A dynamic game approximation for a linear regulator problem with a log-barrier state constraint*

Peter M. Dower† William M. McEneaney‡ Michael Cantoni†

Abstract— **An exact supremum-of-quadratics representation for log-barrier functions is developed for, and subsequently applied in, a state-constrained linear regulator problem. By approximating this representation, it is shown that this regulator problem can be approximated by an unconstrained linear quadratic dynamic game. It is anticipated that this game approximation may facilitate the computation of approximate solutions to such state-constrained regulator problems.**

I. Introduction

Finite horizon linear quadratic regulator (LQR) problems have been extensively studied in the literature over many decades, giving rise to numerous advances in linear systems theory, optimal control, model predictive control (MPC), etc, see for example [10], [3], [2], [12], [4], [6]. In the absence of constraints, the value function defining such problems is guaranteed to be finite everywhere on sufficiently small time horizons, and quadratic with a Hessian that evolves according to the solution of a differential Riccati equation (DRE) initialized with the Hessian of the terminal payoff, see for example [10], [12], [8]. It is well-known that finite horizon linear quadratic regulators admit a linear state feedback characterization of their optimal control, which is defined with respect to the aforementioned DRE solution.

The imposition of constraints in LQR problems fundamentally impacts their solvability. In particular, constraints destroy the quadratic structure in the aforementioned constraint-free case, even though linearity of the underlying model is preserved. In the specific case of linear regulator problems employing state trajectory constraints, the corresponding regulator problem is defined by a non-quadratic value function that cannot be finite for those initial states violating these constraints. This loss of structure means that the value function must be characterized more generally, as the extended real-valued discontinuous viscosity solution of a non-stationary Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), fixed uniquely by a terminal condition determined by the terminal payoff. Explicit solutions to such HJB PDE are exceedingly rare, and numerical approximation schemes are inevitably required. Furthermore, while a state feedback characterization of the optimal control may still exist, it must be numerically approximated using the computed approximation of the unique viscosity solution.

In this paper, a linear regulator problem with a log-barrier type state constraint is considered. By providing a sup(remum)-of-quadratics representation for the attendant log-barrier function, this linear regulator problem is approximated by a linear quadratic dynamic game problem in which one player is the usual control, while the other is an adversary that negotiates an appropriate state penalty. The approach employed is analogous to that recently developed for constructing a fundamental solution to the gravitational N-body problem, see [9]. In contrast to standard MPC [4], [6], it is formulated entirely in continuous time, see also [5]. It is anticipated that this new game problem may assist in the computation of an optimal control that approximately solves the linear regulator problem of interest. It may also provide an continuous time alternative to a recent Lyapunov-based approach to the use of log-barrier functions in MPC [7].

In terms of organization, the state constrained linear regulator problem of interest is formulated in Section II, using a log-barrier function to implement the state constraint involved. A sup-of-quadratics representation for this log-barrier constraint is developed in Section III. This is subsequently applied in Section IV to approximate the constrained linear regulator problem as a game, followed by some brief concluding remarks in Section V. Throughout, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote respectively the natural, integer and real numbers, while $\mathbb{R}_{\geq 0}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ denote respectively the nonnegative real numbers, n-dimensional Euclidean space, and the space of $n\times n$ matrices with real entries. $\mathbb{R}^{\pm}$, etc, denotes the analogous sets defined with respect to extended reals $\mathbb{R}\cup\{\pm\infty\}$. Similarly, $\mathbb{S}^{n\times n}_{\geq 0}$ denotes the space of nonnegative symmetric elements of $\mathbb{R}^{n\times n}$. The transpose of $P\in\mathbb{R}^{n\times n}$ is denoted by $P^T\in\mathbb{R}^{n\times n}$.

II. Linear regulator with a state constraint

Attention is restricted to an example class of linear regulator problems, with a *log-barrier* type hard state constraint (see for example [7]), defined on a finite time horizon $t\in\mathbb{R}_{\geq 0}$ via a value function $\overline{W}_t:\mathbb{R}^n\to\mathbb{R}$. In particular,

$$\overline{W}_t(x)\doteq\inf_{u\in\mathscr{U}[0,t]}\bar{J}_t(x,u), \tag{1}$$

in which $\mathscr{U}[0,t]\doteq\mathscr{L}_2([0,t];\mathbb{R}^m)$, and the total cost $\bar{J}_t$ is defined with respect to the log-barrier and standard quadratic integrated running costs $\bar{I}_t$, I_t, and a terminal cost Ψ_0 by

$$\bar{J}_t,\bar{I}_t,I_t:\mathbb{R}^n\times\mathscr{U}[0,t]\to\mathbb{R}^+,\quad \Psi_0:\mathbb{R}^n\to\mathbb{R},$$

with

$$\bar{J}_t(x,u)\doteq I_t(x,u)+\bar{I}_t(x,u)+\Psi_0(\xi_t), \tag{2}$$

*Research partially supported by AFOSR, Australian Research Council, and NSF. †Dower and Cantoni are with the Department of Electrical & Electronic Engineering, University of Melbourne, Victoria 3010, Australia. [pdower,cantoni]@unimelb.edu.au. ‡McEneaney is with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093, USA. wmceneaney@ucsd.edu.

$$\bar{I}_t(x,u) \doteq \int_0^t \tfrac{1}{2}\,\Phi(|\xi_s|^2)\,ds, \tag{3}$$

$$I_t(x,u) \doteq \int_0^t \tfrac{1}{2}\,|\xi_s|^2 + \tfrac{1}{2}|u_s|^2\,ds, \tag{4}$$

$$\Psi_0(x) \doteq \tfrac{1}{2}x^T P_0\,x, \tag{5}$$

for all $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$. Here, $P_0 \in \mathbb{S}^{n\times n}_{\geq 0}$, while $\xi_s \in \mathbb{R}^n$ denotes the state of the linear dynamics

$$\dot{\xi}_\sigma = A\,\xi_\sigma + B\,u_\sigma, \quad \sigma \in [0,t], \tag{6}$$

evolved to time $s \in [0,t]$ from an initial state $x_0 = x \in \mathbb{R}^n$ via input $u \in \mathscr{U}[0,s]$, given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. In (3), $\Phi : \mathbb{R} \to \mathbb{R}^+_{\geq 0}$ is the extended real-valued *log-barrier* function defined by

$$\Phi(\rho) \doteq \left\{ \begin{array}{ll} -\log(1-\rho/b^2), & \rho \in [0,b^2), \\ +\infty, & \rho \in \mathbb{R}_{<0} \cup \mathbb{R}_{\geq b^2}, \end{array} \right. \tag{7}$$

where $b \in \mathbb{R}_{>0}$ denotes the constraint radius of interest. Its application in the otherwise linear quadratic regulator problem (1) is intended to implement the state trajectory constraint given by

$$\xi_s \in \mathscr{B}[0;b] \doteq \left\{ x \in \mathbb{R}^n \,\middle|\, |x| \leq b \right\} \quad \forall\, s \in [0,t]\,. \tag{8}$$

As alluded to in Section I, the nonlinearity introduced by constraint (8) into the regulator problem (1) renders the value function $\overline{W}_t$ non-quadratic. Consequently, its computation via the solution of a corresponding DRE is no longer possible, with the solution of a general HJB PDE instead required. Here, rather than solving this HJB PDE, a different approach is employed. In particular, by decomposing the log-barrier function (7) into a sup-of-quadratics, it is shown that solution of the regulator problem may instead be approximated via a linear quadratic game.

III. An exact sup-of-quadratics representation for log-barrier function (7), and its approximation

Attention is restricted to a log-barrier function of the form (7). For convenience, write (7) as

$$\Phi(\rho) \doteq \left\{ \begin{array}{ll} \phi(\rho), & \rho \in [0,b^2), \\ +\infty, & \rho \in \mathbb{R}_{<0} \cup \mathbb{R}_{\geq b^2}, \end{array} \right. \tag{9}$$

for all $\rho \in \mathbb{R}$, in which $\phi : [0,b^2) \to \mathbb{R}_{\geq 0}$ is defined by

$$\phi(\rho) \doteq -\log(1-\rho/b^2), \tag{10}$$

for all $\rho \in [0,b^2)$. An exact sup-of-quadratics representation is established using convex duality [1], [13].

A. Exact sup-of-quadratics representation

Lemma 3.1: Given the log-barrier function $\Phi : \mathbb{R} \to \mathbb{R}^+_{\geq 0}$ of (7), there exists a function $A : \mathbb{R} \to \mathbb{R}^+_{\geq 0}$ such that

$$\Phi(\rho) = \sup_{\beta\in\mathbb{R}} \{\beta\,\rho - A(\beta)\}, \tag{11}$$

$$A(\beta) = \sup_{\rho\in\mathbb{R}} \{\beta\,\rho - \Phi(\rho)\}, \tag{12}$$

for all $\rho, \beta \in \mathbb{R}$, in which

$$A(\beta) = \left\{ \begin{array}{ll} 0, & \beta \in \mathbb{R}_{<1/b^2}, \\ a(\beta), & \beta \in \mathbb{R}_{\geq 1/b^2}, \end{array} \right. \tag{13}$$

for all $\beta \in \mathbb{R}$, with $a : [1/b^2, \infty) \to \mathbb{R}_{\geq 0}$ defined by

$$a(\beta) \doteq b^2\,\beta - \log(b^2\,\beta) - 1, \tag{14}$$

for all $\beta \in [1/b^2, \infty)$. Furthermore, the optimizers $\beta^* : \mathbb{R} \to \mathbb{R}^\pm$ and $\rho^* : \mathbb{R} \to \mathbb{R}$ defined by $\beta^*(\rho) \doteq \arg\max_{\beta\in\mathbb{R}}\{\beta\,\rho - A(\beta)\}$ and $\rho^*(\beta) \doteq \arg\max_{\rho\in\mathbb{R}}\{\beta\,\rho - \Phi(\rho)\}$ in (11) and (12) are given respectively by

$$\beta^*(\rho) = \left\{ \begin{array}{ll} -\infty, & \rho \in \mathbb{R}_{<0}, \\ 1/(b^2-\rho), & \rho \in [0,b^2), \\ +\infty, & \rho \in \mathbb{R}_{\geq b^2} \end{array} \right. \tag{15}$$

$$\rho^*(\beta) = \left\{ \begin{array}{ll} 0, & \beta \in \mathbb{R}_{<1/b^2}, \\ b^2 - 1/\beta, & \beta \in \mathbb{R}_{\geq 1/b^2}, \end{array} \right. \tag{16}$$

for all $\beta, \rho \in \mathbb{R}$.

Proof: With $b \in \mathbb{R}_{>0}$ fixed, note that Φ is convex and (lower) closed [1, (3.8), pp.15,17] on $\mathbb{R}$. Hence, [1, Theorem 5] implies that there exists a one-to-one pairing between Φ and its Fenchel transform $A : \mathbb{R} \to \mathbb{R}^+$, as per (11) and (12). By definition (7) of Φ, the supremum in the definition of A will always be achieved via a supremum over $[0,b^2)$, ie.

$$A(\beta) = \sup_{\rho\in[0,b^2)} \pi_\beta(\rho), \quad \pi_\beta(\rho) \doteq \beta\,\rho - \phi(\rho) \tag{17}$$

for all $\beta \in \mathbb{R}$, $\rho \in [0,b^2)$. The supremum is achieved at a stationary point $\rho = \rho^* \in [0,b^2)$ if

$$0 = \pi_\beta'(\rho^*) = \beta - \frac{1}{b^2-\rho^*} \iff \left\{ \begin{array}{l} \beta = \dfrac{1}{b^2-\rho^*} \in \mathbb{R}_{\geq 1/b^2}, \\ \rho^* = b^2 - \dfrac{1}{\beta}\,, \end{array} \right.$$

in which case, the supremum is

$$\pi_\beta(\rho^*) = a(\beta) = b^2\,\beta - \log(b^2\,\beta) - 1.$$

Otherwise, $\beta \in \mathbb{R}_{<1/b^2}$, so that by inspection $\pi_\beta'(\rho) < 0$ for all $\rho \in [0,b^2)$. Hence, the supremum is achieved at $\rho^* = 0$ instead, with $\pi_\beta(\rho^*) = \pi_\beta(0) = 0$. Combining these facts in (12) via (17) reveals that the Fenchel transform A is in fact finite everywhere, and given by (12), (13), (14), with the corresponding optimizer as per (16).

In order to confirm (11) and (15), define $\Gamma : \mathbb{R} \to \mathbb{R}^+$ by

$$\Gamma(\rho) \doteq \sup_{\beta\in\mathbb{R}} \{\beta\,\rho - A(\beta)\} \tag{18}$$

for all $\rho \in \mathbb{R}$, as per the right-hand side of (11). Note by inspection of (13) that

$$\begin{aligned} \Gamma_-(\rho) &\doteq \sup_{\beta\in\mathbb{R}_{<1/b^2}} \{\beta\,\rho - A(\beta)\} \\ &= \sup_{\beta\in\mathbb{R}_{<1/b^2}} \{\beta\,\rho\} = \left\{ \begin{array}{ll} +\infty, & \rho \in \mathbb{R}_{<0}, \\ \rho/b^2, & \rho \in \mathbb{R}_{\geq 0}, \end{array} \right. \end{aligned} \tag{19}$$

for all $\rho \in \mathbb{R}$, with the corresponding optimizer given by

$$\beta_-^*(\rho) = \left\{ \begin{array}{ll} -\infty, & \rho \in \mathbb{R}_{<0}, \\ 1/b^2, & \rho \in \mathbb{R}_{\geq 0}. \end{array} \right. \tag{20}$$

for all $\rho \in \mathbb{R}$. Again by inspection of (13), define

$$\begin{aligned}\Gamma_+(\rho) &\doteq \sup_{\beta\in\mathbb{R}_{\geq 1/b^2}} \{\beta\,\rho - A(\beta)\} \\ &= \sup_{\beta\in\mathbb{R}_{\geq 1/b^2}} \chi_\rho(\beta), \quad \chi_\rho(\beta) \doteq \beta\,\rho - a(\beta)\end{aligned}$$

for all $\rho \in \mathbb{R}$, $\beta \in [1/b^2, +\infty)$. Here, the supremum is achieved at a stationary point $\beta = \beta^* \in [1/b^2, \infty)$ if

$$0 = \chi'_\rho(\beta^*) = \rho - b^2 + \frac{1}{\beta^*} \iff \left\{ \begin{array}{l} b^2 - \rho = \dfrac{1}{\beta^*} \in (0, b^2]\,, \\ \beta^* = \dfrac{1}{b^2 - \rho}\,. \end{array} \right.$$

in which case, the supremum is

$$\chi_\rho(\beta^*) = -\log(1 - \rho/b^2) = \phi(\rho)\,.$$

Otherwise, $\rho \in \mathbb{R}_{<0}$ or $\rho \in \mathbb{R}_{\geq b^2}$. The former case $\rho \in \mathbb{R}_{<0}$ implies by inspection that $\chi'_\rho(\beta) = \rho + \frac{b^2}{\beta}[1/b^2 - \beta] < 0$ for all $\beta \in \mathbb{R}_{\geq 1/b^2}$. Hence, the supremum is achieved at $\beta^* = 1/b^2$, with $\chi_\rho(\beta^*) = \rho/b^2$. Alternatively, the latter case $\rho \in \mathbb{R}_{\geq b^2}$ implies by inspection that $\chi'_\rho(\beta) = [\rho - b^2] + 1/\beta > 0$ for all $\beta \in [1/b^2, +\infty)$. Hence, the supremum is achieved at $\beta^* = +\infty$, with $\chi_\rho(\beta^*) = +\infty$. Consequently,

$$\Gamma_+(\rho) = \left\{ \begin{array}{ll} \rho/b^2, & \rho \in \mathbb{R}_{<0}, \\ \phi(\rho), & \rho \in [0, b^2), \\ +\infty, & \rho \in \mathbb{R}_{\geq b^2}, \end{array} \right. \tag{21}$$

with the corresponding optimizer given by

$$\beta^*_+(\rho) = \left\{ \begin{array}{ll} 1/b^2, & \rho \in \mathbb{R}_{<0}, \\ 1/(b^2 - \rho), & \rho \in [0, b^2), \\ +\infty, & \rho \in \mathbb{R}_{\geq b^2}. \end{array} \right. \tag{22}$$

Combining (18), (19), and (21),

$$\begin{aligned}\Gamma(\rho) &= \max(\Gamma_-(\rho), \Gamma_+(\rho)) \\ &= \left\{ \begin{array}{ll} +\infty, & \rho \in \mathbb{R}_{<0}, \\ \max(\phi(\rho), \rho/b^2), & \rho \in [0, b^2), \\ +\infty, & \rho \in \mathbb{R}_{\geq b^2}, \end{array} \right.\end{aligned}$$

for all $\rho \in \mathbb{R}$. Applying the fact that $\phi(\rho) \geq \rho/b^2$ for all $\rho \in \mathbb{R}_{\geq 0}$ yields that $\Gamma(\rho) = \Phi(\rho)$ for all $\rho \in \mathbb{R}$. That is, (11) holds, with the optimizer selected as per (15). ■

Corollary 3.2: Given $\Phi : \mathbb{R} \to \mathbb{R}^+_{\geq 0}$, $A : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by (7), (13), their corresponding restrictions $\phi : [0, b^2) \to \mathbb{R}_{\geq 0}$, $a : [1/b^2, \infty) \to \mathbb{R}_{\geq 0}$ defined respectively by (10), (14) are strictly monotone increasing, with

$$\phi(\rho) = \sup_{\beta\in[1/b^2,\infty)} \{\beta\,\rho - a(\beta)\}\,, \tag{23}$$

$$a(\beta) = \sup_{\rho\in[0,b^2)} \{\beta\,\rho - \phi(\rho)\}\,, \tag{24}$$

for all $\rho \in [0, b^2)$, $\beta \in [1/b^2, \infty)$.

Proof: Fix $b \in \mathbb{R}_{>0}$. Restricting the domains of Φ, A to $[0, b^2)$, $[1/b^2, +\infty)$ in the proof of Lemma 3.1 immediately yields (23), (24). It follows by inspection of (10), (14) that ϕ, a are strictly monotone increasing. ■

Lemma 3.3: The inverse $a^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1/b^2}$ of a is given explicitly by

$$a^{-1}(\alpha) = -(1/b^2)\, W_{-1}(-\exp(-\alpha - 1)), \tag{25}$$

for all $\alpha \in \mathbb{R}_{\geq 0}$, in which W_{-1} denotes the -1 branch of the *Lambert-W function* (see Appendix A).

Proof: For convenience, write $a \doteq a(\beta)$. Applying the exponential function to both sides of (14),

$$\begin{aligned}\exp(a) &= \exp(b^2\beta)\,\frac{1}{b^2\,\beta}\,\exp(-1) \\ &\iff (-b^2\,\beta)\exp(-b^2\,\beta) = -\exp(-a - 1) \\ &\iff \beta = -(1/b^2)\,W(-\exp(-a - 1)),\end{aligned} \tag{26}$$

where W is the multi-valued Lambert-W function, see (74) in Appendix A. By inspection of (14), $a : \mathbb{R}_{\geq 1/b^2} \to \mathbb{R}_{\geq 0}$ is strictly increasing, asymptotically linear, and satisfies $a(1/b^2) = 0$. Its range is $[0, \infty)$. Hence, the argument of the Lambert-W function satisfies

$$-\exp(-a - 1) \in [-\exp(-1), 0)\,. \tag{27}$$

Furthermore, as $\beta \in \mathbb{R}_{\geq 1/b^2}$ by definition of the domain of a, (26) implies that

$$W(-\exp(-a - 1)) \in (-\infty, -1]\,. \tag{28}$$

Consequently, (27) and (28) together imply that it is the -1 branch of the Lambert-W function, denoted by W_{-1}, that appears in (26), see Figure 3 in Appendix A. Hence, recalling that $a \doteq a(\beta)$, (26) immediately implies (25). ■

Theorem 3.4: Given $b \in \mathbb{R}_{>0}$, the log-barrier term $\Phi(|\cdot|^2)$ appearing in (1) via (2) and (3), and defined by (7), has the sup-of-quadratics representation

$$\Phi(|x|^2) = \sup_{\alpha\in\mathbb{R}_{\geq 0}} \{a^{-1}(\alpha)\,|x|^2 - \alpha\}, \tag{29}$$

for all $x \in \mathbb{R}^n$, where the inverse $a^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1/b^2}$ is as per (25). Furthermore, the optimizer in (29) is

$$\begin{aligned}\alpha^*(|x|^2) &\doteq \operatorname*{arg\,max}_{\alpha\in\mathbb{R}_{\geq 0}} \{a^{-1}(\alpha)\,|x|^2 - \alpha\} \\ &= \left\{ \begin{array}{ll} a \circ \beta^*(|x|^2), & |x| \in \mathbb{R}_{<b}, \\ +\infty, & |x| \in \mathbb{R}_{\geq b}, \end{array} \right.\end{aligned} \tag{30}$$

where a, β^* are as per (14), (15).

Proof: Fix $b \in \mathbb{R}_{>0}$. With $\rho \doteq |x|^2 \in [0, b^2)$, $x \in \mathbb{R}^n$, recall by Lemma 3.1 and Corollary 3.2 that (11) and (23) hold. Note in particular that $a : [1/b^2, \infty) \to \mathbb{R}_{\geq 0}$ appearing in (23) is strictly monotone increasing, with a monotone strictly increasing inverse $a^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1/b^2}$ given by (25). Hence, substituting the change of variable $\beta = a^{-1}(\alpha)$ in (23) yields

$$\Phi(\rho) = \phi(\rho) = \sup_{\alpha\in[0,\infty)} \{a^{-1}(\alpha)\,\rho - \alpha\}\,,\ \rho \in [0, b^2). \tag{31}$$

That is, (29) holds for $\rho = |x|^2 \in [0, b^2)$. Furthermore, the optimizer is given by (15), subject to the change of variable, ie. $\alpha^* = a \circ \beta^*(\rho)$.

Alternatively, with $\rho = |x|^2 \in \mathbb{R}_{\ge b^2}$, $x \in \mathbb{R}^n$, note by definition (7) that

$$\Phi(\rho) = +\infty. \tag{32}$$

Hence, it remains to show that $\sup_{\alpha\in\mathbb{R}_{\ge 0}}\{a^{-1}(\alpha)\,\rho - \alpha\} = +\infty$. To this end, define $\gamma(\alpha) \doteq a^{-1}(\alpha)\,\rho - \alpha$, $\alpha \in \mathbb{R}_{\ge 0}$. Differentiating with respect to α and applying (14) and its derivative,

$$\gamma'(\alpha) = \frac{\rho}{a' \circ a^{-1}(\alpha)} - 1 = \frac{\rho}{b^2 - 1/a^{-1}(\alpha)} - 1.$$

Recall from Corollary 3.2 that $a^{-1}(0) = 1/b^2$, with $a^{-1}(\alpha) \in \mathbb{R}_{>1/b^2}$ strictly increasing for all $\alpha \in \mathbb{R}_{>0}$. Note in particular that $b^2 - 1/a^{-1}(\alpha) \in (0, b^2)$ for all $\alpha \in \mathbb{R}_{>0}$. Hence, recalling that $\rho \in \mathbb{R}_{\ge b^2}$, it is apparent that $\gamma'(0) = +\infty$, and $\gamma'(\alpha) \in \mathbb{R}_{>0}$ for all $\alpha \in \mathbb{R}_{>0}$. Consequently, the supremum over $\alpha \in \mathbb{R}_{\ge 0}$ of $\gamma(\alpha)$ must be achieved at $\alpha = \alpha^* \doteq +\infty$, with $\gamma(\alpha^*) = +\infty$. That is,

$$\gamma(\alpha^*) = \sup_{\alpha\in\mathbb{R}_{\ge 0}} \{a^{-1}(\alpha)\,\rho - \alpha\} = +\infty,\ \alpha^* = +\infty\,. \tag{33}$$

Hence, combining (32) and (33) again yields (29), with the optimizer given by the right-hand equality in (33). ∎

B. Approximate sup-of-quadratics representation

An approximation of the exact sup-of-quadratics representation of Theorem 3.4 can be constructed by restricting the interval over which the supremum is evaluated in the primal-dual form (11). In particular, given $M \in \mathbb{R}_{>0}$, define

$$\Phi^M(\rho) \doteq \sup_{\beta\in(-\infty, a^{-1}(M)]} \{\beta\,\rho - A(\beta)\} \tag{34}$$

for all $\rho \in \mathbb{R}$, in which A , a^{-1} are as per (13), (25).

Lemma 3.5: The following properties hold:

(i) $\Phi^M : \mathbb{R} \to \mathbb{R}^+_{\ge 0}$, $M \in \mathbb{R}_{>0}$, of (34) satisfies

$$\Phi^M(\rho) = \begin{cases} +\infty, & \rho < 0, \\ \phi(\rho), & \rho \in [0, b^2 - 1/a^{-1}(M)], \\ a^{-1}(M)\,\rho - M, & \rho > b^2 - 1/a^{-1}(M), \end{cases} \tag{35}$$

for all $\rho \in \mathbb{R}$, with ϕ, a^{-1} as per (10), (25), and the corresponding optimizer given by

$$\beta^{M*}(\rho) = \begin{cases} -\infty, & \rho < 0, \\ 1/(b^2 - \rho), & \rho \in [0, b^2 - 1/a^{-1}(M)], \\ a^{-1}(M), & \rho > b^2 - 1/a^{-1}(M). \end{cases} \tag{36}$$

(ii) Given $M \in \mathbb{R}_{>0}$, there exists $A^M : \mathbb{R} \to \mathbb{R}^+_{\ge 0}$ s.t.

$$\Phi^M(\rho) = \sup_{\beta\in\mathbb{R}} \left\{\beta\,\rho - A^M(\beta)\right\}, \tag{37}$$

$$A^M(\beta) = \sup_{\rho\in\mathbb{R}} \left\{\beta\,\rho - \Phi^M(\rho)\right\}, \tag{38}$$

for all $\rho, \beta \in \mathbb{R}$, with Φ^M as per (34), (35), with

$$A^M(\beta) = \begin{cases} 0, & \beta < 1/b^2, \\ a(\beta), & \beta \in [1/b^2, a^{-1}(M)], \\ +\infty, & \beta > a^{-1}(M), \end{cases} \tag{39}$$

for all $\beta \in \mathbb{R}$, in which a, a^{-1} are as per (14), (25). Furthermore, the optimizer in (38) that achieves (39) is

$$\rho^{M*}(\beta) = \begin{cases} 0, & \beta < 1/b^2, \\ b^2 - 1/\beta, & \beta \in [1/b^2, a^{-1}(M)], \\ +\infty, & \beta > a^{-1}(M). \end{cases} \tag{40}$$

(iii) Functions Φ^M and A^M of (34), (37) and (38) are pointwise monotone increasing and decreasing in $M \in \mathbb{R}_{>0}$ respectively, and satisfy the respective limit properties

$$\Phi(\rho) = \sup_{M\in\mathbb{R}_{>0}} \Phi^M(\rho) = \lim_{M\to\infty} \Phi^M(\rho), \tag{41}$$

$$A(\beta) = \inf_{M\in\mathbb{R}_{>0}} A^M(\beta) = \lim_{M\to\infty} A^M(\beta) \tag{42}$$

for all $\rho, \beta \in \mathbb{R}$, with Φ, A as per (11), (12).

Proof: *(i)* Definition (34) of Φ^M and (13) imply that

$$\Phi^M(\rho) = \max\left(\Gamma^M_-(\rho),\, \Gamma^M_+(\rho)\right), \tag{43}$$

where

$$\Gamma^M_-(\rho) \doteq \sup_{\beta\in(-\infty,1/b^2)} \{\beta\,\rho\} = \begin{cases} +\infty, & \rho < 0, \\ \rho/b^2, & \rho \ge 0\,, \end{cases} \tag{44}$$

$$\Gamma^M_+(\rho) \doteq \sup_{\beta\in[1/b^2, a^{-1}(M)]} \chi_\rho(\beta), \quad \chi_\rho(\beta) \doteq \beta\,\rho - a(\beta)\,.$$

Modifying the argument preceding (21) in the proof of Lemma 3.1 yields

$$\Gamma^M_+(\rho) = \begin{cases} \rho/b^2, & \rho < 0, \\ \phi(\rho), & \rho \in [0, b^2 - 1/a^{-1}(M)], \\ a^{-1}(M)\,\rho - M & \rho > b^2 - 1/a^{-1}(M). \end{cases} \tag{45}$$

The corresponding optimizers in the definitions of $\Gamma^M_\pm(\rho)$ may be shown to be

$$\beta^{M*}_-(\rho) = \begin{cases} -\infty, & \rho < 0, \\ 1/b^2, & \rho \ge 0, \end{cases} \tag{46}$$

$$\beta^{M*}_+(\rho) = \begin{cases} 1/b^2, & \rho < 0, \\ 1/(b^2 - \rho), & \rho \in [0, b^2 - 1/a^{-1}(M)], \\ a^{-1}(M), & \rho > b^2 - 1/a^{-1}(M)\,. \end{cases}$$

The pointwise maximum in (43) may be evaluated via (44), (45) and the inequalities (76), (77). Indeed, inspection of (43), (44), (45), (76), (77) immediately yields (35). The corresponding optimizer (36) that achieves the supremum in (34) follows by matching the corresponding cases in (46).

(ii) Some straightforward calculations based on *(i)* yield that Φ^M is continuous at $\rho = b^2 - 1/a^{-1}(M)$, and strictly increasing for all $\rho \ge 0$. Furthermore, recalling (10), note that $\phi'(\rho) = 1/(b^2 - \rho)$, so that $\phi'(\rho) \in [1/b^2, a^{-1}(M)]$ for all $\rho \in [0, b^2 - 1/a^{-1}(M)]$. Consequently, $(\Phi^M)'(\rho)$ is non-decreasing for all $\rho \ge 0$. Hence, it may be concluded that Φ^M as per (35) is convex and (lower) closed [1, (3.8), pp.15,17] on $\mathbb{R}$. Hence, [1, Theorem 5] implies that there exists a one-to-one pairing between Φ^M and its Fenchel transform $A^M : \mathbb{R} \to \mathbb{R}^+$, as per (37) and (38). It remains to show that (39) holds.

By inspection of (35), the supremum in (38) will never be achieved at $\rho < 0$. Hence,

$$A^M(\beta) = \max\left(\Lambda^M_-(\beta),\, \Lambda^M_+(\beta)\right), \tag{47}$$

where

$$\Lambda_-^M(\beta) \doteq \sup_{\rho\in[0,b^2-1/a^{-1}(M)]} \pi_\beta(\rho), \tag{48}$$

$$\Lambda_+^M(\beta) \doteq \sup_{\rho>b^2-1/a^{-1}(M)} \left\{(\beta - a^{-1}(M))\,\rho + M\right\}. \tag{49}$$

in which π_β is as per (17) in the proof Lemma 3.1. Replacing b^2 with $b^2-1/a^{-1}(M)$ in the argument following (17) yields

$$\Lambda_-^M(\beta) = \begin{cases} 0, & \beta < 1/b^2, \\ a(\beta), & \beta\in[1/b^2, a^{-1}(M)], \\ \lambda_-^M(\beta), & \beta > a^{-1}(M), \end{cases} \tag{50}$$

$$\lambda_-^M(\beta) \doteq (b^2 - \tfrac{1}{a^{-1}(M)})\,\beta - \log(b^2\,a^{-1}(M)).$$

By inspection of (49),

$$\Lambda_+^M(\beta) = \begin{cases} \lambda_+^M(\beta), & \beta\le a^{-1}(M), \\ +\infty, & \beta > a^{-1}(M), \end{cases} \tag{51}$$

$$\lambda_+^M(\beta) \doteq M - \frac{b^2}{a^{-1}(M)}(a^{-1}(M)-\beta)(a^{-1}(M)-b).$$

The corresponding optimizers in the definitions of $\Lambda_\pm^M(\beta)$ may be shown to be

$$\rho_-^{M*}(\beta) = \begin{cases} 0, & \beta<1/b^2, \\ b^2-1/\beta, & \beta\in[1/b^2, a^{-1}(M)], \\ b^2-1/a^{-1}(M), & \beta > a^{-1}(M), \end{cases}$$

$$\rho_+^{M*}(\beta) = \begin{cases} b^2-1/a^{-1}(M), & \beta\le a^{-1}(M), \\ +\infty, & \beta > a^{-1}(M). \end{cases} \tag{52}$$

The pointwise maximum in (47) may be evaluated via (50), (51), and the inequalities (78), (79), and the fact that $\lambda_-^M(\beta) < +\infty$ for all $\beta > a^{-1}(M)$. Indeed, inspection of (47), (50), (51), (52) and the aforementioned inequalities immediately yields (39), (40).

(iii) Follows by inspection of (9), (13), (35), (39). ■

Theorem 3.6: Given $b\in\mathbb{R}_{>0}$, the following holds:

(i) The approximation Φ^M of the log-barrier function Φ of (7), represented in (34), (35), (37), has the sup-of-quadratics representation

$$\Phi^M(|x|^2) = \sup_{\alpha\in[0,M]} \{a^{-1}(\alpha)\,|x|^2 - \alpha\},\ M>0, \tag{53}$$

for all $x\in\mathbb{R}^n$, where the inverse $a^{-1}:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge 1/b^2}$ is as per (25). Furthermore, the optimizer in (53) is

$$\begin{aligned}\alpha^{M*}(|x|^2) &\doteq \arg\max_{\alpha\in[0,M]} \{a^{-1}(\alpha)\,|x|^2 - \alpha\} \\ &= \begin{cases} a\circ\beta^{M*}(|x|^2), & |x|\in[0, b^2-1/a^{-1}(M)], \\ +M, & |x| > b^2 - 1/a^{-1}(M), \end{cases}\end{aligned} \tag{54}$$

where a, β^{M*} are as per (14), (36); and

(ii) $\Phi^M(|x|^2)$ of (53) is pointwise monotone increasing in $M>0$, and converges to $\Phi(|x|^2)$ of (29) in the limit as $M\to\infty$, for any $x\in\mathbb{R}^n$.

Proof: *(i)* Fix $x\in\mathbb{R}^n$. Recalling the proof of Lemma 3.5*(i)*, and in particular (43), note that for $\rho = |x|^2\in\mathbb{R}_{\ge0}$,

$$\Phi^M(|x|^2) = \Gamma_+^M(|x|^2) = \sup_{\beta\in[1/b^2, a^{-1}(M)]} \{\beta\,|x|^2 - a(\beta)\},$$

where $a:\mathbb{R}_{\ge 1/b^2}\to\mathbb{R}_{\ge0}$ is as per (14). Defining the invertible change of variable $\alpha = a(\beta)$, with $a^{-1}:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge1/b^2}$ as per (25), immediately yields (53). Applying the same change of variable a to the optimizer $\beta^{M*}(|x|^2)$ of (36) yields (54).

(ii) Immediate by inspection of (29) and (53), or via Lemma 3.5 *(iii)* applied for $\rho = |x|^2$, $x\in\mathbb{R}^n$. ■

C. Example

Fix $b\doteq 3$. Applying Theorem 3.4, and in particular, (29), the log-barrier function Φ has the equivalent representation

$$\Phi(\sigma^2) = \sup_{\alpha\in[0,\infty)} \{a^{-1}(\alpha)\,\sigma^2 - \alpha\} \tag{55}$$

for all $\sigma\in[0,3)$. This is illustrated by plotting the family of quadratic functions $s\mapsto\alpha^{-1}(\alpha)s^2-\alpha$ defined by selecting α from a logarithmic grid. In particular, choosing

$$\alpha\in\left\{\alpha_i\ \middle|\ \log_{10}\alpha_i\in[-2,2]\cap\{k/5\,|\,k\in\mathbb{Z}\}\right\}, \tag{56}$$

this family of quadratics is illustrated in Figure 1, along with the desired log-barrier function. It is clear by inspection that the pointwise supremum of the quadratics shown (solid red lines) does indeed yield the log-barrier function of interest (dashed blue line). The function inverse $a^{-1}:\mathbb{R}_{\ge0}\to[1/b^2,\infty)$, as appearing in (55), is illustrated in Figure 2. Note that it is asymptotically linear, as expected from the proof of Lemma 3.1. Also shown there are two maps $\alpha\mapsto\alpha^{-1}(\alpha)[b^2\pm\epsilon]-\alpha$, illustrating the existence of finite supremum in evaluating (55) for $\rho=\sigma^2=b^2-\epsilon$, $\epsilon\doteq0.1$, but not for $\rho=b^2+\epsilon$, as anticipated by (30). □

Remark 3.7: The sup-of-quadratics representation of (29) is not an approximation for the log-barrier function (7), but rather is exact. However, its form does provide scope for approximation, for example by truncating or discretizing the set over which the supremum is computed, as per (56), or by instituting an upper bound on its elements. It is this latter approach that is adopted in formulated an approximating game for (1). □

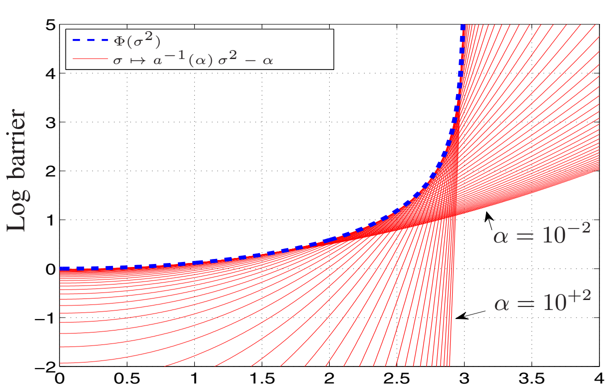

Fig. 1. Log-barrier Φ for $b\doteq3$, and its sup-of-quadratics representation.

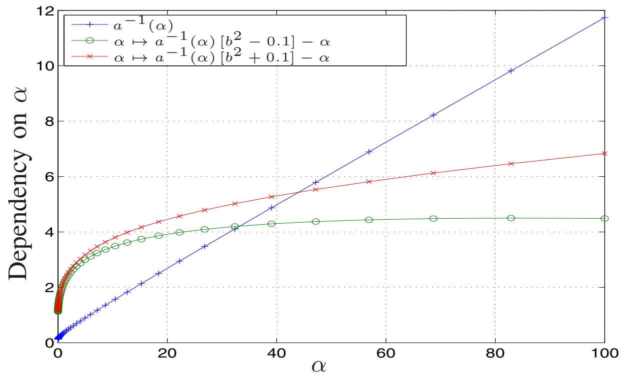

Fig. 2. Role of a^{-1} in representation (55), $b \doteq 3$.

IV. An approximate unconstrained game representation for (1)

In view of the sup-of-quadratics representation (29) and its approximation (53), Theorem 3.6 can be applied to approximate the constrained linear regulator problem of (1) as a game. In order to formulate this game, it is convenient to first define a control problem based on (1) that incorporates the aforementioned sup-of-quadratics approximation (53). In particular, given $M \in \mathbb{R}_{>0}$, define $W_t^M : \mathbb{R}^n \to \mathbb{R}$ by

$$\overline{W}_t^M(x) \doteq \inf_{u \in \mathscr{U}[0,t]} \bar{J}_t^M(x,u) \tag{57}$$

for all $x \in \mathbb{R}^n$, in which the total cost $\bar{J}_t^M$ is defined analogously to (2) but with the log-barrier function Φ replaced with its sup-of-quadratics approximation Φ^M of (34), (35), (53). In particular,

$$\bar{J}_t^M(x,u) \doteq I_t(x,u) + \bar{I}_t^M(x,u) + \Psi_0(\xi_t), \tag{58}$$

$$\bar{I}_t^M(x,u) \doteq \int_0^t \tfrac{1}{2} \sup_{\alpha \in [0,M]} \{a^{-1}(\alpha)\,|\xi_s|^2 - \alpha\}\, ds \tag{59}$$

for all $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$. Motivated by (58), (59), define the signal space

$$\mathscr{A}[0,t] \doteq C([0,t];[0,M]), \tag{60}$$

and a new cost function $\widehat{J}_t^M : \mathbb{R}^n \times \mathscr{U}[0,t] \times \mathscr{A}[0,t] \to \mathbb{R}$ for fixed $t \in \mathbb{R}_{\ge 0}$ by

$$\widehat{J}_t^M(x,u,\alpha) \doteq I_t(x,u) + \widehat{I}_t^M(x,u,\alpha) + \Psi_0(\xi_t), \tag{61}$$

$$\widehat{I}_t^M(x,u,\alpha) \doteq \int_0^t \tfrac{1}{2}[a^{-1}(\alpha_s)\,|\xi_s|^2 - \alpha_s]\, ds \tag{62}$$

for all $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$, $\alpha \in \mathscr{A}[0,t]$. Using (61), (62), define a linear quadratic dynamic game problem (with unconstrained state) via the (upper) value function $\widehat{W}_t : \mathbb{R}^n \to \mathbb{R}$ for fixed $t \in \mathbb{R}_{\ge 0}$ and all $x \in \mathbb{R}^n$ by

$$\widehat{W}_t^M(x) \doteq \inf_{u \in \mathscr{U}[0,t]} \sup_{\alpha \in \mathscr{A}[0,t]} \widehat{J}_t^M(x,u,\alpha). \tag{63}$$

Lemma 4.1: Given any fixed $t \in \mathbb{R}_{>0}$, $u \in \mathscr{U}[0,t]$, $x \in \mathbb{R}^n$, the map $s \mapsto \xi_s$, $s \in [0,t]$, defined by (6) subject to $\xi_0 = x$ is continuous.

Proof: See for example [11, Lemma 3.1.5]. ∎

Theorem 4.2: Given $M \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\ge 0}$, the value functions $\overline{W}_t^M$, $\widehat{W}_t^M$ of (57), (63) are equivalent. Furthermore, these value functions are pointwise monotone non-decreasing with increasing $M \in \mathbb{R}_{>0}$, and satisfy

$$\overline{W}_t^M(x) = \widehat{W}_t^M(x) \le \overline{W}_t^\infty(x) \le \overline{W}_t(x) \tag{64}$$

for all $x \in \mathbb{R}^n$, where $\overline{W}_t$, $\overline{W}_t^\infty : \mathbb{R}^n \to \mathbb{R}_{\ge 0}^+$ are defined for all $x \in \mathbb{R}^n$ by (1) and

$$\overline{W}_t^\infty(x) \doteq \sup_{M \in \mathbb{R}_{>0}} \overline{W}_t^M(x) = \lim_{M \to \infty} \overline{W}_t^M(x). \tag{65}$$

Remark 4.3: By inspection of (57), (63), (64), Theorem 4.2 states that value of the optimal control problem (57) defined with respect to the sup-of-quadratics approximation (53) is identical to the value of the two-player game defined by (63). As the order of the inf and sup in this game can be swapped under mild conditions as per [9], the value (57) can in-principle be computed via a supremum (over $\alpha \in \mathscr{A}[0,t]$) of the values of a family of linear quadratic regulator problems. The details are omitted for brevity.

Proof: [Theorem 4.2] Fix $M \in \mathbb{R}_{>0}$, $t \in \mathbb{R}_{\ge 0}$, and $x \in \mathbb{R}^n$. Let $u \in \mathscr{U}[0,t]$. By inspection of (58), (59), (61), (62), as any input $\alpha \in \mathscr{A}[0,t]$ is pointwise suboptimal in the definition (59) of $\bar{I}_t^M(x,u)$,

$$\bar{J}_t^M(x,u) \ge \sup_{\alpha \in \mathscr{A}[0,t]} \widehat{J}_t^M(x,u,\alpha).$$

As $u \in \mathscr{U}[0,t]$ is arbitrary, it immediately follows that

$$\begin{aligned} \overline{W}_t^M(x) &= \inf_{u \in \mathscr{U}[0,t]} \bar{J}_t^M(x,u) \\ &\ge \inf_{u \in \mathscr{U}[0,t]} \sup_{\alpha \in \mathscr{A}[0,t]} \widehat{J}_t^M(x,u,\alpha) = \widehat{W}_t^M(x) \end{aligned} \tag{66}$$

yielding one of two inequalities required to demonstrate (64).

For the other inequality, first define $\alpha^* : \mathbb{R}^n \to [0,M]$ via Theorem 3.6 by

$$\begin{aligned} \alpha^*(x) &\doteq \arg\max_{\hat{\alpha} \in [0,M]} \left\{ a^{-1}(\hat{\alpha})\,|x|^2 - \hat{\alpha} \right\}, \\ &= \begin{cases} a(1/(b^2 - |x|^2)), & |x|^2 \in [0, b^2 - 1/a^{-1}(M)], \\ +M, & |x|^2 > b^2 - 1/a^{-1}(M), \end{cases} \end{aligned} \tag{67}$$

in which the equality follows by (54). Note in particular that $\alpha^* \in C(\mathbb{R}^n;[0,M])$. With $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$ fixed, let $s \mapsto \xi_s$ denote the corresponding state trajectory map defined by the unique solution of (6) subject to $\xi_0 = x$. As this map is continuous by Lemma 4.1, the composed map defined by $s \mapsto \alpha^*(\xi_s)$ is also continuous. That is, the signal $\hat{\alpha}$ defined by $\hat{\alpha}_s \doteq \alpha^*(\xi_s)$ for all $s \in [0,t]$ satisfies $\hat{\alpha} \in C([0,t];[0,M])$. Hence, recalling (58) and applying Theorem 3.6,

$$\begin{aligned} \bar{J}_t^M(x,u) &= I_t(x,u) + \bar{I}_t^M(x,u) + \Psi_0(\xi_t) \\ &= I_t(x,u) + \int_0^t \tfrac{1}{2}\,\Phi^M(|\xi_s|^2)\, ds + \Phi_0(\xi_t) \\ &= I_t(x,u) + \int_0^t \tfrac{1}{2}\,[a^{-1}(\hat{\alpha}_s)\,|\xi_s|^2 - \hat{\alpha}_s]\, ds + \Psi_0(\xi_t) \\ &= \widehat{J}_t^M(x,u,\hat{\alpha}) \le \sup_{\alpha \in \mathscr{A}[0,t]} \widehat{J}_t^M(x,u,\alpha)\,. \end{aligned}$$

As $u \in \mathscr{U}[0,t]$ is arbitrary, taking the infimum of both sides and recalling (57), (63) yields

$$\overline{W}_t^M(x) \le \widehat{W}_t^M(x). \tag{68}$$

As $x \in \mathbb{R}^n$ is arbitrary, inequalities (66) and (68) together yield the required equivalence.

In order to demonstrate the non-decreasing property, fix $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$, and $M_1, M_2 \in \mathbb{R}_{>0}$, $M_1 \le M_2$. Recall by Theorem 3.6 *(ii)* that $\phi^M(|\cdot|^2)$ defines a pointwise monotone non-decreasing sequence of functions with increasing $M \in \mathbb{R}_{>0}$. Hence, (58), (59) imply that

$$\bar{I}_t^{M_1}(x,u) \le \bar{I}_t^{M_2}(x,u) \implies \bar{J}_t^{M_1}(x,u) \le \bar{J}_t^{M_2}(x,u)\,.$$

As $u \in \mathscr{U}[0,t]$ is arbitrary, it follows immediately by (57) that $\overline{W}_t^{M_1}(x) \le \overline{W}_t^{M_2}(x)$. That is, the pointwise monotone non-decreasing property holds. Furthermore, applying Lemma 3.5 *(iii)* and the definition of $\overline{W}_t^\infty$, in particular (41) and (65),

$$\begin{aligned}
\overline{W}_t^M(x) &\le \sup_{M\in\mathbb{R}_{>0}} \overline{W}_t^M(x) = \overline{W}_t^\infty(x) \\
&\le \inf_{u\in\mathscr{U}[0,t]} \left\{ I_t(x,u) + \sup_{M\in\mathbb{R}_{>0}} \bar{I}_t^M(|\xi_s|^2) + \Psi_0(\xi_t) \right\} \\
&\le \inf_{u\in\mathscr{U}[0,t]} \overline{J}_t(x,u) = \overline{W}_t(x).
\end{aligned}$$

As $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$ are arbitrary, (64) holds. ■

Given $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}_{>0}$, it is useful to define respective sets of all ϵ-optimal inputs in the definitions (1), (57) of $\overline{W}_t(x)$, $\overline{W}_t^M(x)$ for $M \in \mathbb{R}_{>0}$. In particular, define

$$\mathscr{U}_x^\epsilon[0,t] \doteq \left\{ u \in \mathscr{U}[0,t] \,\middle|\, \overline{W}_t(x) + \epsilon > \bar{J}_t(x,u) \right\}, \tag{69}$$

$$\mathscr{U}_x^{M,\epsilon}[0,t] \doteq \left\{ u \in \mathscr{U}[0,t] \,\middle|\, \overline{W}_t^M(x) + \epsilon > \bar{J}_t^M(x,u) \right\}.$$

It is also convenient to define the (possibly empty) set of time intervals for which a trajectory, generated by dynamics (6) corresponding to a particular initial state and input, resides outside the desired state constraint set. That is, given fixed $t \in \mathbb{R}_{\ge 0}$, define $\Delta_t : \mathbb{R}^n \times \mathscr{U} \to \cup_{I\subset[0,t]} I$ by

$$\Delta_t(x,u) \doteq \bigcup_{r\in[0,t],s\in[r,t]} \left\{ [r,s] \,\middle|\, \begin{array}{c} \xi_\sigma \notin \mathscr{B}(0;b)\ \forall\ \sigma \in [r,s], \\ \xi \text{ satisfying (6) given} \\ \xi_0 = x \text{ and } u \end{array} \right\} \tag{70}$$

for all $x \in \mathbb{R}^n$, $u \in \mathscr{U}[0,t]$.

Lemma 4.4: Given $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$ such that $\overline{W}_t(x) < \infty$, and the set $\mathscr{U}_x^\epsilon[0,t]$ of ϵ-optimal inputs (69), $\epsilon \in \mathbb{R}_{>0}$,

$$\sup_{u\in\mathscr{U}_x^\epsilon[0,t]} |\Delta_t(x,u)| = 0\,, \tag{71}$$

where $|\Delta_t(x,u)|$ is the Lebesgue measure of the set (70).

Proof: Fix $t \in \mathbb{R}_{\ge 0}$, $x \in \mathbb{R}^n$ such that $\overline{W}_t(x) < \infty$, and any $\epsilon \in \mathbb{R}_{>0}$. Suppose there exists a $u^\epsilon \in \mathscr{U}_x^\epsilon[0,t]$ such that $|\Delta_t(x,u^\epsilon)| \ge \delta > 0$ for some $\delta \in \mathbb{R}_{>0}$. Recalling (1), (2), (3), (7), and the definition (69) of δ-optimality,

$$\begin{aligned}
\overline{W}_t(x) + \epsilon > \overline{J}_t(x,u^\epsilon) &\ge \int_0^t \tfrac{1}{2}\,\Phi(|\xi_s|^2)\,ds \\
&\ge \delta \lim_{R\to b} \Phi(R) = \infty,
\end{aligned}$$

which is a contradiction of the finiteness of $\overline{W}_t(x)$. Hence, no such $\delta \in \mathbb{R}_{>0}$ exists, so that (71) follows. ■

Lemma 4.5: Given any $t \in \mathbb{R}_{\ge 0}$, $\epsilon, M \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^n$,

$$\sup_{u\in\mathscr{U}_x^{M,\epsilon}[0,t]} |\Delta_t(x,u)| \le \frac{\overline{W}_t^M(x) + \epsilon}{\frac{1}{2}\,\Phi^M(b)}\,. \tag{72}$$

Furthermore, given any $x \in \mathbb{R}^n$ satisfying $\overline{W}_t(x) < \infty$,

$$\lim_{M\to\infty} \sup_{u\in\mathscr{U}_x^{M,\epsilon}[0,t]} |\Delta_t(x,u)| = 0. \tag{73}$$

Proof: Fix $t \in \mathbb{R}_{\ge 0}$, $\epsilon, M \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^n$, and $u \in \mathscr{U}_x^{M,\epsilon}[0,t]$. Let ξ^ϵ denote the trajectory of (6) with initial state $\xi_0^\epsilon = x$ and input u as given. By (69), (70),

$$\begin{aligned}
&\overline{W}_t^M(x) + \epsilon > \bar{J}_t^M(x,u) \\
&= I_t(x,u) + \tfrac{1}{2}\int_0^t \Phi^M(|\xi_s^\epsilon|^2)\,ds + \Psi_0(\xi_t^\epsilon) \\
&\ge \tfrac{1}{2}\int_{\Delta_t(x,u)} \Phi^M(b^2)\,ds = \tfrac{1}{2}\,\Phi^M(b^2)\,|\Delta_t(x,u)|,
\end{aligned}$$

or $|\Delta_t(x,u)| \le [\overline{W}_t^M(x) + \epsilon]/(\frac{1}{2}\,\Phi^M(b^2))$, as $\Phi^M(b^2) \in \mathbb{R}_{>0}$. As the right-hand side of this last inequality is independent of input $u \in \mathscr{U}_x^{M,\epsilon}[0,t]$, the result (72) follows.

Where $\overline{W}_t(x) < \infty$, inequality (64) and (65) of Theorem 4.2, (35) of Lemma 3.5, and (72), together imply that

$$\lim_{M\to\infty} \sup_{u\in\mathscr{U}_x^{M,\epsilon}[0,t]} |\Delta_t(x,u)| \le \lim_{M\to\infty} \left\{ \frac{\overline{W}_t(x) + \epsilon}{\frac{1}{2}\,\Phi^M(b)} \right\} = 0$$

as $\lim_{M\to\infty} \Phi^M(b) = \lim_{M\to\infty}\{a^{-1}(M)\,b - M\} = \infty$. That is, (73) holds. ■

Remark 4.6: Lemma 4.4 indicates that the regulator problem defined by $\overline{W}_t$ of (1) implements that required state constraint for almost every time for those initial states $x \in \mathbb{R}^n$ for which $\overline{W}_t(x) < \infty$. Similarly, Lemma 4.5 indicates that the approximating regulator problem defined by $\overline{W}_t^M$ of (57) implements the same constraint in the limit as $M \to \infty$.

V. Conclusion

By developing a sup-of-quadratics representation for a standard log-barrier constraint, a linear regulator problem with a state constraint is reformulated as an unconstrained dynamic game. Various properties of this representation, and the attendant value functionseq:value-W-M are developed, along with a simple characterization of the behaviour of near-optimal trajectories in relation to the underlying constraint. It is anticipated that the new game formulation, and associated properties, will prove useful in computing solutions to such state constrained regulator problems.

References

[1] R.T. Rockafellar. Conjugate duality and optimization. *SIAM Regional Conf. Series in Applied Math.*, 16, 1974.

[2] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State space solutions to standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$-control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.

[3] C.E. Garcia, D.M. Prett, and M. Morari. Model predictive control: theory and practice—a survey. *Automatica*, 25(3):335–348, 1989.

[4] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[5] L. Wang. Continuous time model predictive control design using orthonormal functions. *International Journal of Control*, 74(16):1588–1600, 2001.

[6] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[7] C. Feller and C. Ebenbauer. Continuous-time linear MPC algorithms based on relaxed logarithmic barrier functions. In *Proc.* 19^{th} *IFAC World Congress (Cape Town, South Africa)*, pages 2481–2488, 2014.

[8] P.M. Dower and H. Zhang. A new fundamental solution for differential Riccati equations arising in $\mathscr{L}_2$-gain analysis. In *proc. Australian Control Conference (Gold Coast, Australia)*, pages 65–68, 2015.

[9] W.M. McEneaney and P.M. Dower. The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems. *SIAM J. Control & Optimization*, 53(5):2898–2933, 2015.

[10] B.D.O. Anderson and J.B. Moore. *Linear optimal control*. Prentice-Hall, Englewood Cliffs, New Jersey, USA, 1971.

[11] R.F. Curtain and H.J. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[12] M. Green and D.J.N. Limebeer. *Linear robust control*. Information and systems sciences. Prentice-Hall, 1995.

[13] R.T. Rockafellar and R.J. Wets. *Variational Analysis*. Springer-Verlag, 1997.

APPENDIX

A. Lambert-W function

The Lambert-W function is a transcendental multi-valued function defined implicitly by

$$W(\eta)\exp(W(\eta)) = \eta \tag{74}$$

for all $\eta \in \mathbb{R}$. It is illustrated in Figure 3, with the -1 branch of Lemma 3.1 explicitly labelled. Note specifically that

$$W_{-1} : [-e^{-1}, 0) \to (-\infty, -1] \tag{75}$$

is monotone decreasing, with infinite gradient at $\eta = -e^{-1}$ and in the limit as $\eta \to 0^-$.

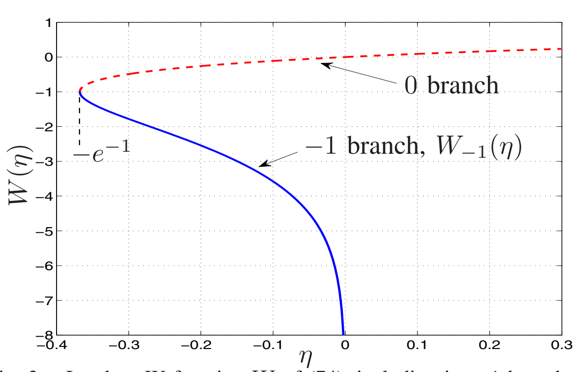

Fig. 3. Lambert-W function W of (74), including its -1 branch.

B. Some useful inequalities

Given $M \in \mathbb{R}_{>0}$,

$$0 \ge \rho/b^2 - \phi(\rho) \qquad \forall\ \rho \ge 0, \tag{76}$$

$$0 \ge \rho/b^2 - a^{-1}(M)\,\rho + M \qquad \forall\ \rho \ge b^2 - \tfrac{1}{a^{-1}(M)}, \tag{77}$$

$$0 \ge \lambda_+^M(\beta) \qquad \forall\ \beta \le a^{-1}(0), \tag{78}$$

$$0 \ge \lambda_+^M(\beta) - a(\beta) \qquad \forall\ \beta \in (a^{-1}(0), a^{-1}(M)], \tag{79}$$

in which $\lambda_+^M : \mathbb{R} \to \mathbb{R}$ is given by

$$\lambda_+^M(\beta) \doteq M - \frac{(a^{-1}(M) - \beta)\,(a^{-1}(M) - a^{-1}(0))}{a^{-1}(0)\,a^{-1}(M)} \tag{80}$$

and $a : \mathbb{R}_{\ge a^{-1}(0)} \to \mathbb{R}_{\ge 0}$, $a^{-1} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge a^{-1}(0)}$ are as per (14), (25), $a^{-1}(0) = 1/b^2$.

Proof: [Inequality (76)] Recalling (10), for all $\rho \in [0, b^2)$, $\phi(\rho) = -\log(1 - \rho/b^2)$, $\phi'(\rho) = \frac{1}{b^2-\rho}$, $\phi''(\rho) = \frac{1}{(b^2-\rho)^2}$. By Taylor's theorem, there exists $c \in [0, \rho]$ such that

$$\begin{aligned}\phi(\rho) &= \phi(0) + \phi'(0)\,\rho + \tfrac{1}{2}\,\phi''(c)\,\rho^2 \\ &= \rho/b^2 + \tfrac{1}{2}/(b-c)^2 \ge \rho/b^2,\end{aligned}$$

as required by (76) for all $\rho \in [0, b^2)$. For $\rho \ge b^2$, (76) holds trivially by inspection of (10).

[Inequality (77)] Fix $M \in \mathbb{R}_{>0}$. Given any

$$\rho \ge b^2 - \frac{1}{a^{-1}(M)} = \frac{a^{-1}(M) - a^{-1}(0)}{a^{-1}(0)\,a^{-1}(M)}$$

it follows immediately that

$$\begin{aligned}\rho/b^2 - a^{-1}(M)\,\rho + M &= M - [a^{-1}(M) - a^{-1}(0)]\,\rho \\ &\le M - \frac{[a^{-1}(M) - a^{-1}(0)]^2}{a^{-1}(0)\,a^{-1}(M)} = \lambda_+^M(a^{-1}(0)),\end{aligned}$$

where λ_+^M is as per (80). Hence, (77) is a special case of inequality (78).

[Inequality (78)] Fix $M \in \mathbb{R}_{>0}$. For any $\beta \le a^{-1}(0)$,

$$\begin{aligned}(\lambda_+^M)'(\beta) &= \frac{a^{-1}(M) - a^{-1}(0)}{a^{-1}(0)\,a^{-1}(M)} > 0 \\ &\implies \lambda_+^M(\beta) \le \lambda_+^M(a^{-1}(0)) = \mu(M)\end{aligned}$$

for all $\beta \le a^{-1}(0)$, in which $\mu : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is defined by

$$\mu(M) \doteq M + 2 - \left[\frac{a^{-1}(M)}{a^{-1}(0)} + \frac{a^{-1}(0)}{a^{-1}(M)}\right]$$

As $(a^{-1})'(M) = \frac{a^{-1}(0)\,a^{-1}(M)}{a^{-1}(0)+a^{-1}(M)}$, differentiation of μ with respect to M yields $\mu'(M) = -\frac{a^{-1}(0)}{a^{-1}(M)} < 0$. Hence,

$$0 = \mu(0) > \mu(M) = \lambda_+^M(a^{-1}(0)) \ge \lambda_+^M(\beta)$$

for all $\beta \le a^{-1}(0)$, as required by (78).

[Inequality (79)] Define $\eta : (0, M] \to \mathbb{R}$ by

$$\eta(\alpha) \doteq \lambda_+^M \circ a^{-1}(\alpha) - \alpha$$

for all $\alpha \in (0, M]$. Differentiation yields that

$$\eta'(\alpha) = \frac{a^{-1}(0)}{a^{-1}(M)}\left[\frac{a^{-1}(M) - a^{-1}(0)}{a^{-1}(\alpha) - a^{-1}(0)} - 1\right] \ge 0$$

for all $\alpha \in (0, M]$. Hence, $\eta(\alpha) \le \eta(M) = 0$ for all $\alpha \in (0, M]$. Setting $\alpha = a(\beta)$ for $\beta \in (a^{-1}(0), a^{-1}(M)]$,

$$0 \ge \eta \circ a(\beta) = \lambda_+^M(\beta) - a(\beta)$$

as required by (79).