MAE 288A
Take-Home Final
Due 11:59pm, Friday, 12 June

Problems to hand in (Not all problems may be graded.)

All work, code and explanations must be included for full credit.

1. (10) Under the assumptions indicated in class, and with the problem definition also given there, prove the following lemma from class:
   There exists $K_V < \infty$ such that
   \[
   |\bar{V}(s, x) - \bar{V}(s, y)| \leq K_V(1 + |x| + |y|)|x - y| \quad \forall x, y \in \mathbb{R}^n, \forall s \in [0, T].
   \]
   A partial proof was provided in class. In your proof, you may use any results obtained prior to that lemma.

2. (10) By hand (intuition should be quite helpful), construct/guess viscosity solutions to the following two problems. Prove that they are, in fact, viscosity solutions. The problems are
   \[
   0 = 4 - |\nabla x W|^2, \quad x \in (-1, 1),
   \]
   \[
   W(-1) = W(1) = 0,
   \]
   and
   \[
   0 = |\nabla x W|^2 - 4, \quad x \in (-1, 1),
   \]
   \[
   W(-1) = W(1) = -2.
   \]
   Take care with the signs.

3. (10) Consider the HJ PDE and boundary condition
   \[
   |\nabla V|^2 - 1 = 0 \quad (x, y) \in G
   \]
   \[
   V(x, y) = 0 \quad (x, y) \in \partial G
   \]
   where
   \[
   G = \left\{ (x, y) \in \mathbb{R}^2 \left| \frac{x^2}{4} + y^2 \leq 1 \right. \right\}.
   \]
Obtain the characteristic equations and their initial conditions. Compute the value at the point
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
2/\sqrt{2} \\
1/\sqrt{2}
\end{pmatrix} - \frac{1}{4} \begin{pmatrix}
1/\sqrt{2} \\
2/\sqrt{2}
\end{pmatrix}.
\]
(The point is given in a form that should make it easy to find the correct characteristic.)

4. (5) Consider the following extremely simple HJ PDE problem:
\[
0 = V_s - V_x, \quad (s, x) \in (0, T) \times \mathbb{R},
\]
\[
V(T, x) = \sqrt{a + x^2}, \quad x \in \mathbb{R},
\]
where \(a = 0.25\) and \(T = 2.5\). (This is in a form analogous to that in the example from class with which we developed the finite element method for this class of problems, i.e., without the multiplication by \(-1\). The solution to this problem is differentiable and hence viscosity-solution issues do not come into play.) Solve the problem over \([0, T] \times \mathbb{R}\) using the method of characteristics.

5. (10) Suppose now that you will use the finite-element method indicated in class to solve the previous problem (Problem 4) over the time interval \([0, T]\) and space region that starts out as \(x \in [-2, 2]\) at time \(T\). (This may need to shrink as time decreases toward \(s = 0\).)

(a) Suppose the space discretization size is \(\Delta x\). Based on your solution to the previous problem, estimate the maximum allowable time discretization size, \(\Delta s\), such that you would expect the method to perform well.

(b) You will use a time-derivative approximation
\[
V_s(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_j) - V(\tau_k, x_j)}{\Delta s}.
\]
Consider the two space-derivative approximations,
\[
V_x(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_{j+1}) - V(\tau_{k+1}, x_j)}{\Delta x} \quad (1)
\]
and
\[ V_x(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_j) - V(\tau_{k+1}, x_{j-1})}{\Delta_x}. \] (2)

Construct the updates for each case. That is, obtain the expressions for the \( V(\tau_k, x_j) \) values from the \( V(\tau_{k+1}, x.) \) values in both cases.

(c) Try solving the problem with each of these methods, using \( \Delta_x = 0.1 \) and \( \Delta_x = 0.08 \). Plot the solutions. By substituting back into the HJ PDE, determine how well each is working. Discuss the reason(s) for the performance difference (if any).

(d) For the better performing version (or either, if they perform equally well/poorly), reduce the space discretization size to \( \Delta_x = 0.05 \), while keeping \( \Delta_x = 0.08 \). Discuss the resulting solution approximation, and the reason(s) for any performance difference.