MAE 289B
Assignment 3
Due 12:30pm, Tuesday, 8 March

Problems to hand in (Not all problems will be graded.)

1. (5) Construct an infinite sequence of piece-wise linear, linearly independent functions in $C[0,1]$.

2. (5) Is the set of continuous, piece-wise linear functions (each consisting of a finite number of segments) a subspace of the normed (vector) space C[0,1]? Is it dense? Support your answers of course.

3. (5) Let $\mathcal{D} \doteq \left\{ x \in \mathbb{R}^2 \,\middle|\, |x| \le 1 \right\}$ and $1 \le p \le q < \infty$. (Note that we are not assuming that p, q are conjugate exponents in this problem.) Show that $L_q(\mathcal{D}) \subseteq L_p(\mathcal{D})$. Also, can you find a $k = k(p,q) \in (0,\infty)$ such that $\|x\|_{L_p} \le k\|x\|_{L_q} \,\forall\, x \in L_q(\mathcal{D})$? If so, what is it? If possible, find the smallest k such that $\|x\|_{L_p} \le k\|x\|_{L_q} \,\forall\, x \in L_q(\mathcal{D})$.

4. (5) Consider $L_2(0,1)$, and let $\bar{B}_1(0)$ denote the elements of the space such that $\|x\| \le 1$. Find a sequence of functions, say $\{x_n\}_{n=1}^{\infty}$, with $x_n \in \bar{B}_1(0) \subset L_2(0,1) \,\forall\, n$, and such that $\|x_n - x_m\|_{L_2} \ge 1/2 \,\forall\, m, n \in \mathbb{N}$.

5. (5) Prove that $C[0,1]$ is a closed, proper subspace of $L_\infty[0,1]$. Consider the application of Riesz's Lemma in the case of $\mathcal{X} = \mathcal{Z} = L_\infty[0,1]$, and $\mathcal{Y} \doteq C[0,1]$. Indicate a $\bar{z} \in \mathcal{Z}$ satisfying the assertions of Riesz's Lemma applied to these spaces/subspaces with $\theta = \frac{1}{2}$. (In particular, you must show that your $\bar{z}$ actually satisfies the assertions.)

6. (2) Let $(\mathcal{X}, \|\cdot\|_X)$ be a Banach space. Let $\mathcal{C} \subset \mathcal{X}$ be compact. Let $f : \mathcal{C} \to \mathbb{R}$ be continuous (but not necessarily linear). Prove that $\min_{x \in \mathcal{C}} f(x)$ exists.

7. (3) Let $\mathcal{S}_1(0) \doteq \{x \in L_2(0,1) \,|\, \|x\| = 1\}$. Prove that $\mathcal{S}_1(0)$ is closed. Prove that a continuous (not necessarily linear) function $f : \mathcal{S}_1(0) \to \mathbb{R}$ does not necessarily achieve its minimum on $\mathcal{S}_1(0)$, i.e., that $\min_{x \in \mathcal{S}_1(0)} f(x)$ may not exist.

8. (3) Let $k \in C([1,3] \times [1,3])$ and $M_k \doteq \max\left\{|k(s,t)| \,\middle|\, (s,t) \in [1,3] \times [1,3]\right\}$. For $x \in \mathcal{X} \doteq L_2(1,3)$, let $[Tx](s) \doteq \int_1^3 k(s,t)x(t)\,dt$ for all $s \in (1,3)$. Show that T is a bounded linear operator from $\mathcal{X}$ into $\mathcal{X}$, i.e., that $T \in \mathcal{B}(\mathcal{X},\mathcal{X})$. What is the induced norm of T, $\|T\|_{\mathcal{B}(\mathcal{X},\mathcal{X})}$?

9. (7) (from Royden) Let $g \in L_1(0,1)$. Show that there exists a bounded, measurable $f : (0,1) \to \mathbb{R}$ such that

$$\int_0^1 f(t)g(t)\,dt = \|g\|_{L_1}\|f\|_{L_\infty}.$$

Also, letting $g : (0,1) \to \mathbb{R}$ be bounded and measurable, and letting $\varepsilon > 0$, show that there exists $f \in L_1(0,1)$ such that

$$\int_0^1 f(t)g(t)\,dt \geq (\|g\|_{L_\infty} - \varepsilon)\|f\|_{L_1}.$$

Note that, as in Royden, the spaces that the space containing f and the space containing g in the second part of the problem are reversed from what they were in the first part of the problem. (Hint: f may be taken to be a suitable characteristic function.) You may freely employ any results from Royden and/or the class notes.