1. (5) Let $X = l_2$. For any $x = \{\xi_i\} \in l_2$, define

$$f(x) = \sum_{i=1}^{\infty} \xi_i.$$ 

Is $f \in X'$? If so, what is $\|f\|$? Address the same questions for

$$g(x) = \sum_{k=1}^{\infty} \frac{\xi_i}{\sqrt{k/2}}.$$ 

2. (5) (from Royden) Let $g \in L_1(0, 1)$. Show that there exists a bounded, measurable $f : (0, 1) \to \mathbb{R}$ such that $\|f\|_\infty \neq 0$ and $\int_{(0,1)} f(t)g(t) \, dt = \|g\|_1 \|f\|_\infty$. 

3. (5) Let $(X, (\cdot, \cdot))$ be a Hilbert space, and let $A \subset X$ be convex. Let $d = \inf_{x \in A} \|x\|$. Prove that if $\{x_n\}_{n=1}^{\infty}$ is such that $\|x_n\| \downarrow d$, then $\{x_n\}_{n=1}^{\infty}$ is Cauchy. 

4. (5) Using the above result, show that any closed, convex subset of a Hilbert space contains a unique element on minimal norm. 

5. (10) Let $[W^{1,2}(0,1)]'$ denote the dual space of Sobolev space $W^{1,2}(0, 1)$. Let $f : W^{1,2}(0, 1) \to \mathbb{R}$ be given by $f(x) = \langle f, x \rangle_{1,2} = x(1/2)$ for all $x \in W^{1,2}(0, 1)$. 

(a) Prove that $f \in [W^{1,2}(0,1)]'$. 
(b) Is it a bounded linear functional on $L_2(0, 1)$? 
(c) Find the induced norm of $f$ (i.e., as an element of $[W^{1,2}(0,1)]'$). 
(d) As $W^{1,2}(0, 1)$ is a Hilbert space, it is reasonable to expect that there is a Riesz representation for $f$ in the form of an element
of $W^{1,2}(0,1)$. That is, one would expect that there exists $v \in W^{1,2}(0,1)$ such that $f(x) = \langle f, x \rangle_{1,2} = (v, x)_{1,2}$ where $\langle \cdot, \cdot \rangle_{1,2}$ denotes the inner product on $W^{1,2}(0,1)$. Find such a Riesz representation for $f$. Hint: Consider a continuous function, $v \in W^{1,2}(0,1)$ where $v(t) = w^-(t)$ for $t \in (0,1/2)$ and $v(t) = w^+(t)$ for $t \in [1/2,1)$, where $w^-$ and $w^+$ are $C^\infty$ on their respective domains. Use integration by parts.

6. (5) Formally, use the calculus of variations to find a stationary point of $G(u(\cdot)) = \int_0^1 |\xi(r)|^2 + |u(r)|^2 \, dr$ where $\xi(0) = x_0$ and $\dot{\xi}(r) = \xi(r) + u(r)$ for $r \in (0,1)$. (You do not need to prove the validity of each step in your analysis.)