Fundamental solutions for two-point boundary value problems in orbital mechanics

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Abstract

We consider a two-point boundary value problem (TPBVP) in orbital mechanics involving a small body (e.g., a spacecraft or asteroid) and N larger bodies. The least action principle TPBVP formulation is converted into an initial value problem via the addition of an appropriate terminal cost to the action functional. The latter formulation is used to obtain a fundamental solution, which may be used to solve the TPBVP for a variety of boundary conditions within a certain class. In particular, the method of convex duality allows one to interpret the least action principle as a differential game, where an opposing player maximizes over an indexed set of quadratics to yield the gravitational potential. In the case where the time duration is less than a specific bound, there exists a unique critical point for the resulting differential game, which yields the fundamental solution given in terms of the solutions of associated Riccati equations.

Keywords Least action, two-point boundary value problem, differential game, Hamilton-Jacobi, optimal control.

Mathematics Subject Classification 49N90, 49Lxx, 93C10,35G20, 35D40.

1 Introduction

We examine the motion of a single body under the influence of the gravitational potential generated by N other celestial bodies, where the mass of the first body is negligible relative to the masses of the other bodies, and we suppose that the N large bodies are on known trajectories. The single, small body follows a trajectory satisfying the principle of stationary action (cf., [3, 4]), where under certain conditions, the stationary-action trajectory coincides with the least-action trajectory. This allows such problems in dynamics to be posed, instead, in terms of optimal control problems with vastly simplified dynamics. Here, we are specifically interested in two-point boundary value problems (TPBVPs). From the solution of certain optimal control problems, we will obtain fundamental solutions for classes of TPBVPs.

In the case of a quadratic potential function, the control problem takes a linear-quadratic form. Although the gravitational potential is not quadratic, one may take a dynamic game approach, where an inner optimization problem is posed in a linear-quadratic form. In particular, the non-quadratic control problem is converted into a differential game where the minimizing, outer player controls the velocity, and the maximizing, inner player controls the potential energy term (cf., [1, 2]). It will be demonstrated that for the case where the time duration is less than a specified bound, the action functional is strictly convex in the velocity control. The action functional is naturally concave in the potential-energy control. We will find that because of the very special form of this problem, one can invert the order of minimum and maximum so that the maximizing player is the outer player For any potential-energy (outer player) control, the minimizing trajectory is the unique stationary point of a quadratic functional, and the least action is obtained by solution of associated Riccati equations. This leaves only a control problem for the outer player. We use

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a numerical method to maximize this concave function. As an aid in this maximization, we also obtain the derivative of the Riccati-equation solution with respect to the potential-energy control.

In Section 2, we define the orbital mechanics problem of interest, and develop its relevant least action principle. In Sections 3 and 4, the problem is reformulated into the aforementioned equivalent differential game in a linear-quadratic form. We provide a bound on the duration that guarantees the existence and uniqueness of the solution. Then, in next two sections, we examine two subproblems of the differential game, separately. Specifically, in Section 5, we demonstrate that the minimizing subproblem in the differential game is solved via associated Riccati equations, and in Section 6, the derivative of the solution of Riccaiti equations with respect to the maximizing player is examined. Further, an approximate subproblem is introduced, and corresponding first-order necessary conditions are obtained, where these are used in the numerical method. An error analysis is also provided. Lastly, in Section 7, an example is given.

2 Problem statement and Fundamental solution

We consider a small body, moving among a set of N other bodies in $I\!R^3$. The only forces to be considered are gravitational. The single body has negligible mass in relation to the masses of the other bodies, and consequently has no effect on their motion. In particular, we suppose that the N bodies are moving along already-known trajectories. We will obtain fundamental solutions of TPBVPs for the motion of the small body. Note that, for a problem involving dynamical systems, we use the term *fundamental solution* to indicate an object, which once obtained for a specific time-horizon, allows solution of the problem for varying input data by an operation on the object and given specific data that does not require re-propagation over time. (See [13] for further discussion.) The concept will become more clear further below.

The set of N bodies may be indexed as $\mathcal{N} \doteq]1, N[\doteq \{1, 2, \cdots, N\}$. Throughout, for integers $a \le b$, we will use $]a, b[$ to denote $\{a, a+1, \cdots, b-1, b\}$. We assume that the larger bodies are spherical with spherically symmetric densities. As for a given total mass, the specific radial density profiles of the bodies do not affect the resulting trajectories (for small-body paths not intersecting the larger bodies), we may, without loss of generality assume that the larger bodies each have uniform density. For $i \in \mathcal{N}$, let ρ_i and R_i denote the (uniform) density and radius of larger body i. Obviously, the mass of each body is given by $m_i = \frac{4}{3}\pi\rho_i R_i^3$.

Let $\zeta_r^i \doteq \zeta^i(r)$ denote the position of the center of body i at time $r \ge 0$. We suppose that $\zeta \doteq \{\zeta^i\}_{i\in\mathcal{N}} \in \widehat{\mathcal{Z}} \doteq \{\{\zeta^i\}_{i\in\mathcal{N}} \,|\, \zeta^i \in C([0,\infty); I\!R^3)\ \forall i \in \mathcal{N}\}$, where $\widehat{\mathcal{Z}}$ will be equipped with the usual (supremum) norm. Assuming that collision between bodies does not occur, we define the subset of $\widehat{\mathcal{Z}}$ given by $\mathcal{Z} \doteq \{\zeta \in \widehat{\mathcal{Z}} \,|\, |\zeta_r^i - \zeta_r^j| > R_i + R_j\ \ \forall r \ge 0, \forall i \ne j \in \mathcal{N}\}$.

For simplicity, the small body is considered as a point particle with mass $\bar{m}$. Suppose that the position of the small body at time r is denoted by ξ_r, where also, we will use $x \in I\!R^3$ to denote generic position values. We model the dynamics of the small body position as

$$\dot{\xi}_r = u_r, \quad \xi_0 = x, \tag{1}$$

where $u = u_\cdot \in \mathcal{U}^\infty \doteq \{u : [0,\infty) \to I\!R^3 \,|\, u_{[0,t)} \in L_2([0,t); I\!R^3)\ \forall t \in [0,\infty)\}$ where $u_{[0,t)}$ denotes the restriction of the function to domain $[0,t)$.

The kinetic energy, $\widehat{T}$, for generic velocity, v, is given by

$$\widehat{T}(v) \doteq \tfrac{1}{2}\bar{m}|v|^2 \quad \forall v \in I\!R^3.$$

Let $\mathcal{Y} \doteq \{\{y^i\}_{i\in\mathcal{N}} \,|\, y^i \in I\!R^3\ \forall i \in \mathcal{N}\}$. Given $i \in \mathcal{N}$ and $Y \in \mathcal{Y}$, the potential energy between the small body at x and body i at y^i, $\widehat{\widehat{V}}_i(x, y^i)$, is given by

$$-\widehat{\widehat{V}}_i(x, y^i) \doteq \begin{cases} Gm_i\bar{m}\frac{3R_i^2 - |x-y^i|^2}{2R_i^3} & \text{if } x \in B_{R_i}(y^i), \\ \frac{Gm_i\bar{m}}{|x-y^i|} & \text{if } x \notin B_{R_i}(y^i), \end{cases} \tag{2}$$

where G is the universal gravitational constant. We define the total potential energy $\widehat{\widehat{V}} : I\!R^3 \times \mathcal{Y} \to I\!R$ as $\widehat{\widehat{V}}(x, Y) \doteq \sum_{i\in\mathcal{N}} \widehat{\widehat{V}}_i(x, y^i)$. We remark that we include the gravitational potential here within the extended bodies as the finiteness and smoothness of the potential are relevant at technical points in the theory, in spite of the infeasibility of small body trajectories that pass through the other bodies.

We remind the reader that we will obtain fundamental solutions for the TPBVPs through a game-theoretic formulation. The game will appear through application of a generalization of convex duality to a control-problem formulation. With that in mind, we define the action functional $J^0 : [0,\infty) \times I\!\!R^3 \times \mathcal{U}^\infty \times \mathcal{Z} \to I\!\!R$ as

$$J^0(t,x,u,\zeta) \doteq \int_0^t T(u_r) - \overline{V}(\xi_r,\zeta_r)\,dr, \tag{3}$$

where

$$\overline{V} \doteq \widehat{\overline{V}}/\bar{m}, \ \ \overline{V}_i \doteq \widehat{\overline{V}_i}/\bar{m} \ \text{ and } \ T \doteq \widehat{T}/\bar{m}, \tag{4}$$

and ξ satisfies (1).

Adding a terminal cost to J^0 will yield a control problem equivalent to a TPBVP, where we can manipulate the terminal condition in the TPBVP by adjusting the terminal cost, and we will have initial condition $\xi_0 = x$. For background on this approach to TPBVPs for conservative systems, see [5, 13, 14]. Given generic terminal cost $\bar{\psi} : I\!\!R^3 \to I\!\!R$, let

$$\bar{J}(t,x,u,\zeta) \doteq J^0(t,x,u,\zeta) + \bar{\psi}(\xi_t), \qquad \overline{W}(t,x,\zeta) \doteq \inf_{u\in\mathcal{U}^\infty} \bar{J}(t,x,u,\zeta). \tag{5}$$

For the development of the fundamental solution, it is useful to introduce a terminal cost that takes the form of a min-plus delta-function. Let $\psi^\infty : I\!\!R^3 \times I\!\!R^3 \to [0,\infty]$ (where throughout we let $[0,\infty] \doteq [0,\infty) \cup \{+\infty\}$) be given by

$$\psi^\infty(y,z) = \delta^-(y-z) \doteq \begin{cases} 0 & \text{if } y = z, \\ \infty & \text{otherwise,} \end{cases}$$

where δ^- denotes the min-plus "delta-function" (cf., [12]). We define the finite time-horizon payoff $\bar{J}^\infty : [0,\infty) \times I\!\!R^3 \times \mathcal{U}^\infty \times \mathcal{Z} \times I\!\!R^3 \to I\!\!R \cup \{\infty\}$ by

$$\bar{J}^\infty(t,x,u,\zeta,z) \doteq J^0(t,x,u,\zeta) + \psi^\infty(\xi_t,z), \tag{6}$$

and the corresponding value function as

$$\overline{W}^\infty(t,x,\zeta,z) = \inf_{u\in\mathcal{U}^\infty} \bar{J}^\infty(t,x,u,\zeta,z), \tag{7}$$

where ξ satisfies (1). The proof of the following is nearly identical to that of Proposition 2.11 in [13], and so is not included.

Theorem 2.1. *For all $t \geq 0$, $x \in I\!\!R^3$ and $\zeta \in \mathcal{Z}$,*

$$\overline{W}^\infty(t,x,\zeta,z) = \inf_{u\in\mathcal{U}^\infty} \{ J^0(t,x,u,\zeta) \,|\, \xi_t = z \}, \quad \textit{and} \quad \overline{W}(t,x,\zeta) \doteq \inf_{z\in I\!\!R^3} \left\{ \overline{W}^\infty(t,x,\zeta,z) + \bar{\psi}(z) \right\}.$$

It is seen that the value function $\overline{W}$ of (5) for terminal cost $\bar{\psi}$ can be evaluated from $\overline{W}^\infty$, and consequently, $\overline{W}^\infty$ may be regarded as a fundamental solution.

3 Optimal control problem

The development in this section and the next is similar to that in [13, 14], where the n-body problem was considered. In this case the state-dimension is substantially reduced, but the presence of the large bodies on known trajectories leads to time-dependent input processes. Because of the time-dependent inputs the results of [13, 14] are not applicable. Nonetheless, the overall structure of the development is formally similar, and where reasonable to do so, the material is condensed.

We will find it helpful to define a value function $\overline{W}^c$ with quadratic terminal cost ψ^c, and demonstrate that the limit property, $\lim_{c\to\infty}\overline{W}^c = \overline{W}^\infty$ holds. For $c \in [0,\infty)$, let $\psi^c : I\!R^3 \times I\!R^3 \to [0,\infty)$ be given by

$$\psi^c(x,z) \doteq \frac{c}{2}|x-z|^2.$$

We define the finite time-horizon payoff, $\bar{J}^c : [0,\infty) \times I\!R^3 \times \mathcal{U}^\infty \times \mathcal{Z} \times I\!R^3 \to I\!R$, by

$$\bar{J}^c(t,x,u,\zeta,z) \doteq J^0(t,x,u,\zeta) + \psi^c(\xi_t,z), \tag{8}$$

where J^0 is given by (3), and corresponding value function,

$$\overline{W}^c(t,x,\zeta,z) \doteq \inf_{u\in\mathcal{U}^\infty} \bar{J}^c(t,x,u,\zeta,z). \tag{9}$$

Lemma 3.1. *The potential energy $\overline{V}(x,Y)$ is globally Lipschitz continuous in x, i.e., for any $Y \doteq \{y^i\}_{i\in\mathcal{N}} \in \mathcal{Y}$, there exists $K_L = K_L(\{m_i,R_i\}_{i\in\mathcal{N}}) < \infty$ such that*

$$|\overline{V}(x,Y) - \overline{V}(\hat{x},Y)| \le K_L|x-\hat{x}| \quad \forall x,\hat{x} \in I\!R^3. \tag{10}$$

Also, there exists $D_V = D_V(\{m_i,R_i\}_{i\in\mathcal{N}}) < \infty$ such that

$$0 < -\overline{V}(x,Y) \le D_V \quad \forall x \in I\!R^3,\ \forall Y \in \mathcal{Y}. \tag{11}$$

Proof. The second assertion is immediate from the definition of $\overline{V}_i$, and so we address only the first. Given $y^i \in I\!R^3$, in the cases where $x,\hat{x} \in B_{R_i}(y^i)$ and $x,\hat{x} \notin B_{R_i}(y^i)$, by (2), (4) and the mean value theorem, we have

$$|\overline{V}_i(x,y^i) - \overline{V}_i(\hat{x},y^i)| \le \frac{Gm_i}{R_i^2}|x-\hat{x}|. \tag{12}$$

Lastly, suppose without loss of generality that $x \notin B_{R_i}(y^i)$ and $\hat{x} \in B_{R_i}(y^i)$. Let $x^\lambda = x(\lambda) \doteq \lambda x + (1-\lambda)\hat{x}$ for $\lambda \in [0,1]$. Then, there exists $\dagger \in (0,1]$ such that $|x^\dagger - y^i| = R_i$. Note that for such $x^\dagger \in I\!R^3$, the potential is given by

$$-\overline{V}_i(x^\dagger,y^i) = Gm_i\frac{3R_i^2 - |x^\dagger - y^i|^2}{2R_i^3} = \frac{Gm_i}{|x^\dagger - y^i|}. \tag{13}$$

Therefore, by the triangle inequality,

$$|\overline{V}_i(x,y^i) - \overline{V}_i(\hat{x},y^i)| \ \le |\overline{V}_i(x,y^i) - \overline{V}_i(x^\dagger,y^i) + |\overline{V}_i(x^\dagger,y^i) - \overline{V}_i(\hat{x},y^i)|,$$

which by using the results of two previous cases with (13),

$$\le \frac{Gm_i}{R_i^2}\left[|x-x^\dagger| + |x^\dagger - \hat{x}|\right] = \frac{Gm_i}{R_i^2}|x-\hat{x}| \tag{14}$$

where the last equality is obtained since $x^\dagger$ is a point on the straight-line between x and $\hat{x}$. Given both (12) and (14), one has the first assertion. □

Theorem 3.2.

$$\overline{W}^\infty(t,x,\zeta,z) = \lim_{c\to\infty}\overline{W}^c(t,x,\zeta,z) = \sup_{c\in[0,\infty)}\overline{W}^c(t,x,\zeta,z).$$

where the convergence is uniform on $\bar{\mathcal{B}}_t \times \bar{\mathcal{B}} \times \mathcal{Z} \times \bar{\mathcal{B}}$ for any compact $\bar{\mathcal{B}}_t \subset [0,\infty)$ and compact $\bar{\mathcal{B}} \subset I\!R^3$.

Proof. Let $t > 0$. Suppose that given $x,z \in I\!R^3$, the "straight-line control" from x to z is given by $u_r^s \doteq (1/t)[z-x]$ for all $r \in [0,t]$, and we let the corresponding trajectory be denoted by ξ^s. Noting that $\xi_t^s = z$, for $c \in [0,\infty)$,

$$\bar{J}^c(t,x,u^s,\zeta,z) \ = J^0(t,x,u^s,\zeta),$$

which by the definition of u^s and Lemma 3.1

$$\leq \tfrac{1}{2t}|z-x|^2 + D_V t \leq D_1(1+|x|^2+|z|^2), \tag{15}$$

for an appropriate choice of $D_1 = D_1(t) < \infty$.

On the other hand, by definition, given $c \in (0,\infty)$ and $\varepsilon \in (0,1]$, there exists $u^{c,\varepsilon} \in \mathcal{U}^\infty$ such that

$$\bar{J}^c(t,x,u^{c,\varepsilon},\zeta,z) \leq \overline{W}^c(t,x,\zeta,z) + \varepsilon. \tag{16}$$

Let $\xi^{c,\varepsilon}$ be the trajectory corresponding to $u^{c,\varepsilon}$. By the non-negativity of T and $-\overline{V}$ and (16),

$$\tfrac{c}{2}|\xi_t^{c,\varepsilon} - z|^2 \ \leq \overline{W}^c(t,x,\zeta,z) + \varepsilon,$$

which by the suboptimality of u^s with respect to $\overline{W}^c$,

$$\leq \bar{J}^c(t,x,u^s,\zeta,z) + \varepsilon,$$

which by (15),

$$\leq D_1(1+|x|^2+|z|^2) + 1 \leq \tfrac{1}{2}[\tilde{D}(1+|x|+|z|)]^2, \tag{17}$$

for an appropriate choice of $\tilde{D} = \tilde{D}(t) < \infty$. This implies that

$$|\xi_t^{c,\varepsilon} - z| \leq \frac{\tilde{D}(1+|x|+|z|)}{\sqrt{c}}. \tag{18}$$

Let

$$\hat{u}_r^{c,\varepsilon} \doteq u_r^{c,\varepsilon} + \tfrac{1}{t}[z - \xi_t^{c,\varepsilon}], \quad \forall r \in [0,t], \tag{19}$$

which yields $\hat{\xi}_t^{c,\varepsilon} = z$ where $\hat{\xi}^{c,\varepsilon}$ denotes the trajectory corresponding to $\hat{u}^{c,\varepsilon}$. Then, by (18) and (19),

$$|\xi_r^{c,\varepsilon} - \hat{\xi}_r^{c,\varepsilon}| \leq \frac{1}{t}\int_0^r |z - \xi_t^{c,\varepsilon}|\, d\rho \leq \frac{r\tilde{D}(1+|x|+|z|)}{t\sqrt{c}} \tag{20}$$

for all $r \in [0,t]$. By (10) and (20),

$$\left|\int_0^t -\overline{V}(\xi_r^{c,\varepsilon},\zeta_r) + \overline{V}(\hat{\xi}_r^{c,\varepsilon},\zeta_r)\, dr\right| \leq K_L \int_0^t |\xi_r^{c,\varepsilon} - \hat{\xi}_r^{c,\varepsilon}|\, dr \leq \frac{K_L \tilde{D}(1+|x|+|z|)t}{2\sqrt{c}}. \tag{21}$$

Now, by (3), (8) and (16),

$$\int_0^t T(u_r^{c,\varepsilon}) - \overline{V}(\xi_r^{c,\varepsilon},\zeta_r)\, dr + \psi^c(\xi_t^{c,\varepsilon}, z) \leq \overline{W}^c(t,x,\zeta,z) + \varepsilon.$$

By the definition of T and the non-negativity of $-\overline{V}$ and ψ^c, this implies

$$\|u^{c,\varepsilon}\|_{L_2(0,t)}^2 \leq \sqrt{2\overline{W}^c(t,x,\zeta,z) + 2\varepsilon} \leq \tilde{D}(1+|x|+|z|), \tag{22}$$

where the last inequality follows by (17). Noting that $\big||a|^2 - |b|^2\big| < |a-b|\,[|a|+|b|]$ for $a,b \in I\!\!R^3$,

$$\left|\int_0^t T(u_r^{c,\varepsilon}) - T(\hat{u}_r^{c,\varepsilon})\, dr\right| \ \leq \frac{1}{2}\int_0^t |u_r^{c,\varepsilon} - \hat{u}_r^{c,\varepsilon}|\,[|u_r^{c,\varepsilon}| + |\hat{u}_r^{c,\varepsilon}|]\ dr,$$

which by (19) and the triangle inequality,

$$\leq \frac{1}{2t}|z - \xi_t^{c,\varepsilon}| \int_0^t 2|u_r^{c,\varepsilon}| + \frac{1}{t}|z - \xi_t^{c,\varepsilon}|\, dr,$$

which by applying Hölder's inequality, (18) and (22),

$$\leq \frac{1}{2t}|z - \xi_t^{c,\varepsilon}| \left[2\sqrt{t}\|u^{c,\varepsilon}\|_{L_2(0,t)} + |z - \xi_t^{c,\varepsilon}|\right] \leq \frac{\hat{D}(t)(1+|x|+|z|)^2}{\sqrt{c}}, \tag{23}$$

for all $x, z \in I\!R^3$ and all $c \in [1, \infty)$, for an appropriate choice of $\hat{D} = \hat{D}(t) < \infty$. Therefore, by (21), (23) and the non-negativity of ψ^c, we have

$$\bar{J}^c(t, x, u^{c,\varepsilon}, \zeta, z) - \bar{J}^c(t, x, \hat{u}^{c,\varepsilon}, \zeta, z) \geq -\frac{D_2(t)(1 + |x| + |z|)^2}{\sqrt{c}} \tag{24}$$

for proper choice of $D_2(t) < \infty$. The suboptimality of $\hat{u}^{c,\varepsilon}$ with respect to $\overline{W}^\infty$ combined with (24) yields

$$\overline{W}^\infty(t, x, \zeta, z) - \frac{D_2(t)(1 + |x| + |z|)^2}{\sqrt{c}} \leq \bar{J}^c(t, x, u^{c,\varepsilon}, \zeta, z) \leq \overline{W}^c(t, x, \zeta, z) + \varepsilon$$

where the last inequality follows by (16). Since this is true for all $\varepsilon \in (0, 1]$,

$$\overline{W}^c(t, x, \zeta, z) \geq \overline{W}^\infty(t, x, \zeta, z) - \frac{D_2(t)(1 + |x| + |z|)^2}{\sqrt{c}}. \tag{25}$$

Next, we examine the monotonicity of $\overline{W}^c$ with respect to c. Given $t > 0$; $x, z \in I\!R^3$; $u \in \mathcal{U}^\infty$ and $\zeta \in \mathcal{Z}$, note that for $c_1 \leq c_2 \leq \infty$, by the definitions of $\bar{J}^\infty$ of (6) and $\bar{J}^c$ of (8), $\bar{J}^{c_1}(t, x, u, \zeta, z) \leq \bar{J}^{c_2}(t, x, u, \zeta, z)$, which easily yields

$$\overline{W}^{c_1}(t, x, \zeta, z) \leq \overline{W}^{c_2}(t, x, \zeta, z) \quad \forall c_1 \leq c_2 \leq \infty.$$

Combining this with (25) implies

$$\overline{W}^\infty(t, x, \zeta, z) - \frac{D_2(t)(1 + |x| + |z|)^2}{\sqrt{c}} \leq \overline{W}^c(t, x, \zeta, z) \leq \overline{W}^\infty(t, x, \zeta, z)$$

for all $x, z \in I\!R^3$, $\zeta \in \mathcal{Z}$, $t > 0$ and $c \in [1, \infty)$. □

4 Differential game formulation

Recall that in the case where the potential energy does take a quadratic form, the fundamental solution may be obtained through the solution of associated differential Riccati equations (DREs) [13, 14]. In order to exploit that Riccati-solution form, we will take a duality-based approach to gravitation. That is, we will express the additive inverse of the gravitational potential as the pointwise maximum over an indexed set of quadratics. Extending this to time-dependent trajectories, the action functional will take the form of a max-plus integral, over potential-energy controls, of quadratic action functionals. We will find that the control problem is converted to a zero-sum differential game where the velocity controller is the minimizing player, and the potential-energy controller is an opposing, maximizing player. Although at first this may appear to lead to additional complications, the ability to exploit the DRE solution form yields significant benefits.

By [13], Lemma 4.1, (see also [14]),

$$-\overline{V}_i(x, y^i) = \sup_{\hat{\bar{\alpha}} \in (0, \sqrt{2/3} R_i^{-1}]} \mu_i \left[\hat{\bar{\alpha}} - \frac{\hat{\bar{\alpha}}^3}{2} |x - y^i|^2 \right] \tag{26}$$

for all $x, y^i \in I\!R^3$ such that $|x - y^i| \geq R_i$, where $\mu_i \doteq G m_i \left(\frac{3}{2}\right)^{3/2}$. Now, let $\bar{\bar{\alpha}} \doteq \sqrt{\frac{2}{3}} R_i^{-1}$ and $|x - y^i| \leq R_i$. Recalling (2), and performing a small calculation yields

$$-\overline{V}_i(x, y^i) = G m_i \frac{3R_i^2 - |x - y^i|^2}{2R_i^3} = \mu_i \Big[\bar{\bar{\alpha}} - \frac{\bar{\bar{\alpha}}^3}{2} |x - y^i|^2 \Big]. \tag{27}$$

Further,

$$\frac{d}{d\hat{\bar{\alpha}}} \Big\{ \hat{\bar{\alpha}} - \tfrac{1}{2} \hat{\bar{\alpha}}^3 |x - y^i|^2 \Big\} = 1 - \tfrac{3}{2} \hat{\bar{\alpha}}^2 |x - y^i|^2 > 0$$

for $\hat{\hat{\alpha}} < \sqrt{2/3}|x-y^i|^{-1}$. Consequently, noting $R_i^{-1} \le |x-y^i|^{-1}$, we see that $\mu_i[\hat{\hat{\alpha}} - \frac{1}{2}\hat{\hat{\alpha}}^3|x-y^i|^2]$ is monotonically increasing on $(0, \sqrt{2/3}R_i^{-1}]$. Therefore, using (27), we see that for $|x-y^i| \le R_i$, we also obtain (26). That is, we have the following.

Theorem 4.1. *Let $\mu_i \doteq Gm_i \left(\frac{3}{2}\right)^{3/2}$. For all $x, y^i \in I\!R^3$,*

$$-\overline{V}_i(x,y^i) = \max_{\hat{\hat{\alpha}} \in (0, \sqrt{2/3}R_i^{-1}]} \mu_i \left[\hat{\hat{\alpha}} - \frac{\hat{\hat{\alpha}}^3}{2}|x-y^i|^2\right].$$

4.1 Revisiting the payoff

Let

$$\mathcal{A} \doteq \{\tilde{\alpha} = \{\tilde{\alpha}^i\}_{i\in\mathcal{N}} \,|\, \tilde{\alpha}^i \in (0, \sqrt{2/3}R_i^{-1}] \quad \forall i \in \mathcal{N}\}.$$

Then, using Theorem 4.1, the potential energy $-\overline{V}$ may be represented by

$$-\overline{V}(x,Y) = -\sum_{i\in\mathcal{N}} \overline{V}_i(x,y^i) \doteq \max_{\tilde{\alpha}\in\mathcal{A}}\{-\widehat{V}(x,Y,\tilde{\alpha})\} \tag{28}$$

where

$$-\widehat{V}(x,Y,\tilde{\alpha}) \doteq \sum_{i\in\mathcal{N}} \mu_i \left[\tilde{\alpha}^i - \frac{(\tilde{\alpha}^i)^3}{2}|x-y^i|^2\right]. \tag{29}$$

Further, the payoff (8) may be written as

$$\bar{J}^c(t,x,u,\zeta,z) \;=\; \int_0^t T(u_r) + \max_{\tilde{\alpha}\in\mathcal{A}}\{-\widehat{V}(\xi_r,\zeta_r,\tilde{\alpha})\}\,dr + \psi^c(\xi_t,z). \tag{30}$$

Given $t > 0$, let

$$\tilde{\mathcal{A}}^t \doteq C([0,t];\mathcal{A}) \;\text{ and }\; \mathcal{A}^t \;\doteq L_\infty([0,t];\mathcal{A}). \tag{31}$$

Also, for $\alpha \in \mathcal{A}^t$, $r \in [0,t]$, $x \in I\!R^3$ and $Y \in \mathcal{Y}$, let

$$-V^\alpha(r,x,Y) \doteq -\widehat{V}(x,Y,\alpha_r) = \sum_{i\in\mathcal{N}} \mu_i \left[\alpha_r^i - \frac{(\alpha_r^i)^3}{2}|x-y^i|^2\right]. \tag{32}$$

Given $c \in [0,\infty]$, let $J^c : [0,\infty) \times I\!R^3 \times \mathcal{U}^\infty \times \mathcal{A}^t \times \mathcal{Z} \times I\!R^3 \to I\!R$ be given by

$$J^c(t,x,u,\alpha,\zeta,z) \;\doteq\; \int_0^t T(u_r) - V^\alpha(r,\xi_r,\zeta_r)\,dr + \psi^c(\xi_t,z) \doteq \check{J}^0(t,x,u,\alpha,\zeta) + \psi^c(\xi_t,z). \tag{33}$$

Let $\bar{\alpha}^* : I\!R^3 \times I\!R^{nN} \to \mathcal{A}$ be given by $\bar{\alpha}^*(x,Y) \doteq \{[\bar{\alpha}^*]^i(x,y^i)\}_{i\in\mathcal{N}}$ where

$$[\bar{\alpha}^*]^i(x,y^i) \doteq \operatorname*{argmax}_{\hat{\alpha}\in(0,\sqrt{2/3}R_i^{-1}]} \mu_i \left[\hat{\alpha} - \frac{\hat{\alpha}^3}{2}|x-y^i|^2\right] = \sqrt{2/3}\min\{R_i^{-1}, |x-y^i|^{-1}\} \tag{34}$$

for all $x, y^i \in I\!R^3$ and all $i \in \mathcal{N}$. Let

$$\alpha_r^* = \alpha^*(r;u_\cdot,\zeta_\cdot) = \{[\alpha_r^*]^i \,|\; i \in \mathcal{N}\}, \qquad [\alpha_r^*]^i = [\bar{\alpha}^*]^i(\xi_r,\zeta_r^i) \quad \forall r \in [0,t], \tag{35}$$

where $\xi_r = x + \int_0^r u_\rho\,d\rho$.

Theorem 4.2. *Let $t > 0$; $c \in [0, \infty)$; $x, z \in I\!R^3$; and $\zeta \in \mathcal{Z}$. For any $u \in \mathcal{U}^\infty$,*

$$\bar{J}^c(t, x, u, \zeta, z) = \max_{\alpha \in \mathcal{A}^t} J^c(t, x, u, \alpha, \zeta, z) = \max_{\alpha \in \tilde{\mathcal{A}}^t} J^c(t, x, u, \alpha, \zeta, z) = J^c(t, x, u, \alpha^*, \zeta, z), \tag{36}$$

where α^*, depending on u, is given by (35). Further,

$$\overline{W}^c(t, x, \zeta, z) = \inf_{u \in \mathcal{U}^\infty} \max_{\alpha \in \mathcal{A}^t} J^c(t, x, u, \alpha, \zeta, z) = \inf_{u \in \mathcal{U}^\infty} \max_{\alpha \in \tilde{\mathcal{A}}^t} J^c(t, x, u, \alpha, \zeta, z). \tag{37}$$

Proof. As (37) is immediate from (36), we need only prove the latter. Fix $t > 0$, $x, z \in I\!R^3$ and $\zeta \in \mathcal{Z}$. Given $u \in \mathcal{U}^\infty$, let ξ denote the state trajectory corresponding to u with $\xi_0 = x$. By (31) and (32), given any $\alpha \in \mathcal{A}^t$, α_r is suboptimal in the maximization in (30) at each $r \in [0, t]$, and in particular,

$$\bar{J}^c(t, x, u, \zeta, z) \geq \int_0^t T(u_r) - \widehat{V}(\xi_r, \zeta_r, \alpha_r)\, dr + \psi^c(\xi_t, z) = J^c(t, x, u, \alpha, \zeta, z).$$

As this is true for all $\alpha \in \mathcal{A}^t$,

$$\bar{J}^c(t, x, u, \zeta, z) \geq \max_{\alpha \in \mathcal{A}^t} J^c(t, x, u, \alpha, \zeta, z). \tag{38}$$

By (28), (32), (34) and (35),

$$-\overline{V}(\xi_r, \zeta_r) = -V^{\alpha^*}(r, \xi_r, \zeta_r) \quad \forall r \in [0, t],$$

and then by (30), (33), this implies

$$\bar{J}^c(t, x, u, \zeta, z) = J^c(t, x, u, \alpha^*, \zeta, z) \leq \sup_{\alpha \in \mathcal{A}^t} J^c(t, x, u, \alpha, \zeta, z). \tag{39}$$

Consequently, combining (38) and (39) yields the first equality of (36).

For $r \in [0, t]$ and $i \in \mathcal{N}$, let $d_r^i \doteq |\xi_r - \zeta_r^i|$. Fix $s \in [0, t]$ and $\varepsilon_d > 0$. By the continuity of ζ and ξ (the latter being guaranteed by $u \in \mathcal{U}^\infty$), there exists $\delta_d > 0$ such that for all $i \in \mathcal{N}$ and all $r \in (s - \delta_d, s + \delta_d) \cap [0, t]$, $|d_r^i - d_s^i| < \varepsilon_d$. Using (34) and (35), one easily sees that this implies $|[\alpha_r^*]^i - [\alpha_s^*]^i| < \varepsilon_d / R_i^2$ for all $r \in (s - \delta_d, s + \delta_d) \cap [0, t]$. Consequently, $\alpha^* \in \tilde{\mathcal{A}}^t$, which implies the second equality of (36). □

Note that the non-quadratic control problem has been converted into a differential game that has a linear-quadratic form of the potential energy.

4.2 Existence and uniqueness of optimal controls: $c < \infty$

We will see that $J^c(t, x, \cdot, \cdot, \zeta, z)$ is strictly convex-concave over the velocity and potential energy control sets within a certain time-horizon bound, and that there exists a unique minimax point over $\mathcal{U}^\infty \times \mathcal{A}^t$.

We first study the question of existence of optimal velocity controls by examining the smoothness, convexity and coercivity of the payoff. We will obtain a bound on the time duration which will be sufficient to guarantee the convexity and coercivity of the payoff J^c and the uniqueness of the stationary-action trajectory.

Let $t > 0$. We define a linear operator $B : L_2(0, t) \to L_2(0, t)$ as

$$[Bv](r) \doteq \int_0^r v_\rho \, d\rho \quad \forall r \in [0, t]. \tag{40}$$

Moreover,

$$\|Bv\|_{L_2(0,t)}^2 = \int_0^t \left| \int_0^r v_\rho \, d\rho \right|^2 dr \leq \int_0^t \left[\int_0^r |v_\rho| \, d\rho \right]^2 dr \leq \int_0^t r \, dr \|v\|_{L_2(0,t)}^2 = \frac{t^2}{2} \|v\|_{L_2(0,t)}^2, \tag{41}$$

where the last bound follows by applying Hölder's inequality to the inner integral.

Let $x, z \in I\!R^3$; $c \in [0, \infty)$; $\alpha \in \mathcal{A}^t$, and $\zeta \in \mathcal{Z}$. Let $u \in \mathcal{U}^\infty$ and ξ be the corresponding trajectory. Then, using (40), we may rewrite (32) as

$$\begin{aligned}\int_0^t -V^\alpha(r, \xi_r, \zeta_r)\, dr &= \sum_{i\in\mathcal{N}} \mu_i \int_0^t \alpha_r^i\, dr - \sum_{i\in\mathcal{N}} \frac{\mu_i}{2} \int_0^t (\alpha_r^i)^3 |\hat{x}_r + [Bu](r) - \zeta_r^i|^2\, dr \\ &\doteq S(\alpha) - \sum_{i\in\mathcal{N}} U^i(u, \alpha, \hat{x}, \zeta) \qquad (42)\end{aligned}$$

where $\hat{x}_r \doteq x$ for all $r \in [0, t]$. Letting

$$[B_\alpha^i u](r) \doteq \mu_i^{1/2} (\alpha_r^i)^{3/2} \int_0^r u_\rho\, d\rho = \mu_i^{1/2} (\alpha_r^i)^{3/2} [Bu](r),$$

we also note that

$$\langle B_\alpha^i u, B_\alpha^i u \rangle_{L_2(0,t)} = \int_0^t \mu_i (\alpha_r^i)^3 |[Bu](r)|^2\, dr,$$

which since $\alpha_r^i \in (0, \sqrt{2/3} R_i^{-1}]$ for all $r \in [0, t]$,

$$\leq \frac{G m_i}{R_i^3} \int_0^t |[Bu](r)|^2\, dr \leq \frac{G m_i}{2 R_i^3} t^2 \|u\|_{L_2(0,t)}^2, \qquad (43)$$

where the last inequality follows by (41). Then, for $i \in \mathcal{N}$, we have

$$\begin{aligned}U^i(u, \alpha, \hat{x}, \zeta) &= \frac{1}{2} \int_0^t \mu_i (\alpha_r^i)^3 |\hat{x}_r - \zeta_r^i|^2\, dr + \int_0^t \mu_i (\alpha_r^i)^3 (\hat{x}_r - \zeta_r^i) \cdot [Bu](r)\, dr + \frac{1}{2} \int_0^t \mu_i (\alpha_r^i)^3 |[Bu](r)|^2\, dr \\ &\doteq \tfrac{1}{2} \langle w^i, w^i \rangle_{L_2(0,t)} + \langle w^i, B_\alpha^i u \rangle_{L_2(0,t)} + \tfrac{1}{2} \langle B_\alpha^i u, B_\alpha^i u \rangle_{L_2(0,t)}\end{aligned}$$

where $w_r^i = w_r^i(\alpha, \hat{x}, \zeta) \doteq \mu_i^{1/2} (\alpha_r^i)^{3/2} (\hat{x}_r - \zeta_r^i)$ for all $i \in \mathcal{N}$ and $r \in [0, t]$. For $\nu \doteq \{\nu^i\}_{i\in\mathcal{N}}, \hat{\nu} \doteq \{\hat{\nu}^i\}_{i\in\mathcal{N}} \subset L_2(0, t)$, define the inner product (with associated norm)

$$\langle \nu, \hat{\nu} \rangle_{L_2(0,t)} \doteq \sum_{i\in\mathcal{N}} \langle \nu^i, \hat{\nu}^i \rangle_{L_2(0,t)}. \qquad (44)$$

Then, letting $w \doteq \{w^i\}_{i\in\mathcal{N}}$ and $B_\alpha u \doteq \{B_\alpha^i u\}_{i\in\mathcal{N}}$, we may rewrite (42) as

$$\int_0^t -V^\alpha(r, \xi_r, \zeta_r)\, dr = S(\alpha) - \tfrac{1}{2} \langle w, w \rangle_{L_2(0,t)} - \langle w, B_\alpha u \rangle_{L_2(0,t)} - \tfrac{1}{2} \langle B_\alpha u, B_\alpha u \rangle_{L_2(0,t)},$$

so that $\check{J}^0$ given in (33) may be rewritten as

$$\check{J}^0(t, x, u, \alpha, \zeta) = \tfrac{1}{2} \langle u, u \rangle_{L_2(0,t)} + \widehat{S}(\alpha) - \langle w, B_\alpha u \rangle_{L_2(0,t)} - \tfrac{1}{2} \langle B_\alpha u, B_\alpha u \rangle_{L_2(0,t)}, \qquad (45)$$

where $\widehat{S}(\alpha) = \widehat{S}(\alpha; \hat{x}, \zeta) \doteq S(\alpha) - \frac{1}{2} \langle w, w \rangle_{L_2(0,t)}$. Further, by (40),

$$\psi^c(\xi_t, z) = \tfrac{c}{2} |x - z + [Bu](t)|^2,$$

which by letting $\hat{z}_r \doteq z$ for all $r \in [0, t]$,

$$= \tfrac{c}{2} |x - z|^2 + \langle c(\hat{x} - \hat{z}), u \rangle_{L_2(0,t)} + \tfrac{c}{2} |[Bu](t)|^2. \qquad (46)$$

Theorem 4.3. *Let $t > 0$; $c \in [0, \infty)$; $x, z \in I\!R^3$; $\alpha \in \mathcal{A}^t$, and $\zeta \in \mathcal{Z}$. Then, $\check{J}^0(t, x, u, \alpha, \zeta)$ and $J^c(t, x, u, \alpha, \zeta, z)$ are Fréchet differentiable with respect to u.*

Proof. Let $u, v \in \mathcal{U}^\infty$. Then, by (33), (45) and (46),

$$\begin{aligned} &J^c(t,x,u+v,\alpha,\zeta,z) - J^c(t,x,u,\alpha,\zeta,z) \qquad (47)\\ &= \langle u,v\rangle_{L_2(0,t)} + \tfrac{1}{2}\langle v,v\rangle_{L_2(0,t)} - \langle w, B_\alpha v\rangle_{L_2(0,t)} - \langle B_\alpha u, B_\alpha v\rangle_{L_2(0,t)} - \tfrac{1}{2}\langle B_\alpha v, B_\alpha v\rangle_{L_2(0,t)}\\ &\quad + c\langle x + [Bu](t) - z, v\rangle_{L_2(0,t)} + \tfrac{c}{2}|B_v(t)|^2, \end{aligned}$$

which by letting B_α^* be the adjoint of B_α (cf., [9]),

$$= \langle c(\xi_t - z) - B_\alpha^* w + (I - B_\alpha^* B_\alpha)u, v\rangle_{L_2(0,t)} + \tfrac{1}{2}\left[\langle v,v\rangle_{L_2(0,t)} - \langle B_\alpha v, B_\alpha v\rangle_{L_2(0,t)} + c|[Bv](t)|^2\right];$$

where I denotes the identity operator. This implies that letting $DJ_\alpha^c(u) \doteq c(\xi_t - z) - B_\alpha^* w + (I - B_\alpha^* B_\alpha)u$,

$$\begin{aligned} &|J^c(t,x,u+v,\alpha,\zeta,z) - J^c(t,x,u,\alpha,\zeta,z) - \langle DJ_\alpha^c(u), v\rangle_{L_2(0,t)}|\\ &\quad \le \tfrac{1}{2}|\langle v,v\rangle_{L_2(0,t)} - \langle B_\alpha v, B_\alpha v\rangle_{L_2(0,t)} + c|[Bv](t)|^2| \le C_u\|v\|^2_{L_2(0,t)} \end{aligned}$$

for proper choice of $C_u = C_u(t,\bar{t}) < \infty$. Since this is true for all $u, v \in \mathcal{U}^\infty$, J^c is Fréchet differentiable with Fréchet derivative representation DJ_α^c. Similarly, using (45), one easily sees that $\check{J}^0$ is Fréchet differentiable with Fréchet derivative representation $D\check{J}_\alpha^0(u) \doteq (I - B_\alpha^* B_\alpha)u - B_\alpha^* w$. □

We also have the following, and do not include the obvious proof.

Lemma 4.4. *Let $t > 0$; $x, z \in I\!R^3$; $\zeta \in \mathcal{Z}$; $\alpha \in \mathcal{A}^t$, and $c \in [0,\infty)$. $J^0(t,x,\cdot,\zeta)$, $\check{J}^0(t,x,\cdot,\alpha,\zeta)$, $\bar{J}^c(t,x,\cdot,\zeta,z)$, and $J^c(t,x,\cdot,\alpha,\zeta,z)$ are continuous on $\mathcal{U}^\infty$.*

Theorem 4.5. *Let*

$$\bar{t} \doteq \left[\sum_{i\in\mathcal{N}} \frac{Gm_i}{2R_i^3}\right]^{-1/2}. \qquad (48)$$

Let $x, z \in I\!R^3$; $c \in [0,\infty)$, and $\zeta \in \mathcal{Z}$. If $t \in (0,\bar{t})$, then $\check{J}^0(t,x,u,\alpha,\zeta)$ and $J^c(t,x,u,\alpha,\zeta,z)$ are strictly convex quadratic and coercive in u for any $\alpha \in \mathcal{A}^t$.

Proof. Considering the quadratic terms in u in the definition (45) of $\check{J}^0$, by (43) and (44),

$$\tfrac{1}{2}\langle u,u\rangle_{L_2(0,t)} - \tfrac{1}{2}\langle B_\alpha u, B_\alpha u\rangle_{L_2(0,t)} \ge \frac{1}{2}\left[1 - \sum_{i\in\mathcal{N}} \frac{Gm_i}{2R_i^3} t^2\right]\|u\|^2_{L_2(0,t)} > 0 \qquad (49)$$

if $t < \bar{t} \doteq \left(\sum_{i\in\mathcal{N}} \frac{Gm_i}{2R_i^3}\right)^{-1/2}$. That is, $\check{J}^0(t,x,\cdot,\alpha,\zeta)$ is coercive and strictly convex if $t \in (0,\bar{t})$. Further, from (46), we note that $\psi^c(\xi_t, z)$ is convex quadratic in u. Consequently, the strict convexity and coercivity of $J^c(t,x,\cdot,\alpha,\zeta,z)$ are guaranteed for $t \in (0,\bar{t})$. □

Remark 4.6. The condition (48) in Theorem 4.5 may be overly restrictive. If one can assume a greater minimum distance from the bodies than their respective R_i, say $\delta_i > R_i$ for $i \in]1,N[$, then this could be relaxed, replacing the R_i with the δ_i. Also, we mention that the condition allows one to seek minima rather than stationary points; consideration of the stationary-point case is an area for future research.

Henceforth, throughout the paper, $\bar{t}$ is used to denote the time upper bound given by (48).

By Theorems 4.2 and 4.5, we have:

Corollary 4.7. *Let $x, z \in I\!R^3$; $c \in [0,\infty)$, and $\zeta \in \mathcal{Z}$. For $t \in (0,\bar{t})$, $\bar{J}^c(t,x,u,\zeta,z)$ is strictly convex in u.*

Lemma 4.8. *Let $x, z \in I\!R^3$; $c \in [0,\infty)$, and $\zeta \in \mathcal{Z}$. Then, for $t > 0$, $\bar{J}^c(t,x,\cdot,\zeta,z)$ is coercive in $\mathcal{U}^\infty$.*

Proof. For any $u \in \mathcal{U}^\infty$, by the non-negativity of $-\overline{V}$ and ψ^c, $\bar{J}^c(t,x,u,\zeta,z) \ge \frac{1}{2}\|u\|^2_{L_2(0,t)}$, which implies the coercivity of $\bar{J}^c(t,x,\cdot,\zeta,z)$. □

Combining Theorem 4.5, Corollary 4.7, Lemmas 4.4 and 4.8 immediately yields the following uniqueness property (cf., [8]).

Theorem 4.9. *Let $t < \bar{t}$; $x, z \in I\!R^3$; $c \in [0, \infty)$. Then, there exists a unique optimal velocity control in the definition (9) of $\overline{W}^c(t, x, \zeta, z)$. For any $\alpha \in \mathcal{A}^t$, there exists a unique optimal velocity control of $J^c(t, x, \cdot, \alpha, \zeta, z)$.*

Next we will examine the concavity of $J^c(t, x, u, \cdot, \zeta, z)$, which guarantees the existence of a unique potential energy control in the maximization in (36).

Lemma 4.10. *For all $t > 0$; $c \in [0, \infty)$; $x, z \in I\!R^3$; $\zeta \in \mathcal{Z}$ and $u \in \mathcal{U}^\infty$, $J^c(t, x, u, \alpha, \zeta, z)$ and $\check{J}^0(t, x, u, \alpha, \zeta)$ are strictly concave in α.*

Proof. Let $t > 0$; $c \in [0, \infty)$; $x, z \in I\!R^3$ and $\zeta \in \mathcal{Z}$. Given $\alpha \in \mathcal{A}^t$, let $\hat{\alpha} \in L_\infty([0, t]; I\!R^3)$ be such that $\alpha \pm \hat{\alpha} \in \mathcal{A}^t$ and $\delta \in [-1, 1]$. Then, for any $u \in \mathcal{U}^\infty$, using (29),

$$\begin{aligned}
&J^c(t, x, u, \alpha + \delta\hat{\alpha}, \zeta, z) + J^c(t, x, u, \alpha - \delta\hat{\alpha}, \zeta, z) - 2J^c(t, x, u, \alpha, \zeta, z) \\
&= \int_0^t -\widehat{V}(\xi_r, \zeta_r, \alpha_r + \delta\hat{\alpha}_r) - \widehat{V}(\xi_r, \zeta_r, \alpha_r - \delta\hat{\alpha}_r) + 2\widehat{V}(\xi_r, \zeta_r, \alpha_r)\, dr, \\
&= \int_0^t \sum_{i \in \mathcal{N}} \mu_i \big[-(\alpha_r^i + \delta\hat{\alpha}_r^i)^3 - (\alpha_r^i - \delta\hat{\alpha}_r^i)^3 + 2(\alpha_r^i)^3 \big] |\xi_r - \zeta_r^i|^2/2\, dr = -3\delta^2 \int_0^t \sum_{i \in \mathcal{N}} \mu_i \alpha_r^i (\hat{\alpha}_r^i)^2 |\xi_r - \zeta_r^i|^2\, dr,
\end{aligned}$$

which completes the proof. □

Theorem 4.11. *Let $t \in (0, \infty)$; $x, z \in I\!R^3$; $c \in [0, \infty)$ and $\zeta \in \mathcal{Z}$. Let $u^\dagger \in \mathcal{U}^\infty$, and let the corresponding trajectory be denoted by $\xi^\dagger$. Let $\alpha_r^* = \alpha^*(r; x, u^\dagger_\cdot, \zeta_\cdot) \doteq \bar{\alpha}^*(\xi_r^\dagger, \zeta_r)$ for all $r \in [0, t]$ where $\bar{\alpha}^*$ is given in (34). Then, $u^\dagger$ is a stationary point of $\bar{J}^c(t, x, \cdot, \zeta, z)$ if and only if $u^\dagger$ is a stationary point of $J^c(t, x, \cdot, \alpha^*, \zeta, z)$.*

Proof. Let $\nu \in \mathcal{U}^\infty$ and $\delta > 0$. Letting $\xi^{\dagger,\nu}$ denote the trajectory corresponding to $u^\dagger + \delta\nu$. We examine differences in the direction ν from $u^\dagger$. Recall from (28) that $-\overline{V}(x, Y) = \max_{\tilde{\alpha} \in \mathcal{A}} \{-\widehat{V}(x, Y, \tilde{\alpha})\}$ where the maximum is uniquely attained at $\bar{\alpha}^*(x, Y)$. Consequently,

$$-\nabla_x \overline{V}(x, Y) = -\nabla_x \widehat{V}(x, Y, \bar{\alpha}^*(x, Y)),$$

and with this, the first-order difference in the potential-energy term is

$$\begin{aligned}
&-\widehat{V}(\xi_r^{\dagger,\nu}, \zeta_r, \alpha_r^*) + \widehat{V}(\xi_r^\dagger, \zeta_r, \alpha_r^*) = -\delta\nabla_x \widehat{V}(\xi_r^\dagger, \zeta_r, \alpha_r^*) \cdot (\xi_r^{\dagger,\nu} - \xi_r^\dagger) + \mathcal{O}(\delta^2) \\
&= -\delta\nabla_x \widehat{V}(\xi_r^\dagger, \zeta_r, \bar{\alpha}_r^*(\xi_r^\dagger, \zeta_r)) \cdot (\xi_r^{\dagger,\nu} - \xi_r^\dagger) + \mathcal{O}(\delta^2) = -\delta\nabla_x \overline{V}(\xi_r^\dagger, \zeta_r) \cdot (\xi_r^{\dagger,\nu} - \xi_r^\dagger) + \mathcal{O}(\delta^2) \\
&= -\overline{V}(\xi_r^{\dagger,\nu}, \zeta_r) + \overline{V}(\xi_r^\dagger, \zeta_r) + \mathcal{O}(\delta^2).
\end{aligned} \tag{50}$$

Now,

$$\begin{aligned}
&J^c(t, x, u^\dagger + \delta\nu, \alpha^*, \zeta, z) - J^c(t, x, u^\dagger, \alpha^*, \zeta, z) \\
&= \int_0^t T(u_r^\dagger + \delta\nu_r) - T(u_r^\dagger) - \widehat{V}(\xi_r^{\dagger,\nu}, \zeta_r, \alpha_r^*) + \widehat{V}(\xi_r^\dagger, \zeta_r, \alpha_r^*)\, dr + \psi^c(\xi_t^{\dagger,\nu}, z) - \psi^c(\xi_t^\dagger, z),
\end{aligned}$$

which by (50),

$$= \bar{J}^c(t, x, u^\dagger + \delta\nu, \zeta, z) - \bar{J}^c(t, x, u^\dagger, \zeta, z) + \mathcal{O}(\delta^2). \tag{51}$$

By (51), we have the desired result. □

Theorem 4.12. *Given $t < \bar{t}$; $x, z \in I\!R^3$; $\zeta \in \mathcal{Z}$ and $c \in [0, \infty)$, let $u^{c,*}$ be the unique minimizer of $\bar{J}^c(t, x, \cdot, \zeta, z)$ over $\mathcal{U}^\infty$, and $\xi^{c,*}$ be the corresponding trajectory. Let $\alpha_r^* \doteq \bar{\alpha}^*(\xi_r^{c,*}, \zeta_r)$ for all $r \in [0, t]$ where $\bar{\alpha}^*$ is given in (34). Then,*

$$\begin{aligned}
\overline{W}^c(t, x, \zeta, z) &= J^c(t, x, u^{c,*}, \alpha^*, \zeta, z) = \min_{u \in \mathcal{U}^\infty} \max_{\alpha \in \mathcal{A}^t} J^c(t, x, u, \alpha, \zeta, z) \\
&= \max_{\alpha \in \mathcal{A}^t} \min_{u \in \mathcal{U}^\infty} J^c(t, x, u, \alpha, \zeta, z) \doteq \max_{\alpha \in \mathcal{A}^t} \mathcal{W}^{\alpha,c}(t, x, \zeta, z).
\end{aligned} \tag{52}$$

Further,

$$u^{c,*} = \operatorname*{argmin}_{u\in\mathcal{U}^\infty} \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) \quad \textit{and} \quad \alpha^* = \operatorname*{argmax}_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,\zeta,z). \tag{53}$$

Proof. Let t, x, z, ζ, c be as indicated in the theorem statement. Note that $\min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,\zeta,z)$ and $\max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z)$ exist for all $u \in \mathcal{U}^\infty$ and $\alpha \in \mathcal{A}^t$ by Theorems 4.2 and 4.9 and Lemma 4.10. By Lemma 4.4 and Theorem 4.11, $u^{c,*}$ is a stationary point of $J^c(t,x,\cdot,\alpha^*,\zeta,z)$. Further, by the uniqueness given in Theorem 4.9, $u^{c,*} = \operatorname{argmin}_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha^*,\zeta,z)$. Also, note that by Theorem 4.5, $J^c(t,x,\cdot,\alpha,\zeta,z)$ is strictly convex in u for all $\alpha \in \mathcal{A}^t$, which implies that $\max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z)$ is strictly convex in u, which yields the first assertion of (53). Similarly, using Lemma 4.10, one obtains the second assertion of (53).

By Lemma 4.10 and Theorem 4.2, $\alpha^* = \operatorname{argmax}_{\alpha\in\mathcal{A}^t} J^c(t,x,u^{c,*},\alpha,\zeta,z)$, which implies that

$$\max_{\alpha\in\mathcal{A}^t} J^c(t,x,u^{c,*},\alpha,\zeta,z) = J^c(t,x,u^{c,*},\alpha^*,\zeta,z) = \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha^*,\zeta,z),$$

which implies

$$\min_{u\in\mathcal{U}^\infty} \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) \le J^c(t,x,u^{c,*},\alpha^*,\zeta,z) \le \max_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,\zeta,z). \tag{54}$$

Also, by the usual reordering inequality, one has

$$\max_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,\zeta,z) \le \min_{u\in\mathcal{U}^\infty} \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z). \tag{55}$$

Combining (54) and (55), one has

$$\min_{u\in\mathcal{U}^\infty} \max_{\alpha\in\mathcal{A}^t} J^c(t,x,u,\alpha,\zeta,z) = J^c(t,x,u^{c,*},\alpha^*,\zeta,z) = \max_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\alpha,\zeta,z).$$

Combining this with (36) completes the proof. □

4.3 Existence and uniqueness of optimal controls: $c = \infty$

Remark 4.13. Recall that the TPBVP corresponds to the case $c = \infty$. Given $x, z \in \mathbb{R}^3$ and $t > 0$, let $\widetilde{\mathcal{U}}^\infty_{t,x,z} \doteq \{u \in \mathcal{U}^\infty \mid \int_0^t u_r\,dr = z - x\}$. By Theorem 2.1,

$$\overline{W}^\infty(t,x,\zeta,z) \;=\; \inf_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} J^0(t,x,u,\zeta).$$

Also, by Lemma 4.4 and Theorem 4.5, $J^0(t,x,u,\zeta)$ is continuous, coercive and strictly convex in $u \in \widetilde{\mathcal{U}}^\infty_{t,x,z}$ if $t < \bar{t}$, which implies that there exists unique optimal velocity control $u^* \in \widetilde{\mathcal{U}}^\infty_{t,x,z} \subset \mathcal{U}^\infty$ in the definition (7) of $\overline{W}^\infty(t,x,\zeta,z)$ (cf., [8]), i.e.,

$$u^* = \operatorname*{argmin}_{u\in\mathcal{U}^\infty} \bar{J}^\infty(t,x,u,\zeta,z) = \operatorname*{argmin}_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \bar{J}^\infty(t,x,u,\zeta,z), \tag{56}$$

where the corresponding trajectory, ξ^*, is the solution of the TPBVP. In an analogous fashion to the proof of Theorem 4.11, u^* is a stationary point of $\check{J}^0(t,x,\cdot,\check{\alpha}^*,\zeta)$ over $\widetilde{\mathcal{U}}^\infty_{t,x,z}$ where $\check{\alpha}^*_r \doteq \bar{\alpha}^*(\xi^*_r,\zeta_r)$ for all $r \in [0,t]$.

Further, we may represent the fundamental solution in terms of $\mathcal{W}^{\alpha,c}$ with a limit property.

Theorem 4.14. *Let $t < \bar{t}$. For $u \in \mathcal{U}^\infty$; $\alpha \in \mathcal{A}^t$; $x, z \in \mathbb{R}^3$, and $\zeta \in \mathcal{Z}$, let*

$$J^\infty(t,x,u,\alpha,\zeta,z) \doteq \check{J}^0(t,x,u,\alpha,\zeta) + \psi^\infty(\xi_t,z). \tag{57}$$

Then,

$$\overline{W}^\infty(t,x,\zeta,z) = \max_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^\infty(t,x,u,\alpha,\zeta,z) \doteq \max_{\alpha\in\mathcal{A}^t} \mathcal{W}^{\alpha,\infty}(t,x,\zeta,z). \tag{58}$$

Further, given $\alpha \in \mathcal{A}^t$,

$$\mathcal{W}^{\alpha,\infty}(t,x,\zeta,z) = \lim_{c\to\infty} \mathcal{W}^{\alpha,c}(t,x,\zeta,z) = \sup_{c>0} \mathcal{W}^{\alpha,c}(t,x,\zeta,z)$$

where the convergence is uniform on $\bar{\mathcal{B}} \times \mathcal{Z} \times \bar{\mathcal{B}}$ for any compact $\bar{\mathcal{B}} \subset I\!R^3$.

Proof. Let $u^* \in \widetilde{\mathcal{U}}^\infty_{t,x,z}$ be as per (56), and $\xi^*_\cdot$ be the corresponding trajectory. Let $\check{\alpha}^*_r \doteq \bar{\alpha}^*(\xi^*_r, \zeta_r)$ for all $r \in [0,t]$. Then, by the definitions of $\widetilde{\mathcal{U}}^\infty_{t,x,z}$ and J^∞,

$$\inf_{u\in\mathcal{U}^\infty} J^\infty(t,x,u,\check{\alpha}^*,\zeta,z) = \inf_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \check{J}^0(t,x,u,\check{\alpha}^*,\zeta),$$

which by Theorem 4.5 and Lemma 4.4,

$$= \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \check{J}^0(t,x,u,\check{\alpha}^*,\zeta) = \check{J}^0(t,x,u^*,\check{\alpha}^*,\zeta). \tag{59}$$

where the last equality follows by Remark 4.13. Noting that the terminal state $\xi_t = z$ corresponding to all $u \in \widetilde{\mathcal{U}}^\infty_{t,x,z}$, by Theorem 4.2, we may have

$$J^0(t,x,u,\zeta) = \max_{\alpha\in\mathcal{A}^t} \check{J}^0(t,x,u,\alpha,\zeta) = \check{J}^0(t,x,u,\alpha^*,\zeta) \quad \forall u \in \widetilde{\mathcal{U}}^\infty_{t,x,z} \tag{60}$$

where $\alpha^*_r \doteq \bar{\alpha}^*(\xi_r,\zeta_r)$ for $r \in [0,t]$ and $\xi_\cdot$ denotes the trajectory corresponding to u. Noting $\xi^*_t = z$, by (57), (59) and (60),

$$\max_{\alpha\in\mathcal{A}^t} J^\infty(t,x,u^*,\alpha,\zeta,z) = J^\infty(t,x,u^*,\check{\alpha}^*,\zeta,z) = \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} J^\infty(t,x,u,\check{\alpha}^*,\zeta,z),$$

which yields in an analogous fashion to the proof of Theorem 4.12 that

$$\min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \max_{\alpha\in\mathcal{A}^t} J^\infty(t,x,u,\alpha,\zeta,z) = \max_{\alpha\in\mathcal{A}^t} \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} J^\infty(t,x,u,\alpha,\zeta,z), \tag{61}$$

and $(u^*, \check{\alpha}^*)$ is the unique solution of (61) over $\widetilde{\mathcal{U}}^\infty_{t,x,z} \times \mathcal{A}^t$. Consequently, by Remark 4.13 and (60),

$$\overline{W}^\infty(t,x,\zeta,z) = \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} J^0(t,x,u,\zeta) = \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \max_{\alpha\in\mathcal{A}^t} \check{J}^0(t,x,u,\alpha,\zeta),$$

which by the definitions of $\widetilde{\mathcal{U}}^\infty_{t,x,z}$ and J^∞ and (61),

$$= \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} \max_{\alpha\in\mathcal{A}^t} J^\infty(t,x,u,\alpha,\zeta,z), = \max_{\alpha\in\mathcal{A}^t} \min_{u\in\widetilde{\mathcal{U}}^\infty_{t,x,z}} J^\infty(t,x,u,\alpha,\zeta,z),$$

which since $J^\infty(t,x,u,\alpha,\zeta,z) = \infty$ for all $u \in \mathcal{U}^\infty \backslash \widetilde{\mathcal{U}}^\infty_{t,x,z}$ and any $\alpha \in \mathcal{A}^t$,

$$= \max_{\alpha\in\mathcal{A}^t} \min_{u\in\mathcal{U}^\infty} J^\infty(t,x,u,\alpha,\zeta,z),$$

which completes the first assertion.

Regarding the last assertion, similar to Theorem 3.2, we have monotonicity of $\mathcal{W}^{\alpha,c}$ in c, and we do not include the analogous proof. □

Recall from Theorem 3.2 that as $c \to \infty$, $\overline{W}^c(t,x,\zeta,z)$ approaches $\overline{W}^\infty(t,x,\zeta,z)$. The following lemma shows that the optimal velocity controls for $c < \infty$ also approach those for $c = \infty$ as $c \to \infty$

Theorem 4.15. *Let $t < \bar{t}$ and $c \in [0,\infty)$. Let $x, z \in I\!R^3$ and $\zeta \in \mathcal{Z}$. Let u^* and $u^{c,*}$ be the least action points in the definitions (7) of $\overline{W}^\infty(t,x,\zeta,z)$ and (9) of $\overline{W}^c(t,x,\zeta,z)$, respectively. Then, there exists $\check{D} = \check{D}(t,\bar{t}) < \infty$ such that*

$$\|u^* - u^{c,*}\|^2_{L_2(0,t)} \le \frac{\check{D}(1+|x|+|z|)^2}{\sqrt{c}}.$$

Proof. Let $u^*, u^{c,*}$ be as per the statement. Let $\xi^{c,*}$ be the trajectory corresponding to $u^{c,*}$, and $\alpha_r^* \doteq \bar{\alpha}^*(\xi_r^{c,*}, \zeta_r)$ for all $r \in [0,t]$ where $\bar{\alpha}^*$ is given in (34). Then, by (24),

$$\frac{D_2(t)(1+|x|+|z|)^2}{\sqrt{c}} \geq \bar{J}^\infty(t,x,u^*,\zeta,z) - \bar{J}^c(t,x,u^{c,*},\zeta,z),$$

which since $\xi_t^* = z$,

$$= \bar{J}^c(t,x,u^*,\zeta,z) - \bar{J}^c(t,x,u^{c,*},\zeta,z),$$

which by the suboptimality and optimality of α^* with respect to $J^c(t,x,u^*,\alpha,\zeta,z)$ and $J^c(t,x,u^{c,*},\alpha,\zeta,z)$, respectively,

$$\geq J^c(t,x,u^*,\alpha^*,\zeta,z) - J^c(t,x,u^{c,*},\alpha^*,\zeta,z),$$

which by (48) and the optimality of $u^{c,*}$ with respect to $J^c(t,x,u,\alpha^*,\zeta,z)$,

$$\geq \tfrac{1}{2}\left[\langle u^* - u^{c,*}, u^* - u^{c,*}\rangle_{L_2(0,t)} - \langle B_{\alpha^*}u^* - u^{c,*}, B_{\alpha^*}u^* - u^{c,*}\rangle_{L_2(0,t)}\right],$$

which by (49),

$$\geq \tfrac{1}{2}(1-(t/\bar{t})^2)\|u^* - u^{c,*}\|^2_{L_2(0,t)},$$

which completes the proof. □

4.4 Hamilton-Jacobi-Bellman PDE

The Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) problem associated with our (ζ,z,α)-indexed control problem is

$$\begin{aligned} 0 &= -\tfrac{\partial}{\partial r}W(r,x,\zeta,z) + \inf_{v\in I\!R^3}\{\tfrac{1}{2}|v|^2 - V^\alpha(t-r,x,\zeta_{t-r}) + v^T\nabla_x W(r,x,\zeta,z)\} \\ &\doteq -\tfrac{\partial}{\partial r}W(r,x,\zeta,z) - \inf_{v\in I\!R^3} H^\alpha(t-r,x,v,\zeta,\nabla_x W(r,x,\zeta,z)) \quad \forall (r,x)\in(0,t)\times I\!R^3, \qquad (62)\\ W(0,x,\zeta,z) &= \psi^c(x,z) \quad \forall x \in I\!R^3, \qquad (63)\end{aligned}$$

where $\nabla_x W$ represents the gradient with respect to the space variable. For $t>0$, let

$$\mathcal{D}_t \doteq C([0,t]\times I\!R^3)\cap C^1((0,t)\times I\!R^3).$$

Suppose that $W \in \mathcal{D}_t$ satisfies (62) and (63). Since

$$\tfrac{1}{2}|v|^2 + v^T\nabla_x W(r,x,\zeta,z) \geq \tfrac{1}{2}|v|^2 - |v||\nabla_x W(r,x,\zeta,z)|,$$

the coercivity and convexity of the Hamiltonian imply that

$$-\inf_{v\in I\!R^3} H^\alpha(t-r,x,v,\zeta,\nabla_x W(r,x,\zeta,z)) = -\min_{v\in I\!R^3} H^\alpha(t-r,x,v,\zeta,\nabla_x W(r,x,\zeta,z)),$$

which since H^α is quadratic in v,

$$= -H^\alpha(t-r,x,v^*,\zeta,\nabla_x W(r,x,\zeta,z)) \qquad (64)$$

where $v^* \doteq -\nabla_x W(r,x,\zeta,z)$.

Theorem 4.16. *Let $t>0$; $c\in[0,\infty)$; $x,z\in I\!R^3$; $\zeta\in\mathcal{Z}$ and $\alpha\in\mathcal{A}^t$. Suppose that $W(\cdot,\cdot,\zeta,z)\in\mathcal{D}_t$ satisfies (62) and (63), and $\nabla_x W(t,\cdot,\zeta,z)$ is globally Lipschitz in x. Then, $W(t,x,\zeta,z) = J^c(t,x,\check{u}^{c,*},\alpha,\zeta,z)$ for the input $\check{u}_r^{c,*} = \tilde{u}(r,\tilde{\xi}_r)$ with $\tilde{\xi}_r$ given by (1) with $\tilde{u}(r,x) = -\nabla_x W(t-r,x,\zeta,z)$ and $\tilde{\xi}_0 = x$. Consequently, $W(t,x,\zeta,z) = \mathcal{W}^{\alpha,c}(t,x,\zeta,z)$.*

Proof. Let W and $\tilde{u}$ be as asserted. Let $\check{u}_r^{c,*} \doteq \tilde{u}(r,\tilde{\xi}_r)$ for all $r\in[0,t]$, and $\check{\xi}^{c,*}$ be the corresponding trajectory. Then, by (64), we may rewrite (62) as

$$\begin{aligned} 0 &= -\tfrac{\partial}{\partial r}W(r,\check{\xi}^{c,*}_{t-r},\zeta,z) - H^\alpha(t-r,\check{\xi}^{c,*}_{t-r},\check{u}^{c,*}_{t-r},\zeta,\nabla_x W(r,\check{\xi}^{c,*}_{t-r},\zeta,z)) \\ &= -\tfrac{\partial}{\partial r}W(r,\check{\xi}^{c,*}_{t-r},\zeta,z) + \nabla_x W(r,\check{\xi}^{c,*}_{t-r},\zeta,z)\cdot\check{u}^{c,*}_{t-r} + \tfrac{1}{2}|\check{u}^{c,*}_{t-r}|^2 - V^\alpha(t-r,\check{\xi}^{c,*}_{t-r},\zeta_{t-r}) \\ &= -\tfrac{d}{dr}W(r,\check{\xi}^{c,*}_{t-r},\zeta,z) + T(\check{u}^*_{t-r}) - V^\alpha(t-r,\check{\xi}^{c,*}_{t-r},\zeta_{t-r}).\end{aligned}$$

Integrating with respect to r over $[0,t]$ yields

$$0 = W(0, \check{\xi}_t^{c,*}, \zeta, z) - W(t, x, \zeta, z) + \int_0^t T(\check{u}_{t-r}^{c,*}) - V^\alpha(t-r, \check{\xi}_{t-r}^{c,*}, \zeta_{t-r})\, dr,$$

or equivalently, by (63) and letting $s = t - r$,

$$W(t,x,\zeta,z) \; = \int_0^t T(\check{u}_s^{c,*}) - V^\alpha(s, \check{\xi}_r^{c,*}, \zeta_s)\, ds + \psi^c(\check{\xi}_t^{c,*}, z) = J^c(t, x, \check{u}^{c,*}, \alpha, \zeta, z),$$

which by Theorem 4.9,

$$= \mathcal{W}^{\alpha,c}(t,x,\zeta,z),$$

which completes the proof. □

5 The fundamental solution in terms of Riccati equation solutions

Given $c \in [0,\infty)$, $r \le t$, $\alpha \in \mathcal{A}^t$, and $\zeta \in \mathcal{Z}$, we look for a solution, $\breve{\mathcal{W}}^{\alpha,c}$, of the form

$$\breve{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) \doteq \tfrac{1}{2}\left[\, p_r^c\, x\cdot x + 2q_r^c\, x\cdot z + r_r^c\, z\cdot z + 2h_r^c\cdot x + 2l_r^c\cdot z + \gamma_r^c \,\right] \tag{65}$$

where $p_\cdot^c$, $q_\cdot^c$, $r_\cdot^c$, $h_\cdot^c$, $l_\cdot^c$, and $\gamma_\cdot^c$ depend implicitly on given α and ζ, and satisfy the respective initial value problems:

$$\begin{aligned}
&\dot{p_r^c} = -[p_r^c]^2 - \textstyle\sum_{i\in\mathcal{N}} \mu_i [\alpha_{t-r}^i]^3, && p_0^c = c,\\
&\dot{q_r^c} = -p_r^c q_r^c, && q_0^c = -c,\\
&\dot{r_r^c} = -[q_r^c]^2, && r_0^c = c,\\
&\dot{h_r^c} = -p_r^c h_r^c + \textstyle\sum_{i\in\mathcal{N}} \mu_i [\alpha_{t-r}^i]^3 \zeta_{t-r}^i, && h_0^c = 0_{n\times 1},\\
&\dot{l_r^c} = -q_r^c h_r^c, && l_0^c = 0_{n\times 1},\\
&\dot{\gamma_r^c} = -[p_r^c]^2 + \textstyle\sum_{i\in\mathcal{N}} \mu_i \left\{ 2\alpha_{t-r}^i - (\alpha_{t-r}^i)^3 |\zeta_{t-r}^i|^2 \right\}, && \gamma_0^c = 0,
\end{aligned} \tag{66}$$

where $0_{m\times k}$ denotes the zero matrix of size $m \times k$.

Lemma 5.1. *Let $t < \bar{t}$. Then, for any $\alpha \in \mathcal{A}^t$ and any $c \in [0,\infty)$, the solution of (66) exists on $[0,t)$.*

Proof. Let $\alpha \in \mathcal{A}^t$ and $c \in [0,\infty)$. Note that since $\dot{p}_r^c < 0$ for all $r \in [0,t]$,

$$p_r^c \le p_0^c = c \quad \forall r \in [0,t]. \tag{67}$$

From (48), we have $\sum_{i\in\mathcal{N}} \mu_i [\alpha_r^i]^3 \le 2\bar{t}^{-2}$, and then $\dot{p}_r^c \ge -[p_r^c]^2 - 2\bar{t}^{-2}$ for all $r \in [0,t]$. Consider

$$\dot{\hat{p}}_r^c = -[\hat{p}_r^c]^2 - 2\bar{t}^{-2}, \quad \hat{p}_0^c = c. \tag{68}$$

Then, $\dot{p}_\cdot^c \ge \dot{\hat{p}}_\cdot^c$, which implies that

$$p_r^c \ge \hat{p}_r^c \quad \forall r \in [0,t]. \tag{69}$$

The analytical solution of (68) is given by $\hat{p}_r^c = -\check{t}^{-1}\tan\big(\check{t}^{-1}(\hat{c}_1 + r)\big)$ where $\check{t} \doteq \bar{t}/\sqrt{2}$ and $\hat{c}_1 \doteq \check{t}\tan^{-1}(-c\check{t})$. Since $\tan^{-1}(-c\check{t}) \in (-\pi/2, 0)$ and $t < \bar{t}$, we see that $\check{t}^{-1}(\hat{c}_1 + r) = \tan^{-1}(-c\check{t}) + r\check{t}^{-1} < \sqrt{2} < \frac{\pi}{2}$ for all $r \in [0,t]$, which implies that there exists $\underline{\hat{D}}_p < \infty$ such that

$$\hat{p}_r^c \ge -\underline{\hat{D}}_p \quad \forall r \in [0,t]. \tag{70}$$

Combining (67), (69) and (70), there exist $\hat{D}_p^0, \hat{D}_p^1 < \infty$ such that $|p_r^c| < \hat{D}_p^0$, $|\dot{p}_r^c| < \hat{D}_p^1$ for all $r \in [0,t]$, which implies by Picard-Lindelöf theorem (cf. [6]), that there exists a unique solution. The existence and uniqueness of the remaining initial value problems follows easily. □

Theorem 5.2. *Let $t \in [0, \bar{t})$ and $c \in [0, \infty)$. Then, $\mathcal{W}^{\alpha,c}(r, x, \zeta, z) = \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z)$ for all $x, z \in \mathbb{R}^3, \zeta \in \mathcal{Z}$, and $r \in [0, t]$.*

Proof. It will be sufficient to show that $\breve{\mathcal{W}}^{\alpha,c}$ satisfies the conditions of Theorem 4.16. Note first that Lemma 5.1 implies $\breve{\mathcal{W}}^{\alpha,c}(\cdot, \cdot, \zeta, z) \in \mathcal{D}_t$. Also by Lemma 5.1, there exists $D_p < \infty$ such that $|p_r^c| < D_p$ for all $r \in [0, t]$. For $x, \hat{x} \in \mathbb{R}^3$, note that

$$|\nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z) - \nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, \hat{x}, \zeta, z)| \leq |p_r^c||x - \hat{x}| \leq D_p |x - \hat{x}|,$$

which implies that $\nabla_x \mathcal{W}^{\alpha,c}(t, \cdot, \zeta, z)$ is globally Lipschitz continuous in x.
Let $\mathcal{P} \doteq \mathbb{R}^{10}$. We define $\widehat{P}_r^c \in \mathcal{P}$ as

$$\widehat{P}_r^c \doteq (p_r^c, q_r^c, r_r^c, (h_r^c)', (l_r^c)', \gamma_r^c)', \tag{71}$$

and accordingly, $\widehat{C} \doteq \widehat{P}_0^c$. For $x, z \in \mathbb{R}^3$, we define $X : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathcal{P}$ as

$$X(x, z) \doteq (x \cdot x, 2x \cdot z, z \cdot z, 2x', 2z', 1)'. \tag{72}$$

Suppose that $\hat{\rho}, \hat{\eta} \in \mathcal{P}$ are given by

$$\hat{\rho} \doteq (\rho_1, \rho_2, \rho_3, \bar{\rho}_4', \bar{\rho}_5', \rho_6)', \quad \hat{\eta} \doteq (\eta_1, \eta_2, \eta_3, \bar{\eta}_4', \bar{\eta}_5', \eta_6)' \tag{73}$$

where $\rho_j, \eta_j \in \mathbb{R}$ for $j \in \{1, 2, 3, 6\}$ and $\bar{\rho}_j, \bar{\eta}_j \in \mathbb{R}^3$ for $j \in \{4, 5\}$. We define $f : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ as

$$f(\hat{\rho}, \hat{\eta}) \doteq -(\rho_1 \eta_1, \rho_1 \eta_2, \rho_2 \eta_2, \rho_1 \bar{\eta}_4', \rho_2 \bar{\eta}_4', \bar{\rho}_4 \cdot \bar{\eta}_4)', \tag{74}$$

and for $\tilde{\alpha} = \{\tilde{\alpha}^i\}_{i \in \mathcal{N}} \in \mathcal{A}$ and $Y \doteq \{y^i\}_{i \in \mathcal{N}} \in \mathcal{Y}$, define $\Gamma : \mathcal{A} \times \mathcal{Y} \to \mathcal{P}$ as

$$\Gamma(\tilde{\alpha}, Y) \doteq \sum_{i \in \mathcal{N}} \mu_i \left(-(\tilde{\alpha}^i)^3, 0, 0, (\tilde{\alpha}^i)^3 (y^i)', 0_{1 \times n}, \Upsilon_i \right)' \tag{75}$$

where $\Upsilon_i \doteq \{2\tilde{\alpha}^i - [\tilde{\alpha}^i]^3 |y^i|^2\}$. Then, we may rewrite (65) as

$$\breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z) = \tfrac{1}{2} X(x, z) \cdot \widehat{P}_r^c, \tag{76}$$

and note that (66) is equivalent to

$$\dot{\widehat{P}}_r^c = f(\widehat{P}_r^c, \widehat{P}_r^c) + \Gamma(\alpha_{t-r}, \zeta_{t-r}) \text{ with } \widehat{P}_0^c = \widehat{C}. \tag{77}$$

Now, from (76), we note that

$$\tfrac{\partial}{\partial r} \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z) = \tfrac{1}{2} X(x, z) \cdot \dot{\widehat{P}}_r^c, \qquad \nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z) = \tfrac{1}{2} \nabla_x X(x, z) \cdot \widehat{P}_r^c, \tag{78}$$

and with a bit of work, one may verify that

$$-|\nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z)|^2 = X(x, z) \cdot f(\widehat{P}_r^c, \widehat{P}_r^c). \tag{79}$$

Also, collecting like terms, we have

$$\begin{aligned} -V^\alpha(t - r, x, \zeta_{t-r}) &= \frac{1}{2} \sum_{i \in \mathcal{N}} \mu_i \left[-(\alpha_{t-r}^i)^3 x \cdot x + x \cdot (\alpha_{t-r}^i)^3 [\zeta_{t-r}^i] + \left\{ 2\alpha_{t-r}^i - (\alpha_{t-r}^i)^3 |\zeta_{t-r}^i|^2 \right\} \right], \\ &= \tfrac{1}{2} X(x, z) \cdot \Gamma(\alpha_{t-r}, \zeta_{t-r}), \end{aligned} \tag{80}$$

where the last equality follows by (72) and (75). Consequently, substituting (78) – (80) in the right-hand side of the PDE (62) yields

$$\begin{aligned} 0 &= -\tfrac{\partial}{\partial r} \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z) - H^\alpha(t - r, x, -\nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z), \zeta, \nabla_x \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z)) \\ &= \tfrac{1}{2} X(x, z) \cdot \left[-\dot{\widehat{P}}_r^c + f(\widehat{P}_r^c, \widehat{P}_r^c) + \Gamma(\alpha_{t-r}, \zeta_{t-r}) \right], \end{aligned}$$

which implies (65) is a solution of HJB PDE (62), and by Theorems 4.5 and 4.16, $\mathcal{W}^{\alpha,c}(r, x, \zeta, z) = \breve{\mathcal{W}}^{\alpha,c}(r, x, \zeta, z)$ for all $r \in [0, t]$, with $t \in [0, \bar{t})$. ∎

Recall from Theorems 3.2 and 4.14 that the fundamental solution of interest is obtained through the $c \to \infty$ limit of $\mathcal{W}^{\alpha,c}$. Consequently, we have that for $t < \bar{t}$, by Theorems 4.14 and 5.2,

$$\overline{W}^{\infty}(t,x,\zeta,z) = \sup_{\alpha \in \mathcal{A}^t} \lim_{c \to \infty} \tfrac{1}{2} X(x,z) \cdot \widehat{P}^c_t(\alpha,\zeta) \doteq \sup_{\alpha \in \mathcal{A}^t} \tfrac{1}{2} X(x,z) \cdot \widehat{P}^{\infty}_t(\alpha,\zeta). \tag{81}$$

Letting $\mathcal{G}_t \doteq \{\widehat{P}^{\infty}_t(\alpha,\zeta) \,|\, \alpha \in \mathcal{A}^t\}$, the fundamental solution (81) can be represented as $\overline{W}^{\infty}(t,x,\zeta,z) = \sup_{P \in \mathcal{G}_t} \frac{1}{2} X(x,z) \cdot P$. Also note that by the linearity in P of the expression inside this supremum,

$$\overline{W}^{\infty}(t,x,\zeta,z) = \sup_{P \in \langle \mathcal{G}_t \rangle} \tfrac{1}{2} X(x,z) \cdot P = \sup_{P \in \partial \langle \mathcal{G}_t \rangle} \tfrac{1}{2} X(x,z) \cdot P, \tag{82}$$

where $\langle \mathcal{G}_t \rangle$ denotes the convex hull of $\mathcal{G}_t$, and $\partial \langle \mathcal{G}_t \rangle$ denotes the boundary of $\langle \mathcal{G}_t \rangle$. Further, by Theorem 4.14, there exists $P^* \in \partial \langle \mathcal{G}_t \rangle$ such that

$$P^* = \operatorname*{argmax}_{P \in \langle \mathcal{G}_t \rangle} \tfrac{1}{2} X(x,z) \cdot P = \operatorname*{argmax}_{P \in \partial \langle \mathcal{G}_t \rangle} \tfrac{1}{2} X(x,z) \cdot P.$$

6 The maximization over α

Recall that by Lemma 4.10, $J^c(t,x,u,\cdot,\zeta,z)$ is strictly concave, and consequently, by definition (52), so is $\mathcal{W}^{\cdot,c}(t,x,\zeta,z)$. Combining this with Theorem 4.12, we see that there exists a unique maximizer. Further, by Theorem 5.2, $\mathcal{W}^{\alpha,c} = \check{\mathcal{W}}^{\alpha,c}$. Using this last representation, in Corollary 6.4 below, we will see that $\mathcal{W}^{\cdot,c}(t,x,\zeta,z)$ is Fréchet differentiable. Consequently, in searching for the unique maximum, we may utilize algorithms which require knowledge of the derivative, through first-order necessary conditions. We next obtain such first-order conditions and, in particular, a useful Riesz representation for the derivative.

Recall from (52) and Theorem 5.2 that

$$\overline{W}^{c}(t,x,\zeta,z) = \max_{\alpha \in \mathcal{A}^t} \check{\mathcal{W}}^{\alpha,c}(r,x,\zeta,z) = \max_{\alpha \in \mathcal{A}^t} \tfrac{1}{2} X(x,z) \cdot \widehat{P}^c_t(\alpha,\zeta).$$

Then, letting

$$\hat{\hat{\alpha}}^{c,*} \doteq \operatorname*{argmax}_{\alpha \in \mathcal{A}^t} \tfrac{1}{2} X(x,z) \cdot \widehat{P}^c_t(\alpha,\zeta),$$

by Theorems 4.12 and 5.2, $\hat{\hat{\alpha}}^{c,*} \equiv \bar{\alpha}^*(\xi^{c,*},\zeta)$ where $\xi^{c,*}$ and $\bar{\alpha}^*$ are indicated in the statement of Theorem 4.12. Further, recall from Theorem 4.14 that the fundamental solution of (58) has a unique solution. Further, by Theorem 4.15, we may see that the solution of $\overline{W}^{c}$ uniformly converges to that of the fundamental solution as $c \to \infty$. That is, letting $\check{\alpha}^*$ be the maximizing solution of J^{∞} of (58), we know from (34) that for proper choice of $\check{D}_\alpha = \check{D}_\alpha(t,\bar{t},\{R_i\}_{i \in \mathcal{N}}) < \infty$,

$$\|\hat{\hat{\alpha}}^{c,*} - \check{\alpha}^*\|^2_{L_2(0,t)} \le \check{D}_\alpha \|u^{c,*} - u^*\|^2_{L_2(0,t)} \le \frac{\check{D}_\alpha \check{D}(1+|x|+|z|)^2}{\sqrt{c}},$$

where the last inequality follows by Theorem 4.15.

We will demonstrate the existence of derivatives of $\mathcal{W}^{\alpha,c}$ with respect to α. Also, in order to develop a tractable numerical scheme, we consider the maximization problem over a finite-dimensional subspace of $\mathcal{A}^t$. In particular, noting the differentiability and concavity, we will seek the maximum through a search for the point where the derivative is zero. We will obtain an efficient means for computing the derivative, and we will also examine the errors induced by the finite-dimensional approximation.

Let $L < \infty$, and suppose $x, z, \zeta^i_r \in \overline{B_L}(0)$ for all $i \in \mathcal{N}$ and all $r \in [0,t]$. Henceforth, we will work on this compact domain. Accordingly, we define the subset of $\mathcal{Z}$ given by $\mathcal{Z}_L \doteq \{\zeta \in \mathcal{Z} \,|\, |\zeta^i_r| \le L \ \forall r \ge 0,\ \forall i \in \mathcal{N}\}$.

We assume that for $t < \bar{t}$, there exists $x, z \in \overline{B_L}(0)$; $\zeta \in \mathcal{Z}_L$; $\bar{c} = \bar{c}(t,x,\zeta,z) < \infty$, and $\bar{\varepsilon} = \bar{\varepsilon}(t,x,\zeta,z)$ such that for any $\varepsilon \in (0,\bar{\varepsilon}]$, $c > \bar{c}$ and any $\varepsilon-$optimal in the definition (9) of $\overline{W}^{c}$, we have

$$|\xi^{\varepsilon}_r - \zeta^i_r| > R_i, \quad \forall r \in [0,t],\ \forall i \in \mathcal{N} \tag{A.N1}$$

where $\xi^{\varepsilon}_{\cdot}$ denotes the corresponding trajectory.

6.1 Derivative of $\mathcal{W}^{\alpha,c}$ with respect to α

We first note some simple miscellaneous bounds that will be used below. For $\hat{\rho}, \hat{\eta} \in \mathcal{P}$ given as (73), using the bilinearity of f in (74), we define linear operators

$$\ell_1(\hat{\eta}) \doteq f_{\hat{\rho}}(\hat{\rho}, \hat{\eta}), \quad \ell_2(\hat{\rho}) \doteq f_{\hat{\eta}}(\hat{\rho}, \hat{\eta}), \tag{83}$$

where the subscripts on f denote differentiation, and ℓ_1, ℓ_2 are introduced to emphasize the dependence on only one variable each. From the definition of f, we see that

$$\|\ell_1(\hat{\eta})\|_F = \|f_{\hat{\rho}}(\hat{\rho}, \hat{\eta})\|_F = \left[|\eta_1|^2 + 2|\eta_2|^2 + 3|\bar{\eta}_4|^2\right]^{1/2} \leq \sqrt{3}|\hat{\eta}|, \tag{84}$$

$$\|\ell_2(\hat{\rho})\|_F = \|f_{\hat{\eta}}(\hat{\rho}, \hat{\eta})\|_F = \left[5|\rho_1|^2 + 4|\rho_2|^2 + |\bar{\rho}_4|^2\right]^{1/2} \leq \sqrt{5}|\hat{\rho}|, \tag{85}$$

where $\|\cdot\|_F$ denotes the Frobenius norm, and hence the Fréchet derivatives involved are bounded linear operators on $\mathcal{P}$. Further, note that for $\sigma, \omega \in \mathcal{P}$,

$$f(\sigma, \sigma) - f(\omega, \omega) = [\ell_1(\sigma) + \ell_2(\omega)](\sigma - \omega), \tag{86}$$

$$\|\ell_1(\sigma) - \ell_1(\omega)\|_F \leq \sqrt{3}|\sigma - \omega|. \tag{87}$$

By Lemma 5.1, given $t < \bar{t}$, $c \in [0, \infty)$ and $\zeta \in \mathcal{Z}_L$, we may choose $K_p = K_p(c, t, \zeta) < \infty$ such that

$$|\widehat{P}_r^c(\alpha, \zeta)| \leq K_p \quad \forall r \in [0, t], \ \alpha \in \mathcal{A}^t \tag{88}$$

where $\widehat{P}_\cdot^c(\cdot, \zeta)$ are the solutions of (77). Further, combining (84), (85) and (88) yields

$$\|\ell_1(\widehat{P}_r^c(\alpha, \zeta))\|_F, \|\ell_2(\widehat{P}_r^c(\alpha, \zeta))\|_F \leq \sqrt{5}K_p \doteq K_f^1 \quad \forall\, r \in [0, t], \ \alpha \in \mathcal{A}^t. \tag{89}$$

Also, by examining (75), we note that for $\tilde{\alpha} \doteq \{\tilde{\alpha}^i\}_{i \in \mathcal{N}} \in \mathcal{A}$ and $Y \in \mathcal{Y}_L \doteq \{\{y^i\}_{i \in \mathcal{N}} \in \mathcal{Y} \,|\, |y^i| \leq L, \ \forall i \in \mathcal{N}\}$,

$$\|\Gamma_\alpha(\tilde{\alpha}, Y)\|_F^2 = 9 \sum_{i \in \mathcal{N}} [(\tilde{\alpha}^i)^2]^2 \left\{1 + |y^i|^2 + |y^i|^4\right\},$$

where the subscripts on Γ indicate differentiation with respect to the first variable α. Then, by the definitions of $\mathcal{A}$ and $\mathcal{Y}_L$, we see that there exists $K_\gamma^1 = K_\gamma^1(\{R_i\}_{i \in \mathcal{N}}, L) < \infty$ such that

$$\|\Gamma_\alpha(\tilde{\alpha}, Y)\|_F \leq K_\gamma^1 \quad \forall\, \tilde{\alpha} \in \mathcal{A}, \ Y \in \mathcal{Y}_L. \tag{90}$$

Further, for $\tilde{\alpha}_1 \doteq \{\tilde{\alpha}_1^i\}_{i \in \mathcal{N}}, \tilde{\alpha}_2 = \{\tilde{\alpha}_2^i\}_{i \in \mathcal{N}} \in \mathcal{A}$ and $Y \in \mathcal{Y}_L$,

$$\|\Gamma_\alpha(\tilde{\alpha}_1, Y) - \Gamma_\alpha(\tilde{\alpha}_2, Y)\|_F^2 = 9 \sum_{i \in \mathcal{N}} [\tilde{\alpha}_1^i + \tilde{\alpha}_2^i]^2 \left\{1 + |y^i|^2 + |y^i|^4\right\} [\tilde{\alpha}_1^i - \tilde{\alpha}_2^i]^2,$$

which implies, by the definitions of $\mathcal{A}$ and $\mathcal{Y}_L$, that there exists $\widehat{K}_\gamma = \widehat{K}_\gamma(\{R_i\}_{i \in \mathcal{N}}, L) < \infty$ such that this is

$$\leq [\widehat{K}_\gamma]^2 \sum_{i \in \mathcal{N}} |\tilde{\alpha}_1^i - \tilde{\alpha}_2^i|^2 = \left\{\widehat{K}_\gamma |\tilde{\alpha}_1 - \tilde{\alpha}_2|\right\}^2. \tag{91}$$

Lemma 6.1. *For $t < \bar{t}$, $c \in [0, \infty)$, $\check{\alpha}, \hat{\alpha} \in \mathcal{A}^t$ and $\zeta \in \mathcal{Z}_L$, there exists $\widehat{C}_1 < \infty$ such that*

$$|\widehat{P}_r^c(\check{\alpha}, \zeta) - \widehat{P}_r^c(\hat{\alpha}, \zeta)| \leq \widehat{C}_1 \|\check{\alpha} - \hat{\alpha}\|_{L_2(0,t)}$$

for all $r \in [0, t]$ where $\widehat{P}_\cdot^c(\check{\alpha}, \zeta)$ and $\widehat{P}_\cdot^c(\hat{\alpha}, \zeta)$ are the solutions of (77) with $\alpha = \check{\alpha}$ and $\alpha = \hat{\alpha}$, respectively.

Proof. By (77) and the triangle inequality,

$$|\widehat{P}^c_r(\check{\alpha},\zeta)-\widehat{P}^c_r(\hat{\alpha},\zeta)| \le \int_0^r |f(\widehat{P}^c_\nu(\check{\alpha},\zeta),\widehat{P}^c_\nu(\check{\alpha},\zeta))-f(\widehat{P}^c_\nu(\hat{\alpha},\zeta),\widehat{P}^c_\nu(\hat{\alpha},\zeta))| + |\Gamma(\check{\alpha}_{t-\nu},\zeta_{t-\nu})-\Gamma(\hat{\alpha}_{t-\nu},\zeta_{t-\nu})|\,d\nu,$$

which by (86) and (90),

$$\le \int_0^r \Big[\|\ell_1(\widehat{P}^c_\nu(\check{\alpha},\zeta))\|_F+\|\ell_2(\widehat{P}^c_\nu(\hat{\alpha},\zeta))\|_F\Big] \cdot|\widehat{P}^c_\nu(\check{\alpha},\zeta)-\widehat{P}^c_\nu(\hat{\alpha},\zeta)|+K^1_\gamma|\check{\alpha}_{t-\nu}-\hat{\alpha}_{t-\nu}|\,d\nu,$$

which by (89) and Hölder's inequality,

$$\le 2K^1_f\int_0^r|\widehat{P}^c_\nu(\check{\alpha},\zeta)-\widehat{P}^c_\nu(\hat{\alpha},\zeta)|\,d\nu+K^1_\gamma\sqrt{t}\|\check{\alpha}-\hat{\alpha}\|_{L_2(0,t)}.$$

Using Gronwall's inequality, this implies

$$|\widehat{P}^c_r(\check{\alpha},\zeta)-\widehat{P}^c_r(\hat{\alpha},\zeta)|\le K^1_\gamma\sqrt{t}\exp(2K^1_f t)\|\check{\alpha}-\hat{\alpha}\|_{L_2(0,t)}$$

for all $r\in[0,t]$. □

Lemma 6.2. *Let $c\in[0,\infty)$. Then $\mathcal{W}^{\alpha,c}(t,x,\zeta,z)$ is Lipschitz continuous in α on bounded subsets of $[0,\bar{t})\times I\!R^3\times\mathcal{Z}_L\times I\!R^3$.*

Proof. Let $t\in(0,\bar{t})$; $x,z\in\overline{B}_L(0)$ and $\zeta\in\mathcal{Z}_L$. Let $\check{\alpha},\hat{\alpha}\in\mathcal{A}^t$. Then, by Theorem 5.2 and Lemma 6.1,

$$|\mathcal{W}^{\check{\alpha},c}(t,x,\zeta,z)-\mathcal{W}^{\hat{\alpha},c}(t,x,\zeta,z)|\le\tfrac{1}{2}|X(x,z)|\,|\widehat{P}^c_t(\check{\alpha},\zeta)-\widehat{P}^c_t(\hat{\alpha},\zeta)|\le\widehat{K}_\alpha\|\check{\alpha}-\hat{\alpha}\|_{L_2(0,t)},$$

for proper choice of $\widehat{K}_\alpha=\widehat{K}_\alpha(\widehat{C}_1,L)<\infty$. □

Letting $\mathcal{A}_o$ denote the interior of $\mathcal{A}$, we define $\mathcal{A}^t_o\doteq L_\infty([0,t];\mathcal{A}_o)$. Given $\zeta\in\mathcal{Z}_L$ and $c\in[0,\infty)$, we will obtain a representation for the derivative of $\widehat{P}^c_\cdot(\alpha,\zeta)$ with respect to $\alpha\in\mathcal{A}^t_o$. For $s\in[0,t)$ and $i\in\mathcal{N}$, consider

$$\dot{\pi}^{s,i}_r\doteq\frac{d\pi^{s,i}_r}{dr}=f(\pi^{s,i}_r,\widehat{P}^c_r)+f(\widehat{P}^c_r,\pi^{s,i}_r)+\Gamma_{\alpha^i}(\alpha_{t-r},\zeta_{t-r}) \tag{92}$$

for all $r\in(s,t)$ with $\pi^{s,i}_s=0_{M\times1}$ where

$$\Gamma_{\alpha^i}(\alpha_{t-r},\zeta_{t-r})\doteq\frac{\partial\Gamma(\alpha_{t-r},\zeta_{t-r})}{\partial\alpha^i}. \tag{93}$$

The following lemma demonstrates the desired representation for the derivative.

Lemma 6.3. *Given $\alpha\in\mathcal{A}^t_o$, let $h\in\mathcal{A}^t_o$ such that $\alpha+h\in\mathcal{A}^t_o$. Let $c\in[0,\infty)$ and $\zeta\in\mathcal{Z}_L$. Then, there exists $\widehat{C}_2<\infty$ such that letting $\widehat{P}^c_{h,r}\doteq\widehat{P}^c_r(\alpha+h,\zeta)$,*

$$\left|\widehat{P}^c_{h,r}-\widehat{P}^c_r-\int_0^r\left(-\frac{d\pi^s_r}{ds}\right)h_{t-s}\,ds\right|\le\widehat{C}_2\|h\|^2_{L_2(0,t)} \tag{94}$$

for all $r\in[0,t]$ where $\widehat{P}^c_\cdot(\cdot,\zeta)$ is the solution of (77).

Proof. Using the integral form of (92) and its initial condition, for $r\in[s,t)$,

$$\pi^{s,i}_r=\int_s^r f(\pi^{s,i}_\nu,\widehat{P}^c_\nu)+f(\widehat{P}^c_\nu,\pi^{s,i}_\nu)+\Gamma_{\alpha^i}(\alpha_{t-\nu},\zeta_{t-\nu})\,d\nu.$$

Differentiating, and using (83), we have

$$\frac{d\pi_r^s}{ds} = -\Gamma_\alpha(\alpha_{t-s},\zeta_{t-s}) + \int_s^r \left[\ell_1(\widehat{P}_\nu^c) + \ell_2(\widehat{P}_\nu^c)\right] \frac{d\pi_\nu^s}{ds}\, d\nu. \tag{95}$$

Letting $\Delta\widehat{P}_r^c \doteq \widehat{P}_{h,r}^c - \widehat{P}_r^c$, we define

$$\phi_r \doteq \Delta\widehat{P}_r^c - \int_0^r -\frac{d\pi_r^s}{ds} h_{t-s}\, ds \quad \forall r\in[0,t]. \tag{96}$$

Letting $\Delta\Gamma_{t-r} \doteq \Gamma(\alpha_{t-r}+h_{t-r},\zeta_{t-r}) - \Gamma(\alpha_{t-r},\zeta_{t-r})$, we note that by (77), differentiation $\Delta\widehat{P}_r^c$ with respect to r is given by

$$\Delta\dot{\widehat{P}}_r^c \doteq \dot{\widehat{P}}^c{}_{h,r} - \dot{\widehat{P}}_r^c = f(\widehat{P}_{h,r}^c,\widehat{P}_{h,r}^c) - f(\widehat{P}_r,\widehat{P}_r) + \Delta\Gamma_{t-r} = [\,\ell_1(\widehat{P}_r^c) + \ell_2(\widehat{P}_r^c)]\Delta\widehat{P}_r^c + \Delta\Gamma_{t-r}, \tag{97}$$

where the last equality follows by (86). Also, note that by (95),

$$\frac{d}{dr}\int_0^r -\frac{d\pi_r^s}{ds} h_{t-s}\, ds = \frac{d}{dr}\int_0^r \Gamma_\alpha(\alpha_{t-s},\zeta_{t-s}) h_{t-s}\, ds - \frac{d}{dr}\int_0^r \int_s^r \left[\ell_1(\widehat{P}_\nu^c) + \ell_2(\widehat{P}_\nu^c)\right] \frac{d\pi_\nu^s}{ds}\, d\nu\, h_{t-s}\, ds,$$

which, using the Leibniz integral rule,

$$\begin{aligned} &= \Gamma_\alpha(\alpha_{t-r},\zeta_{t-r}) h_{t-r} - \int_0^r \frac{d}{dr}\int_s^r \left[\ell_1(\widehat{P}_\nu^c) + \ell_2(\widehat{P}_\nu^c)\right] \frac{d\pi_\nu^s}{ds}\, d\nu\, h_{t-s}\, ds \\ &= \Gamma_\alpha(\alpha_{t-r},\zeta_{t-r}) h_{t-r} + [\,\ell_1(\widehat{P}_r^c) + \ell_2(\widehat{P}_r^c)] \int_0^r -\frac{d\pi_r^s}{ds} h_{t-s}\, ds. \end{aligned} \tag{98}$$

Next, differentiating (96) with respect to r yields

$$\dot{\phi}_r = \Delta\dot{\widehat{P}}_r^c - \frac{d}{dr}\int_0^r -\frac{d\pi_r^s}{ds} h_{t-s}\, ds,$$

which by (97) and (98),

$$\begin{aligned} &= [\,\ell_1(\widehat{P}_{h,r}^c) + \ell_2(\widehat{P}_r^c)]\Delta\widehat{P}_r^c - [\,\ell_1(\widehat{P}_r^c) + \ell_2(\widehat{P}_r^c)] \int_0^r -\frac{d\pi_r^s}{ds} h_{t-s}\, ds + \Delta\Gamma_{t-r} - \Gamma_\alpha(\alpha_{t-r},\zeta_{t-r}) h_{t-r} \\ &= [\,\ell_1(\widehat{P}_r^c) + \ell_2(\widehat{P}_r^c)]\phi_r + [\ell_1(\widehat{P}_r^{c,h}) - \ell_1(\widehat{P}_r^c)]\Delta\widehat{P}_r^c + \Delta\Gamma_{t-r} - \Gamma_\alpha(\alpha_{t-r},\zeta_{t-r}) h_{t-r} \end{aligned} \tag{99}$$

where the last equality follows by (96). Note that by (87),

$$\int_0^r \left| \left[\ell_1(\widehat{P}_r^{c,h}) - \ell_1(\widehat{P}_r^c)\right] \Delta\widehat{P}_\nu^c \right| \nu \le \sqrt{3}\int_0^r |\Delta\widehat{P}_\nu^c|^2\, d\nu,$$

which by Lemma 6.1,

$$\le C_2 \|h\|_{L_2(0,t)}^2 \tag{100}$$

for proper choice of $C_2 < \infty$.

Note that by the integral mean value theorem (cf., Ch. 9 in [11]),

$$\Delta\Gamma_{t-\nu} = \Gamma(\alpha_{t-\nu}+h_{t-\nu},\zeta_{t-\nu}) - \Gamma(\alpha_{t-\nu},\zeta_{t-\nu}) = \left[\int_0^1 \Gamma_\alpha(\alpha_{t-\nu}+sh_{t-\nu})\, ds\right] h_{t-\nu}$$

for all $\nu\in[0,t]$. Then,

$$\begin{aligned} |\Delta\Gamma_{t-\nu} - \Gamma_\alpha(\alpha_{t-\nu},\zeta_{t-\nu}) h_{t-\nu}| &\le \int_0^1 \|\Gamma_\alpha(\alpha_{t-\nu}+sh_{t-\nu}) - \Gamma_\alpha(\alpha_{t-\nu},\zeta_{t-\nu})\|_F\, ds\, |h_{t-\nu}| \\ &\le \int_0^1 \widehat{K}_\gamma s\, ds |h_{t-\nu}|^2 = \tfrac{1}{2}\widehat{K}_\gamma |h_{t-\nu}|^2, \end{aligned}$$

where the last inequality follows by (91). This implies that

$$\int_0^t |\Delta\Gamma_{t-\nu} - \Gamma_\alpha(\alpha_{t-\nu}, \zeta_{t-\nu}) h_{t-\nu}|\, d\nu \le \tfrac{1}{2}\widehat{K}_\gamma \int_0^t |h_{t-\nu}|^2\, d\nu = \tfrac{1}{2}\widehat{K}_\gamma \|h\|^2_{L_2(0,t)} \doteq C_3 \|h\|^2_{L_2(0,t)}. \tag{101}$$

Substituting (100) and (101) into the integration of (99) yields

$$|\phi_r| \le \int_0^r 2K_f^1 |\phi_\nu|\, d\nu + (C_2 + C_3)\|h\|^2_{L_2(0,t)}.$$

By Gronwall's inequality, this implies $|\phi_r| \le (C_2 + C_3) t \exp(2K_f^1 t)\|h\|^2_{L_2(0,t)}$ for all $r \in [0, t]$. □

Corollary 6.4. *Let $c \in [0, \infty)$ and $t \in (0, \bar{t})$. Then, $\mathcal{W}^{\alpha,c}(t, x, \zeta, z)$ is Fréchet differentiable with respect to $\alpha \in \mathcal{A}_o^t$ on bounded subsets of $I\!R^3 \times \mathcal{Z}_L \times I\!R^3$.*

Proof. Let $x, z \in \overline{B}_L(0)$ and $\zeta \in \mathcal{Z}_L$. Given $\alpha \in \mathcal{A}_o^t$, let $h \in \mathcal{A}_o^t$ such that $\alpha + h \in \mathcal{A}_o^t$. Let

$$\phi_r^{\mathcal{W}} \doteq \mathcal{W}^{\alpha+h,c}(r, x, \zeta, z) - \mathcal{W}^{\alpha,c}(r, x, \zeta, z) - \tfrac{1}{2}X(x, z) \cdot \int_0^t \left(-\frac{d\pi_r^s}{ds}\right) h_{t-s}\, ds,$$

where X is given in (72), and which by Theorem 5.2 and (76),

$$= \tfrac{1}{2}X(x, z) \cdot \widehat{P}_r^c(\alpha + h, \zeta) - \tfrac{1}{2}X(x, z) \cdot \widehat{P}_r^c(\alpha, \zeta) - \tfrac{1}{2}X(x, z) \cdot \int_0^t \left(-\frac{d\pi_r^s}{ds}\right) h_{t-s}\, ds.$$

Then, by Lemma 6.3, $|\phi_r^{\mathcal{W}}| \le \frac{1}{2}|X(x, z)|\widehat{C}_2 \|h\|^2_{L_2(0,t)}$, which completes the proof. □

6.2 An approximate solution

We will consider piecewise constant potential energy controls rather than the L_∞ elements of $\mathcal{A}^t$, as this will allow us to compute numerically the derivatives of interest.

Let $t \in (0, \bar{t})$ where $\bar{t}$ is as per (48). Let $K \in I\!N$ denote the number of time intervals contained in the given time duration $[0, t]$, and $\breve{\mathcal{A}}^K$ denote the K^{th} Cartesian power of $\mathcal{A}$. We may choose a norm on $\breve{\mathcal{A}}^K$ as

$$\|\breve{\alpha}\|_2 \doteq \sqrt{\frac{t}{K}} \left[\sum_{i \in \mathcal{K}} |\breve{\alpha}_k|^2\right]^{1/2} \text{ for } \breve{\alpha} \doteq \{\breve{\alpha}_k\}_{k \in \mathcal{K}} \in \breve{\mathcal{A}}^K. \tag{102}$$

Let $\tau \doteq t/K$ denote the length of each time interval. Letting $\tau_0 = 0$, we define

$$\tau_k \doteq k\tau \quad \text{and} \quad \mathcal{I}_k \doteq [\tau_{k-1}, \tau_k) \quad \forall k \in \mathcal{K} \doteq]1, K[.$$

We define the set of piecewise constant functions defined on $[0, t)$ relative to grid $\{\tau_k \,|\, k \in]0, K[\,\}$ as

$$\mathcal{A}_K^t \doteq \{\alpha \in \mathcal{A}^t \,|\, \forall k \in \mathcal{K},\, \exists \breve{\alpha}_k \in \mathcal{A} \text{ s.t. } \alpha_r = \breve{\alpha}_k \ \forall r \in \mathcal{I}_k\}.$$

Further, we define the one-to-one and onto linear mapping $\mathcal{L}^K : \breve{\mathcal{A}}^K \to \mathcal{A}_K^t$ such that for $\breve{\alpha} \doteq \{\breve{\alpha}_k\}_{k \in \mathcal{K}} \in \breve{\mathcal{A}}^K$, letting $\hat{\alpha} \doteq \mathcal{L}^K(\breve{\alpha})$,

$$\hat{\alpha}_r = \breve{\alpha}_k \quad \forall r \in \mathcal{I}_k, \quad \forall k \in \mathcal{K}. \tag{103}$$

From Lemma 4.10 and the definition of $\mathcal{L}^K(\cdot)$, one immediately obtains the following:

Lemma 6.5. *Let $K \in I\!N$. For all $t > 0$, $c \in [0, \infty)$; $x, z \in I\!R^3$, $\zeta \in \mathcal{Z}_L$, and $u \in \mathcal{U}^\infty$, $\mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}(t, x, \zeta, z)$ and $J^c(t, x, u, \mathcal{L}^K(\breve{\alpha}), \zeta, z)$ are stictly concave in $\breve{\alpha} \in \breve{\mathcal{A}}^K$.*

Lemma 6.6. *Let $t < \bar{t}$ and $K \in I\!N$. Let $c \in [0,\infty)$; $x, z \in \overline{B}_L$; $\zeta \in \mathcal{Z}_L$. Then, for any $\breve{u} \in \mathcal{U}^\infty$,*

$$\sup_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, \breve{u}, \mathcal{L}^K(\breve{\alpha}), \zeta, z) = \max_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, \breve{u}, \mathcal{L}^K(\breve{\alpha}), \zeta, z).$$

Further, letting $\breve{\xi}$ be the trajectory corresponding to $\breve{u}$,

$$\breve{\alpha}^* = \{\breve{\alpha}_k^*\}_{k \in \mathcal{K}} \doteq \operatorname*{argmax}_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, \breve{u}, \mathcal{L}^K(\breve{\alpha}), \zeta, z)$$

where

$$\breve{\alpha}_k^{*,i} = \sqrt{2/3} \min \left\{ R_i^{-1}, \sqrt{\tau} \|\breve{\xi} - \zeta^i\|_{L_2(\tau_{k-1}, \tau_k)}^{-1} \right\}$$

for all $i \in \mathcal{N}$ and $k \in \mathcal{K}$

Proof. Let $\breve{u} \in \mathcal{U}^\infty$ and $\breve{\xi}$ be the corresponding trajectory. By the independent sum of integrals over the segments,

$$\begin{aligned}
\sup_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, \breve{u}, \mathcal{L}^K(\breve{\alpha}), \zeta, z) &= \tfrac{1}{2}\|\breve{u}\|_{L_2(0,t)}^2 + \psi^c(\breve{\xi}_t, z) + \sup_{\breve{\alpha} \in \breve{\mathcal{A}}^K} \sum_{k \in \mathcal{K}} \int_{\mathcal{I}_k} -V^{\mathcal{L}^K(\breve{\alpha})}(r, \breve{\xi}_r, \zeta_r)\, dr \\
&= \tfrac{1}{2}\|\breve{u}\|_{L_2(0,t)}^2 + \psi^c(\breve{\xi}_t, z) + \sup_{\breve{\alpha} \in \breve{\mathcal{A}}^K} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{N}} \int_{\mathcal{I}_k} \mu_i \left[\breve{\alpha}_k^i - \tfrac{1}{2}[\breve{\alpha}_k^i]^3 |\breve{\xi}_r - \zeta_r^i|^2 \right] dr, \\
&= \tfrac{1}{2}\|\breve{u}\|_{L_2(0,t)}^2 + \psi^c(\breve{\xi}_t, z) + \sum_{(i,k) \in \mathcal{N} \times \mathcal{K}} \sup_{\breve{\alpha}_k^i \in (0, \sqrt{2/3} R_i^{-1}]} \mathcal{V}_k^i(\breve{\alpha}, \breve{\xi}, \zeta)
\end{aligned}$$

where

$$\mathcal{V}_k^i(\breve{\alpha}, \breve{\xi}, \zeta) \doteq \int_{\mathcal{I}_k} \mu_i \left[\breve{\alpha}_k^i - \tfrac{1}{2}[\breve{\alpha}_k^i]^3 |\breve{\xi}_r - \zeta_r^i|^2 \right] dr.$$

Note that

$$\frac{d}{d\breve{\alpha}_k^i} \mathcal{V}_k^i(\breve{\alpha}, \breve{\xi}, \zeta) = \mu_i \left[\tau - \frac{3}{2}[\breve{\alpha}_k^i]^2 \int_{\mathcal{I}_k} |\breve{\xi}_r - \zeta_r^i|^2\, dr \right].$$

From this, it is not difficult to show that

$$\breve{\alpha}_k^{*,i} = \bar{\breve{\alpha}}_k^i(\breve{\xi}, \zeta) \doteq \operatorname*{argmax}_{\breve{\alpha}_k^i \in (0, \sqrt{2/3} R_i^{-1}]} \mathcal{V}_k^i(\breve{\alpha}, \breve{\xi}, \zeta) = \sqrt{2/3} \min \left\{ R_i^{-1}, \sqrt{\tau}\, \|\breve{\xi} - \zeta_r^i\|_{L_2(\tau_{k-1}, \tau_k)}^{-1} \right\}. \tag{104}$$

□

Let $\bar{\breve{\alpha}} : I\!R^3 \times I\!R^{3N} \to \breve{\mathcal{A}}^K$ be given by

$$\bar{\breve{\alpha}}(\cdot, \cdot) \doteq \{\bar{\breve{\alpha}}_k(\cdot, \cdot)\}_{k \in \mathcal{K}} \tag{105}$$

where $\bar{\breve{\alpha}}_k(\cdot, \cdot) \doteq \{\bar{\breve{\alpha}}_k^i(\cdot, \cdot)\}_{i \in \mathcal{N}}$ and $\bar{\breve{\alpha}}_k^i$ is given by (104) for all $i \in \mathcal{N}$ and $k \in \mathcal{K}$.

Lemma 6.7. *Let $t < \bar{t}$ and $K \in I\!N$. Let $x, z \in I\!R^3$; $\zeta \in \mathcal{Z}_L$ and $c \in [0, \infty)$. Then, there exists a unique stationary point of $J^c(t, x, \cdot, \mathcal{L}^K(\cdot), \zeta, z)$ over $\mathcal{U}^\infty \times \breve{\mathcal{A}}^K$.*

Proof. Recall that by Theorem 4.5 and Lemma 4.4, $J^c(t, x, u, \mathcal{L}^K(\breve{\alpha}), \zeta, z)$ is continuous, coercive and strictly convex in u. Then, by (103), it is easy to see that $\max_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, u, \mathcal{L}^K(\breve{\alpha}), \zeta, z)$ is also continuous, coercive and strictly convex in u, where the existence of the maximum follows from Lemma 6.6. This guarantees the existence of the unique optimal velocity control,

$$\breve{u}^{c,*} = \operatorname*{argmin}_{u \in \mathcal{U}^\infty} \max_{\breve{\alpha} \in \breve{\mathcal{A}}^K} J^c(t, x, u, \mathcal{L}^K(\breve{\alpha}), \zeta, z).$$

This implies

$$\begin{aligned}\min_{u\in\mathcal{U}^\infty}\max_{\breve\alpha\in\breve{\mathcal{A}}^K} J^c(t,x,u,\mathcal{L}^K(\breve\alpha),\zeta,z) &= \max_{\breve\alpha\in\breve{\mathcal{A}}^K} J^c(t,x,\breve u^{c,*},\mathcal{L}^K(\breve\alpha),\zeta,z)\\ &= J^c(t,x,\breve u^{c,*},\mathcal{L}^K(\bar{\breve\alpha}(\xi^{c,*},\zeta)),\zeta,z) \end{aligned}\tag{106}$$

where $\bar{\breve\alpha}$ is given by (105) and $\breve\xi^{c,*}$ denotes the trajectory corresponding to $\breve u^{c,*}$. Then, by an argument similar to that of the proof of Theorem 4.9 and we do not include the repetitive details), letting $\breve\alpha^* \doteq \bar{\breve\alpha}(\breve\xi^{c,*},\zeta)$,

$$J^c(t,x,\breve u^{c,*},\mathcal{L}^K(\breve\alpha^*),\zeta,z) = \min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\mathcal{L}^K(\breve\alpha^*),\zeta,z). \tag{107}$$

Combining (106) and (107) yields in an analogous fashion to the proof of Theorem 4.12 that

$$\min_{u\in\mathcal{U}^\infty}\max_{\breve\alpha\in\breve{\mathcal{A}}^K} J^c(t,x,u,\mathcal{L}^K(\breve\alpha),\zeta,z) = \max_{\breve\alpha\in\breve{\mathcal{A}}^K}\min_{u\in\mathcal{U}^\infty} J^c(t,x,u,\mathcal{L}^K(\breve\alpha),\zeta,z), \tag{108}$$

and $(\breve u^{c,*},\breve\alpha^*)$ is the unique solution of (108) over $\mathcal{U}^\infty\times\breve{\mathcal{A}}^K$. □

Remark 6.8. Let $t>0$ and $K\in I\!N$. For $\breve\alpha=\{\breve\alpha_k\}_{k\in\mathcal{K}}\in\breve{\mathcal{A}}^K$, letting $\hat\alpha\doteq\mathcal{L}^K(\breve\alpha)$,

$$\|\hat\alpha\|^2_{L_2(0,t)} = \int_0^t |\hat\alpha_r|^2\,dr = \sum_{k\in\mathcal{K}}\int_{\mathcal{I}_k}|\hat\alpha_r|^2\,dr = \tau\sum_{k\in\mathcal{K}}|\breve\alpha_k|^2 = \|\breve\alpha\|_2^2,$$

where the last equality follows by (102). Therefore, $\mathcal{L}^K$ is an isomorphism between two normed spaces, $(\breve{\mathcal{A}}^K,\|\cdot\|_2)$ and $(\mathcal{A}^t_K,\|\cdot\|_{L_2(0,t)})$ (cf., [7]). Further, by the linearity of $\mathcal{L}^K$, for $\check{\breve\alpha},\hat{\breve\alpha}\in\breve{\mathcal{A}}^K$,

$$\|\mathcal{L}^K(\check{\breve\alpha})-\mathcal{L}^K(\hat{\breve\alpha})\|_{L_2(0,t)} = \|\mathcal{L}^K(\check{\breve\alpha}-\hat{\breve\alpha})\|_{L_2(0,t)} = \|\check{\breve\alpha}-\hat{\breve\alpha}\|_2.$$

Since $\breve{\mathcal{A}}^K$ and $\mathcal{A}^t_K$ are isomorphic, the next corollaries follow immediately from Lemmas 6.1 and 6.2, respectively.

Corollary 6.9. *For $t<\bar t$, $c\in[0,\infty)$, and $\zeta\in\mathcal{Z}_L$, there exists $\widehat C_1<\infty$ such that*

$$|\widehat P^c_r(\mathcal{L}^K(\check{\breve\alpha}),\zeta)-\widehat P^c_r(\mathcal{L}^K(\hat{\breve\alpha}),\zeta)| \le \widehat C_1\|\check{\breve\alpha}-\hat{\breve\alpha}\|_2$$

for all $r\in[0,t]$ where $\widehat P^c_\cdot(\cdot,\zeta)$ is the solution of (77).

Corollary 6.10. *Let $c\in[0,\infty)$ and $K\in I\!N$. $\mathcal{W}^{\mathcal{L}^K(\breve\alpha),c}(t,x,\zeta,z)$ is Lipschitz continuous in $\breve\alpha\in\breve{\mathcal{A}}^K$ on bounded subsets of $[0,\bar t)\times I\!R^3\times\mathcal{Z}_L\times I\!R^3$.*

6.3 Error analysis

Computationally, we can approximate the maximum over $\mathcal{A}^t$ (equivalently, $\tilde{\mathcal{A}}^t$), which yields $\overline{W}^c$, by taking a maximum over $\mathcal{A}^t_K$, which due to the above-noted isomorphism, is equivalent to a maximum over $\breve{\mathcal{A}}^K$. Letting

$$\overline{W}^c_K(t,x,\zeta,z) \doteq \max_{\breve\alpha\in\breve{\mathcal{A}}^K}\mathcal{W}^{\mathcal{L}^K(\breve\alpha),c}(t,x,\zeta,z) = \min_{u\in\mathcal{U}^\infty}\max_{\breve\alpha\in\breve{\mathcal{A}}^K} J^c(t,x,u,\mathcal{L}^K(\breve\alpha),\zeta,z), \tag{109}$$

for all $t\in[0,\bar t)$; $x,z\in I\!R^3$; $\zeta\in\mathcal{Z}_L$, we will demonstrate that $\overline{W}^c_K\to\overline{W}^c$ as $K\to\infty$. Importantly, we will also demonstrate that the optimal velocity controls converge as $K\to\infty$, which implies that the optimal trajectories converge.

Lemma 6.11. *Let $t < \bar{t}$; $L < \infty$; $x, z \in \overline{B}_L(0)$, $c \in [0,\infty)$ and $\zeta \in \mathcal{Z}_L$. Then,*

$$\overline{W}^c(t,x,\zeta,z) = \lim_{K\to\infty} \overline{W}_K^c(t,x,\zeta,z) \doteq \lim_{K\to\infty} \max_{\breve{\alpha}\in\breve{\mathcal{A}}^K} \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}(t,x,\zeta,z). \tag{110}$$

Further, letting $\alpha^* \doteq \mathrm{argmax}_{\alpha\in\mathcal{A}^t} \mathcal{W}^{\alpha,c}(t,x,\zeta,z)$ and $\breve{\alpha}_K^* \doteq \mathrm{argmax}_{\breve{\alpha}\in\breve{\mathcal{A}}^K} \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}(t,x,\zeta,z)$ for $K \in I\!N$,

$$\overline{W}^c(t,x,\zeta,z) - \overline{W}_K^c(t,x,\zeta,z) \le \widehat{K}_\alpha \|\alpha^* - \mathcal{L}^K(\breve{\alpha}_K^*)\|_{L_2(0,t)},$$

and $\mathcal{L}^K(\breve{\alpha}_K^*) \to \alpha^*$ as $K \to \infty$.

Proof. Given $K \in I\!N$ and $\alpha \in \tilde{\mathcal{A}}^t$, let

$$\beta_k^i(K,\alpha) \doteq \frac{1}{t/K} \int_{\mathcal{I}_k} \alpha_\rho^i \, d\rho \quad \forall i \in \mathcal{N} \quad \forall k \in \mathcal{K}. \tag{111}$$

For $r \in [0,t]$, let $\check{\breve{\alpha}}_K \doteq \{\check{\breve{\alpha}}_K^i\}_{i\in\mathcal{N}} \in \mathcal{A}_K^t$ such that

$$\check{\breve{\alpha}}_K^i(r) \doteq \sum_{k\in\mathcal{K}} \beta_k^i(K,\alpha^*) \mathbf{1}_{\mathcal{I}_k}(r) \quad \forall i \in \mathcal{N} \tag{112}$$

where $\mathbf{1}$. denotes an indicator function. Note that for any $r \in [0,t]$ and $K \in I\!N$, there exist $\hat{k} = \hat{k}(r,K) \in \mathcal{K}$ and $\delta_K^+ = \delta_K^+(r), \delta_K^- = \delta_K^-(r) \ge 0$ such that $r \in \mathcal{I}_{\hat{k}} = [\tau_{\hat{k}-1}, \tau_{\hat{k}}) \doteq [r - \delta_K^-, r + \delta_K^+)$ where $\delta_K^+ + \delta_K^- = t/K$. Also, recalling from Theorem 4.2 that α^* is uniformly continuous in $[0,t]$, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|[\alpha_\rho^*]^i - [\alpha_r^*]^i| < \varepsilon \quad \text{if} \quad \rho \in B_\delta(r) \quad \forall i \in \mathcal{N}.$$

This implies that for any $r \in [0,t]$, there exists $K_\varepsilon < \infty$ such that

$$|[\alpha_\rho^*]^i - [\alpha_r^*]^i| < \varepsilon \quad \forall \rho \in [r - \delta_K^-, r + \delta_K^+) \subset B_\delta(r) \tag{113}$$

for all $K > K_\varepsilon$. Further, by (112), and then (111) and (113),

$$|\check{\breve{\alpha}}_K^i(r) - [\alpha_r^*]^i| = |\beta_{\hat{k}}^i(K,\alpha^*) - [\alpha_r^*]^i| \le \frac{1}{\delta_K^+ + \delta_K^-} \int_{r-\delta_K^-}^{r+\delta_K^+} |[\alpha_\rho^*]^i - [\alpha_r^*]^i| \, d\rho = \varepsilon \tag{114}$$

for all $K > K_\varepsilon$, which implies that $\check{\breve{\alpha}}_K$ converges pointwise to α^* as $K \to \infty$. Further, since $\mathcal{W}^{\alpha,c}$ is Lipschitz continuous in α, given $\varepsilon > 0$, there exists $\widehat{K}_\varepsilon < \infty$ such that for all $K > \widehat{K}_\varepsilon$,

$$\varepsilon \ge \overline{W}^c(t,x,\zeta,z) - \mathcal{W}^{\check{\breve{\alpha}}_K,c}(t,x,\zeta,z),$$

which by the suboptimality of $\check{\breve{\alpha}}_K$ with respect to $\overline{W}_K^c$,

$$\ge \overline{W}^c(t,x,\zeta,z) - \overline{W}_K^c(t,x,\zeta,z), \tag{115}$$

proving the first assertion. The second assertion follows directly from Lemma 6.2.

For the final assertion, let $\check{\alpha}_K \in \breve{\mathcal{A}}^K$ such that $\check{\breve{\alpha}}_K \doteq \mathcal{L}^K(\check{\alpha}_K)$, and let $\breve{\alpha}_K^*$ be as per the Lemma statement. By the optimality of $\breve{\alpha}_K^*$ with respect to $\overline{W}_K^c$,

$$0 \le \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}_K^*),c}(t,x,\zeta,z) - \mathcal{W}^{\mathcal{L}^K(\check{\alpha}_K),c}(t,x,\zeta,z),$$

which by the suboptimality of $\mathcal{L}^K(\breve{\alpha}_K^*)$ with respect to the definition (52) of $\overline{W}^c$,

$$\le \overline{W}^c(t,x,\zeta,z) - \mathcal{W}^{\mathcal{L}^K(\check{\alpha}_K),c}(t,x,\zeta,z),$$

which by (115), for $K > \widehat{K}_\varepsilon$,

$$\le \varepsilon.$$

This implies that

$$\lim_{K\to\infty}\{\mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^*_K),c}(t,x,\zeta,z)-\mathcal{W}^{\mathcal{L}^K(\check{\breve{\alpha}}_K),c}(t,x,\zeta,z)\}=0.$$

Therefore, given $\tilde{\varepsilon}\in(0,1]$, there exists $\widetilde{K}_\varepsilon<\infty$ such that

$$\tilde{\varepsilon}\ \geq \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^*_K),c}(t,x,\zeta,z)-\mathcal{W}^{\mathcal{L}^K(\check{\breve{\alpha}}_K),c}(t,x,\zeta,z)$$

for all $K>\widetilde{K}_\varepsilon$, which by the strong concavity asserted in Lemma 6.5, there exists $C_\alpha>0$ such that

$$\geq C_\alpha\|\breve{\alpha}^*_K-\check{\breve{\alpha}}_K\|_2^2,$$

which by Remark 6.8,

$$= C_\alpha\|\mathcal{L}^K(\breve{\alpha}^*_K)-\check{\breve{\alpha}}_K\|^2_{L_2(0,t)}, \tag{116}$$

which implies that $\|\mathcal{L}^K(\breve{\alpha}^*_K)-\check{\breve{\alpha}}_K\|_{L_2(0,t)}\to 0$ as $K\to\infty$. Further, noting that

$$\|\alpha^*-\mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)}\leq\|\alpha^*-\check{\breve{\alpha}}_K\|_{L_2(0,t)}+\|\check{\breve{\alpha}}_K-\mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)},$$

applying (114) and (116) to the above completes the last assertion. □

Theorem 6.12. *Let $t<\bar{t}$; $c\in[0,\infty)$; $L<\infty$; $x,z\in\overline{B}_L(0)$ and $\zeta\in\mathcal{Z}_L$. Given $K\in I\!N$, suppose that $(u^{c,*},\alpha^*)\in\mathcal{U}^\infty\times\mathcal{A}^t$ and $(\breve{u}^{c,*},\breve{\alpha}^*_K)\in\mathcal{U}^\infty\times\breve{\mathcal{A}}^K$ are the solutions of $\overline{W}^c(t,x,\zeta,z)$ of (52) and $\overline{W}^c_K(t,x,\zeta,z)$ of (109), respectively. Then, there exists $\widehat{D}^u=\widehat{D}^u(t,L)<\infty$ such that*

$$\|u^{c,*}-\breve{u}^{c,*}\|_{L_2(0,t)}\leq\widehat{D}^u\|\alpha^*-\mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)}.$$

Proof. Let α^*,$\breve{\alpha}^*_K$, $u^{c,*}$ and $\breve{u}^{c,*}$ be as asserted, and let $\xi^{c,*}$ and $\breve{\xi}^{c,*}$ denote the trajectories corresponding to $u^{c,*}$ and $\breve{u}^{c,*}$, respectively. By Theorems 4.16 and 5.2 and (78), for all $r\in[0,t]$,

$$u^{c,*}_r\ =-\nabla_x\mathcal{W}^{\alpha^*,c}(t-r,\xi^{c,*}_r,\zeta,z)=-\nabla_x X(\xi^{c,*}_r,z)\cdot\widehat{P}^c_{t-r}(\alpha^*,\zeta), \tag{117}$$

$$\breve{u}^{c,*}_r\ =-\nabla_x\mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^*_K),c}(t-r,\breve{\xi}^{c,*}_r,\zeta,z)=-\nabla_x X(\breve{\xi}^{c,*}_r,z)\cdot\widehat{P}^c_{t-r}(\mathcal{L}^K(\breve{\alpha}^*_K),\zeta). \tag{118}$$

Note that for $r\in[0,t)$, using Hölder's inequality,

$$|\breve{\xi}^{c,*}_r|\ \leq|x|+\int_0^r|\breve{u}^{c,*}_\rho|\,d\rho\leq|x|+\sqrt{t}\|\breve{u}^{c,*}\|_{L_2(0,t)},$$

and by the definition of $\overline{W}^c$, the nonnegativity of $\overline{V}$, and (9) and (15), this is

$$\leq|x|+[2t\,\overline{W}^c(t,x,\zeta,z)]^{1/2}\leq|x|+[2tD_1(1+|x|^2+|z|^2)]^{1/2}\leq\widehat{D}^x \tag{119}$$

for proper choice of $\widehat{D}^x=\widehat{D}^x(t,L)<\infty$. Also, note that by the definition of $X(\cdot,\cdot)$, for all $r\in[0,t]$,

$$\|\nabla_x X(\xi^{c,*}_r,\zeta)\|_F\leq 2(1+|\xi^{c,*}_r|+|z|)\leq 2(1+\widehat{D}^x+L)\doteq\widetilde{D}^x(t,L) \tag{120}$$

where the last bound follows by (119). Similarly,

$$\|\nabla_x X(\xi^{c,*}_r,z)-\nabla_x X(\breve{\xi}^{c,*}_r,z)\|_F=2|\xi^{c,*}_r-\breve{\xi}^{c,*}_r|\leq 2\int_0^r|u^{c,*}_\rho-\breve{u}^{c,*}_\rho|\,d\rho. \tag{121}$$

Then, by (117) and (118),

$$|u^{c,*}_r-\breve{u}^{c,*}_r|=|\nabla_x X(\xi^{c,*}_r,z)\cdot\widehat{P}^c_{t-r}(\alpha^*,\zeta)-\nabla_x X(\breve{\xi}^{c,*}_r,z)\cdot\widehat{P}^c_{t-r}(\mathcal{L}^K(\breve{\alpha}^*_K),\zeta)|,$$

which by the triangle inequality,

$$\leq\|\nabla_x X(\xi^{c,*}_r,z)-\nabla_x X(\breve{\xi}^{c,*}_r,z)\|_F|\widehat{P}^c_{t-r}(\alpha^*,\zeta)|+\|\nabla_x X(\breve{\xi}^{c,*}_r,z)\|_F|\widehat{P}^c_{t-r}(\alpha^*,\zeta)-\widehat{P}^c_{t-r}(\mathcal{L}^K(\breve{\alpha}^*_K),\zeta)|$$

which by (88), (120), (121) and Lemma 6.1,

$$\leq 2K_p \int_0^r |u^{c,*}_\rho - \breve{u}^{c,*}_\rho|\, d\rho + \widetilde{D}^x \widehat{C}_1 \|\alpha^* - \mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)}.$$

By Gronwall's inequality, this implies $|u^{c,*}_r - \breve{u}^{c,*}_r| \leq \widetilde{D}^x \widehat{C}_1 \exp(2K_p t)\|\alpha^* - \mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)}$ for all $r \in [0,t]$, which completes the proof. □

Recall that u^* defined by (56) yields the fundamental solution, $\overline{W}^\infty(t,x,z,\zeta)$. Combining Theorems 4.15 and 6.12, we can obtain a bound on the error in the resulting path, induced by using the approximations $c < \infty$ and piecewise constant α. That is, we have:

Corollary 6.13. *Let $t < \bar{t}$; $c \in [0,\infty)$; $L < \infty$; $x, z \in \overline{B}_L(0)$; and $\zeta \in \mathcal{Z}_L$. Given $K \in I\!N$, suppose that u^* is the least action point in definition (56) of $\overline{W}^\infty(t,x,\zeta,z)$, and that $(\breve{u}^{c,*}, \breve{\alpha}^*_K) \in \mathcal{U}^\infty \times \breve{\mathcal{A}}^K$ is the solution of (109). Then, there exist $\check{D} = \check{D}(t,\bar{t}) < \infty$ and $\widehat{D}^u = \widehat{D}^u(t,L) < \infty$ such that*

$$\|u^* - \breve{u}^{c,*}\|_{L_2(0,t)} \leq \frac{\check{D}(1+|x|+|z|)^2}{\sqrt{c}} + \widehat{D}^u \|\alpha^* - \mathcal{L}^K(\breve{\alpha}^*_K)\|_{L_2(0,t)}.$$

6.4 First-order necessary condition for maximization of $\mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}$

For $K \in I\!N$, let

$$\mathcal{A}^K_o \doteq \{\tilde{\alpha} = \{\tilde{\alpha}_i\}_{i\in\mathcal{N}} \,|\, \tilde{\alpha}_i \in (0, \sqrt{2/3}(R_i + 1/K)^{-1}) \quad \forall\, i \in \mathcal{N}\}.$$

Then, given $t < \bar{t}$, there exists $\widehat{K} = \widehat{K}(t) \in I\!N$ such that

$$t < \left[\sum_{i\in\mathcal{N}} \frac{Gm_i}{2(R_i + 1/K)^3}\right]^{-1/2} < \bar{t} \quad \forall K > \widehat{K}. \tag{122}$$

Letting $\breve{\mathcal{A}}^K_o$ be the K^{th} Cartesian product of $\mathcal{A}^K_o$, we see that for $K \geq \widehat{K}$, the coercivity and strict convexity of $J^c(t,x,\cdot,\mathcal{L}^K(\breve{\alpha}^o),\zeta,z)$ holds for any $\breve{\alpha}^o \doteq \{\breve{\alpha}^o_k\}_{k\in\mathcal{K}} \in \breve{\mathcal{A}}^K_o$. Further, by the arguments similar to that of the proofs of Lemma 6.7 and Lemma 6.11, there exists a unique stationary point of $J^c(t,x,\cdot,\mathcal{L}^K(\cdot),\zeta,z)$ over $\mathcal{U}^\infty \times \breve{\mathcal{A}}^K_o$, and

$$\overline{W}^c(t,x,\zeta,z) = \lim_{K\to\infty} \max_{\breve{\alpha}^o \in \breve{\mathcal{A}}^K_o} \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^o),c}(t,x,\zeta,z),$$

and letting $\breve{\alpha}^{o,*}_K \doteq \mathrm{argmax}_{\breve{\alpha}^o \in \breve{\mathcal{A}}^K_o} \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^o),c}(t,x,\zeta,z)$,

$$\mathcal{L}^K(\breve{\alpha}^{o,*}_K) \to \alpha^* \quad \text{as} \quad K \to \infty.$$

Given $t < \bar{t}$, we fix $K > \widehat{K}(t)$ throughout where $\widehat{K}(\cdot)$ is given in (122), and with a slight abuse of notation, let $\breve{\alpha} = \breve{\alpha}^o$ and $\breve{\alpha}^* = \breve{\alpha}^{o,*}_K$. We will demonstrate the existence of the derivative of $\mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}$ with respect to $\breve{\alpha} \in \breve{\mathcal{A}}^K_o$. Then, the maximum is achieved at the point where this derivative is zero (cf., [10]).

Lemma 6.14. *Given $\breve{\alpha} \in \breve{\mathcal{A}}^K_o$, let $\breve{\delta} \doteq \{\breve{\delta}_k\}_{k\in\mathcal{K}} \in \breve{\mathcal{A}}^K_o$ be such that $\breve{\alpha} + \breve{\delta} \in \breve{\mathcal{A}}^K_o$. Let $c \in [0,\infty)$ and $\zeta \in \mathcal{Z}_L$. Then, there exists $\widehat{C}_2 < \infty$ such that*

$$\left| \widehat{P}^c_r(\mathcal{L}^K(\breve{\alpha} + \breve{\delta}),\zeta) - \widehat{P}^c_r(\mathcal{L}^K(\breve{\alpha}),\zeta) - \sum_{k\in\mathcal{K}} \int_{(0,r)\cap\mathcal{I}_k} \left(-\frac{d\pi^s_r}{ds}\right) ds\, \breve{\delta}_{K+1-k} \right| \leq \widehat{C}_2 \|\breve{\delta}\|_2^2$$

for all $r \in (0,t)$ where $\widehat{P}^c_\cdot(\cdot,\zeta)$ and $\pi^s_\cdot = [\pi^{s,i}_\cdot]_{i\in\mathcal{N}}$ are given by (77) and (92), respectively, driven by $\mathcal{L}^K(\breve{\alpha})$.

Proof. Letting $\hat{\alpha} = \mathcal{L}^K(\breve{\alpha})$ and $h = \mathcal{L}^K(\breve{\delta})$, by the linearity of $\mathcal{L}^K$,

$$\mathcal{L}^K(\breve{\alpha} + \breve{\delta}) = \mathcal{L}^K(\breve{\alpha}) + \mathcal{L}^K(\breve{\delta}) = \hat{\alpha} + h. \tag{123}$$

Note that for $r \in (0,t)$,

$$\int_0^r \left(-\frac{d\pi_r^s}{ds}\right) h_{t-s}\, ds = \sum_{k\in\mathcal{K}} \int_{(0,r)\cap\mathcal{I}_k} \left(-\frac{d\pi_r^s}{ds}\right) ds\, \breve{\delta}_{K+1-k}. \tag{124}$$

Also, by Remark 6.8,

$$\|h\|^2_{L_2(0,t)} = \|\mathcal{L}^K(\hat{\delta})\|_{L_2(0,t)} = \|\breve{\delta}\|_2^2. \tag{125}$$

Substituting (123) – (125) into (94) completes the proof. ☐

By Corollaries 6.9 and 6.14, $\widehat{P}^c_\cdot(\mathcal{L}^K(\cdot),\zeta) \in C([0,t]\times\breve{\mathcal{A}}^K\,;\,\mathcal{P}) \cap C^1((0,t)\times\breve{\mathcal{A}}^K_o\,;\,\mathcal{P})$ (where we recall $\mathcal{P} = \mathbb{R}^{10}$).

For $\breve{\alpha} \in \breve{\mathcal{A}}^K_o$, $i \in \mathcal{N}$ and $k \in \mathcal{K}$, consider

$$\dot{\Pi}_r^{\tau_{k-1},i} = f(\Pi_r^{\tau_{k-1},i}, \widehat{P}_r^{c,k}) + f(\widehat{P}_r^{c,k}, \Pi_r^{\tau_{k-1},i}) + \Gamma_{\breve{\alpha}^i_{K+1-k}}(\breve{\alpha}_{K+1-k}, \zeta_{t-r}) \tag{126}$$

for all $r \in \mathcal{I}_k$ where $\Pi_{\tau_{k-1}}^{\tau_{k-1},i} = 0_{M\times 1}$ (where f is given in (74) and Γ in (93)), and

$$\dot{\widehat{P}}_r^{c,k} = f(\widehat{P}_r^{c,k}, \widehat{P}_r^{c,k}) + \Gamma(\breve{\alpha}_{K+1-k}, \zeta_{t-r}) \tag{127}$$

for all $r \in \mathcal{I}_k$ and $k \in \mathcal{K}$ with $\widehat{P}^{c,k}_{\tau_{k-1}} = \widehat{P}^{c,k-1}_{\tau_{k-1}} = \widehat{P}^c_{\tau_{k-1}}$ where $\widehat{P}^c_\cdot$ is the solution of (77) with $\alpha = \mathcal{L}^K(\breve{\alpha})$.

Proposition 6.15. *For $\breve{\alpha} \in \breve{\mathcal{A}}^K_o$, let $\hat{\alpha} \doteq \mathcal{L}^K(\breve{\alpha})$.*

$$\frac{\partial \widehat{P}_r^{c,k}}{\partial \hat{\alpha}} = \frac{\partial \widehat{P}_r^{c,k}}{\partial \breve{\alpha}_{K+1-k}} = \Pi_r^{\tau_{k-1}} \tag{128}$$

for all $r \in \mathcal{I}_k$ and $k \in \mathcal{K}$ where $\Pi_\cdot^{\tau_{k-1}} \doteq [\Pi_\cdot^{\tau_{k-1},i}]_{i\in\mathcal{N}} \in \mathbb{R}^{M\times N}$, and $\Pi_\cdot^{\tau_{k-1},i}$ and $\widehat{P}_\cdot^{c,k}$ are the solutions of (126) and (127), respectively.

Proof. Let $i \in \mathcal{N}$ and $k \in \mathcal{K}$. Then, we note that $\widehat{P}^c_r(\hat{\alpha},\zeta) = \widehat{P}_r^{c,k}(\mathcal{L}^K(\breve{\alpha}),\zeta)$ and $\hat{\alpha}_{t-r} = \breve{\alpha}_{K+1-k}$ for all $r \in \mathcal{I}_k$. Further, for $r \in \mathcal{I}_k$, by (127),

$$\frac{\partial \widehat{P}_r^{c,k}(\mathcal{L}^K(\breve{\alpha}),\zeta)}{\partial \breve{\alpha}^i_{K+1-k}} = \frac{\partial \widehat{P}_r^{c,k}(\hat{\alpha},\zeta)}{\partial \hat{\alpha}^i} = \frac{\partial \widehat{P}_r^{c}(\hat{\alpha},\zeta)}{\partial \hat{\alpha}^i},$$

which by Corollary 6.14,

$$= \pi_r^{\tau_{k-1},i}$$

where

$$\begin{aligned}\dot{\pi}_r^{\tau_{k-1},i} &= f(\pi_r^{\tau_{k-1},i}, \widehat{P}_r^c) + f(\widehat{P}_r^c, \pi_r^{\tau_{k-1},i}) + \Gamma_{\hat{\alpha}^i}(\hat{\alpha}_{t-r}, \zeta_{t-r})\\ &= f(\pi_r^{\tau_{k-1},i}, \widehat{P}_r^{c,k}) + f(\widehat{P}_r^{c,k}, \pi_r^{\tau_{k-1},i}) + \Gamma_{\breve{\alpha}^i_{K+1-k}}(\breve{\alpha}_{K+1-k}, \zeta_{t-r}).\end{aligned}$$

Replacing $\pi_\cdot^{\tau_{k-1},i}$ with $\Pi_\cdot^{\tau_{k-1},i}$ completes the proof. ☐

We are now in a position to obtain the following identification.

Theorem 6.16. *Let $x, z \in \overline{B}_L(0)$ and $\zeta \in \mathcal{Z}_L$. For $c \in [0,\infty)$,*

$$\breve{\alpha}^* = \underset{\breve{\alpha}\in\mathcal{A}^K_o}{\operatorname{argmax}}\, \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}),c}(t,x,\zeta,z) \quad \textit{if and only if} \quad \widehat{F}(\breve{\alpha}^*) \doteq \nabla_{\breve{\alpha}} \mathcal{W}^{\mathcal{L}^K(\breve{\alpha}^*),c}(t,x,\zeta,z) = 0_{N\times K}$$

where the $(i,j)^{th}$ elements of $\widehat{F}(\breve{\alpha}^*)$ are given by

$$\widehat{F}_{ij}(\breve{\alpha}^*) = \tfrac{1}{2} X(x,z)\cdot \frac{\partial \widehat{P}_t^c(\mathcal{L}^K(\breve{\alpha}^*),\zeta)}{\partial \breve{\alpha}^i_j} = \begin{cases} \tfrac{1}{2} X(x,z)\cdot \Pi_{\tau_K}^{\tau_{K-1},i} & \textit{if } j = 1,\\ \tfrac{1}{2} X(x,z)\cdot \mathcal{D}_{K+2-j}\, \Pi_{\tau_{K+1-j}}^{\tau_{K-j},i} & \textit{if } j \in]2,K[.\end{cases} \tag{129}$$

Proof. By the differentiability and the concavity in $\breve{\alpha}$ given in Corollary 6.14 and Lemma 6.5, we have the assertions with the exception of the last representation for $\widehat{F}_{ij}(\breve{\alpha}^*)$.

For $k \in \mathcal{K}$, note that

$$\frac{\partial \widehat{P}^c_t}{\partial \breve{\alpha}_{K+1-k}} = \frac{\partial \widehat{P}^{c,K}_{\tau_K}}{\partial \widehat{P}^{c,K}_{\tau_{K-1}}} \frac{\partial \widehat{P}^{c,K-1}_{\tau_{K-1}}}{\partial \widehat{P}^{c,K-1}_{\tau_{K-2}}} \cdots \frac{\partial \widehat{P}^{c,k+1}_{\tau_{k+1}}}{\partial \widehat{P}^{c,k+1}_{\tau_k}} \Pi^{\tau_{k-1}}_{\tau_k}, \tag{130}$$

where the last term follows from (128) of Proposition 6.15. Letting

$$\varphi^{k+1}_r \doteq \frac{\partial \widehat{P}^{c,k+1}_r}{\partial \widehat{P}^{c,k+1}_{\tau_k}} \quad \forall r \in \mathcal{I}_{k+1}, \ \forall k \in]1, K-1[, \tag{131}$$

$$\dot{\varphi}^{k+1}_r = \frac{\partial}{\partial r} \frac{\partial \widehat{P}^{c,k+1}_r}{\partial \widehat{P}^{c,k+1}_{\tau_k}} = \frac{\partial}{\partial \widehat{P}^{c,k+1}_{\tau_k}} \frac{\partial \widehat{P}^{c,k+1}_r}{\partial r} = \frac{\partial}{\partial \widehat{P}^{c,k+1}_{\tau_k}} f(\widehat{P}^{c,k+1}_r, \widehat{P}^{c,k+1}_r)$$

with $\varphi^{k+1}_{\tau_k} = I_M$ where I_M denotes the identity matrix of size M and the m^{th} column of $\dot{\varphi}^{k+1}_r$ is given by

$$\dot{\varphi}^{k+1}_{m,r} = f(\varphi^{k+1}_{m,r}, \widehat{P}^{c,k+1}_r) + f(\widehat{P}^{c,k+1}_r, \varphi^{k+1}_{m,r})$$

for all $m \in]1, M[$ where $\widehat{P}^{c,k+1}_{\cdot}$ is given by (127). Substituting (131) into (130), we have

$$\frac{\partial \widehat{P}^c_t}{\partial \breve{\alpha}_{K+1-k}} = \begin{cases} \varphi^K_{\tau_K} \, \varphi^{K-1}_{\tau_{K-1}} \, \varphi^{K-2}_{\tau_{K-2}} \cdots \varphi^{k+1}_{\tau_{k+1}} \, \Pi^{\tau_{k-1}}_{\tau_k} \doteq \mathcal{D}_{k+1} \, \Pi^{\tau_{k-1}}_{\tau_k} & \text{if } k \in]1, K-1[, \\ \Pi^{\tau_{K-1}}_{\tau_K}. & \text{if } k = K. \end{cases}$$

Substituting these expressions into (129), we obtain the last representation. ☐

Remark 6.17. In the above development, we use $c < \infty$, whereas the TPBVP requires $c = \infty$, and the error induced by using $c < \infty$ is indicated in Theorem 4.15. However, in actual computations, when applying standard solution methods (e.g., Runge-Kutta methods) in solution of (66), (71), or equivalently, (77), taking very large c approaching ∞, leads to numerical difficulties. Here, we note a small practical point. An approximation for solution of (77) for $c = \infty$, $\breve{\alpha} \in \breve{\mathcal{A}}^K_o$ and $\zeta \in \mathcal{Z}_L$, on an arbitrarily short initial time segment, say $r \in (0, \check{\tau})$ with $\check{\tau} \ll 1$, is given by

$$\widehat{P}^{\infty}_r(\mathcal{L}^K(\breve{\alpha}), \zeta) = (\, r^{-1}, -r^{-1}, r^{-1}, r\breve{\sigma}^T, -r\breve{\sigma}^T, \tfrac{1}{3}|\breve{\sigma}|^2 r^3 + \breve{\eta} r \,)',$$

where

$$\breve{\sigma} \doteq \tfrac{1}{2} \sum_{i \in \mathcal{N}} \breve{\alpha}^i_K \zeta^i_t, \quad \breve{\eta} = \sum_{i \in \mathcal{N}} \mu_i \{2\breve{\alpha}^i_K - (\breve{\alpha}^i_K)^3 |\zeta^i_t|^2\},$$

which the reader may easily verify, and we do not attempt a bound on the error. Then, for $r > \check{\tau}$, one may continue with a standard method. This is employed in the examples below.

7 Examples

In Section 2, we indicate that $\overline{W}^{\infty}(t, \cdot, \zeta, \cdot)$ acts as a fundamental solution for multiple TPBVPs, given fixed duration, t, and set of large-body trajectories, ζ. The theory in support of the construction of $\overline{W}^{\infty}$ appears in Sections 3–5. Specifically, in (82), we see that for fixed t, ζ, $\overline{W}^{\infty}(t, \cdot, \zeta, \cdot)$ may be represented by a set, $\partial\langle \mathcal{G}_t \rangle \subset I\!R^{10}$. Additional issues regarding the computation of $\overline{W}^{\infty}(t, \cdot, \zeta, \cdot)$ and/or $\partial\langle \mathcal{G}_t \rangle$ are addressed in Section 6.

Below, we will apply $\overline{W}^{\infty}(t, \cdot, \zeta, \cdot)$ in solution of two different TPBVPs. In particular, we will consider one example with the boundary data being the endpoint positions, and one example with the boundary data being the initial position and the terminal velocity. These examples are intended to demonstrate that the fundamental solution applies to multiple boundary-data types.

At the next level, we note that one may precompute $\partial\langle\mathcal{G}_t\rangle \subset I\!R^{10}$, and store this for later use with varying boundary data. Tractable application at this next level will require investigation of means for efficient storage of the high-dimensional set, $\partial\langle\mathcal{G}_t\rangle$. Such storage should also contain data indicating the structure of the set. For example, in the case where one simply stores a closely packed set of points in the set, one should also include data regarding each point's neighbors. Further, one needs to develop efficient means for searching this set for the optimal point corresponding to the specific boundary data, as, for example, in (82). These last steps will require additional development beyond the already extensive material here.

7.1 Specifics of two problem classes

We present two examples: one case where the terminal position, $\xi_t = z$, is specified by choosing $\bar{\psi}(z) = \psi^\infty(\xi_t, z)$ as the terminal cost, and the other where the terminal velocity, $\dot{\xi}_t = \bar{v}$, is specified by taking $\bar{\psi}(z) = -z \cdot \bar{v}$ as the terminal cost (cf., [5, 13, 14]). In both cases, the initial position, x, is also specified. In the case $\bar{\psi}(z) = \psi^\infty(\xi_t, z)$, by (5)–(7) and (82),

$$\overline{W}(t,x,\zeta) = \overline{W}^\infty(t,x,\zeta,z) = \max_{P \in \partial\langle\mathcal{G}_t\rangle} \tfrac{1}{2} X(x,z) \cdot P.$$

For the case of $\bar{\psi}(z) = -z \cdot \bar{v}$, recalling (66), we see that we will have $r_t^\infty > 0$ for sufficiently small $t > 0$. Then, recalling (71) and (81), we see that $\overline{W}^\infty(t,x,\zeta,\cdot)$ will be strictly convex and coercive for such t. Consequently, by Theorem 2.1 and (81), we may rewrite (5) as

$$\begin{aligned}\overline{W}(t,x,\zeta) &= \min_{z \in I\!R^3} \{\overline{W}^\infty(t,x,\zeta,z) + \bar{\psi}(z)\} \\ &= \min_{z \in I\!R^3} \{\max_{\alpha \in \mathcal{A}^t} \tfrac{1}{2} X(x,z) \cdot \widehat{P}_t^\infty(\alpha,\zeta) + \bar{\psi}(z)\} = \min_{z \in I\!R^3} \{\max_{P \in \partial\langle\mathcal{G}_t\rangle} \tfrac{1}{2} X(x,z) \cdot P - \bar{v} \cdot z\},\end{aligned}$$

and by classical results on saddle points, cf. [15], this is

$$= \max_{P \in \partial\langle\mathcal{G}_t\rangle} \min_{z \in I\!R^3} \{\tfrac{1}{2} X(x,z) \cdot P - \bar{v} \cdot z\} = \max_{P \in \partial\langle\mathcal{G}_t\rangle} \{\tfrac{1}{2} X(x,\bar{z}(P)) \cdot P - \bar{v} \cdot \bar{z}(P))\}, \tag{132}$$

where $\bar{z}(P) \doteq r^{-1}(-qx - l + \bar{v})$ where $P \doteq (p,q,r,h',l',\gamma)'$. We see that in both cases, solution of the problem may be obtained from sets of solutions of differential Riccati equations.

7.2 Example 1: $\bar{\psi}(x) = \psi^\infty(x,z)$

In the first example, we solve a TPBVP where initial and terminal positions are specified. Suppose that five large bodies are moving clockwise on circular orbits with radii of 1.5, 3, 4.5, 6 and 7 AU. The masses of bodies are given by $[2,4,9,10,7] \times 10^{31} kg$. In order to ensure a sensible problem, and for reasons of error estimation, we generate a trajectory, $\hat{\xi}$, of the small body by forward propagation of Newton's second law from an initial position/velocity pair given by $x = [0,\ 1,\ 0.02]^T\ AU$, $u_0 = [35,\ 104,\ 0.3]^T\ km/s$ from initial time $r = 0$ to terminal time, $r = t = 11\ days$. This yields a terminal position, $z = [0.179, 5.265, 0.0078]^T$. We reconstruct the trajectory from solution of the TPBVP given by x, z and t. We remark that the body masses and duration form an exaggerated example, constructed such that the dynamics lead to an interesting trajectory. Further, we see that for all $i \in \mathcal{N}$, there exists $\delta_i > 0$ such that $|\hat{\xi}_r - \zeta_r^i| > \delta_i$ for all $r \geq 0$. Assuming the radius of body $R_i < \delta_i$ for all $i \in \mathcal{N}$, by Remark 4.6, the time duration that guarantees the convexity in the velocity control is given by $\tilde{t} = 13.5\ days$. We use piecewise constant potential energy controls over $[0,t]$ with a uniform grid $\{\tau_k | k \in]0,K[\,\}$, with $K = 25$. The small body is required to move through the intermediate points $\hat{\xi}_{t/4}, \hat{\xi}_{t/2}$ and $\hat{\xi}_{3t/4}$ where

$$\hat{\xi}_{t/4} = [0,132,1.708,-0.017],\ \ \hat{\xi}_{t/2} = [0.121,2.681,-.0.011],\ \ \hat{\xi}_{3t/4} = [0.134,3.958,0.0024].$$

The maximizing $\breve{\alpha}^* \in \breve{\mathcal{A}}_o^K$ is obtained numerically by a gradient ascent method using Theorem 6.16 and Remark 6.17. Figure 1 depicts the true trajectory of the small body as well as the trajectories of five large bodies. Figure 2 depicts the true trajectory $\hat{\xi}$ and the approximate solution of the TPBVP. As the bulk of the motion is in the first two coordinates, the trajectories are projected onto the plane generated by these coordinates.

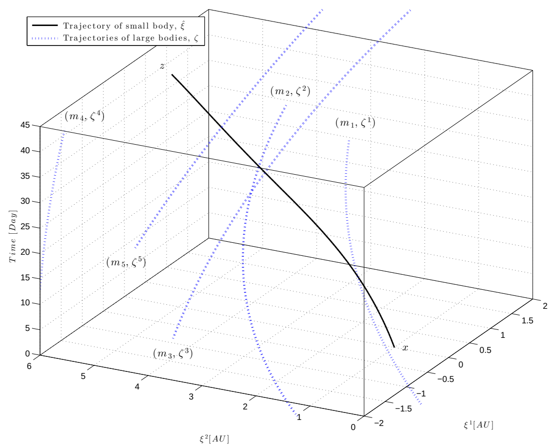

Figure 1: All trajectories with time axis

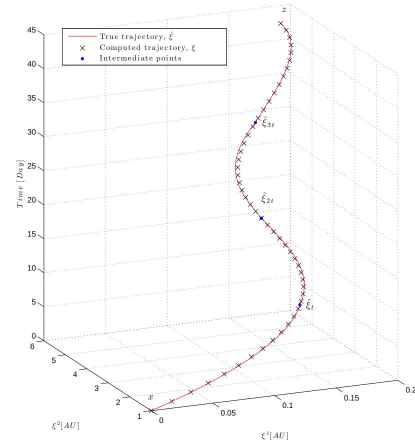

Figure 2: Truth and solution of TPBVP

7.3 Example 2: $\bar{\psi}(x) = -x \cdot \bar{v}$

Consider the same problem as in Example 1, but now we specify the terminal velocity $\bar{v} \in I\!R^3$, rather than the terminal position. Again this is accomplished by taking $\bar{\psi}(x) \doteq -x \cdot \bar{v}$. In this case, once we obtain the minimizer $z^* \doteq \bar{z}$, we have an equivalent initial value problem with boundary conditions $\xi_0 = x$ and $u_0^* = -\nabla_x \overline{W}^\infty(t, x, \zeta, z^*)$, and one may check by integration that this yields $u_t^* = \bar{v}$. More specifically, given $z \in I\!R^3$, the inner maximizing problem is solved via the numerical method introduced in Section 6, while the outer problem of minimization over z was solved via a gradient descent method.

In addition to the initial position specified in Example 1, we specified a terminal velocity of $\dot{\xi}_t = \bar{v} = [40.4, 65.6, 2.9 \times 10^{-5}]^T\, km/s$. The piecewise constant potential energy control approximation over $[0, t]$ used a uniform grid, $\{\tau_k \,|\, k \in]0, K[\,\}$ with $K = 400$. The resulting error in the terminal velocity was $|u_t^* - \bar{v}|/|\bar{v}| \simeq 0.005$.

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