MAE 288A
Take-Home Final
Due Friday, 14 June, 11pm

Complete set of problems. No problems will be added to this.

You are free to use Matlab and/or Python for any of the problems. If you do, you must include copies of the codes.

1. (10) Consider the optimal control problem with dynamics

$$\dot{\xi} = A\xi_t + Bu_t, \qquad \xi_s = x \in I\!R^n$$

and payoff

$$J(s,x;u.) = \int_s^1 \left[\tfrac{1}{2}\xi_t^T C\xi_t + \tfrac{1}{2}u_t^T Du_t\right] + \tfrac{1}{2}\xi_1^T F\xi_1.$$

Let $u.$ be a minimizing control, and let the value function be denoted by $\bar{V}(s,x)$ for $(s,x) \in [0,1] \times I\!R^n$. Let the control take values in $I\!R^m$. Suppose $\bar{V}(s,x)$ takes the form $\bar{V}(s,x) = \frac{1}{2}x^T P_s x$ for all s, x. Find the differential Riccati equation and terminal condition for P. Obtain $V(s,x)$ for $s = 1$, $s = 0.75$, $s = 0.5$, $s = 0.25$ and $s = 0$ for all $x \in I\!R^2$ given problem data

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 2, \quad F = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

2. (10) By hand (intuition should be quite helpful), construct/guess viscosity solutions to the following two problems. Prove that they are, in fact, viscosity solutions. The problems are

$$\begin{aligned} &0 = 4 - |\nabla_x W|^2, \quad x \in (-1,1), \\ &W(-1) = W(1) = 0, \end{aligned}$$

and

$$\begin{aligned} &0 = |\nabla_x W|^2 - 4, \quad x \in (-1,1), \\ &W(-1) = W(1) = -2. \end{aligned}$$

Take care with the signs!

3. (15) Consider the HJ PDE and boundary condition

$$\begin{aligned} &|\nabla V|^2 - 1 = 0 \quad (x,y) \in G \\ &V(x,y) = 0 \quad (x,y) \in \partial G \end{aligned}$$

where

$$G = \left\{(x,y) \in I\!R^2 \,\middle|\, x \in (-2,2),\, y \in (-1,1)\right\}.$$

Obtain the characteristic equations and their initial conditions. Draw (by hand is fine) the state-space components of the characteristics over G. Clearly indicate where they cross (i.e., run into one another), by annotations and/or text. There should be several line segments where this occurs. You may assume that each state-space characteristic curse ceases to exist beyond any point beyond which it crosses another. Compute the solution values $V(0,-0.5)$ and $V(-0.25,0)$.

4. (8) Prove the second half of the viscosity-solution proof from class. That is, prove that the value function, $\bar{V}(\cdot,\cdot)$, is a viscosity supersolution of the associated HJ PDE for a control problem given by

$$\begin{aligned}
&\dot{\xi}_t = f(\xi_t, u_t) \ \text{ for } \ t \in (s,T), \quad \xi_s = x \in I\!R^n, \quad s \in (0,T), \\
&\mathcal{U}_{s,T} = L_2((s,T);U), \quad U \subset I\!R^m, \ \text{ compact}, \\
&J(s,x,u) \doteq \int_s^T L(\xi_t, u_t)\,dt + \psi(\xi_T), \quad \bar{V}(s,x) \doteq \inf_{u \in \mathcal{U}_{s,T}} J(s,x,u)
\end{aligned}$$

(where, of course, the square integrability requirement is redundant for a measurable function from (s,T) into compact U, and you should completely ignore this parenthetical comment if it seems confusing to you). Let f, L, ψ satisfy the assumptions used in class, and feel free to use all the lemmas (without proving them) that were obtained in class while building to the result. Feel free to make any additional helpful assumptions that I might have forgotten.

5. (2) Consider the following extremely simple HJ PDE problem:

$$\begin{aligned}
&0 = -V_s - V_x, \quad (s,x) \in (0,T) \times I\!R, \\
&V(T,x) = \sqrt{a + x^2}, \quad x \in I\!R,
\end{aligned}$$

where $a = 0.25$ and $T = 2.5$. (The solution to this problem is differentiable and hence viscosity-solution issues do not come into play.) Solve the problem over $[0,T] \times I\!R$ using the method of characteristics.

6. (15) Suppose now that you will use the finite-element method indicated in class to solve the previous problem (Problem 5) over the time interval $[0,T]$ and space region that starts out as $x \in [-2,2]$ at time T. (This may need to shrink as time decreases toward $s = 0$.)

   (a) Suppose the space discretization size is Δ_x. Based on your solution to the previous problem, estimate the maximum allowable time discretization size, Δ_s, such that you would expect the method to perform well.

   (b) You will use a time-derivative approximation

   $$V_s(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_j) - V(\tau_k, x_j)}{\Delta_s}.$$

   Consider the two space-derivative approximations,

   $$V_x(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_{j+1}) - V(\tau_{k+1}, x_j)}{\Delta_x} \tag{1}$$

   and

$$V_x(\tau_k, x_j) \simeq \frac{V(\tau_{k+1}, x_j) - V(\tau_{k+1}, x_{j-1})}{\Delta_x}. \tag{2}$$

Construct the updates for each case. That is, obtain the expressions for the $V(\tau_k, x_j)$ values from the $V(\tau_{k+1}, x_\cdot)$ values in both cases.

(c) Try solving the problem with each of these methods, using $\Delta_x = 0.1$ and $\Delta_s = 0.08$. Plot the solutions. By substituting back into the HJ PDE, determine how well each is working. Discuss the reason(s) for the performance difference (if any).

(d) For the better performing version (or either, if they perform equally well/poorly), reduce the space discretization size to $\Delta_x = 0.05$, while keeping $\Delta_s = 0.08$. Discuss the resulting solution approximation, and the reason(s) for any performance difference.