MAE 289C, Introduction to Functional Analysis with Applications
Take-home final
Due 11:59pm, 12 June

Not all problems may be graded.
Your supporting work must be included for full credit.

1. (5 points) Consider the subspace in $\mathcal{U} \doteq L_2(0,1)$ given by

$$\mathcal{L} \doteq \{u \in \mathcal{U} \mid \exists\, k \in \mathbb{R} \text{ s.t. } u(t) = k[t^2 + \sin(\pi t)]\ \forall\, t \in (0,1)\}.$$

What is the dimension of the subspace? Find a subspace containing both $\mathcal{L}$ and the point $\hat{u} \in \mathcal{U}$ given by $\hat{u}(t) = 7$ for almost every $t \in (0,1)$? Recall the geometric form of the Hahn-Banach Theorem, which implies that at least in principle, one can construct a hyperplane containing the subspace you just constructed. In words, indicate, clearly, what steps you might take in this case?

2. (10 points) Consider the subsets of $\mathcal{U} \doteq L_2(0,1)$ given by

$$\mathcal{A} \doteq \left\{u \in \mathcal{U} \,\middle|\, \int_0^1 u^2(t)\,dt \le 1\right\},$$
$$\mathcal{C} \doteq \left\{u \in \mathcal{U} \,\middle|\, \exp\left[1 + \int_0^1 t^2 u(t)\,dt\right] \ge 4\right\}.$$

Can these be separated by a hyperplane such that the sets are strictly on different sides of the hyperplane? If so, find one such hyperplane.

3. (15 points) Using the method indicated in class, try to solve the following problem, or to at least obtain some necessary conditions. The problem is to minimize $\int_0^1 t x^2(t)\,dt$ over $x \in \mathcal{X} \doteq W^{1,2}(0,1)$ subject to the constraint $x(0.5) \ge 2$.

   - Is there an unconstrained minimum, or will it lie on the boundary of the constraint set?
   - What is a representation of the Gateaux derivative of the cost criterion?

- Try writing the Fréchet derivatives of both the cost function and the constraint function as elements of $\mathcal{X}$.
- Try to write down some necessary conditions similar in form to the Lagrange multiplier conditions.
- If you're somehow able, try to solve these necessary conditions by any means at all.

4. (10 points) Complete the first example that I did in class on Tuesday, 3 June, by filling in the steps I skipped. (To be clear, this is the example where $Ax[\omega] \doteq \frac{d^2}{d\omega^2}x(\omega)$ for all $\omega \in (0,1)$.)