MAE 289A: Mathematical Analysis for Applications
Take-Home Final
Due Thursday, 15 June, 5pm

(Complete list of problems.)

1. (5) Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be an inner product space over the real field, $\mathbb{R}$. Let $y, z \in \mathcal{X}$. Show that if $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in \mathcal{X}$, then $y = z$.

2. (10) Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be an inner product space over the real field, $\mathbb{R}$. Let $x, y \in \mathcal{X}$. Show that x and y are orthogonal (i.e., $\langle x, y\rangle = 0$) if and only if $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{R}$.

3. (10) Let $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real field, $\mathbb{R}$. Let $\mathcal{G} \subset \mathcal{X}$ be convex. Consider a sequence, $\{x_n\}$, such that $x_n \in \mathcal{G}$ for all n, and such that $\|x_n\| \to \inf_{x\in\mathcal{G}} \|x\|$. Show that there exists $\bar{x} \in \mathcal{X}$ such that $x_n \to \bar{x}$.

4. (10) Consider the space $(\ell_\infty, \|\cdot\|_\infty)$, which is the space of bounded infinite sequences of real numbers where for $x = \{x_n\}_{n=1}^\infty$, $\|x\|_\infty \doteq \sup_{n\in\mathbb{N}} |x_n|$. Also consider the space $(\ell_1, \|\cdot\|_1)$, which is the space of infinite sequences of real numbers, $x = \{x_n\}_{n=1}^\infty$, such that $\|x\|_1 \doteq \sum_{n=1}^\infty |x_n| < \infty$. Show that ℓ_1 is a subspace of ℓ_∞. Is it a closed subspace? Support your answer.

5. (10) Let our space be $L_2(0,\pi)$, which we recall has inner product $\langle x, y\rangle \doteq \int_0^\pi x(t)y(t)\,dt$. Consider $f : L_2(0,\pi) \to \mathbb{R}$ given by

$$f(x) \doteq \int_0^\pi sin(t)x^2(t)\,dt$$

for all $x \in L_2(0,\pi)$. What is the Fréchet derivative, $Df(x)$? In particular, find a function, say $y \in L_2(0,\pi)$, such that

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - \langle y, h\rangle|}{\|h\|} = 0, \tag{1}$$

in which case $Df(x) = y$. Provide a proof that your asserted function is indeed the Fréchet derivative, i.e., that (1) holds.

6. (10) Recall that the inner product on $H_{1,2}(0,2)$ is given by $\langle x, y \rangle \doteq \int_0^2 x(t)y(t) + \dot{x}(t)\dot{y}(t)\,dt$. Let $g : H_{1,2}(0,2) \to \mathbb{R}$ be given by $g(x) \doteq x(0)$. Prove that this is linear on $H_{1,2}(0,2)$. Can you find a $y \in H_{1,2}(0,2)$ such that $\langle x, y \rangle = g(x)$ for all $x \in H_{1,2}(0,2)$? Prove your claim.

7. (10) Recall that the inner product on $H_{1,2}(0,2)$ is given by $\langle x, y \rangle \doteq \int_0^2 x(t)y(t) + \dot{x}(t)\dot{y}(t)\,dt$. Using the approach indicated in class, minimize

$$J(x) \doteq \int_0^2 [x(t)]^2 + 4[\dot{x}(t)]^2\,dt$$

over $x \in H_{1,2}(0,2)$, subject to the constraint

$$x(0) = 3.$$