1. (10) Let \( x = (x_1, x_2) \) represent a generic element of \( X = \mathbb{R}^2 \). Draw \( B_1((1,1)) \) for the four norms:

(a) the Euclidean norm, i.e. \( \|x\|_2 = [x_1^2 + x_2^2]^{1/2} \),

(b) the norm given by \( \|x\|_1 = |x_1| + |x_2| \),

(c) the norm given by \( \|x\|_\infty = \max\{|x_1|, |x_2|\} \), and

(d) the norm given by \( \|x\|_4 = [x_1^4 + x_2^4]^{1/4} \). (A rough sketch is fine.)

2. (10) On \( \mathbb{R}^2 \), let \( \|x\|_\infty \) and \( \|x\|_2 \) be as indicated in class. (Again, recall that \( \|x\|_2 \) denoted the Euclidean norm of \( x \) and \( \|x\|_\infty = \max\{|x_1|, |x_2|\} \) for all \( x \in \mathbb{R}^2 \).) Write a theorem stating that for all \( x \in \mathbb{R}^2 \), \( c_1 \|x\|_2 \leq \|x\|_\infty \leq c_2 \|x\|_2 \), where you specify specific numbers for \( c_1 \) and \( c_2 \). Also assert in your theorem that your choice of \( c_1 \) is the largest number for which the result holds and that \( c_2 \) is the smallest number for which the result holds. Prove the result.

3. (10) Two norms on \( C([0,1]) \) are \( \|x\|_2 = \sqrt{\int_0^1 x^2(t) \, dt} \) and \( \|x\|_1 = \int_0^1 |x(t)| \, dt \). Find a sequence of continuous functions, say \( \{x_n(\cdot)\}_{n=1}^\infty \) such that \( \|x_n\|_1 \to 0 \) as \( n \to \infty \), but such that there exists \( c > 0 \) such that \( \|x_n\|_2 \geq c \) for all \( n \).

4. (10) Show that the claimed metric in class for a finite, undirected graph does indeed satisfy the triangle inequality requirement of a metric. Specifically, let the graph be denoted by the pair \((\mathcal{G}, \mathcal{E})\), where \( \mathcal{G} = \{1,2,\ldots,n\} \) (for some finite \( n \)) denotes the set of nodes, and \( \mathcal{E} \) denotes some set of edges, where we require \((i,i) \in \mathcal{E}\) for all \( i \in [1,n]\) and \( \{1,2,\ldots,n\} \), and that \((i,j) \in \mathcal{E}\) if and only if \((j,i) \in \mathcal{E}\). You may assume that you also have single-edge costs satisfying \( c(i,j) = c(j,i) > 0 \)
for all \((i, j) \in \mathcal{E}\) such that \(i \neq j\), that \(c(i, i) = 0\) for all \(i \in \mathcal{G}\), and that \(c(i, j) = +\infty\) for all \((i, j) \notin \mathcal{E}\).

5. (10) Let \(S \subset \mathbb{R}, S \neq \emptyset\). Prove that if \(z = \inf S > -\infty\), then there exists \(\{x_n\}_{n=1}^{\infty} \subseteq S\) such that \(x_n \to z\). (This also holds in the case with \(\inf S = -\infty\), but you do not need to prove that.)

6. (10) Protter and Morrey, Sec. 6.3, Problem 9. (You might want to use the fact that the rational numbers are dense in \(\mathbb{R}\), which implies that given any \(a, b \in \mathbb{R}\) with \(a < b\), there exists a rational \(q\) such that \(a < q < b\).) The problem statement is as follows. Let \(G\) be an open set in \(\mathbb{R}\). Show that \(G\) can be represented as the union of open intervals with rational endpoints.

7. (10) Protter and Morrey, Sec. 6.4, Problem 2a. The problem statement is as follows. Show that the intersection of any number of compact sets is compact. (Regarding the definition of compactness, you may use either the first definition given in class or the equivalent definition in terms of sequences.)

You may find one or more of these problems to be quite challenging. Don’t worry – that’s to be expected in a course like this.