represented by a reasonable set of quadratic functions, this method can be expected to perform quite well. It is worth noting that this complexity condition is independent from the dimension of the space; one may have problems with low complexity (by this vaguely defined metric) solutions in high dimensional spaces and vice-versa.

We now begin the analysis of the projection of the V_k^N down to a low-dimensional min-plus subspace. This will be a two-step procedure. First, we indicate a simple reduction from an infinite-dimensional index set, Z_k , to a, possibly very large but finite, subset. Second, we discuss the theory by which, given a desired dimension of the subspace (i.e., index set size), one finds the optimal subspace of that

dimension. We also indicate a computational approach for approximately achieving this projection. We will work entirely in the special case of Section 5.

For the reduction from infinite-dimensional to finite-dimensional, we simply note that by (AQ.2), there exists $D < \infty$ such that $\{(Q_k(z), b_k(z), c_k(z)) \mid z \in Z_k\} \subseteq \overline{B}_{3D}(0)$, which is compact. Therefore, given $\varepsilon > 0$, there exists a finite ε -net for the set, which yields our finite-dimensional min-plus projection.

As indicated above, the key to computational feasibility will be projection of \widehat{V}_k^N down to a low-dimensional min-plus subspace with small error, if such is possible. We will use an approach similar to that in [8]. That is, we will devise an optimization problem to select the small set of quadratics whose pointwise minimum (min-plus sum) will constitute the min-plus projection. The constraint in the optimization will be slightly relaxed from the ideal constraint. With this optimization problem formulation, we will see that our problem reduces to selection of a small set of quadratics out of the set whose min-plus sum is \widehat{V}_k^N .

Let \bar{T} be the set of all triples of coefficients for quadratic functions on \mathbb{R}^n . That is, we take

$$\bar{T} = \{(Q, b, c) \mid Q \in \mathbb{R}^{n \times n}, \text{ symmetric}; b \in \mathbb{R}^n; c \in \mathbb{R}\}.$$

For convenience, we will designate a triple by a single symbol, say $\tau \in \bar{T}$, where τ will correspond to some triple, say $\tau = (Q, b, c) \in \bar{T}$. We let $C^Q(\mathbb{R}^n)$ denote the subset of $C(\mathbb{R}^n)$ consisting of the quadratic functions mapping into \mathbb{R} . We let $\mathcal{G} : \bar{T} \rightarrow C^Q$ be given as follows. For $\tau = (Q, b, c) \in \bar{T}$, let

$$\mathcal{G}[\tau](x) = \mathcal{G}[(Q, b, c)](x) \doteq \frac{1}{2} x^T Q x + b^T x + c.$$

Consider a finite set of quadratic coefficients, $T_{\bar{M}} \subset \bar{T}$ given by

$$T_{\bar{M}} \doteq \{\tau_{\bar{m}} \mid \bar{m} \in \bar{\mathcal{M}}\}, \quad (25)$$

where $\bar{\mathcal{M}} =]1, \bar{M}[\doteq \{1, 2, \dots, \bar{M}\}$ with $\bar{M} \in \mathbb{N}$. (Here, of course, each $\tau_{\bar{m}}$ corresponds to a triple of coefficients.) Note that \bar{M} may be very large. In particular, we will be interested in the specific case where $T_{\bar{M}}$ corresponds to the set of quadratics which generate \widehat{V}_k^N , in which case, $\bar{M} = N_\varepsilon = \#\widehat{Z}_k$. Further, in that case, each triple in $T_{\bar{M}}$ which we now designate as $\tau_{\bar{m}} = (\widehat{Q}^{\bar{m}}, \widehat{b}^{\bar{m}}, \widehat{c}^{\bar{m}})$ corresponds (via bijection) to some triple $(Q_k(\widehat{Z}), b_k(\widehat{Z}), c_k(\widehat{Z}))$. We look for an optimal smaller set of triples, $A_{\bar{N}} = \{\alpha_{\bar{n}} \mid \bar{n} \in \bar{\mathcal{N}}\} \subset \bar{T}$, $\bar{\mathcal{N}} =]1, \bar{N}[$ where $\bar{N} < \bar{M}$.

Let Π be the set of probability measures over $(\mathbb{R}^n, \mathcal{B}^n)$ where \mathcal{B}^n is the collection of Borel sets over \mathbb{R}^n . Integration with respect to a $\pi \in \Pi$ will be designated by $\pi(dx)$. Let λ be a measure over $(\mathbb{R}^n, \mathcal{B}^n)$ such that $\int_{\mathbb{R}^n} \mathcal{G}[\tau](x) \lambda(dx) < \infty$ for all $\tau \in \bar{T}$. The optimization problem will take the form

$$\text{Minimize } J(A_{\bar{N}}) \doteq \int_{\mathbb{R}^n} \min_{\bar{n} \in \bar{\mathcal{N}}} \mathcal{G}[\alpha_{\bar{n}}](x) \lambda(dx) \quad (26)$$

$$\begin{aligned} \text{subject to } & \int_{\mathbb{R}^n} \mathcal{G}[\alpha_{\bar{n}}](x) \pi(dx) \geq \min_{\bar{m} \in \bar{\mathcal{M}}} \int_{\mathbb{R}^n} \mathcal{G}[\tau_{\bar{m}}](x) \pi(dx) \\ & \forall \pi \in \Pi, \forall \bar{n} \in \bar{\mathcal{N}}. \end{aligned} \quad (27)$$

We note that (27) is a relaxation of constraint

$$\mathcal{G}[\alpha_{\bar{n}}](x) \geq \min_{\bar{m} \in \bar{\mathcal{M}}} \mathcal{G}[\tau_{\bar{m}}](x) \quad \forall x \in \mathbb{R}^n. \quad (28)$$

Further, constraint (28) is typical in min-plus expressions. If there exists $\bar{n}' \in \bar{\mathcal{N}}$, $\bar{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that

$$\mathcal{G}[\alpha_{\bar{n}'}](\bar{x}) \leq \min_{\bar{m} \in \bar{\mathcal{M}}} \mathcal{G}[\tau_{\bar{m}}](\bar{x}) - \varepsilon,$$

then addition of more $\alpha_{\bar{n}}$ to $A_{\bar{N}}$ can never correct this error as one will still have

$$\min_{\bar{n} \in \bar{\mathcal{N}}} \mathcal{G}[\alpha_{\bar{n}}](\bar{x}) \leq \min_{\bar{m} \in \bar{\mathcal{M}}} \mathcal{G}[\tau_{\bar{m}}](\bar{x}) - \varepsilon.$$

We will analyze problem (26),(27) to show that this formulation corresponds to minimization of a monotonically increasing, concave function over a cornice set [8], where this class of problems has the property that the minimizing set, $A_{\bar{N}}$, will be a subset of the original set, $T_{\bar{M}}$. That is, the optimal min-plus subspace of dimension N will be a subset of the original set $T_{\bar{M}}$.

For $\tau, \alpha \in \bar{T}$, we say $\alpha \succeq \tau$ if

$$\int_{\mathbb{R}^n} \mathcal{G}[\alpha](x) \pi(dx) \geq \int_{\mathbb{R}^n} \mathcal{G}[\tau](x) \pi(dx) \quad \forall \pi \in \Pi.$$

Note that \succeq is a partial order on \bar{T} . It is also worth noting the following.

Lemma 6.1. $\alpha \succeq \tau$ if and only if $\mathcal{G}[\alpha](x) \geq \mathcal{G}[\tau](x)$ for all $x \in \mathbb{R}^n$.

Given Lemma 6.1, one might wonder why we do not use the condition $\mathcal{G}[\alpha](x) \geq \mathcal{G}[\tau](x)$ for all $x \in \mathbb{R}^n$ to define our partial order. When combined with the minimization operation as on the right-hand side of (27), these are different; the integral definition will be more useful.

For $\pi \in \Pi$, let $\bar{\mathcal{G}}^\pi[\tau] \doteq \int_{\mathbb{R}^n} \mathcal{G}[\tau](x) \pi(dx)$.

Lemma 6.2. For any $\pi \in \Pi$, $\bar{\mathcal{G}}^\pi$ is a linear functional on \bar{T} .

Let

$$S^{\bar{M}} \doteq \left\{ p \in \mathbb{R}^{\bar{M}} \mid p_{\bar{m}} \in [0, 1] \forall \bar{m} \in \bar{M}, \sum_{\bar{m} \in \bar{M}} p_{\bar{m}} = 1 \right\}.$$

For any $p \in S^{\bar{M}}$, define $p \cdot T_{\bar{M}} = \sum_{\bar{m} \in \bar{M}} p_{\bar{m}} \tau_{\bar{m}} \in \bar{T}$. We say

$$\langle T_{\bar{M}} \rangle = \left\{ \tau \in \bar{T} \mid \tau = p \cdot T_{\bar{M}} \text{ for some } p \in S^{\bar{M}} \right\}, \quad (29)$$

which is obviously the convex hull of $T_{\bar{M}}$.

Lemma 6.3. For any $\pi \in \Pi$,

$$\min_{\bar{m} \in \bar{M}} \bar{\mathcal{G}}^\pi[\tau_{\bar{m}}] = \min_{p \in S^{\bar{M}}} \bar{\mathcal{G}}^\pi[p \cdot T_{\bar{M}}],$$

or equivalently,

$$\min_{\bar{m} \in \bar{M}} \int_{\mathbb{R}^n} \mathcal{G}[\tau_{\bar{m}}](x) \pi(dx) = \min_{p \in S^{\bar{M}}} \int_{\mathbb{R}^n} \mathcal{G}[p \cdot T_{\bar{M}}](x) \pi(dx).$$

One then immediately has:

Lemma 6.4. Fix any $\pi \in \Pi$. Let $\alpha \in T_{\bar{M}}$. Then, $\bar{\mathcal{G}}^\pi[\alpha] \geq \min_{\bar{m} \in \bar{M}} \bar{\mathcal{G}}^\pi[\tau_{\bar{m}}]$ if and only if $\bar{\mathcal{G}}^\pi[\alpha] \geq \min_{p \in S^{\bar{M}}} \bar{\mathcal{G}}^\pi[p \cdot T_{\bar{M}}]$.

Using Lemma 6.4, our problem (26),(27) becomes

$$\text{Minimize } J(A_{\bar{N}}) \doteq \int_{\mathbb{R}^n} \min_{\bar{n} \in \bar{\mathcal{N}}} \mathcal{G}[\alpha_{\bar{n}}](x) \lambda(dx) \quad (30)$$

$$\text{subject to } \bar{\mathcal{G}}^\pi[\alpha_{\bar{n}}] \geq \min_{p \in S^{\bar{M}}} \bar{\mathcal{G}}^\pi[p \cdot T_{\bar{M}}]$$

$$\forall \pi \in \Pi, \forall \bar{n} \in \bar{\mathcal{N}}. \quad (31)$$

Given $T \subset \bar{T}$, we define the upper cone of T as

$$\mathcal{U}[T] \doteq \{ \alpha \in \bar{T} \mid \alpha \succeq \tau \text{ for some } \tau \in T \}.$$

We also define the cornice of T as $\mathcal{C}[T] \doteq \mathcal{U}[\langle T \rangle]$ (see [8]), and let $\mathcal{C}^{\bar{N}}[T] \doteq \{ \mathcal{C}[T] \}^{\bar{N}}$ denote the outer product of $\mathcal{C}[T]$, \bar{N} times.

Theorem 6.5. Condition (31) holds if and only if $A_{\bar{N}} \in \mathcal{C}^{\bar{N}}[T_{\bar{M}}]$.

Combining Theorem 6.5 with (30)/(31), we see that problem (26)/(27) is equivalent to:

$$\text{Minimize } J(A_{\bar{N}}) \doteq \int_{\mathbb{R}^n} \min_{\bar{n} \in \bar{\mathcal{N}}} \mathcal{G}[\alpha_{\bar{n}}](x) \lambda(dx) \quad (32)$$

$$\text{subject to } A_{\bar{N}} \in \mathcal{C}^{\bar{N}}[T_{\bar{M}}]. \quad (33)$$

Define the inherited partial order on outer product space $[\bar{T}]^{\bar{N}}$ by $A_{\bar{N}}^1 \preceq A_{\bar{N}}^2$ where $A_{\bar{N}}^j = \{ \alpha_{\bar{n}}^j \mid \alpha_{\bar{n}}^j \in \bar{T} \forall \bar{n} \in \bar{\mathcal{N}} \}$ for $j \in \{1, 2\}$, if $\alpha_{\bar{n}}^1 \preceq \alpha_{\bar{n}}^2$ for all $\bar{n} \in \bar{\mathcal{N}}$.

Lemma 6.6. $J : [\bar{T}]^{\bar{N}} \rightarrow \mathbb{R}$ is monotonically increasing (relative to \preceq) and concave.

Now, given the monotonicity and concavity of J , we can apply [8] Theorem 3.1 to assert:

Theorem 6.7. Let J^* be the optimal value of problem (32),(33), or equivalently, (26),(27). Then, there exists $A_{\bar{N}}^* = \{ \tau_{\bar{m}_{\bar{n}}} \mid \bar{n} \in \bar{\mathcal{N}} \}$ such that $\tau_{\bar{m}_{\bar{n}}} \in T_{\bar{M}}$ for all $\bar{n} \in \bar{\mathcal{N}}$, and such that $A_{\bar{N}}^*$ satisfies (33) (equivalently, (27)) and $J(A_{\bar{N}}^*) = J^*$.

The value of Theorem 6.7 is that we know that the optimal set of quadratics is a subset of the original $T_{\bar{M}}$ set. It is worth noting that condition $A_{\bar{N}} \in \mathcal{C}^{\bar{N}}[T_{\bar{M}}]$ is *not* identical to the condition

$$\mathcal{G}[\alpha_{\bar{n}}](x) \geq \min_{\bar{m} \in \bar{M}} \mathcal{G}[\tau_{\bar{m}}](x) \quad \forall x \in \mathbb{R}^n \forall \bar{n} \in \bar{\mathcal{N}}.$$

Instead, it is somewhat more conservative.

6.1 Tractable Pruning

Now that we understand that pruning is optimal (relative to the chosen criterion and constraints), we consider means for approximate optimal pruning. In particular, solution of even the reduced-complexity pruning problem with the above criterion remains computationally demanding. Consequently, we search instead for a tractable suboptimal pruning method.

Firstly, one would like to remove those quadratics that do not contribute at all to the value function. Let $\hat{\Delta} \geq 0$, and suppose $\hat{m} \in \bar{M}$ is such that

$$\hat{\Delta} + \int_{\mathbb{R}^n} \mathcal{G}[\tau^{\hat{m}}](x) \pi(dx) \geq \min_{\bar{m} \in \bar{M} \setminus \{\hat{m}\}} \int_{\mathbb{R}^n} \mathcal{G}[\tau^{\bar{m}}](x) \pi(dx), \quad (34)$$

for all $\pi \in \Pi$. One obtains:

Proposition 6.8. Suppose

$$\int_{\mathbb{R}^n} \mathcal{G}[\tau^{\hat{m}}](x) \pi(dx) \geq \min_{\bar{m} \in \bar{M} \setminus \{\hat{m}\}} \int_{\mathbb{R}^n} \mathcal{G}[\tau^{\bar{m}}](x) \pi(dx),$$

for all $\pi \in \Pi$. Then,

$$\begin{aligned} & \int_{\mathbb{R}^n} \min_{\bar{m} \in \bar{\mathcal{M}}} \left\{ \mathcal{G}[\tau^{\bar{m}}](x) \right\} \pi(dx) \\ &= \int_{\mathbb{R}^n} \min_{\bar{m} \in \bar{\mathcal{M}} \setminus \{\hat{m}\}} \left\{ \mathcal{G}[\tau^{\bar{m}}](x) \right\} \pi(dx), \quad \forall \pi \in \Pi \end{aligned}$$

and

$$\min_{\bar{m} \in \bar{\mathcal{M}}} \left\{ \mathcal{G}[\tau^{\bar{m}}](x) \right\} = \min_{\bar{m} \in \bar{\mathcal{M}} \setminus \{\hat{m}\}} \left\{ \mathcal{G}[\tau^{\bar{m}}](x) \right\}, \quad \forall x \in \mathbb{R}^n.$$

In other words, $\tau^{\hat{m}}$ does not contribute at all to the pointwise minimum, and so may be removed without affecting the value function approximation. Further, by the monotonicity of the semigroup operator, no progeny of $\tau^{\hat{m}}$ contribute to the value function approximation in subsequent steps, and so removal of $\tau^{\hat{m}}$ does not affect the solution at any time-step.

Now let $\hat{\Delta}_{\hat{m}} \doteq \inf\{\hat{\Delta} \mid (34) \text{ holds}\}$. It is apparent that $\hat{\Delta}_{\hat{m}}$ provides some measure (loosely used) of the merit of $\tau^{\hat{m}}$ relative to the minimum. This leads to the following rough heuristic: To achieve a good subset of size \bar{N} from the original set of \bar{M} quadratics, those with the lowest $\hat{\Delta}_{\bar{m}}$ values should be chosen. This leads to the optimization of a submodular functional, which is an issue we will discuss elsewhere. Here, however, we consider the approximate evaluation of $\hat{\Delta}_{\bar{m}}$.

6.2 Gaussians and Convex Program

Given a set of quadratics, $\{(Q^{\bar{m}}, b^{\bar{m}}, c^{\bar{m}}) \mid \bar{m} \in \bar{\mathcal{M}}'\}$ indexed as above by its coefficients, corresponding to some set $\{\tau^{\bar{m}} \mid \bar{m} \in \bar{\mathcal{M}}'\}$ for some set $\bar{\mathcal{M}}' \subseteq \bar{\mathcal{M}}$, and a specific quadratic, $(Q^{\hat{m}}, b^{\hat{m}}, c^{\hat{m}})$ corresponding to $\tau^{\hat{m}}$, we wish to compute an approximate $\hat{\Delta}_{\hat{m}}$. In order to make the problem tractable, we consider only those $\pi \in \Pi$ corresponding to normal distributions.

Let \mathcal{D} be the set of positive-definite, symmetric $n \times n$ matrices. The appropriate set of normal distributions may be indexed by $(D, \bar{x}) \in \mathcal{D} \times \mathbb{R}^n$. Let $(Q^{\bar{m}}, b^{\bar{m}}, c^{\bar{m}})$ correspond to $\tau^{\bar{m}}$, and $\pi \in \Pi$ correspond to $(D, \bar{x}) \in \mathcal{D} \times \mathbb{R}^n$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{G}[\tau^{\bar{m}}](x) \pi(dx) &= \frac{1}{(2\pi)^{n/2} \sqrt{\det(D)}} \int_{\mathbb{R}^n} \left[\frac{1}{2} x^T Q^{\bar{m}} x \right. \\ &\quad \left. + b^{\bar{m}T} x + c^{\bar{m}} \right] \exp \left\{ -\frac{1}{2} (x - \bar{x})^T D^{-1} (x - \bar{x}) \right\} dx, \end{aligned}$$

and after manipulation, one finds this is

$$= \frac{1}{2} \bar{x}^T Q^{\bar{m}} \bar{x} + b^{\bar{m}T} \bar{x} + c^{\bar{m}} + \frac{1}{2} \text{tr} [Q^{\bar{m}} D].$$

Note that this is linear in D and quadratic in \bar{x} .

Let $\hat{\Pi}$ denote the set of normal probability measures over \mathbb{R}^n . With a little work, one finds that $\hat{\Delta}_{\hat{m}}$ may be computed by solution of:

$$\begin{aligned} & \max_{(D, \bar{x}, z) \in \mathcal{D} \times \mathbb{R}^{n+1}} z - \left[\frac{1}{2} \bar{x}^T Q^{\hat{m}} \bar{x} + b^{\hat{m}T} \bar{x} + c^{\hat{m}} + \frac{1}{2} \text{tr} [Q^{\hat{m}} D] \right] \\ & \text{subject to: } z - \left[\frac{1}{2} \bar{x}^T Q^{\bar{m}} \bar{x} + b^{\bar{m}T} \bar{x} + c^{\bar{m}} \right. \\ & \quad \left. + \frac{1}{2} \text{tr} [Q^{\bar{m}} D] \right] \leq 0 \quad \forall \bar{m} \in \bar{\mathcal{M}}'. \end{aligned} \quad (35)$$

This last problem may be converted to an LMI. Following a similar approach to that used in [10], one obtains a relaxed dual problem:

$$\begin{aligned} & \min_{(\lambda, \zeta) \in S^{\bar{\mathcal{M}}'} \times \mathbb{R}} \left\{ \zeta \right\} \\ & \left[\begin{array}{c} \sum_{\bar{m}} \lambda_{\bar{m}} c^{\bar{m}} - c^{\hat{m}} - \zeta \quad \frac{1}{2} \left(\sum_{\bar{m}} \lambda_{\bar{m}} b^{\bar{m}} - b^{\hat{m}} \right)^T \\ \frac{1}{2} \left(\sum_{\bar{m}} \lambda_{\bar{m}} b^{\bar{m}} - b^{\hat{m}} \right) \quad \frac{1}{2} \left(\sum_{\bar{m}} \lambda_{\bar{m}} Q^{\bar{m}} - Q^{\hat{m}} \right) \end{array} \right] \preceq 0, \\ & \sum_{\bar{m}} \lambda_{\bar{m}} Q^{\bar{m}} - Q^{\hat{m}} \preceq 0 \end{aligned}$$

where $S^{\bar{\mathcal{M}}'} \doteq \{\lambda \in \mathbb{R}^{\bar{\mathcal{M}}'} \mid \lambda_m \in [0, 1] \forall m, \sum_m \lambda_m = 1\}$, which takes the form of an LMI. These LMIs are solved repeatedly to determine the contribution of each quadratic to the overall value function approximation. As noted above, the actual pruning corresponds to optimization of a submodular function on the space of subsets of $\bar{\mathcal{M}}$, and the details are not included due to length restrictions.

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