## MAE 289B <br> Take-Home Final Due 11:59pm, Thursday, 17 March

1. (6) Let $(\mathcal{X},(\cdot, \cdot))$ be a Hilbert space, and let $\mathcal{A} \subset \mathcal{X}$ be convex. Let $d \doteq \inf _{x \in \mathcal{A}}\|x\|$. Prove that if $\left\{x_{n}\right\}_{n=1}^{\infty}$ is such that $\left\|x_{n}\right\| \downarrow d$, then $\left\{x_{n}\right\}_{n=1}^{\infty}$ is Cauchy.
2. (4) Using the above result, show that any closed, convex subset of a Hilbert space contains a unique element on minimal norm.
3. (10) Let $\left[W^{1,2}(0,1)\right]^{\prime}$ denote the dual space of Sobolev space $W^{1,2}(0,1)$. (I have also used the notation $H_{1,2}$ for this Sobolev space; that is, $W^{1,2}$ and $H_{1,2}$ should be considered to be two different notations for the same space.) Let $f: W^{1,2}(0,1) \rightarrow \mathbb{R}$ be given by $f(x)=\langle f, x\rangle_{1,2} \doteq x(1 / 2)$ for all $x \in W^{1,2}(0,1)$.
(a) Prove that $f \in\left[W^{1,2}(0,1)\right]^{\prime}$.
(b) Is it a bounded linear functional on $L_{2}(0,1)$ ?
(c) Find the induced norm of $f$ (i.e., as an element of $\left.\left[W^{1,2}(0,1)\right]^{\prime}\right)$.
(d) As $W^{1,2}(0,1)$ is a Hilbert space, it is reasonable to expect that there is a Riesz representation for $f$ in the form of an element of $W^{1,2}(0,1)$. That is, one would expect that there exists $v \in$ $W^{1,2}(0,1)$ such that $f(x)=\langle f, x\rangle_{1,2}=(v, x)_{1,2}$ where $(\cdot, \cdot)_{1,2}$ denotes the inner product on $W^{1,2}(0,1)$. Find such as Riesz representation for $f$. Hint: Consider a continuous function, $v \in W^{1,2}(0,1)$ where $v(t)=w^{-}(t)$ for $t \in(0,1 / 2)$ and $v(t)=w^{+}(t)$ for $t \in$ $[1 / 2,1)$, where $w^{-}$and $w^{+}$are $C^{\infty}$ on their respective domains. Use integration by parts.
4. (5) Let $F: L^{3}(0,1) \rightarrow \mathbb{R}$ be given by $F(x) \doteq \int_{0}^{1} 8 x^{3}(t) d t$. Does the Gateaux derivative exist for all $x, h \in L^{3}(0,1)$ ? Where it exists, what is it? Rigorously support your answers.
5. (5) Let $F: L^{2}(0, \pi) \rightarrow \mathbb{R}$ be given by $F(x) \doteq \int_{0}^{\pi} \cos (x(t)) d t$. Does the Fréchet derivative exist for all $x \in L^{2}(0, \pi)$ ? Where it exists, obtain
a representation for it as an element of $L^{2}(0, \pi)$. Rigorously support your answers.
6. (5) Formally, use the calculus of variations to find a stationary point of $G(u(\cdot))=\frac{1}{2} \int_{0}^{1}|\xi(r)|^{2}+|u(r)|^{2} d r$ where $\xi(0)=x_{0}$ and $\dot{\xi}(r)=\xi(r)+$ $2 u(r)$ for $r \in(0,1)$. (For this problem, you do not need to prove the validity of each step in your analysis.)
7. (5) Formally, solve the following optimal control problem by following the steps given in the last class. Plot the resulting solution, and comment on the smoothness. (As with the previous problem, you do not need to prove the validity of any of the steps.)

$$
\begin{aligned}
& \text { Minimize } \quad F(\xi) \doteq \frac{1}{2} \int_{0}^{1} \xi^{2}(t)+\dot{\xi}^{2}(t) d t \\
& \text { subject to } \quad \xi \in W^{1,2}(0,1), \quad \xi\left(\frac{1}{2}\right) \geq 3
\end{aligned}
$$

