MAE 289B
Assignment 1
Due 11:59pm, Monday, 24 January

Problems to hand in (Not all problems will be graded.)

1. (3) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be subsets of space $\mathcal{X}$. Prove that $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$.

2. (2) Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that given any $\varepsilon > 0$, there exists $x \in \mathbb{R}^n$ such that $f(x) \leq 2 + \varepsilon$. Prove, by contradiction, that $\inf_{x \in \mathbb{R}^n} f(x) \leq 2$.

3. (5) Let Σ_1 and Σ_2 be two σ-algebras on some set $\mathcal{X}$. Prove that $\Sigma_3 \doteq \Sigma_1 \cap \Sigma_2$ is a σ-algebra.

4. (5) Let $\mathbb{N}$ denote the set of natural numbers. Indicate two distinct σ-algebras on $\mathbb{N}$, neither of which has only a finite number of elements, and further, such that one is contained within the other. (You may want to prove and use the following. *Lemma: Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces, and suppose f maps $\mathcal{X}$ onto $\mathcal{Y}$. Let $\Sigma_{\mathcal{Y}}$ be a σ-algebra on $\mathcal{Y}$, and let $\Sigma_{\mathcal{X}} \doteq \{f^{-1}(\mathcal{A}) \,|\, \mathcal{A} \in \Sigma_{\mathcal{Y}}\}$. Then $\Sigma_{\mathcal{X}}$ is a σ-algebra on $\mathcal{X}$.* There may be other methods.)

5. (5) Royden, Problem 2.35 (2.34 in 2nd ed.): Prove the following proposition (using the Heine-Borel Theorem and De Morgan's laws). Let $\mathcal{C}$ be a collection of closed sets (in $\mathbb{R}$) such that every finite subcollection has a nonempty intersection, and suppose that at least one of the sets is bounded. Then $\bigcap_{\mathcal{F} \in \mathcal{C}} \mathcal{F} \neq \emptyset$.

6. (5) Prove that every accumulation point of the Cantor ternary set is an element of the set. (Recall that x is an accumulation point of set $\mathcal{A}$ if it is in the closure of $\mathcal{A} \setminus \{x\}$.)

7. (5) Royden, Problem 3.7: Prove that m^* is translation invariant.

8. (5) Suppose $\mathcal{E} \in \Sigma_{\mathcal{L}}$ (i.e., Lebesgue measurable) with $m(\mathcal{E}) > 0$. Prove that for any $\alpha \in (0, 1)$, there exists an open interval, $\mathcal{I}$, such that $m(\mathcal{E} \cap \mathcal{I}) = \alpha m(\mathcal{E})$. (You may take intervals such as $(-\infty, b)$ and (a, ∞) if you like.)