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Verifying Fundamental Solution Groups for Lossless Wave Equations via Stationary Action and Optimal Control

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Abstract

A representation of a fundamental solution group for a class of wave equations is constructed by exploiting connections between stationary action and optimal control. By using a Yosida approximation of the associated generator, an approximation of the group of interest is represented for sufficiently short time horizons via an idempotent convolution kernel that describes all possible solutions of a corresponding short time horizon optimal control problem. It is shown that this representation of the approximate group can be extended to longer horizons via a concatenation of such short horizon optimal control problems, provided that the associated initial and terminal conditions employed in concatenating trajectories are determined via a stationarity rather than an optimality based condition. The construction is illustrated by its application to the approximate solution of a two point boundary value problem.

Keywords Optimal control · Stationary action · Dynamic programming · Wave equations · Fundamental solution groups · Two point boundary value problems

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1 Introduction

The *action principle* [1,13–15,17] is a variational principle underpinning modern physics that may be applied to a predefined notion of *action* to yield the equations

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of motion of a physical system and its underlying conservation laws. With a suitable definition of this action, the action principle specializes to *Hamilton's action principle*, an important corollary of which states that

> *any trajectory of an energy conserving system renders the corresponding action functional stationary in the calculus of variations sense.*

Consequently, Hamilton's action principle can be interpreted as providing a characterization of all solutions of an energy conserving or *lossless* system. This interpretation motivates the development summarized in this work, with Hamilton's action principle applied via an optimal control representation to construct the fundamental solution group corresponding to a lossless wave equation. The specific wave equation of interest is given by

$$\ddot{x} = -\Lambda\, x\,, \tag{1}$$

where Λ is a linear, unbounded, positive, self-adjoint operator densely defined in an $\mathscr{L}^2$-space $\mathscr{X}$, with a compact inverse $\Lambda^{-1} \in \mathcal{L}(\mathscr{X})$. The results presented generalize the recent work [9] from the specific Laplacian case $\Lambda \doteq -\partial^2$ to any unbounded operator Λ satisfying the stated assumptions.

In order to apply Hamilton's action principle in this development, compatible notions of kinetic and potential energy are defined with respect to generalized notions of velocity and position corresponding respectively to the input to and solution of an abstract Cauchy problem [3,19]. This allows the integrated action to be rigorously defined as a time horizon parameterized functional of the velocity input. Unlike the finite dimensional case, this action functional is neither convex nor concave for any time horizon, thereby preventing an immediate generalization of the optimal control approach of [17] to its analysis. As a remedy, a corresponding approximate class of wave equations is considered, in which the unbounded linear operator involved is replaced by its (bounded) Yosida approximation. This yields a corresponding action functional that is strictly concave for sufficiently short (but strictly positive) time horizons. The integrated action is subsequently analysed using tools from optimal control theory, semigroup theory, and idempotent analysis, see for example [3,5,9,16,19]. In particular, an *idempotent fundamental solution semigroup* applicable on sufficiently short horizons is used to represent the value function of the attendant optimal control problem as an idempotent convolution of a bivariate kernel with a terminal cost. As the characteristics associated with this optimal control problem correspond to solutions of the approximate wave equation by stationary action, the idempotent fundamental solution semigroup is subsequently used to construct a short horizon prototype for the fundamental solution group for the aforementioned approximate wave equation. These short horizon prototypes are pieced together into longer horizon prototypes using the *stat* operation [18].

In terms of organization, exact and approximate fundamental solutions groups for the lossless wave equation (1) are first established in Sect. 2. Independently, an optimal control problem that encapsulates Hamilton's action principle is introduced in Sect. 3. This control problem is then employed in Sect. 4 to recover the long-horizon group representation of interest, via a concatenation of short horizon prototypes, thereby confirming the groups of Sect. 2. An application of this representation to approximately solving a two point boundary value problem (TPBVP) is considered in Sect. 5. The paper concludes with some brief remarks in Sect. 6. Throughout, $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) denotes the real (nonnegative) numbers, $\overline{\mathbb{R}} \doteq \{\pm\infty\}$ denotes the extended reals, $\mathbb{Q}$ denotes the rationals, and $\mathbb{N}$ denotes the natural numbers.

2 Exact and Approximate Fundamental Solution Groups

Operator Λ specifying the wave equation (1) is linear, unbounded, positive, and self-adjoint, with domain $\mathscr{X}_2 \doteq \mathsf{dom}\,(\Lambda)$ dense in an $\mathscr{L}^2$-space $\mathscr{X}$. For example [9], for a length $L \in \mathbb{R}_{>0}$ vibrating string with Dirichlet boundary data, $-\Lambda$ specializes to the Laplacian, i.e. $-\Lambda \doteq \partial^2$, with (Sobolev) domain $\mathsf{dom}\,(\Lambda) \doteq \mathscr{X}_2 \doteq \mathscr{H}_0^2((0, L); \mathbb{R})$ dense in $\mathscr{X} \doteq \mathscr{L}^2((0, L); \mathbb{R})$. In general, and as per (1), Λ possesses a unique, linear, unbounded, positive, and self-adjoint square-root, and this is denoted throughout by $\Lambda^{\frac{1}{2}}$. The domain of $\Lambda^{\frac{1}{2}}$ defines a Hilbert space $\mathscr{X}_1 \subset \mathscr{X}$, with

$$\mathscr{X}_1 \doteq \mathsf{dom}\,(\Lambda^{\frac{1}{2}})\,, \quad \langle x, \xi \rangle_1 \doteq \langle \Lambda^{\frac{1}{2}}\, x,\ \Lambda^{\frac{1}{2}}\, \xi \rangle\,, \tag{2}$$

for all $x, \xi \in \mathscr{X}_1$, in which $\langle\, ,\, \rangle$ represents the inner product on $\mathscr{X}$. The aforementioned domains and spaces satisfy the relations $\mathscr{X}_2 \subset \mathscr{X}_1 \subset \mathscr{X}$, $\overline{\mathscr{X}_2} = \mathscr{X}_1$, and $\overline{\mathscr{X}_1} = \mathscr{X}$, with the nominated closures being with respect to $\mathscr{X}_1$ and $\mathscr{X}$ respectively. With a view to regularizing Λ, an operator $\mathcal{I}_\mu$, $\mu \in \mathbb{R}_{>0}$, is defined via the resolvent $\mathcal{R}_{-\Lambda}(\frac{1}{\mu^2})$ of $-\Lambda$ by

$$\mathcal{I}_\mu \doteq \tfrac{1}{\mu^2}\mathcal{R}_{-\Lambda}(\tfrac{1}{\mu^2}) = (\mathcal{I} + \mu^2\, \Lambda)^{-1}. \tag{3}$$

Boundedness of $\mathcal{I}_\mu$ follows by the Hille–Yosida theorem [19], and in particular it may be identified as an element of $\mathcal{L}(\mathscr{X})$, $\mathcal{L}(\mathscr{X}_1)$, or indeed $\mathcal{L}(\mathscr{X}; \mathscr{X}_1)$, etc.

Using definitions (2) and (3), wave equation (1) motivates consideration of the linear generators

$$\begin{aligned}
\mathcal{A} &\doteq \begin{pmatrix} 0 & \mathcal{I} \\ -\Lambda & 0 \end{pmatrix}, & \mathsf{dom}\,(\mathcal{A}) &\doteq \mathscr{Y}_1 \doteq \mathscr{X}_2 \times \mathscr{X}_1, \\
\mathcal{A}^\mu &\doteq \begin{pmatrix} 0 & \mathcal{I}_\mu^{\frac{1}{2}} \\ -\Lambda^{\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\, \Lambda^{\frac{1}{2}} & 0 \end{pmatrix}, & \mathsf{dom}\,(\mathcal{A}^\mu) &\doteq \mathscr{Y} \doteq \mathscr{X}_1 \times \mathscr{X},
\end{aligned} \tag{4}$$

in which $\mathscr{Y}$ defines a Hilbert space with $\langle (x, p), (\xi, \pi) \rangle_{\mathscr{Y}} \doteq \langle x, \xi \rangle_1 + \langle p, \pi \rangle$ for all $(x, p), (\xi, \pi) \in \mathscr{Y}$, and $\mathscr{Y}_1$ is dense in $\mathscr{Y}$. As Λ in (1), (4) has a compact inverse, the spectral theorem (see for example, [3, Theorem A.4.25, p.619]) implies that Λ has the representations

$$\begin{aligned} \Lambda\, \xi &= \sum_{n=1}^{\infty} \lambda_n\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \quad \xi \in \mathscr{X}_1, \\ \Lambda\, \pi &= \sum_{n=1}^{\infty} \lambda_n\, \langle \pi,\, \varphi_n \rangle\, \varphi_n\,, \quad \pi \in \mathscr{X}, \end{aligned} \tag{5}$$

on $\mathscr{X}_1$ and $\mathscr{X}$ respectively. Here, the set of all eigenvalues $\{\lambda_n^{-1}\}_{n\in\mathbb{N}}$ of compact Λ^{-1} defines a strictly positive and non-increasing sequence in $\mathbb{R}_{>0}$ satisfying $0 = \lim_{n\to\infty} \lambda_n^{-1}$, while $\{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$, $\{\varphi_n\}_{n\in\mathbb{N}}$ denote respectively the corresponding sets of orthonormal eigenvectors in $\mathscr{X}_1$, $\mathscr{X}$, with $\varphi_n \doteq \sqrt{\lambda_n}\, \tilde{\varphi}_n$.

Operators $(\Lambda\, \mathcal{I}_\mu)^{\frac{1}{2}}$ and $\Lambda\, \mathcal{I}_\mu \equiv \Lambda^{\frac{1}{2}}\, \mathcal{I}_\mu\, \Lambda^{\frac{1}{2}}$, defined using $\mathcal{I}_\mu$ of (3), naturally inherit the spectral form (5) of Λ, see for example the proof of Lemma 5 later. Both operators reside in $\mathcal{L}(\mathscr{X}_1)$, with the former also residing in $\mathcal{L}(\mathscr{X})$, and their corresponding eigenvalues are given by

$$\omega_n^\mu \doteq \sqrt{\lambda_n^\mu}\,, \quad \lambda_n^\mu \doteq \frac{\lambda_n}{1 + \mu^2\, \lambda_n}\,, \quad n \in \mathbb{N}. \tag{6}$$

Theorem 1 *Given $\mu \in \mathbb{R}_{>0}$, operators $\mathcal{A}$ and $\mathcal{A}^\mu$ of (4) satisfy the following:*

(i) *$\mathcal{A}$ is unbounded, closed, and densely defined on $\mathscr{Y}_1 \subset \mathscr{Y}$, with $\overline{\mathscr{Y}_1} = \mathscr{Y}$;*
(ii) *$\mathcal{A}$ generates a strongly continuous group of bounded linear operators $\{\mathcal{U}_t\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathscr{Y}_1)$;*
(iii) *$\mathcal{A}^\mu \in \mathcal{L}(\mathscr{Y})$;*
(iv) *$\mathcal{A}^\mu$ generates a uniformly continuous group of bounded linear operators $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathscr{Y})$, with*

$$\mathcal{U}_t^\mu = \left(\begin{array}{c|c} [\mathcal{U}_t^\mu]_{11} & [\mathcal{U}_t^\mu]_{12} \\ \hline [\mathcal{U}_t^\mu]_{21} & [\mathcal{U}_t^\mu]_{22} \end{array} \right) = \exp(t\, \mathcal{A}^\mu),\ \ t \in \mathbb{R}, \tag{7}$$

in which $[\mathcal{U}_t^\mu]_{11} \in \mathcal{L}(\mathscr{X}_1)$, $[\mathcal{U}_t^\mu]_{12} \in \mathcal{L}(\mathscr{X}; \mathscr{X}_1)$, $[\mathcal{U}_t^\mu]_{21} \in \mathcal{L}(\mathscr{X}_1; \mathscr{X})$, and $[\mathcal{U}_t^\mu]_{22} \in \mathcal{L}(\mathscr{X})$ are given by

$$\begin{aligned} [\mathcal{U}_t^\mu]_{11}\, \xi &\doteq \sum_{n=1}^{\infty} \cos(\omega_n^\mu\, t)\, \langle \xi, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n, \quad [\mathcal{U}_t^\mu]_{12}\, \pi \doteq \sum_{n=1}^{\infty} \sin(\omega_n^\mu\, t)\, \langle \pi, \varphi_n \rangle\, \tilde{\varphi}_n, \\ [\mathcal{U}_t^\mu]_{21}\, \xi &\doteq -\sum_{n=1}^{\infty} \sin(\omega_n^\mu\, t)\, \langle \xi, \tilde{\varphi}_n \rangle_1\, \varphi_n,\ [\mathcal{U}_t^\mu]_{22}\, \pi \doteq \sum_{n=1}^{\infty} \cos(\omega_n^\mu\, t)\, \langle \pi, \varphi_n \rangle\, \varphi_n, \end{aligned} \tag{8}$$

for all $\xi \in \mathscr{X}_1$, $\pi \in \mathscr{X}$. Moreover, there exist $M, \omega \in \mathbb{R}_{\geq 0}$ independent of μ such that $\|\mathcal{U}_s^\mu\|_{\mathcal{L}(\mathscr{Y})} \leq M\, \exp(\omega\, s)$ for all $s \in \mathbb{R}$;

(v) $\mathcal{A}^\mu$ *converges strongly to* $\mathcal{A}$ *on* $\mathscr{Y}_1$ *as* $\mu \to 0$*, i.e.* $\lim_{\mu\to 0} \|\mathcal{A}^\mu\, y - \mathcal{A}\, y\|_{\mathscr{Y}} = 0$ *for all* $y \in \mathscr{Y}_1$*;*

(vi) $\mathcal{U}_t^\mu$ *converges strongly to* $\mathcal{U}_t$*, uniformly in* t *on compact intervals, i.e.* $\lim_{\mu\to 0} \|\mathcal{U}_t^\mu\, y - \mathcal{U}_t\, y\|_{\mathscr{Y}} = 0$ *for all* $y \in \mathscr{Y}$*, uniformly in* $t \in \Omega$ *compact.*

Proof The proof of assertions *(i)–(v)* exploit basic properties of generators and semigroups, see for example [19, Theorem 1.2, p.2, Theorem 2.2, p.4], while the proof of assertion *(vi)* applies the Trotter–Kato theorem, see for example [12, Theorem 4.8, p.209]. The details follow the proofs of [9, Lemma 16, p.2185] and [9, Theorem 17, p.2188], and are omitted. □

Theorem 1 and [19, Theorem 1.3, p.102] imply that there exist unique solutions of the respective abstract Cauchy problems defined via (4) by

$$\begin{aligned} &\begin{pmatrix} \dot{x}_s \\ \dot{p}_s \end{pmatrix} = \mathcal{A} \begin{pmatrix} x_s \\ p_s \end{pmatrix}, && \begin{pmatrix} \dot{\xi}_s \\ \dot{\pi}_s \end{pmatrix} = \mathcal{A}^\mu \begin{pmatrix} \xi_s \\ \pi_s \end{pmatrix}, && s \in \mathbb{R}, \\ &(x_0, p_0) = (x, p) \in \mathscr{Y}_1, && (\xi_0, \pi_0) = (\xi, \pi) \in \mathscr{Y}, && \end{aligned} \tag{9}$$

and that these solutions are continuously differentiable when the initial data is in the domain of the respective generators. Consequently, $\ddot{x}_s$ and $\ddot{\xi}_s$ exist by inspection of (4), (9), and respectively satisfy (1) and

$$\ddot{\xi}_s = -\Lambda\, \mathcal{I}_\mu\, \xi_s, \qquad s \in \mathbb{R}. \tag{10}$$

That is, (1) holds, as does its approximation (10), obtained from (1) by replacing $-\Lambda$ with its Yosida approximation $-\Lambda\, \mathcal{I}_\mu$, see [19, p.9]. Given the groups $\{\mathcal{U}_s\}_{s\in\mathbb{R}}$ and $\{\mathcal{U}_s^\mu\}_{s\in\mathbb{R}}$ generated by $\mathcal{A}$ and $\mathcal{A}^\mu$ as per Theorem 1, these solutions necessarily satisfy

$$\begin{pmatrix} x_s \\ p_s \end{pmatrix} = \mathcal{U}_s \begin{pmatrix} x \\ p \end{pmatrix}, \qquad \begin{pmatrix} \xi_s \\ \pi_s \end{pmatrix} = \mathcal{U}_s^\mu \begin{pmatrix} \xi \\ \pi \end{pmatrix}, \qquad s \in \mathbb{R}. \tag{11}$$

The purpose of the analysis that follows is to *verify* these groups via a construction founded on connections between Hamilton's action principle and optimal control.

Remark 1 In an optimal control setting, the term *verification* refers to the identification of the value function and a feedback representation for the optimal control via solution of the associated Hamilton–Jacobi–Bellman (HJB) partial differential equation (PDE), subject to boundary data determined by the terminal cost. Here, *verification* is intended to inherit this meaning, i.e. the fundamental solution groups of interest are constructed from controls that are verified as optimal in an associated optimal control problem, via a suitable verification theorem (see Theorem 2 and Lemma 7 later).

3 Hamilton's Action Principle as an Optimal Control Problem

The action associated with Hamilton's action principle is defined as the integrated Lagrangian, see for example [17]. Here, a sign change is assumed without loss of generality, as per [9], with the action given by

$$\int_0^t V(x_s) - T(\dot{x}_s)\, ds\,, \tag{12}$$

in which $V(x_s)$ and $T(\dot{x}_s)$ denote the potential and kinetic energies corresponding to a (generalized) position x_s and velocity $\dot{x}_s$, evaluated at time $s \in \mathbb{R}_{\geq 0}$. Explicitly, $V : \mathscr{X}_1 \to \mathbb{R}_{\geq 0}$ and $T : \mathscr{X} \to \mathbb{R}_{\geq 0}$ are defined by

$$V(x) \doteq \tfrac{1}{2}\,\|x\|_1^2\,, \quad T(w) \doteq \tfrac{1}{2}\,\|\Lambda^{-\frac{1}{2}}\, w\|_1^2 = \tfrac{1}{2}\,\|w\|^2\,, \tag{13}$$

for all $x \in \mathscr{X}_1$, $w \in \mathscr{X}$, in which $\|w\|^2 \doteq \langle w,\ w\rangle$, and $\Lambda^{-\frac{1}{2}} \in \mathcal{L}(\mathscr{X};\, \mathscr{X}_1)$, as Λ^{-1} is bounded. Stationarity of (12) may be encapsulated via an optimal control problem as per [6,7,9].

3.1 Optimal Control Problem

The action (12) motivates definition of payoff $J_t : \mathscr{X} \times \mathscr{W}[0,t] \to \mathbb{R}$ for an optimal control problem defined on a horizon $t \in \mathbb{R}_{\geq 0}$, with

$$J_t(x,w) \doteq J_t[\psi](x,w) \doteq \int_0^t V(x_s) - T(w_s)\, ds + \psi(x_t)\,, \tag{14}$$

$$x_s \doteq \chi(x,w)_s \doteq x + \int_0^s w_\sigma\, d\sigma\,, \quad x \in \mathscr{X}_1,\ s \in [0,t]\,, \tag{15}$$

in which $w \in \mathscr{W}[0,t] \doteq \mathscr{L}^2([0,t];\, \mathscr{X})$ is (via an abuse of notation) the velocity as a function of time, $\psi : \mathscr{X} \to \overline{\mathbb{R}}$ is a terminal payoff to be selected later, and $\chi : \mathscr{X} \times \mathscr{W}[0,t] \mapsto C([0,t];\, \mathscr{X})$ is the state trajectory map. In view of (12), (14), the zero terminal payoff $\psi_0 : \mathscr{X}_1 \to \mathbb{R}$ is explicitly defined by

$$\psi_0(x) \doteq 0, \qquad x \in \mathscr{X}_1\,. \tag{16}$$

Unlike finite dimensional problems [17], it may be shown that $J_t[\psi_0](x,\cdot)$ need not be concave for any time horizon $t \in \mathbb{R}_{>0}$, see Lemma 1 below. Consequently, an approximation of (14) that is concave for sufficiently short, strictly positive, time horizons is first considered. This approximation, denoted by $J_t^\mu : \mathscr{X}_1 \times \mathscr{W}_1[0,t] \to \mathbb{R}$, $t \in \mathbb{R}_{>0}$, $\mu \in \mathbb{R}_{>0}$, is defined subject to (15) by

$$J_t^\mu(x,w) = J_t^\mu[\psi](x,w) \doteq \int_0^t V(x_s) - T^\mu(w_s)\, ds + \psi(x_t) \tag{17}$$

for all $x \in \mathscr{X}_1$, $w \in \mathscr{W}_1[0,t] \doteq \mathscr{L}^2([0,t];\, \mathscr{X}_1)$, with $T^\mu : \mathscr{X}_1 \to \mathbb{R}_{\geq 0}$ approximating T in (13), (14) by

$$T^\mu(w) \doteq \tfrac{1}{2}\,\|w\|^2 + \tfrac{\mu^2}{2}\,\|w\|_1^2 = \tfrac{1}{2}\,\|(\Lambda\,\mathcal{I}_\mu)^{-\frac{1}{2}}\, w\|_1^2, \tag{18}$$

for all $w \in \mathscr{X}_1$, in which $\mathcal{I}_\mu \in \mathcal{L}(\mathscr{X}_1)$ is as per (3). Note in particular the increase in regularity of the domain, from $\mathscr{X}$ to $\mathscr{X}_1$. Note further by the asserted properties of Λ, $(\Lambda\,\mathcal{I}_\mu)^{-1} = \Lambda^{-1} + \mu^2\,\mathcal{I}$ is bounded, positive and self-adjoint on $\mathscr{X}_1$, and so has a unique bounded positive self-adjoint square-root as per (18). Similarly, unique $\mathcal{I}_\mu^{\frac{1}{2}} \in \mathcal{L}(\mathscr{X}_1)$ exists and commutes with $\Lambda^{\frac{1}{2}}$ on $\mathscr{X}_1$. Note also that $T^0 \equiv T$.

Lemma 1 *Given $\mu \in \mathbb{R}_{>0}$, horizon $\bar{t}^\mu \doteq \mu\sqrt{2}$, and a concave terminal payoff $\psi : \mathscr{X}_1 \to \overline{\mathbb{R}}$, the approximate payoff $J_t^\mu[\psi](x,\cdot) : \mathscr{W}_1[0,t] \to \mathbb{R}_{\ge 0}$ of (17) is concave for all $t \in [0, \bar{t}^\mu)$, and strongly concave for all $t \in (0, \bar{t}^\mu)$. Alternatively, given $\mu \doteq 0$, zero terminal payoff ψ_0 of (16), and any $t \in \mathbb{R}_{>0}$, there exists $\overline{w} \in \mathscr{W}_1[0,t] \subset \mathscr{W}[0,t]$ such that the restriction $\eta \mapsto J_t^0[\psi_0](x, \eta\,\overline{w}) \in C(\mathbb{R};\mathbb{R})$ defined via the exact payoff (14) is strongly convex.*

Proof The concavity and convexity assertions are demonstrated consecutively.

Concavity assertion: Fix arbitrary $\mu \in \mathbb{R}_{>0}$, $t \in [0, \bar{t}^\mu)$, $x \in \mathscr{X}_1$, $w, h, \tilde{h} \in \mathscr{W}_1[0,t]$, and any concave terminal payoff $\psi : \mathscr{X}_1 \to \overline{\mathbb{R}}$. Given the zero terminal payoff ψ_0 of (16), observe by (17) that

$$J_t^\mu(x,w) = J_t^\mu[\psi](x,w) = J_t^\mu[\psi_0](x,w) + \psi(\chi(x,w)_t), \tag{19}$$

in which χ is as per (15). As ψ is concave and $w \mapsto \chi(x,w)_t$ is affine, see (15), it follows immediately that $w \mapsto \psi(\chi(x,w)_t)$ is concave. Meanwhile, following analogous arguments to [8, Theorem 3.6], the map $w \mapsto J_t^\mu[\psi_0](x,w)$ is twice continuously Fréchet differentiable on $\mathscr{W}_1[0,t]$. In particular, the first Fréchet derivative is $D_w J_t^\mu[\psi_0](x,w)\,h = \langle \nabla_w J_t^\mu[\psi_0](x,w),\, h\rangle_{\mathscr{W}_1[0,t]}$, in which $\nabla_w J_t^\mu[\psi_0](x,w) \in \mathscr{W}_1[0,t]$ is its Riesz representation, and the second Fréchet derivative is $D_w^2 J_t^\mu[\psi_0](x,w)\,h\,\tilde{h} = \langle D_w \nabla_w J_t^\mu[\psi_0](x,w)\,\tilde{h},\, h\rangle_{\mathscr{W}_1[0,t]}$, in which $D_w \nabla_w J_t^\mu[\psi_0](x,w) \in \mathcal{L}(\mathscr{W}_1[0,t])$. Some straightforward calculations yield

$$\begin{aligned}
J_t^\mu[\psi_0](x,w) &= \int_0^t V(x_s) - T^\mu(w_s)\, ds\,, \\
[\nabla_w J_t^\mu[\psi_0](x,w)]_s &\doteq \int_s^t \chi(x,w)_\sigma\, d\sigma - (\Lambda\,\mathcal{I}_\mu)^{-1}\, w_s\,, \\
[D_w \nabla_w J_t^\mu[\psi_0](x,w)\,h]_s &\doteq \int_0^t (t - (\sigma \vee s))\, h_\sigma\, d\sigma - (\Lambda\,\mathcal{I}_\mu)^{-1}\, h_s\,,
\end{aligned} \tag{20}$$

in which $\sigma \vee s \doteq \max(\sigma, s)$. In view of (20), it is useful to define and subsequently bound a functional $I_V : \mathscr{W}_1[0,t] \to \mathbb{R}$, using Cauchy-Schwartz, Young's, and Hölder's inequalities, by

$$\begin{aligned}
I_V(h) &\doteq \int_0^t \left\langle h_s,\, \int_0^t (t - (\sigma \vee s))\, h_\sigma\, d\sigma \right\rangle_1 ds \\
&\le \int_0^t \|h_s\|_1 \left[(t-s) \int_0^s \|h_\sigma\|_1\, d\sigma + \int_s^t (t-\sigma)\, \|h_\sigma\|_1\, d\sigma \right] ds
\end{aligned}$$

$$\leq \int_0^t \|h_s\|_1 \left[(t-s)\sqrt{s} + \tfrac{1}{\sqrt{3}}\,(t-s)^{\frac{3}{2}}\right] ds\ \|h\|_{\mathscr{W}_1[0,t]}$$
$$\leq \left(\int_0^t \left[(t-s)\sqrt{s} + \tfrac{1}{\sqrt{3}}\,(t-s)^{\frac{3}{2}}\right]^2 ds\right)^{\frac{1}{2}} \|h\|^2_{\mathscr{W}_1[0,t]} \leq \tfrac{1}{2}\,t^2\,\|h\|^2_{\mathscr{W}_1[0,t]}. \tag{21}$$

Similarly, define and bound a functional $I_T^\mu : \mathscr{W}_1[0,t] \to \mathbb{R}$ via (3), (5), (6) by

$$I_T^\mu(h) \doteq \int_0^t \left\langle h_s,\ -(\Lambda\,\mathcal{I}_\mu)^{-1}\,h_s\right\rangle_1 ds = -\int_0^t \left\langle h_s,\ \sum_{n=1}^\infty \tfrac{1}{\lambda_n^\mu}\,\langle h_s,\ \tilde{\varphi}_n\rangle_1\,\tilde{\varphi}_n\right\rangle_1 ds$$
$$= -\int_0^t \sum_{n=1}^\infty \tfrac{1}{\lambda_n^\mu}\,|\langle h_s,\ \tilde{\varphi}_n\rangle_1|^2\,ds \leq -\inf_{m\in\mathbb{N}} \tfrac{1}{\lambda_m^\mu} \int_0^t \sum_{n=1}^\infty |\langle h_s,\ \tilde{\varphi}_n\rangle_1|^2\,ds$$
$$= -\mu^2 \int_0^t \|h_s\|_1^2\,ds = -\mu^2\,\|h\|^2_{\mathscr{W}_1[0,t]}. \tag{22}$$

Recalling (20), (21), (22) subsequently yields

$$\left\langle h,\ D_w\nabla_w J_t^\mu[\psi_0](x,w)\,h\right\rangle_{\mathscr{W}_1[0,t]} = I_V(h) + I_T^\mu(h)$$
$$\leq -\tfrac{1}{2}\left[(\bar{t}^\mu)^2 - t^2\right]\|h\|^2_{\mathscr{W}_1[0,t]}\,. \tag{23}$$

By inspection, as $t \in [0, \bar{t}^\mu)$ and $w, h \in \mathscr{W}_1[0,t]$ are arbitrary, the second Fréchet derivative $D_w\nabla_w J_t^\mu[\psi_0](x,w)$ is a non-positive operator, so that the map $w \mapsto J_t^\mu[\psi_0](x,w)$ is concave. Recalling that $w \mapsto \psi(\chi(x,w)_t)$ is also concave, it follows immediately that $w \mapsto J_t^\mu(x,w)$ is also concave. Moreover, if $t \in (0, \bar{t}^\mu)$, the second Fréchet derivative above is a negative operator, so that the stated strongly concave assertion follows.

Convexity assertion: Fix any $t \in \mathbb{R}_{>0}$, and let $\omega \doteq \frac{\pi}{2t}$. Define $\alpha : [0,t] \to \mathbb{R}$ by $\alpha_s \doteq \cos(\omega\,s)$, $s \in [0,t]$. Fix an arbitrary $m \in \mathbb{N}$ and define $w_m^\alpha \in \mathscr{W}_1[0,t]$ by $[w_m^\alpha]_s \doteq \alpha_s\,\tilde{\varphi}_m$ for all $s \in [0,t]$, with $\tilde{\varphi}_m \in \mathscr{X}_1$ as per (5). Note that

$$\|w_m^\alpha\|^2_{\mathscr{W}_1[0,t]} = \int_0^t |\alpha_s|^2\,\|\tilde{\varphi}_m\|_1^2\,ds = \|\alpha\|^2_{\mathscr{L}^2([0,t];\mathbb{R})} = \tfrac{1}{2}\,t\,.$$

For convenience, let $\widehat{w}_m^\alpha \in \mathscr{W}_1[0,t]$ be defined by

$$[\widehat{w}_m^\alpha]_s \doteq \int_0^t (t-\sigma\vee s)\,[w_m^\alpha]_\sigma\,d\sigma = (t-s)\int_0^s [w_m^\alpha]_\sigma\,d\sigma + \int_s^t (t-\sigma)\,[w_m^\alpha]_\sigma\,d\sigma$$
$$= \left[(t-s)\int_0^s \alpha_\sigma\,d\sigma + \int_s^t (t-\sigma)\,\alpha_\sigma\,d\sigma\right]\tilde{\varphi}_m = \tfrac{1}{\omega^2}\,\alpha_s\,\tilde{\varphi}_m = \tfrac{1}{\omega^2}\,[w_m^\alpha]_s\,,$$

for all $s \in [0, t]$, in which the second last equality follows by integration by parts and the definition of α. Hence, recalling (21),

$$\begin{aligned} I_V(w_m^\alpha) &= \int_0^t \left\langle [w_m^\alpha]_s,\, \int_0^t (t - \sigma \vee s)\, [w_m^\alpha]_\sigma \, d\sigma \right\rangle_1 ds \\ &= \int_0^t \langle [w_m^\alpha]_s,\, [\widehat{w}_m^\alpha]_s \rangle_1 \, ds = \tfrac{1}{\omega^2} \, \| w_m^\alpha \|_{\mathscr{W}_1[0,t]}^2 = \tfrac{2}{\pi^2} \, t^3 \, . \end{aligned}$$

Fix $\mu \in \mathbb{R}_{\geq 0}$. Recalling (22),

$$\begin{aligned} I_T^\mu(w_m^\alpha) &= \int_0^t \left\langle [w_m^\alpha]_s,\, -(\Lambda\, \mathcal{I}_\mu)^{-1} \, [w_m^\alpha]_s \right\rangle_1 ds = -\int_0^t \sum_{n=1}^\infty \tfrac{1}{\lambda_n^\mu} \, |\langle \alpha_s \, \tilde{\varphi}_m,\, \tilde{\varphi}_n \rangle_1|^2 \, ds \\ &= -\tfrac{1}{\lambda_m^\mu} \, \|\alpha\|_{\mathscr{L}^2([0,t];\mathbb{R})}^2 = -\tfrac{1}{2\, \lambda_m^\mu} \, t \, . \end{aligned}$$

Consequently, recalling (23),

$$\begin{aligned} \left\langle w_m^\alpha,\, D_w \nabla_w J_t^\mu[\psi_0](x, w) \, w_m^\alpha \right\rangle_{\mathscr{W}_1[0,t]} &= I_V(w_m^\alpha) + I_T^\mu(w_m^\alpha) = \tfrac{2}{\pi^2} \, (t^2 - \tfrac{\pi^2}{4\, \lambda_m^\mu}) \, t \\ &= \tfrac{2}{\pi^2} \, (t^2 - \tfrac{\pi^2}{4} (\mu^2 + \tfrac{1}{\lambda_m})) \, t \, . \end{aligned}$$

With $\mu \in \mathbb{R}_{>0}$ and $t < \bar{t}^\mu < (\frac{\pi}{2}) \, \mu$, the right-hand side is strictly negative, and concavity is guaranteed. However, with $\mu = 0$ and $t \in \mathbb{R}_{>0}$, observe that there exists an $\widehat{m}_t \in \mathbb{N}$ such that for any $m \geq \widehat{m}_t$,

$$\left\langle w_m^\alpha,\, D_w \nabla_w J_t^0[\psi_0](x, w) \, w_m^\alpha \right\rangle_{\mathscr{W}_1[0,t]} \geq \tfrac{1}{\pi^2} \, t^3 > 0 \, ,$$

with $\|w_m^\alpha\|_{\mathscr{W}_1[0,t]} = \frac{1}{2} \, t$. That is, there exists $\overline{w} \doteq w_m^\alpha \in \mathscr{W}_1[0, t]$ such that the map $\eta \mapsto J_t^0[\psi_0](x, \eta \, \overline{w})$ is strongly convex. □

The concavity property provided by the first assertion of Lemma 1 implies that the value function $W_t^\mu : \mathscr{X}_1 \to \overline{\mathbb{R}}$ corresponding to (17) is well-defined for $\mu \in \mathbb{R}_{>0}$ and short horizons $t \in [0, \bar{t}^\mu)$ by

$$W_t^\mu(\xi) \doteq \sup_{w \in \mathscr{W}_1[0,t]} J_t^\mu[\psi](\xi, w) \, , \tag{24}$$

for all $\xi \in \mathscr{X}_1$. As a linear choice for the terminal payoff ψ (i.e. lacking strong concavity) will ultimately be of some interest, see (59), it is important to emphasize that the Yosida approximation and positivity of the associated approximation parameter μ are essential, by the second assertion of Lemma 1. The optimal control problem defined by (24) naturally admits a verification theorem, posed with respect to an attendant Hamilton–Jacobi–Bellman (HJB) partial differential equation (PDE); see also [9, Theorem 6] for a special case.

Theorem 2 *Given any* $\mu \in \mathbb{R}_{>0}$ *and* $t \in (0, \bar{t}^\mu)$*, suppose there exists a functional* $(s, x) \mapsto W_s(x) \in C([0, t] \times \mathscr{X}_1; \mathbb{R}) \cap C^1((0, t) \times \mathscr{X}_1; \mathbb{R})$ *such that*

$$0 = -\frac{\partial W_s}{\partial s}(x) + H(x, \nabla_x W_s(x)), \qquad W_0(x) = \psi(x), \tag{25}$$

for all $s \in (0, t)$*,* $x \in \mathscr{X}_1$*, where* $\nabla_x W_s(x) \in \mathscr{X}_1$ *denotes the Riesz representation of the Fréchet derivative of* $x \mapsto W_s(x)$*, defined with respect to the inner product* $\langle \cdot, \cdot \rangle_1$ *on* $\mathscr{X}_1$*, and* $H : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ *is the Hamiltonian*

$$H(x, p) \doteq \tfrac{1}{2} \|x\|_1^2 + \tfrac{1}{2} \|\mathcal{I}_\mu^{\frac{1}{2}} \Lambda^{\frac{1}{2}} p\|_1^2 \tag{26}$$

for all $x, p \in \mathscr{X}_1$*. Then,* $W_t(x) \geq J_t^\mu(x, w)$ *for all* $w \in \mathscr{W}_1[0, t]$*. Furthermore, if there exists a solution* $s \mapsto \xi_s^*$ *of* (15) *with*

$$\xi_s^* = \xi + \int_0^s w_\sigma^* \, d\sigma, \quad w_\sigma^* = k_\sigma^{\mu,t}(\xi_\sigma^*), \tag{27}$$

$$k_\sigma^{\mu,t}(y) \doteq \mathcal{I}_\mu^{\frac{1}{2}} \mathcal{E}_\mu \nabla_x W_{t-\sigma}^\mu(y), \quad \sigma \in [0, s], \; s \in [0, t], \; y \in \mathscr{X}_1,$$

$$\mathcal{E}_\mu \doteq \Lambda^{\frac{1}{2}} \mathcal{I}_\mu^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \in \mathcal{L}(\mathscr{X}_1; \mathscr{X}), \; \mathcal{I}_\mu^{\frac{1}{2}} \in \mathcal{L}(\mathscr{X}; \mathscr{X}_1), \tag{28}$$

such that $\xi_s^* \in \mathscr{X}_1$ *for all* $s \in [0, t]$*, then* $W_s(x) = J_s^\mu(x, w^*) = W_s^\mu(x)$ *for all* $s \in [0, t]$*,* $x \in \mathscr{X}_1$*.*

Proof Fix $\mu \in \mathbb{R}_{>0}$, $t \in (0, \bar{t}^\mu)$, and let $(s, x) \mapsto W_s(x) \in C([0, t] \times \mathscr{X}_1; \mathbb{R}) \cap C^1((0, t) \times \mathscr{X}_1; \mathbb{R})$ satisfy (25). Fix $x \in \mathscr{X}_1$, $\overline{w} \in \mathscr{W}_1[0, t]$, and let $s \mapsto \bar{\xi}_s \in C([0, t]; \mathscr{X}_1)$ denote the solution (15) with $\bar{\xi}_0 = x$ and $w = \overline{w}$. With $s \in [0, t]$, let $\bar{p}_s \doteq \nabla_x W_{t-s}(\bar{\xi}_s) \in \mathscr{X}_1$. By (18) and completion of squares,

$$\begin{aligned}
\langle \bar{p}_s, w \rangle_1 - T^\mu(w) &= \langle \bar{p}_s, w \rangle_1 - \tfrac{1}{2} \|(\Lambda \mathcal{I}_\mu)^{-\frac{1}{2}} w\|_1^2 \\
&= \tfrac{1}{2} \|(\Lambda \mathcal{I}_\mu)^{\frac{1}{2}} \bar{p}_s\|_1^2 - \tfrac{1}{2} \|(\Lambda \mathcal{I}_\mu)^{-\frac{1}{2}} [w - \mathcal{I}_\mu^{\frac{1}{2}} \mathcal{E}_\mu \bar{p}_s]\|_1^2 \leq \tfrac{1}{2} \|\mathcal{I}_\mu^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \bar{p}_s\|_1^2.
\end{aligned} \tag{29}$$

Consequently, applying the chain rule and (25), (26),

$$\begin{aligned}
\tfrac{d}{ds}[W_{t-s}(\bar{\xi}_s)] &= -\tfrac{\partial}{\partial s} W_{t-s}(\bar{\xi}_s) + \langle \nabla_x W_{t-s}(\bar{\xi}_s), \overline{w}_s \rangle_1 \\
&= \big[-\tfrac{\partial}{\partial s} W_{t-s}(\bar{\xi}_s) + H(\bar{\xi}_s, \nabla_x W_{t-s}(\bar{\xi}_s)\big] + \langle \bar{p}_s, \overline{w}_s \rangle_1 - \tfrac{1}{2} \|\bar{\xi}_s\|_1^2 \\
&\quad - \tfrac{1}{2} \|\mathcal{I}_\mu^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \bar{p}_s\|_1^2 \\
&= \langle \bar{p}_s, \overline{w}_s \rangle_1 - \tfrac{1}{2} \|\bar{\xi}_s\|_1^2 - \tfrac{1}{2} \|\mathcal{I}_\mu^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \bar{p}_s\|_1^2 \leq T^\mu(\overline{w}_s) - V(\bar{\xi}_s),
\end{aligned}$$

in which the final inequality follows by application of (29) with $w = \overline{w}_s$. Integrating with respect to $s \in [0, t]$ and recalling the initial condition in (25),

$$\psi(\bar{\xi}_t) - W_t(x) = \int_0^t \tfrac{d}{ds}[W_{t-s}(\bar{\xi}_s)] \, ds \leq \int_0^t T^\mu(\overline{w}_s) - V(\bar{\xi}_s) \, ds$$

$$\Longrightarrow \quad W_t(x) \geq \int_0^t V(\bar{\xi}_s) - T^\mu(\overline{w}_s)\, ds + \psi(\bar{\xi}_t) = J_t^\mu(x, \overline{w})\,. \tag{30}$$

As $x \in \mathscr{X}_1$ and $\overline{w} \in \mathscr{W}[0, t]$ are arbitrary, the first assertion follows. Moreover, if w^* exists as per (27), selecting $\overline{w}_s \doteq w_s^*$ yields equality in (29), (30), yielding the second assertion. □

Verification Theorem 2 is particularly useful in establishing an idempotent convolution representation for the value function (24), and in providing a feedback characterization of the optimal control considered in Sect. 4.

3.2 Idempotent Convolution Representation for (24)

As illustrated in [4,9,10,17], the value function of an optimal control problem can be expressed as an idempotent convolution of an element of the attendant idempotent fundamental solution semigroup with the terminal payoff of interest. In the specific case of optimal control problem (24) for $\mu \in \mathbb{R}_{>0}$, this yields the value function representation

$$W_t^\mu(\xi) = \sup_{\zeta \in \mathscr{X}_1} \left\{ G_t^\mu(\xi, \zeta) + \psi(\zeta) \right\}, \tag{31}$$

for all $t \in (0, \bar{t}^\mu)$, $\xi \in \mathscr{X}_1$, in which $G_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \overline{\mathbb{R}}$ is the bivariate idempotent convolution kernel associated with the *max-plus primal space fundamental solution semigroup* corresponding to the optimal control problem (24), see for example [9, Theorem 2] or [4, Theorem 6]. Given any $t \in (0, \bar{t}^\mu)$, this kernel is defined via an optimal TPBVP by

$$G_t^\mu(\xi, \zeta) \doteq \sup_{w \in \mathscr{W}_1[0,t]} \left\{ \int_0^t V(x_s) - T^\mu(w_s)\, ds \,\middle|\, x_0 = \xi,\ x_t = \zeta \right\}, \tag{32}$$

for all $\xi, \zeta \in \mathscr{X}_1$. As anticipated by the special case described by [9, Theorem 11], this kernel also has a quadratic representation.

Theorem 3 *Given any $\mu \in (0, 1]$ and $t \in (0, \bar{t}^\mu)$, the idempotent convolution kernel $G_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \overline{\mathbb{R}}$ of (31), (32) has the quadratic representation*

$$G_t^\mu(\xi, \zeta) = \tfrac{1}{2} \left\langle \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \begin{pmatrix} \mathcal{P}_t^\mu & \mathcal{Q}_t^\mu \\ \mathcal{Q}_t^\mu & \mathcal{P}_t^\mu \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_\sharp, \tag{33}$$

for all $\xi, \zeta \in \mathscr{X}_1$, in which $\langle (x, z), (\xi, \zeta) \rangle_\sharp \doteq \langle x, \xi \rangle_1 + \langle z, \zeta \rangle_1$ for all $(x, z), (\xi, \zeta) \in \mathscr{X}_1 \times \mathscr{X}_1$, and $\mathcal{P}_t^\mu, \mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ are self-adjoint operators. Moreover, these operators also have the spectral form (5), with

$$\mathcal{P}_t^\mu\, \xi \doteq \sum_{n=1}^\infty [p_t^\mu]_n\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \qquad \mathcal{Q}_t^\mu\, \xi \doteq \sum_{n=1}^\infty [q_t^\mu]_n\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \tag{34}$$

for all $\xi \in \mathscr{X}_1$*, in which the respective eigenvalues* $[p_t^\mu]_n$*,* $[q_t^\mu]_n$ *are defined by*

$$[p_t^\mu]_n \doteq \frac{-1}{\omega_n^\mu \, \tan(\omega_n^\mu \, t)}, \quad [q_t^\mu]_n \doteq \frac{1}{\omega_n^\mu \, \sin(\omega_n^\mu \, t)}, \tag{35}$$

for all $n \in \mathbb{N}$.

Proof of a special case [9, Theorem 11] of Theorem 3 employs a homotopy argument to verify the corresponding quadratic representation analogous to (33). Here, motivated by [11, Theorem 2] and [4, Theorem 6], an alternative approach to the proof of Theorem 3 is developed by exploiting semiconvex duality. This development commences with the definition of a parameterized terminal cost $\varphi : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ by

$$\varphi(x, z) \doteq \tfrac{1}{2} \langle x - z, \, \mathcal{M}\,(x - z)\rangle_1 \tag{36}$$

for all $x, z \in \mathscr{X}_1$, in which $\mathcal{M} \in \mathcal{L}(\mathscr{X}_1)$ is a negative self-adjoint operator of spectral form (5), with

$$\mathcal{M}\,\xi \doteq \sum_{n=1}^{\infty} m_n \, \langle \xi, \, \tilde{\varphi}_n\rangle_1 \, \tilde{\varphi}_n, \;\; m_n \in [-\overline{m}, \, -\underline{m}], \;\; 0 < \tfrac{1}{\omega_1^1} \tan\sqrt{2} < \underline{m} \le \overline{m} < \infty, \tag{37}$$

for all $\xi \in \mathscr{X}_1$. Observe by (6) that, for all $\mu \in (0, 1]$, $n \in \mathbb{N}$,

$$\begin{aligned} &\omega_1^1 \le \omega_1^\mu \le \omega_n^\mu \le \omega_\infty^\mu \doteq \frac{1}{\mu}, && 0 < \frac{\mu}{\overline{m}} < \frac{-1}{\omega_n^\mu \, m_n} < \frac{1}{\omega_1^1 \, \underline{m}} < \frac{1}{\tan\sqrt{2}}, \\ &\theta_n^\mu \doteq \tan^{-1}\left(\frac{-1}{\omega_n^\mu \, m_n}\right), && 0 < \theta_n^\mu < \tan^{-1}\left(\frac{1}{\tan\sqrt{2}}\right) = \frac{\pi}{2} - \sqrt{2}. \end{aligned} \tag{38}$$

Note by definition of $\bar{t}^\mu$ and (38) that

$$\mu \in (0, 1], \; t \in (0, \bar{t}^\mu) \quad \Longrightarrow \quad \omega_n^\mu \, t + \theta_n^\mu \in \left(0, \, \frac{\pi}{2}\right) \quad \forall \, n \in \mathbb{N}. \tag{39}$$

Given $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$, and with φ as per (36), it is useful to define an auxiliary optimal control problem with value function $S_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \overline{\mathbb{R}}$ by

$$S_t^\mu(\xi, \zeta) \doteq \sup_{w \in \mathscr{W}_1[0,t]} J_t^\mu[\varphi(\cdot, \zeta)](\xi, w) \tag{40}$$

for all $\xi, \zeta \in \mathscr{X}_1$, in which J_t^μ is as per (17).

Lemma 2 *Given* $\mu \in (0, 1]$*,* $t \in (0, \bar{t}^\mu)$*, the value function* $S_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ *of* (40) *has the explicit quadratic representation*

$$S_t^\mu(\xi, \zeta) = \tfrac{1}{2} \langle \xi, \, \mathcal{X}_t^\mu \, \xi\rangle_1 + \langle \xi, \, \mathcal{Y}_t^\mu \, \zeta\rangle_1 + \tfrac{1}{2} \langle \zeta, \, \mathcal{Z}_t^\mu \, \zeta\rangle_1 \tag{41}$$

for all $\xi, \zeta \in \mathscr{X}_1$, *in which* $\mathcal{X}_t^\mu, \mathcal{Y}_t^\mu, \mathcal{Z}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ *are bounded linear operators of the spectral form* (5), *with respective eigenvalues given by*

$$
\begin{aligned}
[x_t^\mu]_n &\doteq -\frac{1}{\omega_n^\mu} \cot\left(\omega_n^\mu\, t + \theta_n^\mu\right), \quad [y_t^\mu]_n \doteq +\frac{1}{\omega_n^\mu} \cos\theta_n^\mu \csc(\omega_n^\mu\, t + \theta_n^\mu), \\
[z_t^\mu]_n &\doteq -\frac{1}{\omega_n^\mu} \cos^2\theta_n^\mu \left[\tan\theta_n^\mu + \cot(\omega_n^\mu\, t + \theta_n^\mu)\right],
\end{aligned} \tag{42}
$$

for all $n \in \mathbb{N}$, *and satisfying* $\dot{\mathcal{X}}_t^\mu, \dot{\mathcal{Y}}_t^\mu, \dot{\mathcal{Z}}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$.

Proof Fix $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$. Let $\mathcal{X}_t^\mu, \mathcal{Y}_t^\mu, \mathcal{Z}_t^\mu$ be linear operators of the spectral form (5) with eigenvalues (42), as per (41), and note that

$$
\begin{aligned}
[x_t^\mu]_n &= m_n + \frac{1}{\omega_n^\mu}\left[\cot\theta_n^\mu - \cot(\omega_n^\mu\, t + \theta_n^\mu)\right], \\
[y_t^\mu]_n &= -m_n - \frac{1}{\omega_n^\mu} \cos\theta_n^\mu \left[\csc\theta_n^\mu - \csc(\omega_n^\mu\, t + \theta_n^\mu)\right], \\
[z_t^\mu]_n &= m_n + \frac{1}{\omega_n^\mu} \cos^2\theta_n^\mu \left[\cot\theta_n^\mu - \cot(\omega_n^\mu\, t + \theta_n^\mu)\right],
\end{aligned} \tag{43}
$$

for all $n \in \mathbb{N}$. Bounds (38), (39) imply that the corresponding eigenvalue sequences are bounded. Moreover, elements of these eigenvalue sequences are differentiable with respect to t, and satisfy

$$
\begin{aligned}
&[\dot{x}_t^\mu]_n = 1 + \lambda_n^\mu\, [x_t^\mu]_n^2, && [\ddot{x}_t^\mu]_n = 2\,\lambda_n^\mu\, [x_t^\mu]_n\, [\dot{x}_t^\mu]_n, && [x_0^\mu]_n = m_n, \\
&[\dot{y}_t^\mu]_n = \lambda_n^\mu\, [x_t^\mu]_n\, [y_t^\mu]_n, && [\ddot{y}_t^\mu]_n = \lambda_n^\mu\, ([\dot{x}_t^\mu]_n\, [y_t^\mu]_n + [x_t^\mu]_n\, [\dot{y}_t^\mu]_n), && [y_0^\mu]_n = -m_n, \\
&[\dot{z}_t^\mu]_n = \lambda_n^\mu\, [y_t^\mu]_n^2, && [\ddot{z}_t^\mu]_n = 2\,\lambda_n^\mu\, [y_t^\mu]_n\, [\dot{y}_t^\mu]_n, && [z_0^\mu]_n = m_n,
\end{aligned} \tag{44}
$$

so that the sequences of corresponding derivatives are also bounded. Hence, the linear operators $\mathcal{X}_t^\mu, \mathcal{Y}_t^\mu, \mathcal{Z}_t^\mu$ are bounded and Fréchet differentiable with bounded derivatives $\dot{\mathcal{X}}_t^\mu, \dot{\mathcal{Y}}_t^\mu, \dot{\mathcal{Z}}_t^\mu$. That is, $\mathcal{X}_t^\mu, \mathcal{Y}_t^\mu, \mathcal{Z}_t^\mu, \dot{\mathcal{X}}_t^\mu, \dot{\mathcal{Y}}_t^\mu, \dot{\mathcal{Z}}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$, as asserted. By inspection of (5), (6), (44), note further that these operators satisfy the respective Cauchy problems

$$
\begin{aligned}
\dot{\mathcal{X}}_t^\mu &= \mathcal{I} + \mathcal{X}_t^\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{I}_\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{X}_t^\mu, \quad \dot{\mathcal{Y}}_t^\mu = \mathcal{X}_t^\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{I}_\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{Y}_t^\mu, \\
\dot{\mathcal{Z}}_t^\mu &= \mathcal{Y}_t^\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{I}_\mu\, \Lambda^{\frac{1}{2}}\, \mathcal{Y}_t^\mu, \quad \mathcal{X}_0^\mu = +\mathcal{M} = \mathcal{Z}_0^\mu,\ \mathcal{Y}_0^\mu = -\mathcal{M},
\end{aligned} \tag{45}
$$

in which $\mathcal{I} \in \mathcal{L}(\mathscr{X}_1)$ denotes the identity. Define $\widehat{S}_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ as per the quadratic form in the lemma statement, i.e.

$$
\widehat{S}_t^\mu(\xi, \zeta) \doteq \tfrac{1}{2}\, \langle \xi,\, \mathcal{X}_t^\mu\, \xi \rangle_1 + \langle \xi,\, \mathcal{Y}_t^\mu\, \zeta \rangle_1 + \tfrac{1}{2}\, \langle \zeta,\, \mathcal{Z}_t^\mu\, \zeta \rangle_1 \tag{46}
$$

for all $\xi, \zeta \in \mathscr{X}_1$. Observe that $t \mapsto S_t^\mu(\xi, \zeta)$ is Fréchet differentiable, as is $\xi \mapsto S_t^\mu(\xi, \zeta)$, with derivatives given via their Riesz representation by

$$\begin{aligned} \frac{\partial \widehat{S}_t^\mu}{\partial t}(\xi, \zeta) &= \tfrac{1}{2}\langle \xi, \dot{\mathcal{X}}_t^\mu \xi \rangle_1 + \langle \xi, \dot{\mathcal{Y}}_t^\mu \zeta \rangle_1 + \tfrac{1}{2}\langle \zeta, \dot{\mathcal{Z}}_t^\mu \zeta \rangle_1, \\ \nabla_\xi \widehat{S}_t^\mu(\xi, \zeta) &= \mathcal{X}_t^\mu \xi + \mathcal{Y}_t^\mu \zeta, \end{aligned} \tag{47}$$

for all $\xi, \zeta \in \mathscr{X}_1$. Consequently, applying (45), (47) in (25), (26) yields

$$\begin{aligned} &-\frac{\partial \widehat{S}_t^\mu}{\partial t}(\xi, \zeta) + H(\xi, \nabla_\xi \widehat{S}_t^\mu(\xi, \zeta)) = -\tfrac{1}{2}\langle \xi, \dot{\mathcal{X}}_t^\mu \xi \rangle_1 - \langle \xi, \dot{\mathcal{Y}}_t^\mu \zeta \rangle_1 - \tfrac{1}{2}\langle \zeta, \dot{\mathcal{Z}}_t^\mu \zeta \rangle_1 \\ &\quad + \tfrac{1}{2}\|\xi\|_1^2 + \tfrac{1}{2}\langle \mathcal{X}_t^\mu \xi + \mathcal{Y}_t^\mu \zeta, \Lambda^{\frac{1}{2}} \mathcal{I}_\mu \Lambda^{\frac{1}{2}} \left(\mathcal{X}_t^\mu \xi + \mathcal{Y}_t^\mu \zeta\right)\rangle_1 \\ &= \tfrac{1}{2}\langle \xi, (-\dot{\mathcal{X}}_t^\mu + \mathcal{I} + \mathcal{X}_t^\mu \Lambda^{\frac{1}{2}} \mathcal{I}_\mu \Lambda^{\frac{1}{2}} \mathcal{X}_t^\mu)\, \xi \rangle_1 \\ &\quad + \langle \xi, (-\dot{\mathcal{Y}}_t^\mu + \mathcal{X}_t^\mu \Lambda^{\frac{1}{2}} \mathcal{I}_\mu \Lambda^{\frac{1}{2}} \mathcal{Y}_t^\mu)\, \zeta \rangle_1 \\ &\quad + \tfrac{1}{2}\langle \zeta, (-\dot{\mathcal{Z}}_t^\mu + \mathcal{Y}_t^\mu \Lambda^{\frac{1}{2}} \mathcal{I}_\mu \Lambda^{\frac{1}{2}} \mathcal{Y}_t^\mu)\, \zeta \rangle_1 = 0. \end{aligned}$$

Meanwhile, the initial data in (45), along with (36), (46), yield

$$\widehat{S}_0^\mu(\xi, \zeta) = \tfrac{1}{2}\langle \xi, \mathcal{M}\xi \rangle_1 - \langle \xi, \mathcal{M}\zeta \rangle_1 + \tfrac{1}{2}\langle \zeta, \mathcal{M}\zeta \rangle_1 = \varphi(\xi, \zeta).$$

That is, $\widehat{S}_t^\mu(\cdot, \zeta) : \mathscr{X}_1 \to \mathbb{R}$, $\zeta \in \mathscr{X}_1$, satisfies the HJB PDE (25), (26). Moreover, (unique) solution $s \mapsto \xi_s^*$ of (27) may also be shown to exist, with $\xi_s \in \mathscr{X}_1$ for all $s \in [0, t]$, using a fixed point argument. The details parallel [9, Theorem 13], and are omitted. Hence, the conditions of verification Theorem 2 are satisfied, so that $\widehat{S}_t^\mu(\cdot, \zeta)$ of (46) is the value of an optimal control problem of the form (24) with terminal cost $\psi \doteq \varphi(\cdot, \zeta)$. That is, recalling (40), $\widehat{S}_t^\mu(\xi, \zeta) = \sup_{w \in \mathscr{W}_1[0,t]} J_t^\mu[\varphi(\cdot, \zeta)](\xi, w) = S_t^\mu(\xi, \zeta)$ for all $\xi, \zeta \in \mathscr{X}_1$ and $t \in (0, \bar{t}^\mu)$, as required. □

Coercivity of $\mathcal{Z}_t^\mu - \mathcal{M}$ is useful in preparing for the proof of Theorem 3.

Lemma 3 *Given $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$, the operator $\mathcal{Z}_t^\mu - \mathcal{M} \in \mathcal{L}(\mathscr{X}_1)$ of (41), (42) is coercive.*

Proof Fix $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$. The asserted boundedness, i.e. $\mathcal{Z}_t^\mu - \mathcal{M} \in \mathcal{L}(\mathscr{X}_1)$, is immediate by (37) and Lemma 2, see (42), (43). Moreover, this operator has the spectral form (5), with

$$\langle \zeta, (\mathcal{Z}_t^\mu - \mathcal{M})\, \zeta \rangle_1 = \sum_{n=1}^\infty ([z_t^\mu]_n - m_n)\, |\langle \zeta, \tilde{\varphi}_n \rangle_1|^2 \tag{48}$$

Recalling the last equality in (43), $[z_t^\mu]_n - m_n = [\cos^2 \theta_n^\mu]\, f_n^\mu(t)$, with $\cos^2 \theta_n^\mu \geq \cos^2\left(\frac{\pi}{2} - \sqrt{2}\right) = \sin^2 \sqrt{2} > 0$, $f_n^\mu(t) \doteq \frac{1}{\omega_n^\mu}[\cot \theta_n^\mu - \cot(\omega_n^\mu t + \theta_n^\mu)]$, $f_n^\mu(0) = 0$, and $(f_n^\mu)'(t) = \csc^2(\omega_n^\mu t + \theta_n^\mu) > 1$ for all $n \in \mathbb{N}$. Hence, $[z_t^\mu]_n - m_n \geq t \sin^2 \sqrt{2}$

for all $n \in \mathbb{N}$, so that by (48), $\langle \zeta,\, (\mathcal{Z}_t^\mu - \mathcal{M})\,\zeta \rangle_1 \geq \sum_{n=1}^\infty t \,\sin^2 \sqrt{2}\, |\langle \zeta,\, \tilde{\varphi}_n \rangle_1|^2 = t\, \sin^2 \sqrt{2}\, \|\zeta\|_1^2$, for all $\zeta \in \mathscr{X}_1$. That is, $\mathcal{Z}_t^\mu - \mathcal{M}$ is coercive, as required. □

In continuing the preparations for the proof of Theorem 3, some definitions relating to semiconvex duality are required. In particular, a function $\psi : \mathscr{X}_1 \to \overline{\mathbb{R}}$ is convex if its epigraph $\{(x, \alpha) \in \mathscr{X}_1 \times \mathbb{R} \,|\, \psi(x) \leq \alpha\}$ is convex. It is lower closed if $\psi = \mathrm{cl}^- \,\psi$, in which the lower closure cl^- is

$$\mathrm{cl}^- \,\psi(x) \doteq \begin{cases} \mathrm{lsc}\,\psi(x), & \mathrm{lsc}\,\psi(x) > -\infty \text{ for all } x \in \mathscr{X}_1, \\ -\infty, & \text{otherwise}, \end{cases}$$

for all $x \in \mathscr{X}_1$, and lsc is the lower semicontinuous envelope, see [20]. Following [4,5], uniformly semiconvex and semiconcave extended real valued function spaces $\mathscr{S}_+^{-\mathcal{M}}$ and $\mathscr{S}_-^{-\mathcal{M}}$ are defined with respect to $\mathcal{M}$ of (36), (37) by

$$\begin{aligned} \mathscr{S}_+^{-\mathcal{M}} &\doteq \left\{ \psi : \mathscr{X}_1 \to \overline{\mathbb{R}} \;\middle|\; \psi + \tfrac{1}{2}\langle \cdot\,, -\mathcal{M}\,\cdot \rangle_1 \text{ convex, lower closed} \right\}, \\ \mathscr{S}_-^{-\mathcal{M}} &\doteq \left\{ \psi : \mathscr{X}_1 \to \overline{\mathbb{R}} \;\middle|\; -\psi \in \mathscr{S}_+^{-\mathcal{M}} \right\}. \end{aligned} \tag{49}$$

Semiconvex duality is a duality between the spaces of (49), defined via the semiconvex transform. The semiconvex transform is a generalization of the Legendre-Fenchel transform, in which convexity is weakened to semiconvexity by relaxing affine support to quadratic support. The quadratic support functions involved are defined here via the bivariate quadratic basis function $\varphi : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ of (36). The semiconvex transform and its inverse, denoted by $\mathcal{D}_\varphi : \mathscr{S}_+^{-\mathcal{M}} \to \mathscr{S}_-^{-\mathcal{M}}$ and $\mathcal{D}_\varphi^{-1} : \mathscr{S}_-^{-\mathcal{M}} \to \mathscr{S}_+^{-\mathcal{M}}$, are given by [4,5]

$$\mathcal{D}_\varphi\, \Psi \doteq -\sup_{\xi \in \mathscr{X}_1} \{\varphi(\xi, \cdot) - \Psi(\xi)\}, \qquad \mathcal{D}_\varphi^{-1}\, a \doteq \sup_{z \in \mathscr{X}_1} \{\varphi(\cdot, z) + a(z)\}, \tag{50}$$

for all $\Psi \in \mathscr{S}_+^{-\mathcal{M}}$ and $a \in \mathscr{S}_-^{-\mathcal{M}}$. It is also useful to define

$$\delta^-(\xi, \zeta) \doteq \begin{cases} 0 & \|\xi - \zeta\|_1 = 0, \\ -\infty & \|\xi - \zeta\|_1 \neq 0, \end{cases} \tag{51}$$

for all $\xi, \zeta \in \mathscr{X}_1$.

These definitions and concepts may now be used to establish a representation for the convolution kernel G_t^μ of (31).

Lemma 4 *Given $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$, the auxiliary value function S_t^μ of (40), (41) and the convolution kernel G_t^μ of (31), (32) satisfy*

$$S_t^\mu(\xi, \cdot) \in \mathscr{S}_+^{-\mathcal{M}}, \qquad G_t^\mu(\xi, \zeta) = [\mathcal{D}_\varphi\, S_t^\mu(\xi, \cdot)](\zeta) \tag{52}$$

for all $\xi, \zeta \in \mathscr{X}_1$, in which $\mathcal{D}_\varphi$ is the semiconvex dual operation (50).

Proof Fix $\mu \in (0,1]$, $t \in (0, \bar{t}^\mu)$, and $\xi, \zeta \in \mathscr{X}_1$. Applying (36) and Lemma 2,

$$S_t^\mu(\xi,\zeta) + \tfrac{1}{2}\langle \zeta,\, -\mathcal{M}\,\zeta\rangle_1 = \tfrac{1}{2}\langle \xi,\, \mathcal{X}_t^\mu\,\xi\rangle_1 + \langle \xi,\, \mathcal{Y}_t^\mu\,\zeta\rangle_1 + \tfrac{1}{2}\langle \zeta,\, (\mathcal{Z}_t^\mu - \mathcal{M})\,\zeta\rangle_1.$$

As $\mathcal{Z}_t^\mu - \mathcal{M}$ is coercive by Lemma 3, it follows immediately that $\zeta \mapsto S_t^\mu(\xi,\zeta) + \frac{1}{2}\langle \zeta,\, -\mathcal{M}\,\zeta\rangle_1$ is convex. Hence, $S_t^\mu(\xi,\cdot) \in \mathscr{S}_+^{-\mathcal{M}}$ by (49), yielding the first assertion in (52).

For the remaining assertion in (52), note by (17), (36), (40), (50) that

$$\begin{aligned}
S_t^\mu(\xi,\zeta) &= \sup_{w\in\mathscr{W}_1[0,t]} J_t^\mu[\varphi(\cdot,\zeta)](\xi,w) = \sup_{w\in\mathscr{W}_1[0,t]}\left\{\int_0^t V(\xi_s) - T^\mu(w_s)\,ds + \varphi(\xi_t,\zeta)\right\}\\
&= \sup_{w\in\mathscr{W}_1[0,t]}\left\{\int_0^t V(\xi_s) - T^\mu(w_s)\,ds + \sup_{y\in\mathscr{X}_1}\left\{\delta^-(\xi_t,y) + \varphi(y,\zeta)\right\}\right\}\\
&= \sup_{y\in\mathscr{X}_1}\left\{\sup_{w\in\mathscr{W}_1[0,t]}\left\{\int_0^t V(\xi_s) - T^\mu(w_s)\,ds + \delta^-(\xi_t,y)\right\} + \varphi(y,\zeta)\right\}\\
&= \sup_{y\in\mathscr{X}_1}\left\{G_t^\mu(\xi,y) + \varphi(y,\zeta)\right\} = \sup_{y\in\mathscr{X}_1}\left\{\varphi(\zeta,y) + G_t^\mu(\xi,y)\right\}\\
&= [\mathcal{D}_\varphi^{-1}\, G_t^\mu(\xi,\cdot)](\zeta),
\end{aligned}$$

in which δ^- is as per (51), and the second last equality follows by symmetry of φ, i.e. $\varphi(\xi,\zeta) = \varphi(\zeta,\xi)$. Hence, by semiconvex duality and the first assertion,

$$G_t^\mu(\xi,\zeta) = [\mathcal{D}_\varphi\,\mathcal{D}_\varphi^{-1}\, G_t^\mu(\xi,\cdot)](\zeta) = [\mathcal{D}_\varphi\, S_t^\mu(\xi,\cdot)](\zeta),$$

yielding the second assertion. □

It remains to prove Theorem 3, using Lemma 4.

Proof [*Theorem* 3] Fix $\mu \in (0,1]$, $t \in (0,\bar{t}^\mu)$, $\xi,\zeta \in \mathscr{X}_1$. Applying (50),

$$\begin{aligned}
G_t^\mu(\xi,\zeta) &= [\mathcal{D}_\varphi\, S_t^\mu(\xi,\cdot)](\zeta) = \inf_{y\in\mathscr{X}_1}\left\{S_t^\mu(\xi,y) - \varphi(y,\zeta)\right\}\\
&= \inf_{y\in\mathscr{X}_1}\left\{\tfrac{1}{2}\langle \xi,\,\mathcal{X}_t^\mu\,\xi\rangle_1 + \langle \xi,\,\mathcal{Y}_t^\mu\,y\rangle_1 + \tfrac{1}{2}\langle y,\,\mathcal{Z}_t^\mu\,y\rangle_1 - \tfrac{1}{2}\langle y-\zeta,\,\mathcal{M}\,(y-\zeta)\rangle_1\right\}\\
&= \tfrac{1}{2}\langle \xi,\,\mathcal{X}_t^\mu\,\xi\rangle_1 - \tfrac{1}{2}\langle \zeta,\,\mathcal{M}\,\zeta\rangle_1\\
&\quad + \inf_{y\in\mathscr{X}_1}\left\{\langle y,\,(\mathcal{Y}_t^\mu)'\,\xi + \mathcal{M}\,\zeta\rangle_1 + \tfrac{1}{2}\langle y,\,(\mathcal{Z}_t^\mu - \mathcal{M})\,y\rangle_1\right\}.
\end{aligned}$$

Applying Lemma 3, $\mathcal{Z}_t^\mu - \mathcal{M}$ is coercive and so boundedly invertible. Hence, the infimum is achieved at $y = y^* \in \mathscr{X}_1$, with $y^* \doteq -(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,[(\mathcal{Y}_t^\mu)'\,\xi + \mathcal{M}\,\zeta]$. By substitution,

$$\begin{aligned}
G_t^\mu(\xi,\zeta) &= \tfrac{1}{2}\langle \xi,\,\mathcal{X}_t^\mu\,\xi\rangle_1 - \tfrac{1}{2}\langle \zeta,\,\mathcal{M}\,\zeta\rangle_1 + \langle y^*,\,(\mathcal{Y}_t^\mu)'\,\xi + \mathcal{M}\,\zeta\rangle_1\\
&\quad + \tfrac{1}{2}\langle y^*,\,(\mathcal{Z}_t^\mu - \mathcal{M})\,y^*\rangle_1
\end{aligned}$$

$$= \tfrac{1}{2}\,\langle \xi,\, [\mathcal{X}_t^\mu - \mathcal{Y}_t^\mu\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,(\mathcal{Y}_t^\mu)']\,\xi\rangle_1 - \langle \xi,\, \mathcal{Y}_t^\mu\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,\mathcal{M}\,\zeta\rangle_1$$
$$+ \tfrac{1}{2}\,\langle \zeta,\, [-\mathcal{M} - \mathcal{M}\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,\mathcal{M}]\,\zeta\rangle_1$$
$$\doteq \tfrac{1}{2}\,\langle \xi,\, \widehat{\mathcal{X}}_t^\mu\,\xi\rangle + \langle \xi,\, \widehat{\mathcal{Y}}_t^\mu\,\zeta\rangle_1 + \tfrac{1}{2}\,\langle \zeta,\, \widehat{\mathcal{Z}}_t^\mu\,\zeta\rangle_1 = \tfrac{1}{2}\left\langle \begin{pmatrix} \xi \\ \zeta \end{pmatrix},\, \begin{pmatrix} \widehat{\mathcal{X}}_t^\mu & \widehat{\mathcal{Y}}_t^\mu \\ (\widehat{\mathcal{Y}}_t^\mu)' & \widehat{\mathcal{Z}}_t^\mu \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_\sharp, \tag{53}$$

in which $\widehat{\mathcal{X}}_t^\mu, \widehat{\mathcal{Y}}_t^\mu, \widehat{\mathcal{Z}}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ are defined by

$$\widehat{\mathcal{X}}_t^\mu \doteq \mathcal{X}_t^\mu - \mathcal{Y}_t^\mu\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,(\mathcal{Y}_t^\mu)', \quad \widehat{\mathcal{Y}}_t^\mu \doteq -\mathcal{Y}_t^\mu\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,\mathcal{M},$$
$$\widehat{\mathcal{Z}}_t^\mu \doteq -\mathcal{M} - \mathcal{M}\,(\mathcal{Z}_t^\mu - \mathcal{M})^{-1}\,\mathcal{M},$$

and the inner product $\langle \cdot\,, \cdot\rangle_\sharp$ is as per the theorem statement. Recalling (37), (42), these operators are necessarily also of the spectral form (5), with their respective eigenvalues given by inspection by

$$[\hat{x}_t^\mu]_n \doteq [x_t^\mu]_n - \frac{[y_t^\mu]_n^2}{[z_t^\mu]_n - m_n}, \quad [\hat{y}_t^\mu]_n \doteq -\frac{[y_t^\mu]_n\,m_n}{[z_t^\mu]_n - m_n},$$
$$[\hat{z}_t^\mu]_n \doteq -m_n - \frac{m_n^2}{[z_t^\mu]_n - m_n},$$

for all $n \in \mathbb{N}$. After applying (42), (43), sum-of-angle manipulations yield

$$[\hat{x}_t^\mu]_n = [p_t^\mu]_n, \quad [\hat{y}_t^\mu]_n = [q_t^\mu]_n, \quad [\hat{z}_t^\mu]_n = [p_t^\mu]_n, \tag{54}$$

for all $n \in \mathbb{N}$, where $[p_t^\mu]_n, [q_t^\mu]_n$ are as per (35). For example, for the second equality,

$$[\hat{y}_t^\mu]_n \doteq -\frac{[y_t^\mu]_n\,m_n}{[z_t^\mu]_n - m_n} = -\frac{m_n\,\cos\theta_n^\mu\,\csc(\omega_n^\mu\,t + \theta_n^\mu)}{\cos^2\theta_n^\mu\,[\,\cot\theta_n^\mu - \cot(\omega_n^\mu\,t + \theta_n^\mu)\,]}$$
$$= \frac{1}{\omega_n^\mu}\,\frac{\csc(\omega_n^\mu\,t + \theta_n^\mu)}{(\frac{-1}{\omega_n^\mu\,m_n})\,\cos\theta_n^\mu\,[\,\cot\theta_n^\mu - \cot(\omega_n^\mu\,t + \theta_n^\mu)\,]}$$
$$= \frac{1}{\omega_n^\mu}\,\frac{1}{\sin(\omega_n^\mu\,t + \theta_n^\mu)\,\cos\theta_n^\mu - \cos(\omega_n^\mu\,t + \theta_n^\mu)\,\sin\theta_n^\mu}$$
$$= \frac{1}{\omega_n^\mu\,\sin(\omega_n^\mu\,t)} = [q_t^\mu]_n$$

for all $n \in \mathbb{N}$. The other equalities in (54) follow similarly. Hence, $\widehat{\mathcal{X}}_t^\mu = \mathcal{P}_t^\mu = \widehat{\mathcal{Z}}_t^\mu$ and $\widehat{\mathcal{Y}}_t^\mu = \mathcal{Q}_t^\mu = (\mathcal{Q}_t^\mu)'$ by (34), (54), so that (33) follows by (53). □

Remark 2 The Hessian of the idempotent convolution kernel (33) may also be interpreted as the solution of an operator differential Riccati equation [5] that arises in an optimal control problem of the form (24) with $\psi \doteq \delta^-(\cdot, \zeta)$, i.e. the optimal TPBVP (32), where δ^- is as per (51). □

Some useful properties of the operators $\mathcal{E}_\mu$ and $\mathcal{P}_t^\mu$, $\mathcal{Q}_t^\mu$ of (28) and (34) follow by generalizing results from [9, Appendix B]. These properties find application in the group constructions to follow.

Lemma 5 *Given $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$, operators $\mathcal{E}_\mu$, $\mathcal{P}_t^\mu$, $\mathcal{Q}_t^\mu$ of (28), (34) are bounded and boundedly invertible, i.e.*

$$\mathcal{E}_\mu \in \mathcal{L}(\mathscr{X}_1; \mathscr{X}),\ \mathcal{E}_\mu^{-1} \in \mathcal{L}(\mathscr{X}; \mathscr{X}_1), \quad \mathcal{P}_t^\mu, \mathcal{Q}_t^\mu, (\mathcal{P}_t^\mu)^{-1}, (\mathcal{Q}_t^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1).$$

Moreover, given ω_n^μ as per (6),

$$\begin{aligned}
\mathcal{E}_\mu\, \xi &= \sum_{n=1}^\infty \omega_n^\mu\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \varphi_n, \\
(\mathcal{P}_t^\mu)^{-1}\, \xi &= -\sum_{n=1}^\infty \omega_n^\mu\, \tan(\omega_n^\mu\, t)\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \\
\mathcal{E}_\mu^{-1}\, \pi &= \sum_{n=1}^\infty \tfrac{1}{\omega_n^\mu}\, \langle \pi,\, \varphi_n \rangle\, \tilde{\varphi}_n, \\
(\mathcal{Q}_t^\mu)^{-1}\, \xi &= \sum_{n=1}^\infty \omega_n^\mu\, \sin(\omega_n^\mu\, t)\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \\
-(\mathcal{Q}_t^\mu)^{-1}\, \mathcal{P}_t^\mu\, \xi &= \sum_{n=1}^\infty \cos(\omega_n^\mu\, t)\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n\,, \\
(\mathcal{Q}_t^\mu)^{-1}\, (\mathcal{E}_t^\mu)^{-1}\, \pi &= \sum_{n=1}^\infty \sin(\omega_n^\mu\, t)\, \langle \pi,\, \varphi_n \rangle\, \tilde{\varphi}_n\,, \\
-\mathcal{E}_\mu\, \mathcal{Q}_t^\mu\, (\mathcal{I} - [(\mathcal{Q}_t^\mu)^{-1}\, \mathcal{P}_t^\mu]^2)\, \xi &= -\sum_{n=1}^\infty \sin(\omega_n^\mu\, t)\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \varphi_n\,, \\
-\mathcal{E}_\mu\, \mathcal{P}_t^\mu\, (\mathcal{Q}_t^\mu)^{-1}\, \mathcal{E}_\mu^{-1}\, \pi &= \sum_{n=1}^\infty \cos(\omega_n^\mu\, t)\, \langle \pi,\, \varphi_n \rangle\, \varphi_n\,,
\end{aligned} \tag{55}$$

for all $\xi \in \mathscr{X}_1$, $\pi \in \mathscr{X}$.

Proof The first and last equalities in (55) and associated boundedness properties are demonstrated below. The remaining equalities and bounds follow using analogous arguments.

First equality in (55): Fix $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$, $\xi \in \mathscr{X}_1$, $\pi \in \mathscr{X}$. By (3), (5), (6), (28), $\Lambda^{\frac{1}{2}} : \mathscr{X}_1 \to \mathscr{X}$, $\mathcal{I}_\mu^{\frac{1}{2}} : \mathscr{X} \to \mathscr{X}_1$, and $\mathcal{I}_\mu^{\frac{1}{2}}\, \Lambda^{\frac{1}{2}} : \mathscr{X}_1 \to \mathscr{X}_1$ satisfy

$$\Lambda^{\frac{1}{2}}\, \xi = \sum_{n=1}^\infty \sqrt{\lambda_n}\, \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n = \sum_{n=1}^\infty \langle \xi,\, \tilde{\varphi}_n \rangle_1\, \varphi_n\,,$$

$$\mathcal{I}_\mu^{\frac{1}{2}}\,\pi = \sum_{n=1}^\infty \frac{1}{\sqrt{1+\mu^2\,\lambda_n}}\,\langle \pi,\,\varphi_n\rangle\,\varphi_n = \sum_{n=1}^\infty \omega_n^\mu\,\langle \pi,\,\varphi_n\rangle\,\tilde{\varphi}_n\,,$$
$$\mathcal{I}_\mu^{\frac{1}{2}}\,\Lambda^{\frac{1}{2}}\,\xi = \sum_{n=1}^\infty \omega_n^\mu\,\langle \Lambda^{\frac{1}{2}}\,\xi,\,\varphi_n\rangle\,\tilde{\varphi}_n = \sum_{n=1}^\infty \omega_n^\mu\,\langle \xi,\,\tilde{\varphi}_n\rangle_1\,\tilde{\varphi}_n\,.$$

Hence, $\mathcal{E}_\mu = \Lambda^{\frac{1}{2}}\,\mathcal{I}_\mu^{\frac{1}{2}}\,\Lambda^{\frac{1}{2}} : \mathscr{X}_1 \to \mathscr{X}$ satisfies

$$\begin{aligned}\mathcal{E}_\mu\,\xi &= \Lambda^{\frac{1}{2}}\,(\mathcal{I}_\mu^{\frac{1}{2}}\,\Lambda^{\frac{1}{2}})\,\xi = \sum_{n=1}^\infty \langle \mathcal{I}_\mu^{\frac{1}{2}}\,\Lambda^{\frac{1}{2}}\,\xi,\,\tilde{\varphi}_n\rangle_1\,\varphi_n \\ &= \sum_{n=1}^\infty \left\langle \sum_{k=1}^\infty \omega_k^\mu\,\langle \xi,\,\tilde{\varphi}_k\rangle_1\,\tilde{\varphi}_k,\,\tilde{\varphi}_n \right\rangle_1 \varphi_n = \sum_{n=1}^\infty \omega_n^\mu\,\langle \xi,\,\tilde{\varphi}_n\rangle_1\,\varphi_n\,, \end{aligned} \tag{56}$$

as per the first equality in (55). Note by inspection that $\|\mathcal{E}_\mu\|_{\mathcal{L}(\mathscr{X}_1;\mathscr{X})} \le \sup_{n\in\mathbb{N}} |\omega_n^\mu| = \frac{1}{\mu} < \infty$.

Last equality in (55): Fix $\mu \in (0, 1]$, $t \in (0, \bar{t}^\mu)$. By inspection of (6), (34), (35), note that $|([q_t^\mu]_n)^{-1}| = |\omega_n^\mu|\,|\sin(\omega_n^\mu\,t)| \le \frac{1}{\mu}$ for all $n \in \mathbb{N}$. Consequently, a bounded linear operator of the spectral form (5) is defined by

$$\mathcal{R}_t^\mu\,\xi \doteq \sum_{n=1}^\infty ([q_t^\mu]_n)^{-1}\,\langle \xi,\,\tilde{\varphi}_n\rangle_1\,\tilde{\varphi}_n$$

for all $\xi \in \mathscr{X}_1$, with $\|\mathcal{R}_t^\mu\|_{\mathcal{L}(\mathscr{X}_1)} \le \frac{1}{\mu}$. Fix any $\xi \in \mathscr{X}_1$, $\pi \in \mathscr{X}$. Recalling (34),

$$\begin{aligned}\mathcal{R}_t^\mu\,\mathcal{Q}_t^\mu\,\xi &= \sum_{n=1}^\infty ([q_t^\mu]_n)^{-1} \left\langle \sum_{k=1}^\infty [q_t^\mu]_k\,\langle \xi,\,\tilde{\varphi}_k\rangle_1\,\tilde{\varphi}_k,\,\tilde{\varphi}_n \right\rangle_1 \tilde{\varphi}_n \\ &= \sum_{n=1}^\infty \sum_{k=1}^\infty \frac{[q_t^\mu]_k}{[q_t^\mu]_n}\,\langle \xi,\,\tilde{\varphi}_k\rangle_1\,\langle \tilde{\varphi}_k,\,\tilde{\varphi}_n\rangle_1\,\tilde{\varphi}_n = \sum_{n=1}^\infty \langle \xi,\,\tilde{\varphi}_n\rangle_1\,\tilde{\varphi}_n = \xi, \end{aligned}$$

so that $(\widetilde{\mathcal{Q}}_t^\mu)^{-1} \doteq \widetilde{\mathcal{R}}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$. Similarly, recalling (56),

$$\mathcal{E}_\mu^{-1}\,\pi = \sum_{n=1}^\infty \frac{1}{\omega_n^\mu}\,\langle \pi,\,\varphi_n\rangle\,\tilde{\varphi}_n\,, \tag{57}$$

and $\|\mathcal{E}_\mu^{-1}\|_{\mathcal{L}(\mathscr{X};\mathscr{X}_1)} \le \sup_{n\in\mathbb{N}} 1/|\omega_n^\mu| = 1/|\omega_1^\mu| < \infty$. Applying (34), (35), (56), (57), and the definition of $\mathcal{R}_t^\mu = (\mathcal{Q}_t^\mu)^{-1}$ above, analogous calculations yield

$$-\mathcal{E}_\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}\,\pi = -\sum_{n=1}^\infty \omega_n^\mu\,\frac{[p_t^\mu]_n}{[q_t^\mu]_n}\,\frac{1}{\omega_n^\mu}\,\langle \pi,\,\varphi_n\rangle\,\varphi_n$$

$$= \sum_{n=1}^{\infty} \frac{\sin(\omega_n^\mu\, t)}{\tan(\omega_n^\mu\, t)}\, \langle \pi,\, \varphi_n \rangle\, \varphi_n = \sum_{n=1}^{\infty} \cos(\omega_n^\mu\, t)\, \langle \pi,\, \varphi_n \rangle\, \varphi_n\,,$$

as per the last equality in (55), and $\| -\mathcal{E}_\mu\, \mathcal{P}_t^\mu\, (\mathcal{Q}_t^\mu)^{-1}\, \mathcal{E}_\mu^{-1} \|_{\mathcal{L}(\mathscr{X})} \le 1$. □

Remark 3 By inspection of (34), (35), along with Lemma 5, the respective eigenvalues of operators $\mathcal{P}_t^\mu$, $(\mathcal{P}_t^\mu)^{-1}$, $\mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ experience finite escape behaviour, with

$$\lim_{t \to (\frac{j\,\pi}{\omega_n^\mu})} |[p_t^\mu]_n| = \infty = \lim_{t \to (\frac{j\,\pi}{\omega_n^\mu})} |[q_t^\mu]_n|\,, \qquad \lim_{t \to (\frac{(j-\frac{1}{2})\,\pi}{\omega_n^\mu})} \frac{1}{|[p_t^\mu]_n|} = \infty, \quad n, j \in \mathbb{N}. \tag{58}$$

The first of these escape times is $\inf_{n \in \mathbb{N}} \frac{\pi}{2\,\omega_n^\mu} = (\frac{\pi}{2\sqrt{2}})\, \bar{t}^\mu \approx 1.11\, \bar{t}^\mu$, which is accompanied by an anticipated loss of concavity of $J_t^\mu(\xi, \cdot)$ for horizons beyond $\bar{t}^\mu$, for any $\xi \in \mathscr{X}_1$. □

4 Group Construction via Optimal Control

Hamilton's action principle suggests that the characteristic system associated with the optimal control problem (24) may be used to represent all solutions of the wave equation (1) via its approximation (10). This motivates the construction and validation of a prototype fundamental solution group for (10). As per the special case documented in [9, Section 4.2], the finite escape behaviour identified in Remark 3 indicates that this construction is limited to short horizons, see Lemma 1. However, propagation to longer horizons can proceed via the temporal concatenation of sufficiently many sufficiently short horizons using the aforementioned short horizon prototype [9].

4.1 Short Horizon Prototype

With $\mu \in (0, 1]$, a prototype element of the fundamental solution group $\{\mathcal{U}_t^\mu\}_{t \in \mathbb{R}}$ for the approximate wave equation (10) may be constructed [6,7] on a short horizon via a special case of the optimal control problem (24), using the idempotent representation (31), (33). In particular, a fixed short horizon $t \in (0, \bar{t}^\mu)$ and specific terminal payoff $\psi = \psi_v : \mathscr{X}_1 \to \mathbb{R}$ are considered in (24), with

$$\psi(\xi) = \psi_v(\xi) \doteq \langle \xi,\, \mathcal{E}_\mu^{-1}\, v \rangle_1, \quad \xi \in \mathscr{X}_1, \tag{59}$$

for any a priori fixed $v \in \mathscr{X}$, in which $\mathcal{E}_\mu^{-1} \in \mathcal{L}(\mathscr{X}; \mathscr{X}_1)$ is as per Lemma 5. Recalling (31), as $G_t^\mu(\cdot, \zeta)$, $G_t^\mu(\xi, \cdot)$, and $\psi = \psi_v$ are Fréchet differentiable for any $\xi, \zeta \in \mathscr{X}_1$, the supremum in (31) must be achieved where the Riesz representation of the Fréchet derivative of $G_t^\mu(\xi, \cdot) + \psi_v(\cdot)$ is zero. That is,

$$\zeta_\xi^* \in \operatorname*{arg\,max}_{\zeta \in \mathscr{X}_1} \{ G_t^\mu(\xi, \zeta) + \psi_v(\zeta) \}$$

$$\iff \quad 0 = \nabla_\zeta [G_t^\mu(\xi, \zeta) + \psi_v(\zeta)]_{\zeta = \zeta_\xi^*} = \mathcal{Q}_t^\mu \, \xi + \mathcal{P}_t^\mu \, \zeta_\xi^* + \mathcal{E}_\mu^{-1} \, v \,, \tag{60}$$

and $\zeta_\xi^* \in \mathscr{X}_1$ is the terminal state achieved, given the initial state $\xi \in \mathscr{X}_1$. As $\mathcal{P}_t^\mu$ is boundedly invertible for $t \in (0, \bar{t}^\mu)$ by Lemma 5, ζ_ξ^* is defined uniquely by (60), and a representation of $W_t^\mu(\xi)$ subsequently follows. In particular,

$$\zeta_\xi^* = -(\mathcal{P}_t^\mu)^{-1} \left(\mathcal{Q}_t^\mu \, \xi + \mathcal{E}_\mu^{-1} \, v \right), \qquad W_t^\mu(\xi) = G_t^\mu(\xi, \zeta_\xi^*) + \psi_v(\zeta_\xi^*) \,, \tag{61}$$

for all $\xi \in \mathscr{X}_1$. With a view to computing a candidate optimal trajectory via (27), note by (31), (60) and the chain rule that

$$\begin{aligned} \nabla_x W_t^\mu(\xi) &= \nabla_x [G_t^\mu(\xi, \zeta_\xi^*) + \psi_v(\zeta_\xi^*)] \\ &= \nabla_\xi G_t^\mu(\xi, \zeta)|_{\zeta = \zeta_\xi^*} + (D_\xi \zeta_\xi^*)' \, \nabla_\zeta [G_t^\mu(\xi, \zeta) + \psi_v(\zeta)]_{\zeta = \zeta_\xi^*} = \nabla_\xi G_t^\mu(\xi, \zeta)|_{\zeta = \zeta_\xi^*}, \end{aligned}$$

in which $D_\xi \zeta_\xi^* = -(\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ is the Frechet derivative of the map $\xi \mapsto \zeta_\xi^*$, $\xi \in \mathscr{X}_1$, see (61), and the final equality follows by (60). Hence, recalling (31), (33), (61),

$$\nabla_x W_t^\mu(\xi) = \mathcal{P}_t^\mu \xi + \mathcal{Q}_t^\mu \zeta_\xi^* = (\mathcal{P}_t^\mu - \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu) \, \xi - \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, v.$$

Motivated by (27), and again recalling (61), define

$$\begin{aligned} \hat{\xi}_0 &\doteq \xi \,, \\ \hat{\xi}_t &\doteq \zeta_\xi^* = -(\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu \, \xi - (\mathcal{P}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, v \,, \\ \hat{\pi}_0 &\doteq \mathcal{E}_\mu \, \nabla_x W_t^\mu(\xi) = \mathcal{E}_\mu \, (\mathcal{P}_t^\mu - \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu) \, \xi - \mathcal{E}_\mu \, \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, v \,, \\ \hat{\pi}_t &\doteq \mathcal{E}_\mu \, \nabla_x W_0^\mu(\zeta_\xi^*) = \mathcal{E}_\mu \, \nabla_x \psi_v(\zeta_\xi^*) = \mathcal{E}_\mu \, \mathcal{E}_\mu^{-1} \, v = v \,, \end{aligned} \tag{62}$$

in which $\hat{\xi}_0, \hat{\xi}_t \in \mathscr{X}_1$ denote the initial and terminal states of a trajectory corresponding to a candidate optimal control $s \mapsto \hat{w}_s$ satisfying $\hat{w}_0 = \mathcal{I}_\mu^{\frac{1}{2}} \, \hat{\pi}_0$ and $\hat{w}_t = \mathcal{I}_\mu^{\frac{1}{2}} \, \hat{\pi}_t$. By inspection, eliminating ξ and v in (62) yields

$$\begin{aligned} \hat{\xi}_t &= -(\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu \, \hat{\xi}_0 - (\mathcal{P}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, \hat{\pi}_t, \\ \hat{\pi}_0 &= \mathcal{E}_\mu \, (\mathcal{P}_t^\mu - \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{Q}_t^\mu) \, \hat{\xi}_0 - \mathcal{E}_\mu \, \mathcal{Q}_t^\mu \, (\mathcal{P}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, \hat{\pi}_t. \end{aligned} \tag{63}$$

By exploiting invertibility of the operators involved, see Lemma 5, and some straightforward manipulations, see Remark 4 below, it follows by (63) that

$$\begin{pmatrix} \hat{\xi}_t \\ \hat{\pi}_t \end{pmatrix} = \widehat{\mathcal{U}}_t^\mu \begin{pmatrix} \hat{\xi}_0 \\ \hat{\pi}_0 \end{pmatrix}, \quad \widehat{\mathcal{U}}_t^\mu \doteq \left(\begin{array}{c|c} [\widehat{\mathcal{U}}_t^\mu]_{11} & [\widehat{\mathcal{U}}_t^\mu]_{12} \\ \hline [\widehat{\mathcal{U}}_t^\mu]_{21} & [\widehat{\mathcal{U}}_t^\mu]_{22} \end{array} \right), \tag{64}$$

with elements $[\widehat{\mathcal{U}}_t^\mu]_{11} \in \mathcal{L}(\mathscr{X}_1)$, $[\widehat{\mathcal{U}}_t^\mu]_{12} \in \mathcal{L}(\mathscr{X};\mathscr{X}_1)$, $[\widehat{\mathcal{U}}_t^\mu]_{21} \in \mathcal{L}(\mathscr{X}_1;\mathscr{X})$, and $[\widehat{\mathcal{U}}_t^\mu]_{22} \in \mathcal{L}(\mathscr{X})$ given for $t \in (0, \bar{t}^\mu)$ by

$$\begin{aligned}
&[\widehat{\mathcal{U}}_t^\mu]_{11} \doteq -(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{P}_t^\mu, && [\widehat{\mathcal{U}}_t^\mu]_{12} \doteq (\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1},\\
&[\widehat{\mathcal{U}}_t^\mu]_{21} \doteq -\mathcal{E}_\mu\,\mathcal{Q}_t^\mu\left(\mathcal{I}-[(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{P}_t^\mu]^2\right), && [\widehat{\mathcal{U}}_t^\mu]_{22} \doteq -\mathcal{E}_\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}.
\end{aligned} \tag{65}$$

Remark 4 Manipulating the second equation in (63) to solve for $\hat{\pi}_t$ yields

$$\begin{aligned}
\hat{\pi}_t &= -\mathcal{E}_t^\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,(\mathcal{Q}_t^\mu\,(\mathcal{P}_t^\mu)^{-1}\,\mathcal{Q}_t^\mu - \mathcal{P}_t^\mu)\,\hat{\xi}_0 - \mathcal{E}_\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}\,\hat{\pi}_0\\
&= -\mathcal{E}_t^\mu\,(\mathcal{Q}_t^\mu - \mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{P}_t^\mu)\,\hat{\xi}_0 - \mathcal{E}_\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}\,\hat{\pi}_0\\
&= -\mathcal{E}_t^\mu\,\mathcal{Q}_t^\mu\,(\mathcal{I} - [(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{P}_t^\mu]^2)\,\hat{\xi}_0 - \mathcal{E}_\mu\,\mathcal{P}_t^\mu\,(\mathcal{Q}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}\,\hat{\pi}_0\\
&\doteq [\widehat{\mathcal{U}}_t^\mu]_{21}\,\hat{\xi}_0 + [\widehat{\mathcal{U}}_t^\mu]_{22}\,\hat{\pi}_0\,,
\end{aligned} \tag{66}$$

which is as per (64), (65). A similar manipulation of the first equation in (63) via (66) completes the calculation of (64), (65). □

Lemma 6 *Given $\mu \in (0,1]$, $t \in (0,\bar{t}^\mu)$, the operators $\mathcal{U}_t^\mu, \widehat{\mathcal{U}}_t^\mu \in \mathcal{L}(\mathscr{Y})$ of (7), (8) and (64), (65) are equivalent, i.e. $\widehat{\mathcal{U}}_t^\mu = \mathcal{U}_t^\mu$.*

Proof Fix $\mu \in (0,1]$, $t \in (0,\bar{t}^\mu)$, $\xi \in \mathscr{X}_1$, $\pi \in \mathscr{X}$. The assertion follows by comparing (7), (8) with (64), (65), via Lemma 5. □

The candidate optimal control $\hat{w}$ alluded to in the prototype group construction above can be represented explicitly via Lemma 6, and subsequently confirmed to be optimal. This confirmation can proceed via Theorem 2, or directly by substitution using (17), (24), or alternatively by substitution using (31), (32). The last approach is formalized as follows. With $\xi, \zeta \in \mathscr{X}_1$ fixed as the initial and terminal states of the candidate optimal trajectory, observe by (64), (65), i.e. (63), and Lemma 6, that $\zeta = \hat{\xi}_t = [\mathcal{U}_t^\mu]_{11}\,\xi + [\mathcal{U}_t^\mu]_{12}\,\hat{\pi}_0$, or

$$\hat{\pi}_0 = \hat{\pi}_0(\xi,\zeta) \doteq [\mathcal{U}_t^\mu]_{12}^{-1}\,(\zeta - [\mathcal{U}_t^\mu]_{11}\,\xi)\,, \tag{67}$$

in which the inverse involved exists by inspection of (65). Again applying Lemma 6, the candidate optimal control $\hat{w} \in \mathscr{W}_1[0,t]$ and corresponding trajectory $\hat{\xi} \in C([0,t];\mathscr{X}_1)$ are defined by

$$\hat{w}_s \doteq \mathcal{I}_\mu^{\frac{1}{2}}\,\hat{\upsilon}(\xi,\zeta)_s\,, \quad \begin{pmatrix}\hat{\xi}_s\\ \hat{\pi}_s\end{pmatrix} \doteq \begin{pmatrix}\hat{\chi}(\xi,\zeta)_s\\ \hat{\upsilon}(\xi,\zeta)_s\end{pmatrix} \doteq \mathcal{U}_s^\mu \begin{pmatrix}\xi\\ \hat{\pi}_0(\xi,\zeta)\end{pmatrix}, \tag{68}$$

for all $s \in [0,t]$, in which the form of $s \mapsto \hat{w}_s$ follows from the generator $\mathcal{A}^\mu$ of the group $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}}$, see (4) and Theorem 1.

Lemma 7 *Given $\mu \in (0,1]$, $t \in (0,\bar{t}^\mu)$, and $\xi, \zeta \in \mathscr{X}_1$, the candidate optimal control $\hat{w} = \mathcal{I}_\mu^{\frac{1}{2}}\,\hat{\upsilon}(\xi,\zeta)$ of (68) is optimal in the definition (32) of $G_t^\mu(\xi,\zeta)$, and $\hat{\xi} = \hat{\chi}(\xi,\zeta)$ of (68) is the corresponding optimal trajectory satisfying (15).*

Proof Fix $\mu \in \mathbb{R}_{>0}$, $t \in (0, \bar{t}^\mu)$, and $\xi, \zeta \in \mathscr{X}_1$, and let $\hat{w} = \mathcal{I}_\mu^{\frac{1}{2}}\, \hat{\pi} = \mathcal{I}_\mu^{\frac{1}{2}}\, \hat{\upsilon}(\xi, \zeta)$ and $\hat{\xi} = \hat{\chi}(\xi, \zeta)$ be the candidate optimal control and trajectory as per (68). Note in particular that $\hat{\xi}_0 = \xi$ and $\hat{\xi}_t = \zeta$ by definition, i.e. (67), (68), while $\hat{w}, \hat{\xi}$ satisfies (15) by inspection of (9), (68). Hence, the candidate optimal control and trajectory satisfy the constraints in (32), and in particular

$$
\begin{aligned}
G_t^\mu(\xi, \zeta) \geq J_t^\mu[\delta^-(\cdot, \zeta)](\xi, \hat{w}) &= J_t^\mu[\psi_0](\xi, \hat{w}) \\
&= \int_0^t \tfrac{1}{2}\, \|\hat{\xi}_s\|_1^2 - \tfrac{1}{2}\, \|\hat{\pi}_s\|^2\, ds > -\infty
\end{aligned}
$$

by inspection of (17), (18), (51), (68). Moreover, Theorem 1, (4), (9), (10), (28), (68), and the chain rule together imply that

$$
\begin{aligned}
\|\hat{\xi}_s\|_1^2 - \|\hat{\pi}_s\|^2 &= \langle \hat{\xi}_s,\, \hat{\xi}_s\rangle_1 - \langle \mathcal{I}_\mu^{-\frac{1}{2}}\, \mathcal{I}_\mu^{\frac{1}{2}}\, \hat{\pi}_s,\, \hat{\pi}_s\rangle = \langle \hat{\xi}_s,\, \hat{\xi}_s\rangle_1 - \langle \mathcal{I}_\mu^{\frac{1}{2}}\, \hat{\pi}_s,\, \mathcal{E}_\mu^{-1}\, \hat{\pi}_s\rangle_1 \\
&= \langle \hat{\xi}_s,\, -\mathcal{E}_\mu^{-1}\, \dot{\hat{\pi}}_s\rangle_1 + \langle \dot{\hat{\xi}}_s,\, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_s\rangle_1 = \tfrac{d}{ds}\, \langle \hat{\xi}_s,\, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_s\rangle_1\,,
\end{aligned}
$$

in which the second equality follows by identifying the adjoint of $\mathcal{I}_\mu^{-\frac{1}{2}}$ with $\mathcal{E}_\mu^{-1}$, i.e. $\langle \mathcal{I}_\mu^{-\frac{1}{2}}\, y,\, z\rangle = \langle y,\, \mathcal{E}_\mu^{-1}\, z\rangle_1$ for all $y \in \mathscr{X}_1$, $z \in \mathscr{X}$, see (28). Consequently, by integration and (65), (67), (68),

$$
\begin{aligned}
&J_t^\mu[\delta^-(\cdot, \zeta)](\xi, \hat{w}) = J_t^\mu[\psi_0](\xi, \hat{w}) \\
&= \tfrac{1}{2} \int_0^t \tfrac{d}{ds}\, \langle \hat{\xi}_s,\, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_s\rangle_1\, ds = \tfrac{1}{2}\, \langle \zeta, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_t\rangle_1 - \tfrac{1}{2}\, \langle \xi, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_0\rangle_1 \\
&= \tfrac{1}{2}\, \langle \zeta, -\mathcal{E}_\mu^{-1}\, ([\mathcal{U}_t^\mu]_{21}\, \xi + [\mathcal{U}_t^\mu]_{22}\, \hat{\pi}_0)\rangle_1 - \tfrac{1}{2}\, \langle \xi, -\mathcal{E}_\mu^{-1}\, \hat{\pi}_0\rangle_1 \\
&= \tfrac{1}{2}\, \langle \zeta, -\mathcal{E}_\mu^{-1}\, ([\mathcal{U}_t^\mu]_{21}\, \xi + [\mathcal{U}_t^\mu]_{22}\, [\mathcal{U}_t^\mu]_{12}^{-1}\, (\zeta - [\mathcal{U}_t^\mu]_{11}\, \xi))\rangle_1 \\
&\quad - \tfrac{1}{2}\, \langle \xi, -\mathcal{E}_\mu^{-1}\, [\mathcal{U}_t^\mu]_{12}^{-1}\, (\zeta - [\mathcal{U}_t^\mu]_{11}\, \xi)\rangle_1 \\
&= \tfrac{1}{2}\, \langle \xi,\, -\mathcal{E}_\mu^{-1}\, [\mathcal{U}_t^\mu]_{12}^{-1}\, [\mathcal{U}_t^\mu]_{11}\, \xi\rangle_1 + \tfrac{1}{2}\, \langle \zeta,\, -\mathcal{E}_\mu^{-1}\, [\mathcal{U}_t^\mu]_{22}\, [\mathcal{U}_t^\mu]_{12}^{-1}\, \zeta\rangle_1 \\
&\quad + \tfrac{1}{2}\, \langle \zeta, \Big[([\mathcal{U}_t^\mu]_{12}^{-1})'\, \mathcal{E}_\mu^{-1} - \mathcal{E}_\mu^{-1}\, [\mathcal{U}_t^\mu]_{21} + \mathcal{E}_\mu^{-1}\, [\mathcal{U}_t^\mu]_{22}\, [\mathcal{U}_t^\mu]_{12}^{-1}\, [\mathcal{U}_t^\mu]_{11}\Big]\, \xi\rangle_1 \\
&= \tfrac{1}{2}\, \langle \xi,\, \mathcal{P}_t^\mu\, \xi\rangle_1 + \tfrac{1}{2}\, \langle \zeta,\, \mathcal{P}_t^\mu\, \zeta\rangle_1 + \tfrac{1}{2}\, \langle \zeta,\, 2\, \mathcal{Q}_t^\mu\, \zeta\rangle_1 \\
&= G_t^\mu(\xi, \zeta)\,,
\end{aligned}
$$

thereby yielding the claimed optimality. □

Lemma 6 confirms that the one parameter family of prototypes $\{\widehat{\mathcal{U}}_t^\mu\}_{t\in(0,\bar{t}^\mu)}$ constructed from the candidate optimal control (68), yielding (64), (65), corresponds to a subset of the fundamental solution group $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}}$ for the approximate wave equation (10). Lemma 7 demonstrates that this candidate optimal control is indeed optimal in the required sense.

Remark 5 Given any terminal payoff ψ and any initial state $\xi \in \mathscr{X}_1$, if the corresponding terminal state $\zeta_\xi^* \in \arg\max_{\zeta\in\mathscr{X}_1}\{G_t^\mu(\xi,\zeta)+\psi(\zeta)\}$ exists, then the candidate optimal control (68) is likewise optimal in the definition of the corresponding value function (24). That is,

$$W_t^\mu(\xi) = \sup_{w\in\mathscr{W}_1[0,t]} J_t^\mu[\psi](\xi,w) = J_t^\mu[\psi](\xi,\hat{w})\,, \quad \hat{w} = \mathcal{I}_\mu^{\frac{1}{2}}\,\hat{v}(\xi,\zeta_\xi^*)\,.$$

□

4.2 Longer Horizons

The correspondence between stationary action and optimal control can break down for longer time horizons due to a loss of concavity of the payoff (14), see Lemma 1 and the finite escape property (58) associated with the idempotent representation (31), (33), (34), (35). Consequently, for longer horizons, a modified approach is required. In particular, by directly replacing the *sup* operation in (24), (31) with a *stat* operation [9, 17,18], a value function corresponding to a *stationary* payoff may be defined, without the need to assume that stationarity is achieved at a maximum. Less generally, by retaining the sup operation in (24) on shorter horizons, longer time horizons may be accumulated by concatenating these short horizons, with the stat operation used to relax the constraints associated with the intermediate states joining adjacent horizons, via a generalization of G_t^μ in (31), (33). This latter approach is considered here. An appropriate definition of the *stat* operation given by

$$\begin{aligned} \operatorname*{stat}_{\zeta\in\mathscr{X}_1} F(\zeta) &\doteq \left\{ F(\bar\zeta) \,\middle|\, \bar\zeta \in \operatorname*{arg\,stat}_{\zeta\in\mathscr{X}_1} F(\zeta) \right\}, \quad F:\mathscr{X}_1\to\mathbb{R}, \\ \operatorname*{arg\,stat}_{\zeta\in\mathscr{X}_1} F(\zeta) &\doteq \left\{ \zeta\in\mathscr{X}_1 \,\middle|\, 0 = \lim_{y\to\zeta} \frac{|F(y)-F(\zeta)|}{\|y-\zeta\|_1} \right\}. \end{aligned} \tag{69}$$

With a view to formalizing the aforementioned concatenation approach, with $\mu \in (0,1]$ and $x \in \mathscr{X}_1$ fixed, consider any longer horizon $t \in [\bar{t}^\mu, \infty)$ of interest for which the payoff $J_t^\mu(x,\cdot)$ of (17) may not be concave. The key idea is to select a sufficiently large number $n_t \in \mathbb{N}$ of shorter horizons $\tau \doteq t/n_t \in (0,\bar{t}^\mu)$ such that concavity of $J_\tau^\mu(\zeta_k,\cdot)$ is retained on every subinterval $[(k-1)\,\tau,\ k\,\tau]$, $k\in[1,n_t]\cap\mathbb{N}$, where $\zeta_k = \xi_{k\tau} \in \mathscr{X}_1$ denotes an intermediate state.

In further formalizing this approach, a candidate generalization of the value function W_t^μ of (24) is proposed, via a corresponding generalization of (31), that is applicable on longer horizons. Note by Remark 3 that finite escape times must be avoided. To this end, fix an arbitrary horizon t^μ satisfying

$$t^\mu \in \mathbb{R}_{>0}\setminus \Xi^\mu\,, \quad \Xi^\mu \doteq \bigcup_{n,j,k\in\mathbb{N}} \{t_{n,j,k}^\mu,\ t_{\infty,j,k}^\mu\}\,, \tag{70}$$

in which

$$\begin{cases} t^{\mu}_{n,j,k} \doteq \frac{j}{k}\,(\frac{\pi}{2})\,\frac{1}{\omega^{\mu}_n}, & \frac{1}{\omega^{\mu}_n} = (\frac{1}{\lambda^{\mu}_n})^{\frac{1}{2}} = (\frac{1}{\lambda_n} + \mu^2)^{\frac{1}{2}}, \\ t^{\mu}_{\infty,j,k} \doteq \frac{j}{k}\,(\frac{\pi}{2})\,\mu\,, & n, j, k \in \mathbb{N}\,, \end{cases}$$

and note that this is always possible, as Ξ^{μ} is a countable subset of $\mathbb{R}_{>0}$. Define an unbounded set of longer horizons with respect to this t^{μ} by

$$\Omega^{\mu} \doteq \{\gamma\, t^{\mu} \,|\, \gamma \in \mathbb{Q}_{>0}\}\,. \tag{71}$$

Lemma 8 *Ω^{μ} of (71) is a countable and dense subset of $\mathbb{R}_{\geq 0}$, and closed under addition and rational multiplication.*

Proof Immediate, by (71) and the corresponding properties of $\mathbb{Q}_{>0}$. □

In view of representations (32), (33) of the short horizon bivariate convolution kernel G^{μ}_t for $t \in (0, \bar{t}^{\mu})$, and (71), define a corresponding long horizon bivariate convolution kernel $\widetilde{G}^{\mu}_t : \mathscr{X}_1 \times \mathscr{X}_1 \to \mathbb{R}$ by

$$\widetilde{G}^{\mu}_t(\xi, \zeta) \doteq \frac{1}{2} \left\langle \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \begin{pmatrix} \widetilde{\mathcal{P}}^{\mu}_t & \widetilde{\mathcal{Q}}^{\mu}_t \\ \widetilde{\mathcal{Q}}^{\mu}_t & \widetilde{\mathcal{P}}^{\mu}_t \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} \right\rangle_{\sharp}, \tag{72}$$

for all $t \in \Omega^{\mu}$, in which $\widetilde{\mathcal{P}}^{\mu}_t$, $\widetilde{\mathcal{Q}}^{\mu}_t$ are defined analogously to (34) by

$$\begin{aligned} \widetilde{\mathcal{P}}^{\mu}_t\,\xi &\doteq \sum_{n=1}^{\infty} [\tilde{p}^{\mu}_t]_n\, \langle \xi, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n, \\ \widetilde{\mathcal{Q}}^{\mu}_t\,\xi &\doteq \sum_{n=1}^{\infty} [\tilde{q}^{\mu}_t]_n\, \langle \xi, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n, \quad t \in \Omega^{\mu},\ \xi \in \mathscr{X}_1, \end{aligned} \tag{73}$$

in which the respective eigenvalues are defined analogously to (35) by

$$[\tilde{p}^{\mu}_t]_n \doteq \frac{-1}{\omega^{\mu}_n\ \tan(\omega^{\mu}_n\, t)}, \quad [\tilde{q}^{\mu}_t]_n \doteq \frac{1}{\omega^{\mu}_n\ \sin(\omega^{\mu}_n\, t)}, \tag{74}$$

for all $n \in \mathbb{N}$, $t \in \Omega^{\mu}$. Given (33), (72), note that for $\mu \in (0, 1]$, $t \in (0, \bar{t}^{\mu}) \cap \Omega^{\mu}$,

$$\widetilde{G}^{\mu}_t(\xi, \zeta) = G^{\mu}_t(\xi, \zeta) \quad \forall\, \xi, \zeta \in \mathscr{X}_1. \tag{75}$$

Lemma 9 *Given any $\mu \in (0, 1]$, $t \in \Omega^{\mu}$, there exists an $L^{\mu}_t \in \mathbb{R}_{>0}$ such that*

$$\max(|[\tilde{p}^{\mu}_t]_n|,\ |[\tilde{q}^{\mu}_t]_n|,\ |[\tilde{p}^{\mu}_t]_n|^{-1},\ |[\tilde{q}^{\mu}_t]_n|^{-1}) \leq L^{\mu}_t < \infty \tag{76}$$

for all $n \in \mathbb{N}$, *with* $[\tilde{p}_t^\mu]_n, [\tilde{q}_t^\mu]_n \in \mathbb{R}$ *as per* (74). *Consequently,* $\widetilde{\mathcal{P}}_t^\mu, \widetilde{\mathcal{Q}}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ *and* $(\widetilde{\mathcal{P}}_t^\mu)^{-1}, (\widetilde{\mathcal{Q}}_t^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1)$.

Proof Fix any $\mu \in (0, 1]$, $t \in \Omega^\mu$. Recalling (6), $\{\omega_n^\mu\}_{n\in\mathbb{N}}$ is a positive non-decreasing convergent sequence with $\omega_\infty^\mu \doteq \lim_{n\to\infty} \omega_n^\mu = \frac{1}{\mu}$. Recalling (70), $\omega_\infty^\mu\, t \neq j\,(\frac{\pi}{2}) \neq \omega_n^\mu\, t$, for all $n \in \mathbb{N}$, $j \in \mathbb{Z}_{\geq 0}$. Let $\epsilon_t^\mu \doteq \frac{1}{2} \inf_{j\in\mathbb{Z}_{\geq 0}} |\omega_\infty^\mu\, t - j\,(\frac{\pi}{2})|$, and note that $\epsilon_t^\mu \in (0, \frac{\pi}{4})$. By definition of ω_∞^μ, there exists an $N_t^\mu \in \mathbb{N}$ such that $|\omega_n^\mu\, t - \omega_\infty^\mu\, t| < \epsilon_t^\mu$ for all $n > N_t^\mu$. Moreover, by the triangle inequality,

$$\begin{aligned}\inf_{j\in\mathbb{Z}_{\geq 0}} \inf_{n>N_t^\mu} |\omega_n^\mu\, t - j\,(\tfrac{\pi}{2})| &\geq \inf_{j\in\mathbb{Z}_{\geq 0}} |\omega_\infty^\mu\, t - j\,(\tfrac{\pi}{2})| - \sup_{n>N_t^\mu} |\omega_\infty^\mu\, t - \omega_n^\mu\, t| \\ &\geq 2\,\epsilon_t^\mu - \epsilon_t^\mu = \epsilon_t^\mu > 0\,.\end{aligned}$$

Meanwhile, $\hat{\epsilon}_t^\mu \doteq \min_{j\in\mathbb{Z}_{\geq 0}} \min_{n\in[1,N_t^\mu]\cap\mathbb{N}} |\omega_n^\mu\, t - j\,(\frac{\pi}{2})| > 0$, so that

$$|\omega_n^\mu\, t - j\,(\tfrac{\pi}{2})| \geq \bar{\epsilon}_t^\mu \doteq \min(\hat{\epsilon}_t^\mu, \epsilon_t^\mu) > 0 \tag{77}$$

for all $n \in \mathbb{N}$, $j \in \mathbb{Z}_{\geq 0}$. Let $j_n^{\mu,t} \doteq \lfloor \frac{2\,\omega_n^\mu\, t}{\pi} \rfloor \in \mathbb{Z}_{\geq 0}$, and note that

$$\tan(\omega_n^\mu\, t) = \begin{cases} \tan(\omega_n^\mu\, t - j_n^{\mu,t}\,(\frac{\pi}{2}))\,, & j_n^{\mu,t} \text{ even,} \\ -\cot(\omega_n^\mu\, t - j_n^{\mu,t}\,(\frac{\pi}{2}))\,, & j_n^{\mu,t} \text{ odd,} \end{cases} \tag{78}$$

in which $\omega_n^\mu\, t - j_n^\mu\,(\frac{\pi}{2}) \in (0, \frac{\pi}{2})$. With $f(\epsilon) \doteq \tan^2(\epsilon)$, $g(\epsilon) \doteq \cot^2(\epsilon)$, $\epsilon \in (0, \frac{\pi}{2})$, let $f^{(4)}$, $g^{(4)}$ denote the fourth derivatives, and note that $f^{(4)}(\epsilon), g^{(4)}(\epsilon) \in \mathbb{R}_{\geq 0}$ for all $\epsilon \in (0, \frac{\pi}{2})$. By Taylor's theorem,

$$\begin{aligned} f(\epsilon) &= \epsilon^2 + \tfrac{1}{4!}\, f^{(4)}(\tilde{\epsilon})\, \epsilon^4 \geq \epsilon^2\,, & \tilde{\epsilon} \in (0, \epsilon)\,, \\ g(\epsilon) &= (\epsilon - \tfrac{\pi}{2})^2 + \tfrac{1}{4!}\, g^{(4)}(\hat{\epsilon})\,(\epsilon - \tfrac{\pi}{2})^4 \geq (\epsilon - \tfrac{\pi}{2})^2\,, & \hat{\epsilon} \in (\epsilon, \tfrac{\pi}{2})\,, \end{aligned}$$

for all $\epsilon \in (0, \frac{\pi}{2})$. In particular, as $\omega_n^\mu\, t - j_n^\mu\,(\frac{\pi}{2}) \in (0, \frac{\pi}{2})$,

$$\begin{aligned} \tan^2(\omega_n^\mu\, t - j_n^{\mu,t}\,(\tfrac{\pi}{2})) &\geq |\omega_n^\mu\, t - j_n^{\mu,t}\,(\tfrac{\pi}{2})|^2 \geq (\bar{\epsilon}_t^\mu)^2\,, \\ \cot^2(\omega_n^\mu\, t - j_n^{\mu,t}\,(\tfrac{\pi}{2})) &\geq |\omega_n^\mu\, t - (j_n^{\mu,t} + 1)\,(\tfrac{\pi}{2})|^2 \geq (\bar{\epsilon}_t^\mu)^2\,, \end{aligned} \tag{79}$$

in which the final inequalities follow by (77), irrespective of whether $j_n^{\mu,t}$ is odd or even. Hence, by (78), (79), $\tan^2(\omega_n^\mu\, t) \geq (\bar{\epsilon}_t^\mu)^2$, $\cot^2(\omega_n^\mu\, t) \geq (\bar{\epsilon}_t^\mu)^2$, and so $\tan^2(\omega_n^\mu\, t),\ \cot^2(\omega_n^\mu\, t) \in [(\bar{\epsilon}_t^\mu)^2,\ \frac{1}{(\bar{\epsilon}_t^\mu)^2}]$. Consequently, (35) yields

$$\left(\frac{\bar{\epsilon}_t^\mu}{\omega_\infty^\mu}\right)^2 \leq |[\tilde{p}_t^\mu]_n|^2 = \frac{1}{(\omega_n^\mu)^2\, \tan^2(\omega_n^\mu\, t)} \leq \frac{1}{(\omega_1^\mu\, \bar{\epsilon}_t^\mu)^2}\,,$$

with $|[\tilde{p}_t^\mu]_n|^{-2}$ likewise bounded, uniformly in $n \in \mathbb{N}$. Note further by (35) that $|[\tilde{q}_t^\mu]_n|^2 = \frac{1}{(\omega_n^\mu)^2} + |[\tilde{p}_t^\mu]_n|^2$, so that $|[\tilde{q}_t^\mu]_n|^2$ and $|[\tilde{q}_t^\mu]_n|^{-2}$ are similarly bounded, uniformly in $n \in \mathbb{N}$. Existence of $L_t^\mu < \infty$ subsequently follows, as per the hypothesis, and $\widetilde{\mathcal{P}}_t^\mu$, $\tilde{\mathcal{Q}}_t^\mu$, $(\widetilde{\mathcal{P}}_t^\mu)^{-1}$, $(\widetilde{\mathcal{Q}}_t^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1)$. □

In order to apply $\widetilde{G}_t^\mu$, it is crucial to show that the operators $\widetilde{\mathcal{P}}_t^\mu$, $\widetilde{\mathcal{Q}}_t^\mu$ can be propagated to arbitrary longer horizons in Ω^μ via concatenations of horizons. This can be achieved using standard Schur complement operations.

Lemma 10 *Given any $\mu \in (0, 1]$, $s, \sigma \in \Omega^\mu$,*

$$\widetilde{\mathcal{P}}_s^\mu,\ \widetilde{\mathcal{P}}_{s+\sigma}^\mu,\ (\widetilde{\mathcal{P}}_s^\mu)^{-1},\ \widetilde{\mathcal{Q}}_s^\mu,\ \widetilde{\mathcal{Q}}_{s+\sigma}^\mu,\ (\widetilde{\mathcal{Q}}_s^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1)\,, \tag{80}$$

$$[\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_s^\mu,\ [\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_\sigma^\mu \in \mathcal{L}(\mathscr{X}_1)\,,$$

$$\widetilde{\mathcal{P}}_{s+\sigma}^\mu = \widetilde{\mathcal{P}}_s^\mu - \widetilde{\mathcal{Q}}_s^\mu\, [\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_s^\mu\,,\quad \widetilde{\mathcal{Q}}_{s+\sigma}^\mu = -\widetilde{\mathcal{Q}}_s^\mu\, [\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_\sigma^\mu\,. \tag{81}$$

Proof Fix $\mu \in (0, 1]$, $s, \sigma \in \Omega^\mu$, and note that $s + \sigma \in \Omega^\mu$ by Lemma 8.

Boundedness assertions (80): Lemma 9 immediately yields that

$$\widetilde{\mathcal{P}}_s^\mu,\quad \widetilde{\mathcal{P}}_\sigma^\mu,\quad \widetilde{\mathcal{P}}_{s+\sigma}^\mu,\quad (\widetilde{\mathcal{P}}_s^\mu)^{-1},\quad \widetilde{\mathcal{Q}}_s^\mu,\quad \widetilde{\mathcal{Q}}_\sigma^\mu,\quad \widetilde{\mathcal{Q}}_{s+\sigma}^\mu,\quad (\widetilde{\mathcal{Q}}_s^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1)\,.$$

Recall by definition of $s, \sigma, s + \sigma \in \Omega^\mu$ that

$$\omega_n^\mu\, s \neq j\,(\tfrac{\pi}{2}),\ \omega_n^\mu\, \sigma \neq j\,(\tfrac{\pi}{2}),\qquad \omega_n^\mu\,(s + \sigma) \neq j\,(\tfrac{\pi}{2}),\ n, j \in \mathbb{N}, \tag{82}$$

so that for any $n, j \in \mathbb{N}$,

$$\begin{aligned} &|\sec(\omega_n^\mu\, s)| < \infty,\ \ |\csc(\omega_n^\mu\, s)| < \infty, |\cot(\omega_n^\mu\, s)| < \infty,\ \ |\tan(\omega_n^\mu\, \sigma)| < \infty,\\ &\ \tan(\omega_n^\mu\, s) + \tan(\omega_n^\mu\, \sigma) = \tan(\omega_n^\mu\,(s + \sigma) - \omega_n^\mu\, \sigma) + \tan(\omega_n^\mu\, \sigma)\\ &\quad \neq \tan(j\,\pi - \omega_n^\mu\, \sigma) + \tan(\omega_n^\mu\, \sigma) = 0. \end{aligned} \tag{83}$$

Define

$$\widehat{\mathcal{Q}}_{s,\sigma}^\mu \doteq [\mathcal{U}_s^\mu]_{11} + (\widetilde{\mathcal{Q}}_s^\mu)^{-1}\, \widetilde{\mathcal{P}}_{s+\sigma}^\mu \in \mathcal{L}(\mathscr{X}_1), \tag{84}$$

in which $[\mathcal{U}_s^\mu]_{11} \in \mathcal{L}(\mathscr{X}_1)$ is as per (7), (8), and $\widetilde{\mathcal{P}}_{s+\sigma}^\mu$, $(\widetilde{\mathcal{Q}}_s^\mu)^{-1} \in \mathcal{L}(\mathscr{X}_1)$ as demonstrated above. Note that $\widehat{\mathcal{Q}}_{s,\sigma}^\mu$ may be represented in the form (5), with

$$\widehat{\mathcal{Q}}_{s,\sigma}^\mu\, \xi = \sum_{n=1}^\infty [\hat{q}_{s,\sigma}^\mu]_n\, \langle \xi, \tilde{\varphi}_n \rangle_1\, \tilde{\varphi}_n, \tag{85}$$

in which $[\hat{q}_{s,\sigma}^\mu]_n$ is well-defined via (8), (73) by

$$[\hat{q}_{s,\sigma}^\mu]_n \doteq \cos(\omega_n^\mu\, s) + ([\tilde{q}_s^\mu]_n)^{-1}\, [\tilde{p}_{s+\sigma}^\mu]_n. \tag{86}$$

Observe by (74) and standard trigonometric identities (including sum-of-angles for tan) that

$$
\begin{aligned}
[\tilde{p}^{\mu}_{s+\sigma}]_n &= \frac{-1}{\omega^{\mu}_n \, \tan(\omega^{\mu}_n (s+\sigma))} \\
&= \frac{-1}{\omega^{\mu}_n \, \tan(\omega^{\mu}_n \, s)} + \frac{1}{\omega^{\mu}_n \, \sin^2(\omega^{\mu}_n \, s)} \frac{\tan(\omega^{\mu}_n \, s) \, \tan(\omega^{\mu}_n \, \sigma)}{\tan(\omega^{\mu}_n \, s) + \tan(\omega^{\mu}_n \, \sigma)},
\end{aligned}
\tag{87}
$$

in which all terms are finite by (82), (83). Substituting (87) in (86) yields

$$
\begin{aligned}
[\hat{q}^{\mu}_{s,\sigma}]_n &= \cos(\omega^{\mu}_n \, s) + \omega^{\mu}_n \, \sin(\omega^{\mu}_n \, s) \, [\tilde{p}^{\mu}_{s+\sigma}]_n \\
&= \frac{\sec(\omega^{\mu}_n \, s) \, \tan(\omega^{\mu}_n \, \sigma)}{\tan(\omega^{\mu}_n \, s) + \tan(\omega^{\mu}_n \, \sigma)} = -\frac{[\tilde{q}^{\mu}_s]_n}{[\tilde{p}^{\mu}_s]_n + [\tilde{p}^{\mu}_\sigma]_n},
\end{aligned}
$$

in which all terms are again finite by (82), (83). Hence, recalling (84), (85), it follows that $\widehat{\mathcal{Q}}^{\mu}_{s,\sigma} \equiv -[\widetilde{\mathcal{P}}^{\mu}_s + \widetilde{\mathcal{P}}^{\mu}_\sigma]^{-1} \, \widetilde{\mathcal{Q}}^{\mu}_s$, so that $-[\widetilde{\mathcal{P}}^{\mu}_s + \widetilde{\mathcal{P}}^{\mu}_\sigma]^{-1} \, \widetilde{\mathcal{Q}}^{\mu}_s$, $-[\widetilde{\mathcal{P}}^{\mu}_s + \widetilde{\mathcal{P}}^{\mu}_\sigma]^{-1} \, \widetilde{\mathcal{Q}}^{\mu}_\sigma \in \mathcal{L}(\mathscr{X}_1)$. Therefore, (80) holds.

Semigroup properties (81): Observe by (73), (74), (84), (85) that

$$
(\widetilde{\mathcal{P}}^{\mu}_s - \widetilde{\mathcal{Q}}^{\mu}_s \, [\widetilde{\mathcal{P}}^{\mu}_s + \widetilde{\mathcal{P}}^{\mu}_\sigma]^{-1} \, \widetilde{\mathcal{Q}}^{\mu}_s) \, \xi = (\widetilde{\mathcal{P}}^{\mu}_s - \widetilde{\mathcal{Q}}^{\mu}_s \, \widehat{\mathcal{Q}}^{\mu}_{s,\sigma}) \, \xi = \sum_{n=1}^{\infty} [\hat{p}^{\mu}_{s,\sigma}]_n \langle \xi, \tilde{\varphi}_n \rangle_1 \, \tilde{\varphi}_n \, ,
\tag{88}
$$

where $[\hat{p}^{\mu}_{s,\sigma}]_n \in \mathbb{R}$, $n \in \mathbb{N}$, is well-defined via (82), (83), (87) by

$$
\begin{aligned}
[\hat{p}^{\mu}_{s,\sigma}]_n &\doteq [\tilde{p}^{\mu}_s]_n - [\tilde{q}^{\mu}_s]_n^2 ([\tilde{p}^{\mu}_s]_n + [\tilde{p}^{\mu}_\sigma]_n)^{-1} \\
&= \frac{-1}{\omega^{\mu}_n \, \tan(\omega^{\mu}_n \, s)} + \frac{1}{\omega^{\mu}_n \, \sin^2(\omega^{\mu}_n \, s)} \frac{\tan(\omega^{\mu}_n \, s) \, \tan(\omega^{\mu}_n \, \sigma)}{\tan(\omega^{\mu}_n \, s) + \tan(\omega^{\mu}_n \, \sigma)} = [\tilde{p}^{\mu}_{s+\sigma}]_n \, .
\end{aligned}
$$

Hence, recalling (73), (74), (88) yields the first equality in (81). A similar calculation (via sum-of-angles for sin) yields the second equality in (81). □

Lemma 11 *Given* $\mu \in (0, 1]$*, and any* $s, \sigma \in \Omega^{\mu}$*,*

$$
\widetilde{G}^{\mu}_{s+\sigma}(\xi, \zeta) = \operatorname*{stat}_{\eta \in \mathscr{X}_1} \{\widetilde{G}^{\mu}_s(\xi, \eta) + \widetilde{G}^{\mu}_\sigma(\eta, \zeta)\}
\tag{89}
$$

for all $\xi, \zeta \in \mathscr{X}_1$*, in which* $\widetilde{G}^{\mu}_s$ *is as per* (72)*. Furthermore,*

$$
\eta^* \doteq -[\widetilde{\mathcal{P}}^{\mu}_s + \widetilde{\mathcal{P}}^{\mu}_\sigma]^{-1} \, (\widetilde{\mathcal{Q}}^{\mu}_s \, \xi + \widetilde{\mathcal{Q}}^{\mu}_\sigma \, \zeta)
\tag{90}
$$

is well-defined and satisfies

$$
\eta^* \in \operatorname*{arg\,stat}_{\eta \in \mathscr{X}_1} \{\widetilde{G}^{\mu}_s(\xi, \eta) + \widetilde{G}^{\mu}_\sigma(\eta, \zeta)\}.
\tag{91}
$$

Proof Given $\mu \in (0, 1]$, fix any $s, \sigma \in \Omega^\mu$. Applying Lemma 10,

$$[\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_s^\mu, \qquad [\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]^{-1}\, \widetilde{\mathcal{Q}}_\sigma^\mu \in \mathcal{L}(\mathscr{X}_1)\,,$$

so that η^* is well-defined by (90), given any $\xi, \zeta \in \mathscr{X}_1$. By inspection of (72),

$$\nabla_\eta \{\widetilde{G}_s^\mu(\xi, \eta) + \widetilde{G}_\sigma^\mu(\eta, \zeta)\} = [\widetilde{\mathcal{P}}_s^\mu + \widetilde{\mathcal{P}}_\sigma^\mu]\,\eta + \widetilde{\mathcal{Q}}_s^\mu\, \xi + \widetilde{\mathcal{Q}}_\sigma^\mu\, \zeta$$

for all $\xi, \eta, \zeta \in \mathscr{X}_1$, so that $0 = \nabla_\eta \{\widetilde{G}_s^\mu(\xi, \eta) + \widetilde{G}_\sigma^\mu(\eta, \zeta)\}\big|_{\eta=\eta^*}$. Hence, η^* of (90) also satisfies (91), and (89) subsequently follows by definition (69). □

The above lemmas, culminating in Lemma 11, allow the proposed long horizon bivariate convolution kernel $\widetilde{G}_t^\mu$ defined by (72) to be represented as an idempotent convolution of corresponding short horizon bivariate convolution kernels, i.e. via G_τ^μ of (32), (33) for a sufficiently small horizon $\tau \in (0, \bar{t}^\mu)$.

Theorem 4 *Given any $\mu \in (0, 1]$, $t \in \Omega^\mu \cap [\bar{t}^\mu, \infty)$, and $n_t \in \mathbb{N}$ sufficiently large such that $\tau \doteq t/n_t \in (0, \bar{t}^\mu)$, the long horizon extension $\widetilde{G}_t^\mu$ of G_τ^μ, see (72), (33), satisfies*

$$\widetilde{G}_t^\mu(\xi, \zeta) = \operatorname*{stat}_{\eta \in (\mathscr{X}_1)^{n_t - 1}} \left\{ G_\tau^\mu(\xi, \eta_1) + \sum_{k=2}^{n_t - 1} G_\tau^\mu(\eta_{k-1}, \eta_k) + G_\tau^\mu(\eta_{n_t - 1}, \zeta) \right\} \tag{92}$$

for all $\xi, \zeta \in \mathscr{X}_1$, in which $(\mathscr{X}_1)^{n_t-1}$ denotes the product space $\mathscr{X}_1 \times \cdots \times \mathscr{X}_1$, $n_t - 1$ times. Furthermore,

$$\begin{aligned} \widetilde{G}_t^\mu(\xi, \zeta) &= \operatorname*{stat}_{\eta \in \mathscr{X}_1} \left\{ \widetilde{G}_{k\,\tau}^\mu(\xi, \eta) + \widetilde{G}_{(n_t - k)\,\tau}^\mu(\eta, \zeta) \right\} \\ &= \widetilde{G}_{k\,\tau}^\mu(\xi, \eta_k^*) + \widetilde{G}_{(n_t-k)\,\tau}^\mu(\eta_k^*, \zeta), \quad k \in \mathbb{N}_{<n_t}, \end{aligned} \tag{93}$$

in which the stat is achieved at $\eta_k^* \in \mathscr{X}_1$, where

$$\eta_k^* = \eta_k^*(\xi, \zeta) \doteq -[\widetilde{\mathcal{P}}_{k\,\tau}^\mu + \widetilde{\mathcal{P}}_{(n_t-k)\,\tau}^\mu]^{-1}\, (\widetilde{\mathcal{Q}}_{k\,\tau}^\mu\, \xi + \widetilde{\mathcal{Q}}_{(n_t-k)\,\tau}^\mu\, \zeta),\ k \in \mathbb{N}_{<n_t}. \tag{94}$$

Proof Fix $\mu \in (0, 1]$, $t \in \Omega^\mu \cap [\bar{t}^\mu, \infty)$, and $n_t \in \mathbb{N}$, $\tau \in (0, \bar{t}^\mu)$ as per the theorem statement. By Lemma 8, $k\,\tau \in \Omega^\mu$ for all $k \in \mathbb{N}$. Hence, given $k \in [2, n_t] \cap \mathbb{N}$, applying Lemma 11 with $s \doteq (k-1)\,\tau$ and $\sigma \doteq \tau$ yields

$$\begin{aligned} \widetilde{G}_{k\,\tau}^\mu(\xi, \zeta) &= \operatorname*{stat}_{\eta_{k-1} \in \mathscr{X}_1} \{\widetilde{G}_{(k-1)\,\tau}^\mu(\xi, \eta_{k-1}) + \widetilde{G}_\tau^\mu(\eta_{k-1}, \zeta)\} \\ &= \operatorname*{stat}_{\eta_{k-1} \in \mathscr{X}_1} \{\widetilde{G}_{(k-1)\,\tau}^\mu(\xi, \eta_{k-1}) + G_\tau^\mu(\eta_{k-1}, \zeta)\} \\ &= \operatorname*{stat}_{\eta_{k-1}, \eta_{k-2} \in \mathscr{X}_1} \{\widetilde{G}_{(k-2)\,\tau}^\mu(\xi, \eta_{k-2}) + G_\tau^\mu(\eta_{k-2}, \eta_{k-1}) + G_\tau^\mu(\eta_{k-1}, \zeta)\}, \end{aligned}$$

which yields (92) by induction, for $k = n_t$. Alternatively, applying Lemma 11 with $s \doteq k\,\tau$, $\sigma \doteq (n_t - k)\,\tau$ for $k \in \mathbb{N}_{<n_t}$ subsequently yields (93), (94). □

In view of (31), (69), (72), (75), and Theorem 4, the optimal control problem with value function W_t^μ of (24), (31) may be relaxed to a stationary control problem with value function $\widetilde{W}_t^\mu : \mathscr{X}_1 \to \overline{\mathbb{R}}$ defined for $t \in \Omega^\mu$ by

$$\widetilde{W}_t^\mu(\xi) \doteq \operatorname*{stat}_{\zeta \in \mathscr{X}_1} \{\widetilde{G}_t^\mu(\xi, \zeta) + \psi(\zeta)\} \tag{95}$$

for all $\xi \in \mathscr{X}_1$. With $\psi = \psi_v$ as per (59), selecting $n_t \in \mathbb{N}$ as indicated in Theorem 4, and generalizing (61), note that the stat in (95) is achieved at

$$\zeta_\xi^* = -(\widetilde{\mathcal{P}}_t^\mu)^{-1}(\widetilde{\mathcal{Q}}_t^\mu\, \xi + \mathcal{E}_\mu^{-1}\, v)\,. \tag{96}$$

Theorem 4 and Lemma 7 subsequently imply the existence of a sequence of intermediate states $\eta_k^* = \eta_k^*(\xi, \zeta_\xi^*) \in \mathscr{X}_1$, defined for $k \in \mathbb{N}_{<n_t}$ by (94), and optimal controls $\hat{w}_k^* \in \mathscr{W}_1[0, \tau]$ and corresponding trajectories $\hat{\xi}_k^* \in C([0, \tau]; \mathscr{X}_1)$, defined via (68) and Lemma 7 for $k \in \mathbb{N}_{\le n_t}$, such that

$$\begin{aligned}
&\hat{w}_k^* \doteq \mathcal{I}_\mu^{\frac{1}{2}} \begin{cases} \hat{v}(\xi, \eta_1^*)\,, & k = 1\,, \\ \hat{v}(\eta_{k-1}^*, \eta_k^*)\,, & k \in [2, n_t - 1]\,, \\ \hat{v}(\eta_{n_t-1}^*, \zeta_\xi^*)\,, & k = n_t\,, \end{cases} \quad \hat{\xi}_k^* \doteq \begin{cases} \hat{\chi}(\xi, \eta_1^*)\,, & k = 1\,, \\ \hat{\chi}(\eta_{k-1}^*, \eta_k^*)\,, & k \in [2, n_t - 1]\,, \\ \hat{\chi}(\eta_{n_t-1}^*, \zeta_\xi^*)\,, & k = n_t\,, \end{cases} \\
&G_\tau^\mu([\hat{\xi}_k^*]_0, [\hat{\xi}_k^*]_\tau) = J_\tau^\mu[\delta^-(\cdot, [\hat{\xi}_k^*]_\tau)]([\hat{\xi}_k^*]_0, \hat{w}_k)\,,
\end{aligned} \tag{97}$$

for all $k \in \mathbb{N}_{\le n_t}$. These optimal controls and trajectories may be pieced together in time to yield a concatenated control $w^* \in \mathscr{W}[0, t]$ and corresponding trajectories $\xi^* \in C([0, t]; \mathscr{X}_1)$ and $\pi^* \in C([0, t]; \mathscr{X})$, defined on the longer horizon $[0, t]$ by

$$\begin{aligned}
&w_s^* \doteq \left[\hat{w}_{k_s}^*\right]_{s-(k_s-1)\,\tau}, \quad \xi_s^* \doteq \left[\hat{\xi}_{k_s}^*\right]_{s-(k_s-1)\,\tau}, \\
&k_s \doteq \min(n_t, \lfloor \tfrac{s}{\tau} \rfloor + 1)\,, \quad s \in [0, t]\,.
\end{aligned} \tag{98}$$

As every constituent short horizon control is optimal with respect to its corresponding payoff, see (97), each renders that payoff stationary. Furthermore, as the intermediate states (94) are selected to achieve stationarity in the idempotent convolution (92), see Theorem 4, the concatenated control $w^* \in \mathscr{W}_1[0, t]$ of (98) must render the longer horizon action $J_t^\mu[\psi_0](\xi, \cdot)$ of (17) stationary.

By construction, the concatenated trajectory (98) also satisfies the boundary conditions $\xi_0^* = \xi$ and $\xi_t^* = \zeta_\xi^*$ via (96). Fix any $k \in \mathbb{N}_{<n_t}$ and, in an abuse of notation in (94), let $\eta_0^* \doteq \xi$, $\eta_{n_t}^* \doteq \zeta_\xi^*$. Recalling Theorems 3 and 4, i.e. (33), (92), note by inspection that $\eta_{k-1}^*, \eta_k^*, \eta_{k+1}^*$ satisfy

$$0 = \nabla_\beta[G_\tau^\mu(\eta_{k-1}^*, \beta) + G_\tau^\mu(\beta, \eta_{k+1}^*)]_{\beta = \eta_k^*} = \widetilde{\mathcal{Q}}_\tau^\mu\, \eta_{k-1}^* + 2\, \widetilde{\mathcal{P}}_\tau^\mu\, \eta_k^* + \widetilde{\mathcal{Q}}_\tau^\mu\, \eta_{k+1}^*,$$

so that by (65), (67), and Lemma 6,

$$
\begin{aligned}
&- \mathcal{E}_\mu \, (\widetilde{\mathcal{Q}}_\tau^\mu \, \eta_{k-1}^* + \widetilde{\mathcal{P}}_\tau^\mu \, \eta_k^*) \\
&\quad = \mathcal{E}_\mu \, \widetilde{\mathcal{Q}}_\tau^\mu \, (\eta_{k+1}^* - [-(\widetilde{\mathcal{Q}}_\tau^\mu)^{-1} \, \widetilde{\mathcal{P}}_\tau^\mu] \, \eta_k^*) = [\mathcal{U}_\tau^\mu]_{12}^{-1} \, (\eta_{k+1}^* - [\mathcal{U}_\tau^\mu]_{11} \, \eta_k^*) \\
&\quad = \hat{\pi}_0(\eta_k^*, \eta_{k+1}^*) \, .
\end{aligned}
$$

With $\pi_s^* \doteq \mathcal{I}_\mu^{-\frac{1}{2}} \, w_s^*$ for all $s \in [0, t]$, subsequently observe by (68), (98) that

$$
\begin{aligned}
\mathcal{U}_\tau^\mu \begin{pmatrix} \eta_{k-1}^* \\ \hat{\pi}_0(\eta_{k-1}^*, \eta_k^*) \end{pmatrix} &= \mathcal{U}_\tau^\mu \begin{pmatrix} \eta_{k-1}^* \\ [\mathcal{U}_\tau^\mu]_{12}^{-1} \, (\eta_k^* - [\mathcal{U}_\tau^\mu]_{11} \, \eta_{k-1}^*) \end{pmatrix} \\
&= \begin{pmatrix} \eta_k^* \\ ([\mathcal{U}_\tau^\mu]_{21} - [\mathcal{U}_\tau^\mu]_{22} \, [\mathcal{U}_\tau^\mu]_{12}^{-1} \, [\mathcal{U}_\tau^\mu]_{11}) \, \eta_{k-1}^* + [\mathcal{U}_\tau^\mu]_{22} \, [\mathcal{U}_\tau^\mu]_{12}^{-1} \, \eta_k^* \end{pmatrix} \\
&= \begin{pmatrix} \eta_k^* \\ -\mathcal{E}_\mu \, (\widetilde{\mathcal{Q}}_\tau^\mu \, \eta_{k-1}^* + \widetilde{\mathcal{P}}_\tau^\mu \, \eta_k^*) \end{pmatrix} = \begin{pmatrix} \eta_k^* \\ \hat{\pi}_0(\eta_k^*, \eta_{k+1}^*) \end{pmatrix} .
\end{aligned}
$$

Hence, the concatenated trajectory (98) also describes a solution of the approximate wave equation (10) on the longer horizon, by Theorem 1. Moreover, the boundary conditions involved also follow via (94), with

$$
\eta_1^* = -[\widetilde{\mathcal{P}}_\tau^\mu + \widetilde{\mathcal{P}}_{(n_t-1)\tau}^\mu]^{-1} (\widetilde{\mathcal{Q}}_\tau^\mu \, \xi + \widetilde{\mathcal{Q}}_{(n_t-1)\tau}^\mu \, \zeta_\xi^*) \, ,
$$

so that by (67),

$$
\begin{aligned}
\pi_0 &\doteq \hat{\pi}_0(\eta_0^*, \eta_1^*) = \hat{\pi}_0(\xi, \eta_1^*) = [\mathcal{U}_\tau^\mu]_{12}^{-1} \, (\eta_1^* - [\mathcal{U}_\tau^\mu]_{11} \, \xi) \\
&= \mathcal{E}_\mu \, \widetilde{\mathcal{Q}}_\tau^\mu \left(-[\widetilde{\mathcal{P}}_\tau^\mu + \widetilde{\mathcal{P}}_{(n_t-1)\tau}^\mu]^{-1} (\widetilde{\mathcal{Q}}_\tau^\mu \, \xi + \widetilde{\mathcal{Q}}_{(n_t-1)\tau}^\mu \, \zeta_\xi^*) + (\widetilde{\mathcal{Q}}_\tau^\mu)^{-1} \, \widetilde{\mathcal{P}}_\tau^\mu \, \xi \right) \\
&= \mathcal{E}_\mu \, (\widetilde{\mathcal{P}}_\tau^\mu - \widetilde{\mathcal{Q}}_\tau^\mu \, [\widetilde{\mathcal{P}}_\tau^\mu + \widetilde{\mathcal{P}}_{(n_t-1)\tau}^\mu]^{-1} \, \widetilde{\mathcal{Q}}_\tau^\mu) \xi \\
&\quad - \mathcal{E}_\mu \, \widetilde{\mathcal{Q}}_\tau^\mu \, [\widetilde{\mathcal{P}}_\tau^\mu + \widetilde{\mathcal{P}}_{(n_t-1)\tau}^\mu]^{-1} \, \widetilde{\mathcal{Q}}_{(n_t-1)\tau}^\mu \, \zeta_\xi^* \qquad (99) \\
&= \mathcal{E}_\mu \, (\widetilde{\mathcal{P}}_t^\mu \, \xi + \widetilde{\mathcal{Q}}_t^\mu \, \zeta_\xi^*) = \mathcal{E}_\mu \, (\widetilde{\mathcal{P}}_t^\mu - \widetilde{\mathcal{Q}}_t^\mu \, (\widetilde{\mathcal{P}}_t^\mu)^{-1} \, \widetilde{\mathcal{Q}}_t^\mu) \, \xi - \mathcal{E}_\mu \, \widetilde{\mathcal{Q}}_t^\mu \, (\widetilde{\mathcal{P}}_t^\mu)^{-1} \, \mathcal{E}_\mu^{-1} \, \pi_t ,
\end{aligned}
$$

in which the second last and last equalities follow by Lemma 10 and (94), (96), and $\pi_t \doteq [\mathcal{U}_\tau^\mu]_{21} \, \eta_{n_t-1}^* + [\mathcal{U}_\tau^\mu]_{22} \, \hat{\pi}_0(\eta_{n_t-1}^*, \eta_{n_t}^*) = v$, following similar steps. By inspection, the obtained relationship between boundary conditions is of exactly the same form as (63).

In this way, the short horizon prototype $\widehat{\mathcal{U}}_t^\mu$ of (64), (65) extends to all longer horizons in Ω^μ, which is dense in $\mathbb{R}_{>0}$. Explicitly, the elements of the corresponding long horizon prototype $\{\widetilde{\mathcal{U}}_t^\mu\}_{t \in \Omega^\mu}$ are

$$\begin{pmatrix} \xi_t \\ \pi_t \end{pmatrix} = \widetilde{\mathcal{U}}_t^\mu \begin{pmatrix} \xi_0 \\ \pi_0 \end{pmatrix}, \quad \widetilde{\mathcal{U}}_t^\mu \doteq \left(\begin{array}{c|c} [\widetilde{\mathcal{U}}_t^\mu]_{11} & [\widetilde{\mathcal{U}}_t^\mu]_{12} \\ \hline [\widetilde{\mathcal{U}}_t^\mu]_{21} & [\widetilde{\mathcal{U}}_t^\mu]_{22} \end{array}\right), \quad t \in \Omega^\mu, \tag{100}$$

in which $[\widetilde{\mathcal{U}}_t^\mu]_{11} \in \mathcal{L}(\mathscr{X}_1)$, $[\widetilde{\mathcal{U}}_t^\mu]_{12} \in \mathcal{L}(\mathscr{X};\mathscr{X}_1)$, $[\widetilde{\mathcal{U}}_t^\mu]_{21} \in \mathcal{L}(\mathscr{X}_1;\mathscr{X})$, $[\widetilde{\mathcal{U}}_t^\mu]_{22} \in \mathcal{L}(\mathscr{X})$ are given by

$$\begin{aligned} &[\widetilde{\mathcal{U}}_t^\mu]_{11} \doteq -(\widetilde{\mathcal{Q}}_t^\mu)^{-1}\,\widetilde{\mathcal{P}}_t^\mu, && [\widetilde{\mathcal{U}}_t^\mu]_{12} \doteq (\widetilde{\mathcal{Q}}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}, \\ &[\widetilde{\mathcal{U}}_t^\mu]_{21} \doteq -\mathcal{E}_\mu\,\widetilde{\mathcal{Q}}_t^\mu\left(\mathcal{I} - [(\widetilde{\mathcal{Q}}_t^\mu)^{-1}\,\widetilde{\mathcal{P}}_t^\mu]^2\right), && [\widetilde{\mathcal{U}}_t^\mu]_{22} \doteq -\mathcal{E}_\mu\,\widetilde{\mathcal{P}}_t^\mu\,(\widetilde{\mathcal{Q}}_t^\mu)^{-1}\,\mathcal{E}_\mu^{-1}, \end{aligned} \tag{101}$$

for all $t \in \Omega^\mu$. Recalling (73), (74), the corresponding long horizon extension of Lemma 5 implies that these operators exhibit the spectral representation (5), with corresponding eigenvalues given by

$$\begin{aligned} &[[\tilde{u}_t^\mu]_{11}]_n \doteq -\frac{[\tilde{p}_t^\mu]_n}{[\tilde{q}_t^\mu]_n} = \cos(\omega_n^\mu\,t), && [[\tilde{u}_t^\mu]_{12}]_n \doteq \sin(\omega_n^\mu\,t), \\ &[[\tilde{u}_t^\mu]_{21}]_n \doteq -\sin(\omega_n^\mu\,t), && [[\tilde{u}_t^\mu]_{22}]_n \doteq \cos(\omega_n^\mu\,t). \end{aligned} \tag{102}$$

The prototype (101) is extended to negative horizons t, that is $-t \in \Omega^\mu$, via

$$\widetilde{\mathcal{U}}_t^\mu \doteq \widetilde{\mathcal{U}}_{-t}^\mu = \left(\begin{array}{c|c} [\widetilde{\mathcal{U}}_{-t}^\mu]_{11} & [\widetilde{\mathcal{U}}_{-t}^\mu]_{12} \\ \hline [\widetilde{\mathcal{U}}_{-t}^\mu]_{21} & [\widetilde{\mathcal{U}}_{-t}^\mu]_{22} \end{array}\right), \quad -t \in \Omega^\mu.$$

Theorem 5 *Given any $\mu \in (0,1]$, the set of prototypes $\{\widetilde{\mathcal{U}}_t^\mu\}_{t\in\Omega^\mu} \cup \{\mathcal{I}\} \cup \{\widetilde{\mathcal{U}}_{-t}^\mu\}_{t\in\Omega^\mu}$ populated via (100), (101) defines a uniformly continuous group, and is equivalent to the subgroup $\{\mathcal{U}_t^\mu\}_{t\in\Omega^\mu} \cup \{\mathcal{I}\} \cup \{\mathcal{U}_{-t}^\mu\}_{t\in\Omega^\mu}$ populated via (7), (8) that is generated by $\mathcal{A}^\mu$ of (4).*

Proof Immediate by comparison of (7), (8) with (100), (101), (102). □

5 Application to Solving a TPBVP

Given $\mu \in (0,1]$, $x, z \in \mathscr{X}_1$, and $t \in \Omega^\mu$, consider the TPBVPs defined with respect to (t, x, z) and the exact and approximate wave equations (1), (10) by

$$\text{(TPBVP)} \qquad \begin{cases} \text{Find } p_0 = \dot{x}_0 \in \mathscr{X} \text{ s.t. (1) holds} \\ \qquad \text{with } x_0 = x,\ x_t = z. \end{cases}$$

$$\text{(TPBVP-}\mu) \qquad \begin{cases} \text{Find } \pi_0^\mu = \mathcal{I}_\mu^{-\frac{1}{2}}\,\dot{\xi}_0^\mu \in \mathscr{X} \text{ s.t. (10) holds} \\ \qquad \text{with } \xi_0^\mu = x,\ \xi_t^\mu = z. \end{cases}$$

These TPBVPs are reminiscent of mass transport and Schrödinger bridge problems [1,2] insofar as solutions define a continuous evolution between elements of an infinite dimensional function space, via an underlying PDE. The associated groups $\{\mathcal{U}_t\}_{t\in\Omega^\mu}$, $\{\mathcal{U}_t^\mu\}_{t\in\Omega^\mu}$ generated by $\mathcal{A}$, $\mathcal{A}^\mu$ explicitly propagate solutions of (1), (10) for any initial data, see (4), (7), (11). These groups can be used to facilitate solution of (TPBVP) and (TPBVP-μ). In particular, the latter solution follows from (99), by setting $\xi = x$ and replacing the achieved terminal state ζ_ξ^* with the desired terminal state z, i.e.

$$\pi_0^\mu = \mathcal{E}_\mu\,(\widetilde{\mathcal{P}}_t^\mu\, x + \widetilde{\mathcal{Q}}_t^\mu\, z). \tag{103}$$

The spectral representation (73) further implies the equivalent form

$$\pi_0^\mu = \sum_{n=1}^{\infty} \frac{\lambda_n}{1+\mu^2\,\lambda_n} \left[[\tilde{p}_t^\mu]_n \,\langle x,\, \tilde{\varphi}_n\rangle_1 + [\tilde{q}_t^\mu]_n\, \langle z,\, \tilde{\varphi}_n\rangle_1 \right] \tilde{\varphi}_n\,, \tag{104}$$

in which λ_n^{-1} and $\tilde{\varphi}_n$ are the eigenvalues and eigenvectors of $\Lambda^{-1} \in \mathcal{L}(\mathscr{X})$. The solution of the approximate wave equation (10) subsequently follows by applying Theorem 1 and (11) to the initial conditions (x, π_0^μ). In particular,

$$\begin{pmatrix} \xi_s^\mu \\ \pi_s^\mu \end{pmatrix} = \widetilde{\mathcal{U}}_s^\mu \begin{pmatrix} x \\ \pi_0^\mu \end{pmatrix}, \quad s \in (0, t) \cap \Omega^\mu. \tag{105}$$

Together, (103), (104), (105) solve (TPBVP-μ), and so approximately solve (TPBVP). As an illustration, consider the specific problem defined with respect to (1), evolving in two spatial dimensions, by

$$\begin{aligned} &X \doteq [0,1]^2 \subset \mathbb{R}^2, \quad \mathscr{X} \doteq \mathscr{L}^2(X;\mathbb{R}), \\ &\Lambda \doteq -\partial_1^2 - \partial_2^2, \quad \mathrm{dom}\,(\Lambda) = \mathscr{X}_2 \doteq \mathscr{H}_0^2(X;\mathbb{R}), \end{aligned} \tag{106}$$

in which ∂_1 and ∂_2 denote partial derivatives with respect to the first and second spatial variables, and $-\Lambda$ is the Laplacian operator on X. As required, Λ is linear, unbounded, positive, self-adjoint, and possesses a compact inverse. The eigenvalues and eigenvectors of this compact inverse are $\lambda_{n,m}^{-1} \in \mathbb{R}_{>0}$ and $\tilde{\varphi}_{n,m} \in \mathscr{X}_1$, with $\lambda_{n,m} \doteq (n^2+m^2)\,\pi^2$ and $\tilde{\varphi}_{n,m}(x_1,x_2) \doteq (2/\sqrt{\lambda_{n,m}})\,\sin(n\,\pi\,x_1)\,\sin(m\,\pi\,x_2)$ for all $n, m \in \mathbb{N}$, $(x_1,x_2) \in X$. These eigenvalues and eigenvectors may be enumerated as per (5), yielding $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\tilde{\varphi}_n\}_{n\in\mathbb{N}}$, with the latter defining an orthonormal basis for $\mathscr{X}_1$. The corresponding eigenvalues $\{[\tilde{p}_t^\mu]_n\}_{n\in\mathbb{N}}$ and $\{[\tilde{q}_t^\mu]_n\}_{n\in\mathbb{N}}$ required for computation of π_0^μ using (104) follow from (6), (74).

For illustration, an initial state $x \in \mathscr{X}_1$ is chosen (arbitrarily) to be the zero function on X, while the terminal state $z \in \mathscr{X}_1$ is as per Fig. 1. With approximation parameter $\mu \doteq 10^{-3}$ fixed, a (long) horizon $t \doteq (\pi/3)^{\frac{1}{2}} \in \Omega^\mu$, $t \gg \bar{t}^\mu$, is selected so as to avoid finite escape times as per (70), (71). The solution $\pi_0^\mu \in \mathscr{X}$ of (TPBVP-μ) is given by (103), (104). For computational purposes, and without an error analysis, the sum involved is truncated to the first 400 terms. This yields an approximation of the solution

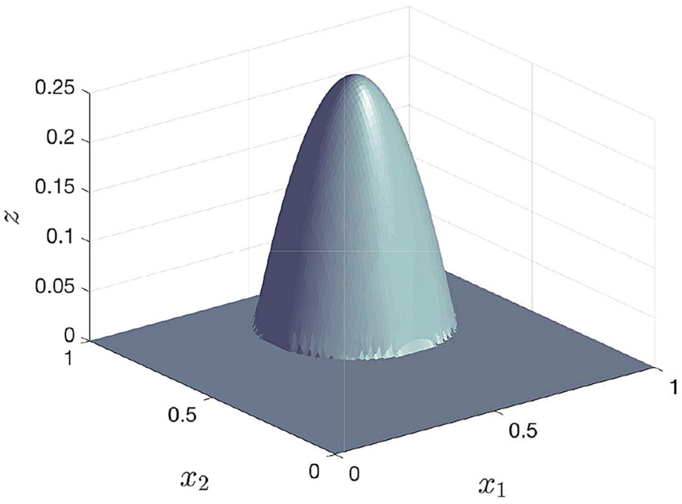

Fig. 1 Desired terminal state $z \in \mathscr{X}_1$ for all $(x_1, x_2) \in X$ for (TPBVP), (106)

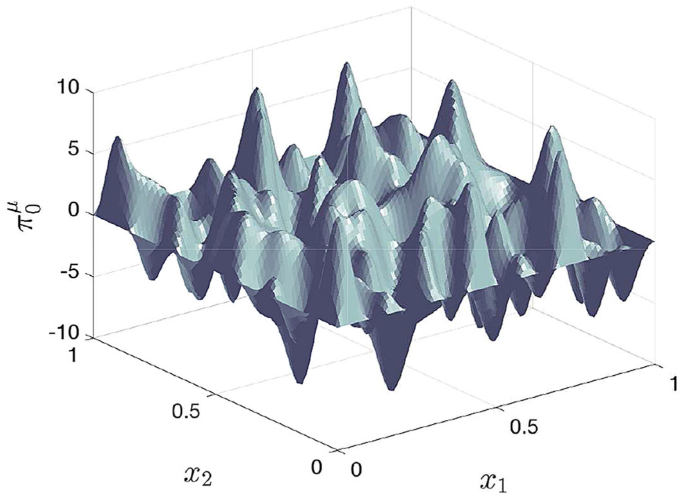

Fig. 2 Computed solution $\dot{\pi}_0^\mu \in \mathscr{X}_1$ of (104) for (TPBVP-μ), (106)

$\pi_0^\mu \in \mathscr{X}$ of (TPBVP-μ) that is illustrated in Fig. 2. By propagating (x, π_0^μ) forward in time, using the corresponding truncated representation of the exact group $\{\mathcal{U}_s\}_{s\in\mathbb{R}}$ via (11), it is observed in Fig. 3 that the desired terminal state z is approximately achieved. That is, (103), (104) provides an approximate solution to (TPBVP), (106).

All computations were performed using MATLAB (R2018a, 64-bit) on a MacBook Pro (2017, 2.9GHz Intel Core i7, MacOS High Sierra 10.13.6). The approximate solution was computed in 20 s, and all plots rendered in 6 s.

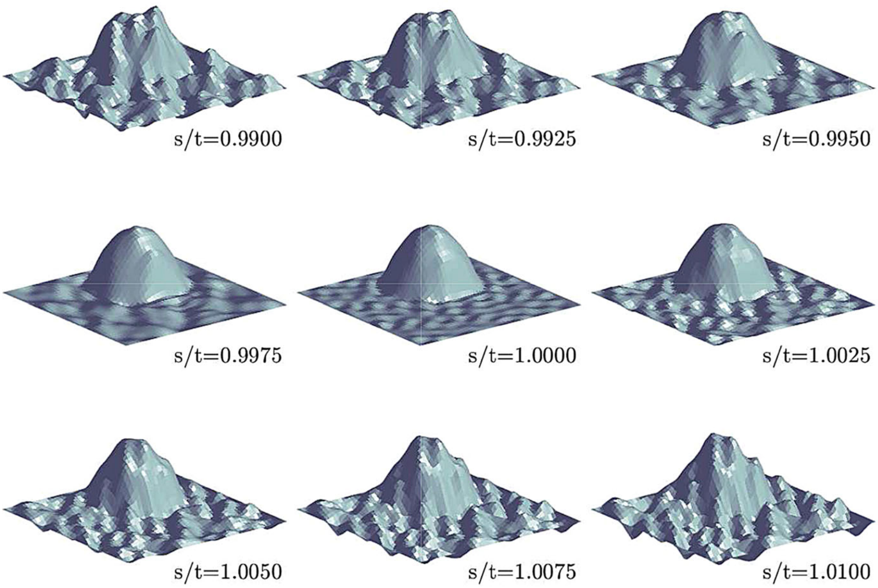

Fig. 3 Propagated solution of (1), from (x, π_0^μ) at $s = 0$ to $s/t \in [0.98, 1.02]$

6 Conclusions

A representation for the fundamental solution group for a class of wave equations is constructed via Hamilton's action principle and an optimal control problem. In particular, solutions of a wave equation in the class of interest are identified as rendering a corresponding action functional stationary. By encapsulating this action functional in an optimal control problem, these solutions are expressed as the corresponding optimal dynamics involved. By employing a idempotent convolution kernel to equivalently represent the value of the optimal control problem, a prototype of an approximation of the fundamental solution group involved is obtained. However, as the action functional loses concavity (in this case) for longer time horizons, the prototype fundamental solution group is restricted to short time horizons. This restriction is subsequently avoided via a relaxation of the optimal control problem to include stationary (rather than exclusively optimal) payoffs. The approximate fundamental solution group obtained, and its limit, are verified via the Trotter–Kato theorem to correspond to that of the class of approximating wave equations, and the exact wave equation respectively, of interest. They are applied in posing and solving a TPBVP.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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