MAE 289C, Introduction to Functional Analysis with Applications
Assignment 3
Due 11:59pm, 5 June

Not all problems may be graded.
Your supporting work must be included for full credit.
You may submit either a hardcopy (in class) or an electronic version by the deadline.
More problems will likely be added to the assignment.

1. (10 points) For $x \in L_2(\mathcal{B}_1)$, let $F(x) \doteq \int_{\mathcal{B}_1} x^2(\omega)\, d\omega$, where $\mathcal{B}_1$ denotes the unit ball centered at the origin. Is the range of F contained in $\mathbb{R}$? If so, try to obtain representations of the Gateaux and Fréchet derivatives as elements of $L_2(\mathcal{B}_1)$.

2. (15 points) Consider $f : L_2(0,1) \to \mathbb{R}$ given by
$$\int_{(0,1)} \sqrt{1 + x^2(t)}\, dt.$$
Try to obtain representations of the Gateaux and Fréchet derivatives as elements of $L_2(0,1)$, if they exist. Repeat the process (skipping redundant details, of course!), but replacing the domain space with $L_4(0,1)$, and using the appropriate $L_q(0,1)$ space for the derivatives (if they exist).

3. (10 points) For $x : [0,1] \to \mathbb{R}$, let $f(x) \doteq x(\frac{1}{2})$. Is f a bounded linear functional on $L_2(0,1)$? Is it a bounded linear functional on $W^{1,2}(0,1)$? In each case for which it is indeed a bounded linear functional, try to find a representation for the functional as an element of the space.

4. (10 points) With $x_0 \in \mathbb{R}$, let
$$F(u) = F(u; x_0) \doteq \int_0^2 (2-t)\xi^2(t) + u^2(t)\, dt,$$
$$\text{where} \;\; \xi(t) \doteq x_0 + \int_0^t r + e^{-r} u(r)\, dr \quad \forall\, t \in [0,2].$$
Using the approach from class, try to find $\inf_{u \in L_2(0,2)} F(u)$. If a minimizer exists, what is it?

5. (5 points) Give an example of a sublinear functional on $C[0,1]$ that is neither a norm nor a linear functional.

6. For $n \in \mathbb{N}$, let $x_n \in \ell_2$ be given by $x_n = \xi_j^n$ for $j \in \mathbb{N}$, where $\xi_j^x \doteq \delta_{n,j}$ for all n, j. Does the [strong] limit exist, and if so, what is it? Does the [weak] limit exist, and if so, what is it?

7. (10 points) This problem is a slight simplification of Taylor and Lay, Problem 3.2.3. Let $\mathcal{X}$ be a vector space. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be nonempty, convex, open subsets of $\mathcal{X}$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Show that there exists a hyperplane, $\mathcal{H}$, such that $\mathcal{C}_1$ lies on one side of the hyperplane, and $\mathcal{C}_2$ lies on the other side. This is a **separation** result. You do not need to prove that they lie **strictly** on one side of the hyperplane or the other. (Hint: The set $\mathcal{C} = \{x : x = x_1 - x_2, x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2\}$ is convex (show this) and open (skip showing this), and $0 \notin \mathcal{C}$. Use a theorem to show that there exists a hyperplane, $\mathcal{H}$, passing through the origin such that $\mathcal{C}$ lies on one side of $\mathcal{H}$. Use the linear functional associated with this hyperplane to obtain another, parallel hyperplane.)