A max-plus fundamental solution semigroup for a class of lossless wave equations*

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Abstract

A new max-plus fundamental solution semigroup is presented for a class of lossless wave equations. This new semigroup is developed by employing the action principle to encapsulate the propagation of all possible solutions of a given wave equation in the evolution of the value function of an associated optimal control problem. The max-plus fundamental solution semigroup for this optimal control problem is then constructed via dynamic programming, and used to formulate the fundamental solution semigroup for the original wave equation. An application of this semigroup to solving two-point boundary value problems is discussed via an example.

1 Introduction

The *action principle* postulates that any trajectory generated by a system that conserves energy must render the *action functional* stationary in the sense of the calculus of variations [7, 8, 9]. In previous work by the authors [4, 11, 12], connections between the action principle and optimal control have been exploited to solve two-point boundary value problems constrained by energy conserving systems. In that work, the action functional is interpreted as the integrated running payoff in an optimal control problem, in which a fictitious terminal payoff is introduced to capture boundary data. By considering sufficiently short time horizons, it is shown that the total payoff involved is either concave or convex, so that stationarity of the action functional can be achieved as an extremum in the optimal control problem. Consequently, the optimal control problem can be solved, and the state feedback characterization of the optimal control obtained (via dynamic programming) can be used to propagate solutions of the conservative system of interest to meet the boundary conditions required. By formulating a *fundamental solution* to this optimal control problem, i.e. one that captures solutions of the optimal control problem for *any* terminal payoff, it is possible to solve any two point boundary value problems formulated in this way, see [4, 11, 12].

In this paper, attention is restricted to two-point boundary value problems for energy conserving infinite dimensional systems, and their solution via stationary action and optimal control. The objective is to further generalize recent work in this direction, beyond the simple scalar wave equation that is used to model a vibrating string, see [4]. In particular, attention is expanded to consider abstract second order partial differential equations (PDEs) of the form

$$\ddot{x} = -\mathcal{A}\, x\,, \tag{1.1}$$

in which x and $\dot{x}$ may (for example) be interpreted respectively as the distributed position (or deflection) and velocity of some vibrating mechanical structure. Operator $\mathcal{A}$ is assumed to have some general properties that are summarized as follows (see [2] for definitions).

Assumption 1.1.

1) *$\mathcal{A}$ is linear, unbounded, positive, and self-adjoint on a subset $\mathscr{X}_0 \doteq dom(\mathcal{A})$ of an $\mathcal{L}_2$-space $\mathscr{X}$ of real valued functions;*
2) *$-\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on $\mathscr{X}$; and*
3) *$\mathcal{A}$ has a compact inverse.*

Note that $\mathcal{A}$ is densely defined, i.e. $\overline{\mathscr{X}_0} \equiv \mathscr{X}$, and closed by 2) and the Hille-Yosida Theorem (see for example [14, Theorem 5.3]). The closed property also follows by 3), as $\mathcal{A}^{-1} \in \mathscr{L}(\mathscr{X})$ exists. In a mechanical setting, domain $\mathscr{X}_0$ may be interpreted as the space of sufficiently smooth functions that describe (for example) admissible deflections of a vibrating beam or structure, subject to its boundary data. Typically, $\mathscr{X}_0$ is a Sobolev space. By definition, $\mathcal{A}$ has a unique, positive, self-adjoint, and boundedly invertible square root $\mathcal{A}^{\frac{1}{2}}$, whose domain $\mathscr{X}_{\frac{1}{2}} \doteq \mathrm{dom}\,(\mathcal{A}^{\frac{1}{2}})$ defines a real Hilbert space with inner product $\langle x,\, \xi\rangle_{\frac{1}{2}} \doteq \langle \mathcal{A}^{\frac{1}{2}}\, x,\, \mathcal{A}^{\frac{1}{2}}\, \xi\rangle$ and

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associated norm $\|x\|_{\frac{1}{2}}$ defined for all $x, \xi \in \mathscr{X}_{\frac{1}{2}}$. Using this Hilbert space, potential and kinetic energy functionals V and T may be defined and associated with (1.1), with

$$V(x) \doteq \tfrac{1}{2}\, \|x\|_{\frac{1}{2}}^2\,, \quad T(\dot x) \doteq \tfrac{1}{2}\, \|\dot x\|^2 = \|\mathcal{J}\, \dot x\|_{\frac{1}{2}}^2\,, \tag{1.2}$$

for all $x \in \mathscr{X}_{\frac{1}{2}}$, $\dot x \in \mathscr{X}$, in which $\mathcal{J} \doteq (\mathcal{A}^{\frac{1}{2}})^{-1} \in \mathscr{L}(\mathscr{X})$. In order to see that system (1.1) is energy conserving, note that the instantaneous total energy associated with a deflection x and velocity $\dot x$ in (1.1) is $E(x, \dot x) \doteq V(x) + T(\dot x)$. Hence, differentiating along trajectories and recalling that $\mathcal{J}^2 = \mathcal{A}^{-1}$,

$$\begin{aligned} &\tfrac{d}{ds} E(x(s), \dot x(s)) \\ &= \langle \nabla_x V(x(s)),\, \dot x(s)\rangle_{\frac{1}{2}} + \langle \nabla_{\dot x} T(\dot x(s)),\, \ddot x(s)\rangle_{\frac{1}{2}} \\ &= \langle x(s) + \mathcal{J}^2\, \ddot x(s),\, \dot x(s)\rangle_{\frac{1}{2}} = 0\,, \end{aligned}$$

for all $s \in \mathbb{R}_{\ge 0}$. That is, the abstract second order PDE (1.1) is conservative with respect to the potential and kinetic energies (1.2). Hence, the action principle is applicable with action functional

$$\int_0^t V(x(s)) - T(\dot x(s))\, ds \tag{1.3}$$

for all $t \in \mathbb{R}_{\ge 0}$, see [4]. Together, the characteristic equations corresponding to the calculus of variations problem defined by the action principle applied to (1.3) yield the abstract Cauchy problem (see Remark 2.2)

$$\begin{pmatrix} \dot x(s) \\ \dot p(s) \end{pmatrix} = \mathcal{A}^\vee \begin{pmatrix} x(s) \\ p(s) \end{pmatrix}, \quad s \in \mathbb{R}_{\ge 0}\,, \tag{1.4}$$

in which p denotes the costate variable,

$$\mathcal{A}^\vee \doteq \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{A} & 0 \end{pmatrix}, \quad \mathrm{dom}\,(\mathcal{A}^\vee) = \mathscr{Y}_0 \doteq \mathscr{X}_0 \vee \mathscr{X}_{\frac{1}{2}}\,,$$

and $\vee$ is used to unambiguously denote the direct sum. By inspection, any classical solution of (1.4) satisfies $\ddot x = \dot p = -\mathcal{A}\, x$, which is precisely (1.1).

2 Action principle and optimal control problem

In order to formulate an optimal control problem that encapsulates the action principle, define the abstract Cauchy problem [2, 14]

$$\dot\xi(s) = w(s)\,, \qquad \xi(0) = x \in \mathscr{X}_{\frac{1}{2}}\,, \tag{2.5}$$

in which $\xi(s)$ denotes the infinite dimensional state at time $s \in [0, t]$, evolved from initial state $x \in \mathscr{X}_{\frac{1}{2}}$ via the velocity input $w \in \mathscr{W}[0, s] \doteq \mathcal{L}_2([0, s]; \mathscr{X}_{\frac{1}{2}})$. In view of (1.3), see also [4], define the payoff functional $J_\psi^\mu(t, \cdot, \cdot) : \mathscr{X}_{\frac{1}{2}} \times \mathscr{W}[0, t] \to \mathbb{R}$ given $\mu, t \in \mathbb{R}_{\ge 0}$ and concave terminal payoff $\psi : \mathscr{X}_{\frac{1}{2}} \to \mathbb{R}$ by

$$J_\psi^\mu(t, x, w) \doteq \int_0^t V(\xi(s)) - T^\mu(w(s))\, ds + \psi(\xi(t)) \tag{2.6}$$

where the perturbed potential $T^\mu(w) \doteq T(w) + \frac{\mu^2}{2}\, \|w\|_{\frac{1}{2}}^2$ approximates the actual potential $T(w)$ for all $\mu \ne 0$ and $w \in \mathscr{X}_{\frac{1}{2}}$ (with $T^0 = T$). It may be shown [4, Theorem 2.1] w.l.o.g. that $J_\psi^\mu(t, x, \cdot)$ is strictly concave for all $t \in [0, \bar t^\mu)$, where

$$\bar t^\mu \doteq \mu\, \sqrt{2}\,. \tag{2.7}$$

Note in particular that the action functional J_0^0 as per (1.3) need not be concave (or convex), but its approximation J_0^μ is, for $t \in [0, \bar t^\mu)$. Hence, it is useful to consider an *approximating* optimal control problem defined via the value function $W^\mu : \mathbb{R}_{\ge 0} \times \mathscr{X}_{\frac{1}{2}} \to \mathbb{R}$ defined for $\mu \in \mathbb{R}_{>0}$ by

$$W^\mu(t, x) \doteq \sup_{w \in \mathscr{W}[0,t]} J_\psi^\mu(t, x, w) \tag{2.8}$$

for all $t \in [0, \bar t^\mu)$, $x \in \mathscr{X}_{\frac{1}{2}}$. A standard application of dynamic programming yields that W^μ is the unique solution of the corresponding Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE)

$$\begin{aligned} &0 = -\frac{\partial W}{\partial t}(t, x) + H(x, \nabla_x W(t, x))\,, \\ &W(0, x) = \psi(x) \end{aligned} \tag{2.9}$$

for all $t \in [0, \bar t^\mu)$, $x \in \mathscr{X}_{\frac{1}{2}}$, where

$$H(x, p) \doteq \tfrac{1}{2}\, \|x\|_{\frac{1}{2}}^2 + \tfrac{1}{2}\, \|\mathcal{I}_\mu^{\frac{1}{2}}\, \mathcal{A}^{\frac{1}{2}}\, p\|_{\frac{1}{2}}^2 \tag{2.10}$$

for all $x, p \in \mathscr{X}_{\frac{1}{2}}$, with $\mathcal{I}_\mu : \mathscr{X} \to \mathscr{X}_0$ defined by

$$\mathcal{I}_\mu \doteq (\mathcal{I} + \mu^2\, \mathcal{A})^{-1}\,. \tag{2.11}$$

(Relevant properties of $\mathcal{I}_\mu$, including existence of its square root, are catalogued in Lemma A.1, see Appendix A.) In terms of the unique solution $W = W^\mu$ of (2.9), the optimal input in (2.8) is

$$w^*(s) = k(s, \xi^*(s))\,, \tag{2.12}$$

in which ξ^* denotes the trajectory of system (2.5) in feedback with $k(s, x) \doteq \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \nabla_x W(t - s, x)$. The characteristic equations corresponding to the Hamiltonian H of (2.10) define the abstract Cauchy problem

$$\begin{pmatrix} \dot\xi(s) \\ \dot\pi(s) \end{pmatrix} = \mathcal{A}_\mu^\vee \begin{pmatrix} \xi(s) \\ \pi(s) \end{pmatrix}, \quad s \in \mathbb{R}_{\ge 0}\,, \tag{2.13}$$

in which

$$\mathcal{A}_\mu^\vee \doteq \begin{pmatrix} 0 & \mathcal{I}_\mu^{\frac12} \\ -\mathcal{A}^{\frac12}\,\mathcal{I}_\mu^{\frac12}\,\mathcal{A}^{\frac12} & 0 \end{pmatrix}, \quad \begin{array}{l} \operatorname{dom}(\mathcal{A}_\mu^\vee) \\ \quad = \mathscr{Y}_{\frac12} \doteq \mathscr{X}_{\frac12} \vee \mathscr{X}\,. \end{array}$$

Here, $\mathscr{Y}_{\frac12}$ is a Hilbert space equipped with the inner product defined by $\langle (x,p),\,(\xi,\,\pi)\rangle_\vee = \langle x,\,\xi\rangle_{\frac12} + \langle p,\,\pi\rangle$ for all $x,\xi\in\mathscr{X}_{\frac12}$, $p,\pi\in\mathscr{X}$.

In order for the optimal control problem (2.8) to be useful in encapsulating (1.1), it is essential that $\mathcal{A}^\vee$ and $\mathcal{A}_\mu^\vee$ of (1.4) and (2.13) both generate semigroups, and that these semigroups converge in an appropriate sense as $\mu\to 0$. These properties are established via the following lemma and theorem, the proofs of which are deferred to Appendix B.

Lemma 2.1. *Given $\mu\in(0,1]$, the operators $\mathcal{A}_\mu^\vee$ and $\mathcal{A}^\vee$ of (2.13) and (1.4) satisfy the following properties:*

1) $\mathcal{A}_\mu^\vee \in \mathscr{L}(\mathscr{Y}_{\frac12})$;

2) $\mathcal{A}_\mu^\vee$ generates a uniformly continuous semigroup of bounded linear operators $\mathcal{T}_\mu^\vee(t)\in\mathscr{L}(\mathscr{Y}_{\frac12})$, $t\in\mathbb{R}_{\ge 0}$;

3) $\mathcal{A}^\vee$ is unbounded, closed, and densely defined on $\mathscr{Y}_0\doteq\mathscr{X}_0\vee\mathscr{X}_{\frac12}$ (with $\overline{\mathscr{Y}_0}=\mathscr{Y}_{\frac12}$);

4) $\mathcal{A}^\vee$ generates a strongly continuous semigroup of bounded linear operators $\mathcal{T}^\vee(t)\in\mathscr{L}(\mathscr{Y}_{\frac12})$, $t\in\mathbb{R}_{\ge 0}$;

5) $\mathcal{A}_\mu^\vee$ converges strongly to $\mathcal{A}^\vee$ as $\mu\to 0$, i.e. $\lim_{\mu\to 0}\|\mathcal{A}_\mu^\vee\,y - \mathcal{A}^\vee\,y\|_\vee = 0$ for all $y\in\mathscr{Y}_0$.

Theorem 2.1. *$\mathcal{T}_\mu^\vee(t)$ converges strongly to $\mathcal{T}^\vee(t)$ as $\mu\to 0$, uniformly for $t\in\mathbb{R}_{>0}$ in compact intervals. In particular, $\lim_{\mu\to 0}\left\|\mathcal{T}_\mu^\vee(t)\,y - \mathcal{T}^\vee(t)\,y\right\|_\vee = 0$ for all $y\in\mathscr{Y}_{\frac12}$, $t\in\mathscr{I}$, $\mathscr{I}\subset\mathbb{R}_{\ge 0}$ compact.*

The strong convergence property set out in Theorem 2.1 states that *any* solution of the approximate Cauchy problem (2.13) defined by the characteristics of the optimal control problem (2.8) converges to the corresponding solution of the exact Cauchy problem (2.13) defined by the characteristics of the action principle. It is in this sense that the optimal control problem (2.8) approximates solutions of (1.1).

Remark 2.1. While the approximation $\mathcal{T}_\mu^\vee(t)$ of $\mathcal{T}^\vee(t)$ improves with decreasing $\mu\in\mathbb{R}_{>0}$, it is important to note that the time horizon $t\in[0,\bar t^\mu)$ of the associated optimal control problem (2.8) must similarly decrease via (2.7). Indeed, in the limit as $\mu\to 0$, the time horizon on which stationarity of action is achieved as a maximum converges to zero. Hence, in order for this optimal control approach to be useful, it is crucial that there exist a mechanism for extending the horizon beyond the bound defined by (2.7). As will be demonstrated, one such mechanism involves the concatenation of control horizons $[0,t]$, $t\in(0,\bar t^\mu)$, using a *max-plus fundamental solution semigroup* for the optimal control problem (2.8).

Remark 2.2. The abstract Cauchy problems (1.4) and (2.13) are constructed via a transformation of the characteristic equations defined by the Hamiltonian H of (2.10). Differentiating H, the characteristic equations involved are given by

$$\begin{aligned} \dot\xi(s) &= \nabla_p H(\xi(s),\hat\pi(s)) = \mathcal{A}^{\frac12}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac12}\,\hat\pi(s)\,, \\ \dot{\hat\pi}(s) &= -\nabla_x H(\xi(s),\hat\pi(s)) = -\xi(s)\,, \end{aligned} \tag{2.14}$$

for $s\in\mathbb{R}_{\ge 0}$. The transformation of interest is defined by $\pi(s)\doteq\mathcal{A}^{\frac12}\,\mathcal{I}_\mu^{\frac12}\,\mathcal{A}^{\frac12}\,\hat\pi(s)$, $s\in\mathbb{R}_{\ge 0}$. Applying this transformation in (2.14) yields (2.13), with (1.4) defined so as to agree in the limit as $\mu\to 0$. This construction ensures that the domains of operators $\mathcal{A}^\vee$ and $\mathcal{A}_\mu^\vee$ are as per (1.4) and (2.13), see [2, Example 2.2.5, p.34], so that Lemma 2.1 and Theorem 2.1 may be applied.

3 Max-plus fundamental solution semigroup

The max-plus algebra is a commutative idempotent semifield over $\mathbb{R}^-\doteq\mathbb{R}\cup\{-\infty\}$ equipped with the addition and multiplication operations $\oplus$ and $\otimes$ defined by $a\oplus b\doteq\max(a,b)$ and $a\otimes b\doteq a+b$. Max-plus integration of a functional $f:\mathscr{X}\to\mathbb{R}^-$ is defined by $\int_{\mathscr{X}}^\oplus f(x)\,dx\doteq\sup_{x\in\mathscr{X}} f(x)$. A max-plus linear max-plus integral operator is an operator of the form $\mathcal{F}\psi\doteq\int_{\mathscr{X}}^\oplus F(\cdot,z)\otimes\psi(z)\,dz$, in which the kernel $F:\mathscr{X}\times\mathscr{X}\to\mathbb{R}^-$ is a bi-functional, and ψ is any functional for which the associated supremum exists everywhere.

A max-plus fundamental solution semigroup corresponding to the optimal control problem (2.8) is a semigroup of horizon indexed max-plus linear max-plus integral operators, with associative binary operation defined by operator composition, from which the value function W^μ of (2.8) can be computed for any terminal payoff $\psi:\mathscr{X}_{\frac12}\to\mathbb{R}^-$. There are two types of max-plus fundamental solution semigroups, called *dual* and *primal* space semigroups [3, 5, 15, 16], where the type is determined by whether the Legendre-Fenchel transform is involved in its definition. Their definition is motivated by the Lax-Oleinik semigroup of max-plus linear dynamic programming evolution operators, see [10].

With a view to defining the max-plus primal space fundamental solution semigroup for the optimal control problem (2.8), define the auxiliary value function $G_t^\mu(\cdot,z):\mathscr{X}_{\frac12}\to\mathbb{R}^-$ for each $t\in[0,\bar t^\mu)$ and $z\in\mathscr{X}_{\frac12}$ by

$$G_t^\mu(x,z) = \sup_{w\in\mathscr{W}[0,t]} J_{\delta_z}^\mu(t,x,w)\,, \tag{3.15}$$

in which $J^\mu_{\delta_z}$ is the payoff (2.6), which in turn is defined in terms of the max-plus delta functional $\delta_z : \mathscr{X}_{\frac{1}{2}} \to \mathbb{R}^-$, defined for each $z \in \mathscr{X}_{\frac{1}{2}}$ by

$$\delta_z(x) \doteq \left\{ \begin{array}{ll} 0\,, & x = z\,, \\ -\infty\,, & x \neq z\,. \end{array} \right. \tag{3.16}$$

Using the kernel G_t defined by (3.15), define the max-plus linear max-plus integral operator $\mathcal{G}^\mu_t$ by

$$\mathcal{G}^\mu_t\,\psi = \int^\oplus_{\mathscr{X}_{\frac{1}{2}}} G^\mu_t(\cdot, z) \otimes \psi(z)\, dz \tag{3.17}$$

for any $\psi : \mathscr{X}_{\frac{1}{2}} \to \mathbb{R}^-$ semiconvex such that $\mathcal{G}^\mu_t\,\psi$ is semiconvex (for further details, see [5]). Without loss of generality, it may be shown [4, 5, 15, 16] that

$$\begin{gathered} W^\mu(t,x) = (\mathcal{G}^\mu_t\,\psi)(x)\,, \\ \mathcal{G}^\mu_{t+\tau}\,\psi = \mathcal{G}^\mu_t\,\mathcal{G}^\mu_\tau\,\psi\,, \quad \mathcal{G}^\mu_0\,\psi = \psi\,, \end{gathered} \tag{3.18}$$

for all $x \in \mathscr{X}_{\frac{1}{2}}$, and all $t, \tau \in [0, \bar{t}^\mu)$ such that $t + \tau \in [0, \bar{t}^\mu)$. In particular, the first identity follows analogously to [4, Theorem 3.1], while the second and third identities follow by dynamic programming and by inspection respectively. These last two identities indicate that $\{\mathcal{G}^\mu_t\}_{t\in[0,\bar{t}^\mu)}$ defines a semigroup with composition as the associative binary operation. This defines the max-plus primal space fundamental solution semigroup for the optimal control problem (2.8).

As indicated by (3.15), kernel G^μ_t defining operator $\mathcal{G}^\mu_t$ in (3.17) is the unique solution to the HJB (2.9) with terminal payoff ψ replaced with δ_z. It may be found via a limiting argument, see [4, Section 3.3]. In particular,

$$G^\mu_t(x,z) = \lim_{c\to\infty} W^{\mu,c}(t,x,z) \tag{3.19}$$

where $W^{\mu,c}(t,x,\cdot) \doteq \sup_{w\in\mathscr{W}[0,t]} J^\mu_{\psi^{\mu,c}(\cdot,z)}(t,x,w)$,

$$\psi^{\mu,c}(x,z) \doteq -\tfrac{c}{2}\,\|\mathcal{K}_\mu\,(x-z)\|^2_{\frac{1}{2}}\,, \quad c \in \mathbb{R}_{\geq 0}\,,$$

and $\mathcal{K}_\mu \in \mathscr{L}(\mathscr{X}_{\frac{1}{2}})$ is a positive, self-adjoint, and boundedly invertible operator, see [4]. Following the aforementioned argument of [4], this limit is given by

$$\begin{aligned} G^\mu_t(x,z) = \tfrac{1}{2}\,\langle x,\, \breve{\mathcal{P}}^\mu(t)\,x\rangle_{\frac{1}{2}} + \langle x,\, \breve{\mathcal{Q}}^\mu(t)\,z\rangle_{\frac{1}{2}} \\ + \tfrac{1}{2}\,\langle z,\, \breve{\mathcal{R}}^\mu(t)\,z\rangle_{\frac{1}{2}}\,, \end{aligned} \tag{3.20}$$

in which $\breve{\mathcal{P}}^\mu, \breve{\mathcal{Q}}^\mu, \breve{\mathcal{R}}^\mu : (\delta, \bar{t}^\mu) \to \mathscr{L}(\mathscr{X}_{\frac{1}{2}})$ are the solutions of the operator differential equations

$$\begin{aligned} \dot{\breve{\mathcal{P}}}^\mu(t) &= \mathcal{I} + \breve{\mathcal{P}}^\mu(t)\,\mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac{1}{2}}\,\breve{\mathcal{P}}^\mu(t)\,, \\ \dot{\breve{\mathcal{Q}}}^\mu(t) &= \breve{\mathcal{P}}^\mu(t)\,\mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac{1}{2}}\,\breve{\mathcal{Q}}^\mu(t)\,, \\ \dot{\breve{\mathcal{R}}}^\mu(t) &= \breve{\mathcal{Q}}^m u(t)'\,\mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac{1}{2}}\,\breve{\mathcal{Q}}^\mu(t)\,, \end{aligned} \tag{3.21}$$

for $t \in [0, \bar{t}^\mu)$, defined via the limit

$$\breve{\mathcal{P}}^\mu(0) = -\breve{\mathcal{Q}}^\mu(0) = \breve{\mathcal{R}}^\mu(0) = \lim_{c\to\infty} -c\,(\mathcal{K}^\mu)^2\,,$$

and subsequently restricted to domain $(\delta, \bar{t}^\mu)$. Assertion 3) of Assumption 1.1 facilitates a representation of these operator-valued functions via the spectral theorem. In particular, [2, Theorem A.4.25, p.619] implies that $\mathcal{A}$ has the spectral decomposition

$$\mathcal{A}\,x = \sum_{n=1}^\infty \lambda_n\,\langle x,\, \tilde{\varphi}_n\rangle_{\frac{1}{2}}\,\tilde{\varphi}_n \tag{3.22}$$

for all $x \in \mathscr{X}_0 = \mathrm{dom}\,(\mathcal{A})$, in which λ_n^{-1} and $\tilde{\varphi}_n$ are the eigenvalues and eigenvectors of $\mathcal{A}^{-1} \in \mathscr{L}(\mathscr{X})$, and $\widetilde{\mathcal{B}} \doteq \{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$ defines an orthonormal basis for $\mathscr{X}_{\frac{1}{2}}$. (Note that $\lambda_n \in \mathbb{R}_{>0}$ by positivity of $\mathcal{A}$, see Assumption 1.1.) Furthermore,

$$\mathrm{dom}\,(\mathcal{A}) = \left\{ x \in \mathscr{X}_{\frac{1}{2}} \,\middle|\, \sum_{n=1}^\infty |\lambda_n|\,|\langle x,\, \tilde{\varphi}_n\rangle_{\frac{1}{2}}|^2 < \infty \right\}.$$

Operators $\mathcal{A}^{\frac{1}{2}}$ and $\mathcal{I}_\mu$ inherit corresponding representations by definition, leading to similarly represented solutions $\breve{\mathcal{P}}^\mu$, $\breve{\mathcal{Q}}^\mu$, $\breve{\mathcal{R}}^\mu$ of (3.21) of the form

$$\breve{\mathcal{P}}^\mu(t)\,x = \sum_{n=1}^\infty p^\mu_n(t)\,\langle x,\, \tilde{\varphi}_n\rangle_{\frac{1}{2}}\,\tilde{\varphi}_n \tag{3.23}$$

for all $t \in (\delta, \bar{t}^\mu)$ and $x \in \mathscr{X}_{\frac{1}{2}}$. For each $t \in (\delta, \bar{t}^\mu)$, the respective eigenvalues $p^\mu_n(t)$, $q^\mu_n(t)$, $r^\mu_n(t)$ are given by

$$\begin{gathered} p^\mu_n(t) = r^\mu_n(t) \doteq -\tfrac{1}{\omega^\mu_n}\,\frac{1}{\tan(\omega^\mu_n\,t)}\,, \\ q^\mu_n(t) \doteq +\tfrac{1}{\omega^\mu_n}\,\frac{1}{\sin(\omega^\mu_n\,t)}\,, \end{gathered} \tag{3.24}$$

in which $\omega^\mu_n \doteq \sqrt{\lambda^\mu_n}$, $\lambda^\mu_n \doteq \frac{\lambda_n}{1+\mu^2\,\lambda_n}$, with $\{\lambda_n\}_{n\in\mathbb{N}}$ enumerated in non-decreasing order. Note in particular that $\{\lambda_n\}_{n\in\mathbb{N}}$ is strictly positive and unbounded (as $\mathcal{A}$ is unbounded and positive by Assumption 1.1). Consequently, applying (2.7), $\omega^\mu_n\,t \in (0, \bar{t}^\mu/\mu) = (0, \sqrt{2}) \subset (0, \pi/2)$ for all $t \in (0, \bar{t}^\mu)$, so that the eigenvalues (3.24) are well-defined for each $t \in (0, \bar{t}^\mu)$.

The correspondence between stationary action and optimal control exploited for horizons $t \in [0, \bar{t}^\mu)$ may break down for longer horizons, due to loss of concavity of the action (and hence payoff) functional. That is, for longer horizons, stationarity of the action functional is no longer achieved as a maximum. However, for any sufficiently short horizon within that longer horizon, concavity is retained. Hence, it is possible to accumulate longer horizons via a concatenation of sufficiently short horizons, *provided* maximization over the intermediate states that connect adjacent horizons is relaxed to a stationarity condition. In order to formalize this rationale, given a fixed longer horizon $t \in [\bar{t}^\mu, \infty)$ of interest, select a sufficiently large number $n_t \in \mathbb{N}$ of shorter horizons $\tau \doteq t/n_t$ such that $\tau \in [0, \bar{t}^\mu)$. By definition of τ, the payoff in (2.8) or (3.15) is concave on each of the subintervals $[(k-1)\tau, k\tau]$, $k \in [1, n_t] \cap \mathbb{N}$. Consequently, the loss of concavity for the longer horizon must occur at the intermediate states $\zeta_k \doteq \xi(k\tau) \in \mathscr{X}_{\frac{1}{2}}$. Motivated by this observation, and paying particular attention to the kernel G_t^μ defined via (3.15), this concatenation of horizons can be written as

(3.25)
$$G_t^\mu(x,z) = \operatorname*{stat}_{\zeta \in (\mathscr{X}_{\frac{1}{2}})^{n_t - 1}} \left\{ \bigotimes_{k=1}^{n_t} G_\tau^\mu(\zeta_{k-1}, \zeta_k) \,\middle|\, \begin{array}{c} \zeta_0 = x \\ \zeta_{n_t} = z \end{array} \right\},$$

for all $x, z \in \mathscr{X}_{\frac{1}{2}}$, in which the *stat* operation [13] is defined generally for differentiable $F : \mathscr{X}_{\frac{1}{2}} \to \mathbb{R}$ by

$$\operatorname*{stat}_{x \in \mathscr{X}_{\frac{1}{2}}} F(x) \doteq \left\{ F(\bar{x}) \,\middle|\, \bar{x} \in \operatorname*{argstat}_{x \in \mathscr{X}_{\frac{1}{2}}} F(x) \right\},$$
$$\operatorname*{argstat}_{x \in \mathscr{X}_{\frac{1}{2}}} F(x) \doteq \left\{ x \in \mathscr{X}_{\frac{1}{2}} \,\middle|\, 0 = \lim_{y \to x} \frac{|F(y) - F(x)|}{\|y - x\|_{\frac{1}{2}}} \right\}.$$

Crucially, it may be shown that this construction *preserves* the explicit representation (3.20), see Appendix C. That is, (3.20) is valid for both shorter and longer horizons, defined with respect to $\bar{t}^\mu$ of (2.7). Consequently, the collection of max-plus linear max-plus integral operators $\{\mathcal{G}_t^\mu\}_{t \in \mathbb{R}_{\ge 0}}$ for all horizons does indeed define a semigroup.

The max-plus fundamental solution semigroup $\{\mathcal{G}_t^\mu\}_{t \in \mathbb{R}_{\ge 0}}$ can also be used to write down the approximating semigroup $\{\mathcal{T}_\mu^\vee(t)\}_{t \in \mathbb{R}_{\ge 0}}$ for the wave equation (1.1). To see how, choose a specific terminal payoff ψ in (2.8) and (3.18) defined by

$$\psi(x) = \psi_v(x) \doteq \langle (\mathcal{A}^{-1} + \mu^2 \mathcal{I})\, v,\, x \rangle_{\frac{1}{2}} \tag{3.26}$$

for all $x \in \mathscr{X}_{\frac{1}{2}}$, where $v \in \mathscr{X}_{\frac{1}{2}}$ represents a target terminal velocity $\dot{x}(t)$ in (1.1). Given an initial state $x \in \mathscr{X}_{\frac{1}{2}}$, the corresponding final optimal input $w^*(t)$ is, according to (2.12),

$$\begin{aligned} w^*(t) &= \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \nabla_x W^\mu(0,x) = \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \nabla_x \psi_v(x) \\ &= \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \mathcal{J}\, \mathcal{I}_\mu^{-1}\, \mathcal{J}\, v = v\,, \end{aligned}$$

where $\mathcal{J} \doteq (\mathcal{A}^{\frac{1}{2}})^{-1}$. Similarly, the initial optimal input $w^*(0)$ required to achieve this final velocity is

$$w^*(0) = \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \nabla_x W^\mu(t,x)\,. \tag{3.27}$$

In order to find the gradient required here, note from (3.17) and (3.18) that

$$W^\mu(t,x) = G_t^\mu(x, z^*(x)) + \psi_v(z^*(x)) \tag{3.28}$$

where $z^*(x) = \operatorname{argmax}_{z \in \mathscr{X}_{\frac{1}{2}}} \{G_t(x,z) + \psi_v(z)\}$. As G_t and ψ_v are respectively quadratic and linear functionals, see (3.20) and (3.26), they are differentiable. Consequently, $z^*(x)$ can be found explicitly, with

$$z^*(x) = -\breve{\mathcal{R}}^\mu(t)^{-1} [\breve{\mathcal{Q}}^\mu(t)' x + (\mathcal{A}^{-1} + \mu^2 \mathcal{I})\, v]\,, \tag{3.29}$$

where the inverse guaranteed to exist for all $t \in (\delta, \bar{t}^\mu)$, see [4]. Hence, applying (3.28), the definition of $z^*(x)$, and (3.20) in (3.27),

(3.30)
$$\begin{aligned} w^*(0) &= \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \nabla_x [G_t^\mu(x, z^*(x)) + \psi_v(z^*(x))] \\ &= \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, [\nabla_x G_t^\mu(x,z)]\big|_{z = z^*(x)} \\ &= \mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, [\breve{\mathcal{P}}^\mu(t)\, x + \breve{\mathcal{Q}}^\mu(t)\, z^*(x)]\,, \end{aligned}$$

where the second equality follows as $0 = \nabla_z [G_t(x,z) + \psi_v(z)]\big|_{z = z^*(x)}$ by definition of $z^*(x)$. Note further that the terminal state of the dynamics (2.5), with optimal input w^* applied, must be $\xi(t) = z^*(x)$. Solving (3.29) and (3.30) for the terminal position and velocity yields

$$\begin{pmatrix} z^*(x) \\ v \end{pmatrix} = \mathcal{T}_\mu^\vee(t) \begin{pmatrix} x \\ w^*(0) \end{pmatrix}, \tag{3.31}$$

where $\mathcal{T}_\mu^\vee(t) = \left(\begin{array}{c|c} [\mathcal{T}_\mu^\vee(t)]_{11} & [\mathcal{T}_\mu^\vee(t)]_{12} \\ \hline [\mathcal{T}_\mu^\vee(t)]_{21} & [\mathcal{T}_\mu^\vee(t)]_{22} \end{array} \right)$, with

$$\begin{aligned} &[\mathcal{T}_\mu^\vee(t)]_{11} \doteq -\breve{\mathcal{Q}}^\mu(t)^{-1}\, \breve{\mathcal{P}}^\mu(t) \\ &[\mathcal{T}_\mu^\vee(t)]_{12} \doteq \breve{\mathcal{Q}}^\mu(t)^{-1}\, (\mathcal{A}^{-1} + \mu^2 \mathcal{I}) \\ &[\mathcal{T}_\mu^\vee(t)]_{21} \doteq -\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, (\breve{\mathcal{Q}}^\mu(t)' - \breve{\mathcal{R}}^\mu(t)\, \breve{\mathcal{Q}}^\mu(t)^{-1}\, \breve{\mathcal{P}}^\mu(t)) \\ &[\mathcal{T}_\mu^\vee(t)]_{22} \doteq -\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu\, \mathcal{A}^{\frac{1}{2}}\, \breve{\mathcal{R}}^\mu(t)\, \breve{\mathcal{Q}}^\mu(t)^{-1}\, (\mathcal{A}^{-1} + \mu^2 \mathcal{I}) \end{aligned}$$

That is, (3.31) provides a representation for the uniformly continuous semigroup generated by $\mathcal{A}_\mu^\vee$ as per Lemma 2.1.

4 Example

The max-plus fundamental solution semigroup $\{\mathcal{T}_\mu^\vee(t)\}_{t \in \mathbb{R}_{\ge 0}}$ defined by (3.31) explicitly propagates solutions of (1.1) for any initial data. Its construction also facilitates the solution of two-point boundary value problems constrained by (1.1). For example, given fixed $t \in \mathbb{R}_{>0}$, $x, z \in \mathscr{X}_{\frac{1}{2}}$, it is possible to compute the initial velocity $\dot{x}(0)$ such that the propagated wave equation dynamics satisfy $x(t) = z$.

Indeed, by definition of the kernel G_t^μ of (3.15), this initial velocity is given by

$$\begin{aligned}\dot x(0) = w^*(0) &= \mathcal{A}^{\frac12}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac12}\,\nabla_x G_t^\mu(x,z)\\ &= \mathcal{A}^{\frac12}\,\mathcal{I}_\mu\,\mathcal{A}^{\frac12}\,\left(\breve{\mathcal{P}}^\mu(t)\,x + \breve{\mathcal{Q}}^\mu(t)\,z\right),\end{aligned}$$

where the second equality follows by (3.20). Applying the decomposition for $\mathcal{P}^\mu(t)$ and $\mathcal{Q}^\mu(t)$ defined by (3.22), (3.23), (3.24) yields

(4.32)
$$\dot x(0) = \sum_{n=1}^\infty \frac{\lambda_n}{1+\mu^2\,\lambda_n}\left[p_n\,\langle x,\,\tilde\varphi_n\rangle_{\frac12} + q_n\,\langle z,\,\tilde\varphi_n\rangle_{\frac12}\right]\tilde\varphi_n$$

where λ_n^{-1} and $\tilde\varphi_n$ denote the eigenvalues and eigenvectors of $\mathcal{A}^{-1}\in\mathscr{L}(\mathscr{X})$. In order to illustrate the application of (4.32), a specific example is considered. In particular, select $\Omega \doteq [0,1]^2\subset\mathbb{R}^2$, and define

$$\begin{aligned}&\mathscr{X} \doteq \mathcal{L}_2(\Omega;\mathbb{R})\,,\\ &\mathscr{X}_0 \doteq \left\{x\in\mathscr{X}\;\middle|\; \begin{array}{c} x,\,\partial_1 x,\,\partial_2 x \text{ absolutely continuous}\\ x\big|_{\partial\Omega}=0\,,\ \partial_1^2 x,\,\partial_2^2 x\in\mathscr{X}\end{array}\right\},\\ &\mathcal{A}\doteq -\partial_1^2-\partial_2^2\,,\qquad \mathrm{dom}\,(\mathcal{A}) = \mathscr{X}_0\,,\end{aligned}$$

in which ∂_1 and ∂_2 denote the partial derivative operators defined with respect to the first and second cartesian coordinates in $\mathbb{R}^2$ respectively, and $\partial\Omega$ denotes the boundary of Ω. It may be noted that $-\mathcal{A}$ is the Laplacian operator on Ω, with $\mathcal{A}$ satisfying Assumption 1.1. For example, positivity and self-adjointness of $\mathcal{A}$ follow by Green's first identity, while [2, Corollary 2.2.3, p.33] implies that $-\mathcal{A}$ generates a contraction semigroup on $\mathscr{X}$. Furthermore, $\mathcal{A}^{-1}$ is compact, with eigenvalues $\lambda_{n,m}^{-1}\in\mathbb{R}_{>0}$ and eigenvectors $\tilde\varphi_{n,m}\in\mathscr{X}_{\frac12}$ defined respectively by $\lambda_{n,m}\doteq(n^2+m^2)\,\pi^2$ and $\tilde\varphi_{n,m}(x_1,x_2)\doteq(2/\sqrt{\lambda_{n,m}})\,\sin(n\,\pi\,x_1)\,\sin(m\,\pi\,x_2)$ for all $n,m\in\mathbb{N}$, $(x_1,x_2)\in\Omega$. It may be noted that $\widetilde{\mathcal{B}}\doteq\{\tilde\varphi_{n,m}\}_{n,m\in\mathbb{N}}$ defines an orthonormal basis for $\mathscr{X}_{\frac12}$. As $\mathbb{N}^2$ is countable, these eigenvalues and eigenvectors may be enumerated as per (3.22).

For illustrative purposes, the specific initial state $x\in\mathscr{X}_0$ is chosen (arbitrarily) to be the zero function on Ω, while the terminal state $z\in\mathscr{X}_0$ is selected to be as per Figure 1. A horizon $t\doteq\pi/3$ is assumed. The initial velocity $\dot x(0)$ obtained in the $\mu=0$ limit in (4.32) is illustrated in Figure 2. By propagating the initial state $x(0)=x$ and velocity $\dot x(0)$ forward in time, it may be seen that (4.32) does indeed solve the two-point boundary value problem of interest, see Figure 3.

5 Conclusion

By exploiting a correspondence between stationary action and optimal control, a max-plus fundamental solution semigroup can be constructed for a class of lossless

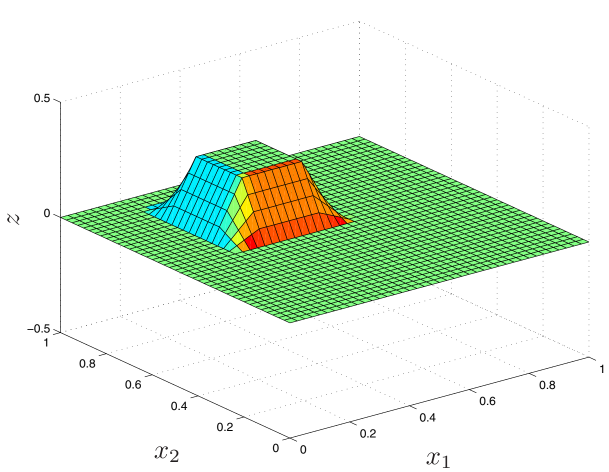

Figure 1: Terminal state $z\in\mathscr{X}_0$ for all $(x_1,x_2)\in\Omega$.

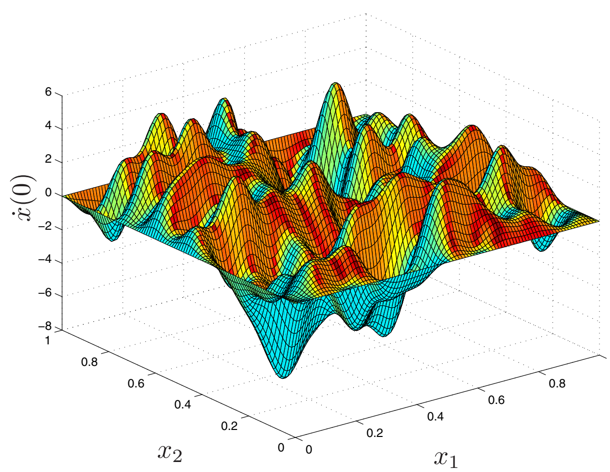

Figure 2: Two-point boundary value problem solution (4.32).

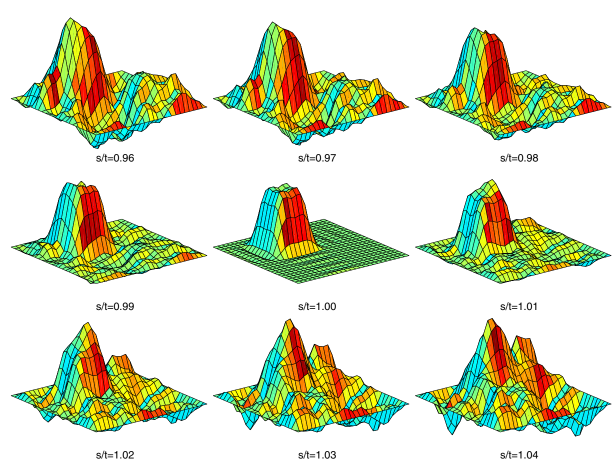

Figure 3: Solution $x(s)$ of (1.1) propagated forward from $x(0)=x$, $\dot x(0)$ as per (4.32), to $s/t\in[0.96,1.04]$.

wave equations. This construction relies on the development of a semigroup of max-plus linear max-plus integral operators that collectively describes all possible solutions to the corresponding optimal control problem for different terminal payoffs and time horizons. The max-plus fundamental solution semigroup obtained can be used to propagate the dynamics of the lossless wave equation, and to solve two-point boundary value problems constrained by it. Its application to a specific wave equation is illustrated via an example.

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A Properties of $\mathcal{I}_\mu$

LEMMA A.1. *The following properties concerning operator $\mathcal{I}_\mu$ of (2.11) hold on $\mathscr{X}$ for any $\mu \in \mathbb{R}_{>0}$:*

1) $\mathcal{I}_\mu = \frac{1}{\mu^2}\,\mathcal{R}_{-\mathcal{A}}(\frac{1}{\mu^2}) \in \mathscr{L}(\mathscr{X})$, where $\mathcal{R}_{-\mathcal{A}}(\cdot)$ denotes the resolvent of $-\mathcal{A}$;

2) $\mathcal{I}_\mu$ is self-adjoint and positive, with $\mathcal{I}_\mu\, x \in \mathscr{X}_0$ for all $x \in \mathscr{X}$;

3) $\mathcal{I}_\mu$ has a unique, bounded, linear, self-adjoint, and positive square root $\mathcal{I}_\mu^{\frac{1}{2}}$, with

$$\mathcal{I}_\mu^{\frac{1}{2}}\, x \in \mathscr{X}_{\frac{1}{2}}\,,\ \mathcal{I}_\mu^{\frac{1}{2}}\,\mathcal{I}_\mu^{\frac{1}{2}}\, x = \mathcal{I}_\mu\, x\,, \quad x \in \mathscr{X}\,;$$

4) $\mathcal{I}_\mu$, $\mathcal{I}_\mu^{\frac{1}{2}}$, $\mathcal{A}$, and $\mathcal{A}^{\frac{1}{2}}$ commute, with

$$\begin{aligned} &\mathcal{I}_\mu^{\frac{1}{2}}\,\mathcal{A}^{\frac{1}{2}}\, x = \mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu^{\frac{1}{2}}\, x\,,\ \mathcal{I}_\mu\,\mathcal{A}^{\frac{1}{2}}\, x = \mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu\, x\,, && x \in \mathscr{X}_{\frac{1}{2}}\,, \\ &\mathcal{I}_\mu\,\mathcal{A}\, x = \mathcal{A}\,\mathcal{I}_\mu\, x\,,\ \mathcal{I}_\mu^{\frac{1}{2}}\,\mathcal{A}\, x = \mathcal{A}\,\mathcal{I}_\mu^{\frac{1}{2}}\, x\,, && x \in \mathscr{X}_0\,; \end{aligned}$$

5) Selected compositions of $\mathcal{I}_\mu$, $\mathcal{I}_\mu^{\frac{1}{2}}$, $\mathcal{A}$, and $\mathcal{A}^{\frac{1}{2}}$ define bounded linear operators, with

$$\begin{aligned} &\mathcal{A}\,\mathcal{I}_\mu,\ \mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu^{\frac{1}{2}} \in \mathscr{L}(\mathscr{X})\,, \quad \mathcal{A}\,\mathcal{I}_\mu \in \mathscr{L}(\mathscr{X}_{\frac{1}{2}}) \\ &\mathcal{A}^{\frac{1}{2}}\,\mathcal{I}_\mu^{\frac{1}{2}}\,\mathcal{A}^{\frac{1}{2}} \in \mathscr{L}(\mathscr{X}_{\frac{1}{2}};\mathscr{X})\,. \end{aligned}$$

Proof. 1) By definition (2.11) and that of the resolvent, $\mathcal{I}_\mu = \frac{1}{\mu^2}\,(\frac{1}{\mu^2}\,\mathcal{I} - (-\mathcal{A}))^{-1} = \frac{1}{\mu^2}\,\mathcal{R}_{-\mathcal{A}}(\frac{1}{\mu^2})$ as required. Assertion 2) of Assumption 1.1 and the Hille-Yosida Theorem (e.g. [14, Theorem 5.2]) then imply that $\mathcal{I}_\mu \in \mathscr{L}(\mathscr{X})$.

2) Following on from 1), $\mathcal{A}\,\mathcal{I}_\mu$ defines a Yosida approximation of $\mathcal{A}$, so that $\mathcal{A}\,\mathcal{I}_\mu \in \mathscr{L}(\mathscr{X})$. Hence, $\operatorname{ran}(\mathcal{I}_\mu) = \mathscr{X}_0$. The fact that $\mathcal{I}_\mu$ is positive and self-adjoint follows by (2.11) and the corresponding properties of $\mathcal{A}$.

3) The existence of a unique, positive, and bounded square root $\mathcal{I}_\mu^{\frac{1}{2}}$ is guaranteed by 1) and 2), see for example [1, Theorem 4].

4) The fact that $\mathcal{A}$ and $\mathcal{I}_\mu$ commute follows by Assertion 1). As $\mathcal{A}$ and $\mathcal{I}_\mu$ are both closed (the former as it is boundedly invertible by the third assertion of Assumption 1.1, and the latter as it is bounded and defined on the entirety of $\mathscr{X}$), the remaining commutations follow (for example) by repeated applications of [1, Theorem 10].

5) Follows identically to [4, Lemma A.4*(iv)*]. $\square$

B Proof of Lemma 2.1 and Theorem 2.1 [4]

Proofs of a special case of these results appear in [4].

Proof. [Lemma 2.1] *1)* Fix any $y \doteq \begin{pmatrix} \xi \\ \pi \end{pmatrix} \in \mathscr{Y}_{\frac{1}{2}}$. Consequently,

$$\begin{aligned} \|\mathcal{A}_\mu^\vee\, y\|_\vee^2 &= \left\|\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\, \pi\right\|^2 + \left\|\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\, (\mathcal{A}^{\frac{1}{2}}\, \xi)\right\|^2 \\ &\le \left\|\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\right\|^2 \|\pi\|^2 + \left\|\mathcal{A}^{\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\right\|^2 \|\xi\|_{\frac{1}{2}}^2 = \|\mathcal{A}\,\mathcal{I}_\mu\|\; \|y\|_\vee^2\,. \end{aligned}$$

The assertion follows by Lemma A.1, assertion 5).

2) is immediate by *1)* and [14, Theorem 1.2, p.2].

3) and *4)* follow by [2, Example 2.2.5, p.34].

5) Fix any $y \doteq \begin{pmatrix} \xi \\ \pi \end{pmatrix} \in \mathrm{dom}\,(\mathcal{A}^\vee)$. Consequently,

$$\text{(B.1)} \qquad \left\|\mathcal{A}_\mu^\vee\, y - \mathcal{A}^\vee\, y\right\|_\vee^2 = \|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\, \mathcal{A}^{\frac{1}{2}}\, \pi\|^2 + \|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\, \mathcal{A}\,\xi\|^2\,,$$

by definition of $\|\cdot\|_{\frac{1}{2}}$ and Lemma A.1. Note further that $\mathcal{A}^{\frac{1}{2}}\,\pi,\ \mathcal{A}\,\xi \in \mathscr{X}$ by definition of $y \in \mathscr{Y}_0$. Consequently, it remains to be shown that $\mathcal{I}_\mu^{\frac{1}{2}}$ converges strongly to $\mathcal{I}$ on $\mathscr{X}$ as $\mu \to 0$. To this end, fix any $x \in \mathscr{X}_0$, so that $\|\mathcal{A}\,x\| < \infty$. Assumption 1.1 and definition (2.11) imply that $\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I}$ is an operator with a decomposition of the form (3.22) on $\mathscr{X}$, with $\mathrm{dom}\,(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I}) = \mathscr{X}$. Hence,

$$\text{(B.2)} \qquad \left\|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\, x\right\|^2 = \sum_{n=1}^\infty \beta_{\lambda_n}(\mu^2)\, |\langle x,\, \varphi_n\rangle|^2\,,$$

where $\beta_\lambda : \mathbb{R}_{\ge 0} \to [0,1)$ is defined for each $\lambda \in \mathbb{R}_{>0}$ by $\beta_\lambda(\epsilon) \doteq [1 - \frac{1}{\sqrt{1+\epsilon\,\lambda}}]^2$, and $\{\varphi_n\}_{n\in\mathbb{N}}$ is the orthonormal basis for $\mathscr{X}$ defined by $\varphi_n \doteq \sqrt{\lambda_n}\,\tilde{\varphi}_n$. Taylor's theorem implies that for any $\epsilon \in \mathbb{R}_{\ge 0}$, there exists an $c_\epsilon \in (0,\epsilon)$ such that $\beta_\lambda(\epsilon) = [\frac{d^2\beta_\lambda}{d\epsilon^2}(c_\epsilon)]\frac{\epsilon^2}{2} \le \frac{7}{4}\,\lambda^2\,\epsilon^2$ for all $\lambda \in \mathbb{R}_{>0}$. Substitution in (B.2) yields that $\|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\,x\|^2 \le \frac{7}{4}\,\mu^4\,\sum_{n=1}^\infty |\lambda_n|^2\, |\langle x,\,\varphi_n\rangle|^2 = \frac{7}{4}\,\mu^4\,\|\mathcal{A}\,x\|^2$. Recalling that $x \in \mathscr{X}_0$, so that $\|\mathcal{A}\,x\| < \infty$, it follows immediately that $\lim_{\mu\to 0} \|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\,x\| = 0$ for any $x \in \mathscr{X}_0$. As $\mathcal{I}_\mu^{\frac{1}{2}} \in \mathscr{L}(\mathscr{X})$ by Lemma A.1, and $\mathscr{X}_0$ is dense in $\mathscr{X}$, it may also be concluded that $\lim_{\mu\to 0} \|(\mathcal{I}_\mu^{\frac{1}{2}} - \mathcal{I})\,x\| = 0$ for any $x \in \mathscr{X}$. Recalling (B.1) completes the proof. $\square$

Proof. [Theorem 2.1] The proof follows by application of the First Trotter-Kato Approximation Theorem (see for example [6, Theorem 4.8, p.209]), via Lemma 2.1. $\square$

C Invariance of representation (3.20)

In order to justify the claim that the explicit representation (3.20) is invariant under concatenation of short horizons, fix an arbitrary pair of horizons $t, s \in [0, \bar{t}^\mu)$, and states $x, z \in \mathscr{X}_{\frac{1}{2}}$. Using (3.25), define

$$\text{(C.3)} \qquad \widehat{G}_{t,s}^\mu(x,z) \doteq \operatorname*{stat}_{\zeta\in\mathscr{X}_{\frac{1}{2}}} \left\{G_t^\mu(x,\zeta) \otimes G_s^\mu(\zeta,z)\right\}\,.$$

Note by (3.20) that $G_t^\mu(x,\cdot)$ and $G_s^\mu(\cdot,z)$ are quadratic functionals, and hence differentiable. Applying the definition of stat [13], the *staticizing* intermediate state $\zeta^* \in \mathscr{X}_{\frac{1}{2}}$ is given by $0 = \nabla_\zeta[G_t^\mu(x,\zeta)\otimes G_s^\mu(\zeta,z)]\big|_{\zeta=\zeta^*} = \breve{\mathcal{Q}}^\mu(t)'x + [\breve{\mathcal{R}}^\mu(t) + \breve{\mathcal{P}}^\mu(s)]\,\zeta^* + \breve{\mathcal{Q}}^\mu(s)\,z$, or $\zeta^* = -[\breve{\mathcal{P}}^\mu(s) + \breve{\mathcal{R}}^\mu(t)]^{-1}[\breve{\mathcal{Q}}^\mu(t)'x + \breve{\mathcal{Q}}^\mu(s)\,z]$. Back substitution in (C.3) and some straightforward manipulations yield

$$\text{(C.4)} \qquad \begin{aligned} &\widehat{G}_{t,s}^\mu(x,z) = G_t^\mu(x,\zeta^*) + G_s^\mu(\zeta^*,z) \\ &= \tfrac{1}{2}\,\langle x,\, \mathcal{X}(t,s)\,x\rangle_{\frac{1}{2}} + \langle x,\, \mathcal{Y}(t,s)\,z\rangle + \tfrac{1}{2}\,\langle z,\, \mathcal{Z}(t,s)\,z\rangle_{\frac{1}{2}} \end{aligned}$$

where

$$\begin{aligned} \mathcal{X}(t,s) &\doteq \breve{\mathcal{P}}^\mu(t) - \breve{\mathcal{Q}}^\mu(t)\,[\breve{\mathcal{P}}^\mu(s) + \breve{\mathcal{R}}^\mu(t)]^{-1}\,\breve{\mathcal{Q}}^\mu(t)'\,, \\ \mathcal{Y}(t,s) &\doteq -\breve{\mathcal{Q}}^\mu(t)\,[\breve{\mathcal{P}}^\mu(s) + \breve{\mathcal{R}}^\mu(t)]^{-1}\,\breve{\mathcal{Q}}(s)\,, \\ \mathcal{Z}(t,s) &\doteq \breve{\mathcal{R}}^\mu(s) - \breve{\mathcal{Q}}(s)'\,[\breve{\mathcal{P}}^\mu(s) + \breve{\mathcal{R}}^\mu(t)]^{-1}\,\breve{\mathcal{Q}}^\mu(s)\,, \end{aligned}$$

where existence of the inverse involved follows by definition of horizons s, t. As $\breve{\mathcal{P}}^\mu$, $\breve{\mathcal{Q}}^\mu$, $\breve{\mathcal{R}}^\mu$ are operator-valued functions of the form (3.23), operators $\mathcal{X}(t,s)$, $\mathcal{Y}(t,s)$, $\mathcal{Z}(t,s)$ are of the same form. Their respective eigenvalues are given by

$$\begin{aligned} X_n(t,s) &= p_n^\mu(t) - \frac{q_n^\mu(t)^2}{p_n^\mu(s) + r_n^\mu(t)} = Z_n(s,t)\,, \\ Y_n(t,s) &= -\frac{q_n^\mu(t)\,q_n^\mu(s)}{p_n^\mu(s) + r_n^\mu(t)}\,, \end{aligned}$$

in which p_n^μ, q_n^μ, r_n^μ are defined by (3.24). Substituting accordingly, applying sum-of-angle formulae for sin and tan, and manipulating algebraically, yields

$$\begin{aligned} X_n(t,s) &= -\tfrac{1}{\omega_n^\mu}\,\frac{1}{\tan(\omega_n^\mu\,(s+t))} = Z_n(s,t)\,, \\ Y_n(t,s) &= +\tfrac{1}{\omega_n^\mu}\,\frac{1}{\sin(\omega_n^\mu\,(s+t))}\,. \end{aligned}$$

Recalling (3.24), it is evident that $\mathcal{X}^\mu(t,s) = \breve{\mathcal{P}}^\mu(t+s)$, $\mathcal{Y}^\mu(t,s) = \breve{\mathcal{Q}}^\mu(t+s)$, and $\mathcal{Z}^\mu(t,s) = \breve{\mathcal{R}}^\mu(t+s)$. Hence, (C.4) implies that

$$\begin{aligned} \widehat{G}_{t,s}^\mu(x,z) &= \tfrac{1}{2}\,\langle x,\, \breve{\mathcal{P}}^\mu(t+s)\,x\rangle_{\frac{1}{2}} + \langle x,\, \breve{\mathcal{Q}}^\mu(t+s)\,z\rangle \\ &\quad + \tfrac{1}{2}\,\langle z,\, \breve{\mathcal{R}}^\mu(t+s)\,z\rangle_{\frac{1}{2}} = G_{t+s}^\mu(x,z)\,, \end{aligned}$$

as required. $\square$