MAE 289C, Introduction to Functional Analysis with Applications
Assignment 2
due 13 May

Not all problems may be graded.
Your supporting work must be included for full credit.

1. (10 points) Let $x^i \doteq \{\xi_k^i\}$ with $\xi_k^i = 1$ if $k \le i$ and $\xi_k^i = 0$ otherwise. Let $\mathcal{A} \subset \ell_2$ be given by $\mathcal{A} \doteq \{x^i \,|\, i \in \mathbb{N}\}$. Is $\mathsf{span}\,(\mathcal{A}) = \ell_2$?

2. (10 points) Prove that a set contained in a Banach space is compact (open cover based definition) if and only if it is sequentially compact.

3. (10 points) Let $\mathcal{X} \doteq L_2(\mathcal{D})$ with $\mathcal{D} \doteq \{z \in \mathbb{R}^2 \,|\, |z|_e < 1\}$ where in this context, $|z|_e$ denotes the Euclidean norm, and for $x \in \mathcal{X}$, let $|x|_2$ denote the usual L_2 norm. Let $\mathcal{Y} \doteq L_2(0,1)$ with the usual norm. Let $T : \mathcal{X} \to \mathcal{Y}$ be given by $[Tx](r) \doteq \int_{B_r(0)} x(z)\,dz$ for all $r \in (0,1)$, and where $B_r(0) \doteq \{z \in \mathbb{R}^2 \,|\, |z|_e < r\}$. (Throughout the course, unless otherwise indicated, all integrals are Lebesgue.) Is this a linear operator? Is it a bounded linear operator? If so, try to determine the bound.

4. (5 points) Let $x \in L_p(\mathcal{B})$, $y \in L_q(\mathcal{B})$, $z \in L_r(\mathcal{B})$, where $\mathcal{B} = \mathcal{B}_1(0)$ denotes the open unit disc in $\mathbb{R}^2$. Suppose $1/p + 1/q + 1/r = 1$. Is

$$\int_{\mathcal{B}} |x(\omega)y(\omega)z(\omega)|\,d\omega \le \Big(\int_{\mathcal{B}} |x(\omega)|^p\,d\omega\Big)^{1/p}\Big(\int_{\mathcal{B}} |y(\omega)|^q\,d\omega\Big)^{1/q}\Big(\int_{\mathcal{B}} |z(\omega)|^r\,d\omega\Big)^{1/r}?$$

5. (10 points) Let $\mathcal{B}$ denote the open unit ball in $\mathbb{R}^n$. For $x \in L_1(\mathcal{B};\mathbb{R})$ and $\epsilon > 0$, let

$$[f_\epsilon\, x](z) \doteq \int_{\mathcal{B}} \nu_\epsilon(z-\zeta)x(\zeta)\,d\zeta \quad \forall\, z \in \mathbb{R}^n,$$

where

$$\nu_\epsilon(z) \doteq \epsilon^{-n}\,\bar\nu(\tfrac{1}{\epsilon}z), \quad \bar\nu(z) \doteq \begin{cases} c\,\exp\left\{\frac{-1}{1-|z|^2}\right\} & \text{if } |z|<1, \\ 0 & \text{otherwise} \end{cases} \quad \forall\, z \in \mathbb{R}^n,$$

and $c \in \mathbb{R}$ is chosen such that $\int_{\mathcal{B}} \bar\nu(z)\,dz = 1$. For simplicity of the proofs, let $n = 1$. Is $f_\epsilon\, x \in C^\infty(\mathbb{R})$? Is f_ϵ linear? Show that if $x \in L_p(\mathbb{R};\mathbb{R})$, then $|f_\epsilon\, x|_p \le |x|_p$. Show that (for $x \in L_1(\mathbb{R};\mathbb{R})$) $|f_\epsilon\, x - x|_\infty \to 0$ as $\epsilon \downarrow 0$.

6. (10 points) For $a, b \in \mathbb{R} \cup \{-\infty\}$, define $a \oplus b \doteq a \vee b$ and $a \otimes b \doteq a + b$. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be convex and let $\mathcal{X}$ denote the set of convex functions on $\mathcal{C}$ (taking values in $\mathbb{R} \cup \{-\infty\}$). Show that with these definitions, $\mathcal{X}$ is closed under addition and under multiplication by a constant. Letting $\mathcal{C}$ be the closed unit ball in $\mathbb{R}^2$, $x(\omega) \doteq |\omega|^2$ and $y(\omega) \doteq 3\omega_1$, where ω_k denotes the k^{th} component of ω. Plot, possibly very roughly, $z \doteq x \oplus (-1 \otimes y)$.

7. (5 points) Consider the space $C([-1,1];\mathbb{R})$, but now with (L_2-type) inner product $(x, y) \doteq \int_{[-1,1]} x(t)y(t)\,dt$. Let $\mathcal{M}$ be the subset consisting of the odd continuous functions. Is this a subspace? What is the orthogonal complement of $\mathcal{M}$? (The orthogonal complement of a subset is the subspace consisting of functions that are orthogonal to all the elements of $\mathcal{M}$.)

8. (10 points) It was claimed in class that every infinite-dimensional separable Hilbert space is congruent to ℓ_2 in the specific sense indicated in class for inner-product spaces. Try to prove this.