MAE 288A
Take-Home Final
Due Thursday, 15 June, 5pm

(Complete list of problems.)

You are free to use Matlab, Octave, Python, C, C++, Java, Fortran or Mathematica for any problems in the course. If you do, please include copies of the codes.

1. (10) Consider the optimal control problem with dynamics

\[ \dot{\xi} = A\xi_t + Bu, \quad \xi_s = x \in \mathbb{R}^n \]

and payoff

\[ J(s, x; u) = \int_s^1 \left[ \frac{1}{2}(\xi_t - \hat{x})^T C(\xi_t - \hat{x}) + \frac{1}{2} u_t^T Du_t \right] + \frac{1}{2} \xi_1^T F \xi_1. \]

(Note that \( \hat{x} \) is a fixed vector – not time-dependent.) Let \( u \) be a minimizing control, and let the value function be denoted by \( V(s, x) \) for all \( s \in [0, 1] \). Let the control take values in \( \mathbb{R}^m \). You may assume that all the matrices are as nice as needed. For example, you may assume that \( C, D, \) and \( F \) are positive definite, symmetric matrices. Suppose \( V(s, x) \) takes the form \( V(s, x) = \frac{1}{2}(x - \bar{x}_s)^T P_s(x - \bar{x}_s) + r_s \) for all \( s, x \). Find differential equations for \( P_s, \bar{x}_s \) and \( r_s \).

2. (5) Consider the special case

\[ A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 1, \quad F = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

Find \( V(0, x) \).

3. (5) By hand (intuition should be quite helpful), construct/guess the viscosity solution to the following problem. Prove that it is, in fact, the viscosity solution.

\[ 0 = |\nabla_x W|^2 - 3, \quad x \in (-1, 1), \]
\[ W(-1) = W(1) = 0. \]

4. (10) Under the assumptions indicated in class, and with the problem definition also given there, prove the following lemma from class on June 1:

There exists \( K_V < \infty \) such that

\[ |\hat{V}(s, x) - \hat{V}(s, y)| \leq K_V (1 + |x| + |y|)|x - y| \quad \forall x, y \in \mathbb{R}^n, \forall s \in [0, T]. \]

A partial proof was provided in class. In your proof, you may use any results obtained prior to that lemma.
5. (10) Consider the HJB PDE and boundary condition

\[ |\nabla V|^2 - 1 = 0 \quad (x, y) \in G, \]
\[ V(x, y) = 0 \quad (x, y) \in \partial G, \]
\[ G = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in (0, 1), y \in (0, 1) \right\}. \]

Obtain the characteristic equations and their initial conditions. We specifically wish to obtain the viscosity solution of the HJB PDE in the form \( 0 = -\{1 - |\nabla V|^2\} = |\nabla V|^2 - 1 \) as a function of \((x, y)\) from the characteristics, where you will need to consider that the characteristics cross, and note the viscosity-solution condition at such points.

6. (10) Consider the following HJB PDE problem:

\[ 0 = W_s(s, x) - \pi \arctan(W_x(s, x)), \quad (s, x) \in (0, T) \times \mathbb{R}, \]  
\[ W(T, x) = 2|x|, \quad x \in \mathbb{R}, \]  

where \( T = 1.5 \). (Equation (1) is in a form analogous to that in the example from class for which we developed the finite element method for such PDEs, i.e., without the multiplication by \(-1\) that is needed for the viscosity solution check.)

(a) Given a space step size, \( \Delta x \), what would be the corresponding range of time step sizes, \( \Delta t \), for which we could expect an appropriate finite-element method to function properly.

(b) Construct two finite-element schemes, where these employ two different approximations to the space derivative, \( W_x(s, x) \). In particular, construct one where you expect the method to work, and one where you expect it to fail. Indicate why one is expected to work while the other is expected to fail.

(c) Using these finite-element methods, attempt to propagate the solution backward in time for at least five time steps. You should perform this only over a limited interval in the \( x \)-direction, say for \( x \in [-3, 3] \). Note that you will not be able to maintain this width in \( x \) as you propagate backwards; the region will need to shrink by \( \Delta x \) at each time-step.

(d) Try to determine whether your expectations of success and failure of the methods are correct. You’ll likely need to be a bit clever in this last bit, as we did not discuss any means for determining such in class.