1. (10) Protter and Morrey, Sec. 6.4, Problem 1. (That is, Show that the union of a compact set and a finite set is compact.)

2. (10) Consider the normed vector space \((\ell^2, \| \cdot \|_2)\), i.e., the set of infinite sequences of real numbers, say \(\{x_n\} = \{x_n\}_{n=1}^\infty\), such that \(\sum_{n=1}^\infty x_n^2 < \infty\), and where \(\|\{x_n\}\|_2 = [\sum_{n=1}^\infty x_n^2]^{1/2}\). Let \(e^i = \{e^i_n\}\) denote the element of \(\ell^2\) such that \(e^i_i = 1\) and \(e^i_n = 0\) if \(n \neq i\). Let \(A = \{e^i \mid i \in \mathbb{N}\}\). Is \(A\) closed? Is \(A\) bounded? Is \(A\) compact? In any cases where the answer is affirmative, provide a proof. Where your answer is negative, provide a counterexample.

3. (10) Let \((\mathcal{X}, \| \cdot \|_x)\) and \((\mathcal{Y}, \| \cdot \|_y)\) denote normed vector spaces. Suppose \(f : \mathcal{X} \rightarrow \mathcal{Y}\) is continuous. Suppose \(A \subset \mathcal{X}\) is compact. Prove that \(f(A) \doteq \{y \in \mathcal{Y} \mid \exists x \in A \text{ s.t. } y = f(x)\}\) is compact.

4. (10) Disprove the claim that a continuous function, \(f : \mathbb{R}^n \rightarrow \mathbb{R}\), maps closed sets onto closed sets, i.e., that if \(C\) is closed, then \(f(C) = \{y \in \mathbb{R} \mid \exists x \in C \text{ s.t. } y = f(x)\}\) is closed.

5. (10) In the proof of the Banach Fixed Point Theorem, we saw that if \(F : \mathcal{X} \rightarrow \mathcal{X}\) is a contraction, then starting from any \(x_0 \in \mathcal{X}\), the iteration \(x_{n+1} = F(x_n)\) converges to the unique fixed point. Try applying five iterations of this algorithm (as applied to initial value problems with ODE dynamics), to obtain a solution approximation of \(\dot{\xi} = -2\xi, \xi_0 = 1\). You may start from any function satisfying the initial condition that you like. Comment on the resulting solution approximation. (N.B.: This is not generally a computationally efficient algorithm!)

6. (10) Suppose \(f : \mathbb{R}^n \rightarrow \mathbb{R}, x \in \mathbb{R}^n\) and \(\beta \in \mathbb{R}^n\) with \(\|\beta\| \neq 0\). Prove that if the one-sided directional derivatives \(\partial_\beta f(x)\) and \(\partial_{-\beta} f(x)\) exist,
and if $\partial_{-\beta}f(x) = -\partial_{\beta}f(x)$, then the (two-sided) directional derivative, $D_\beta f(x)$, exists and $D_\beta f(x) = \partial_{\beta}f(x)$. You should use an “$\varepsilon - \delta$” proof to show that the required limit exists.