MAE 289B
Assignment 2
Due 11:59pm, Monday, 7 February

Problems to hand in (Not all problems will be graded.)

1. (5) (from Royden) Prove that if $\mathcal{E} \subset \mathbb{R}$ is (Lebesgue) measurable, then for any $y \in \mathbb{R}$, $y + \mathcal{E}$ is also (Lebesgue) measurable.

2. (5) Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz, with constant $K < \infty$. Suppose $m(\mathcal{A}) = 0$. Show that the image of $\mathcal{A}$ under f, say $\mathcal{B}$, satisfies $m(\mathcal{B}) = 0$. Hint: First prove that for any interval, (a, b), $m^*(\{f(x) \,|\, x \in (a, b)\}) \leq K(b-a)$. Then, use Prop. 3.15 (2nd edition) to approximate $\mathcal{A}$ by a countable set of open intervals whose union contains $\mathcal{A}$.

3. (5) (from Royden) Let $f : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ be a measurable, nonnegative function. Show that if $\int_{\mathcal{D}} f = 0$, then $f = 0$ a.e. on $\mathcal{D}$.

4. (5) Let f measurable, with $f(x) > 0$ a.e. on $\mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}$ be measurable. Show that if $\int_{\mathcal{D}} f = 0$, then $m(\mathcal{D}) = 0$.

5. (5) Recall the Bounded Convergence Theorem. Demonstrate that both the boundedness assumption and the assumption of the finiteness of the measure of the domain are each necessary.

6. (5) (from Royden) Show that the Monotone Convergence Theorem need not hold for monotonically decreasing sequences of nonnegative functions.

7. (5) Suppose $f_n : \mathcal{D} \to [0, \infty)$ are measurable for all n, with $f_n(x) \to f(x)$ for all $x \in \mathcal{D}$. Suppose also that $\int_{\mathcal{D}} f_n \to \int_{\mathcal{D}} f$ where $\int_{\mathcal{D}} f < \infty$. Show that $\int_{\mathcal{A}} f_n \to \int_{\mathcal{A}} f$ for all $\mathcal{A} \in \Sigma_{\mathcal{L}}$ such that $\mathcal{A} \subseteq \mathcal{D}$.

8. (5) Consider $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sqrt{x}\cos(\pi/x) & \text{otherwise.} \end{cases}$$

   Is f of bounded variation on $[0, 1]$? Is f absolutely continuous on $[0, 1]$? Is f uniformly continuous on $[0, 1]$?

9. (5) (from Royden) Suppose $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Show that the total variation is given by $T(a, b, f) = \int_a^b |f'(x)|\, dx$.