UCSD MAE288A
Optimal Control

CLASS WILL START AT 5PM

Spring 2024

Lecture 18

Viscosity Solutions and the Method of Characteristics

Recall Continuous-Time/Continuous-Space Deterministic Control Problem

- Problem definition:

$$\dot{\xi}_t = f(\xi_t, u_t), \qquad (D)$$
$$\xi_s = x \in \boldsymbol{R}^n, \qquad (IC)$$
$$U \subseteq \boldsymbol{R}^m, \quad \mathcal{U}_{s,T} \doteq L_2\big((s,T);U\big),$$
$$J(s,x,u_\cdot) \doteq \int_s^T L(\xi_t, u_t)\,dt + \psi(\xi_T), \qquad (P)$$
$$\bar{V}(s,x) \doteq \inf_{u \in \mathcal{U}_{s,T}} J(s,x,u) \;\; \forall (s,x) \in [0,T] \times \boldsymbol{R}^n, \qquad (V).$$

- Assumed f, L, Ψ continuous (stronger than necessary) and:

$$\exists K_f < \infty \text{ s.t. } |f(x,v) - f(y,v)| \le K_f|x-y| \;\; \forall x,y \in \boldsymbol{R}^n,\, v \in U, \qquad (A.1)$$
$$\exists C_f < \infty \text{ s.t. } |f(x,v)| \le C_f(1+|v|) \;\; \forall x \in \boldsymbol{R}^n,\, v \in U. \qquad (A.2)$$
$$0 \le L(x,v) \le C_L(1+|x|^2+|v|^2) \;\; \forall x \in \boldsymbol{R}^n,\, v \in U, \qquad (A.3)$$
$$0 \le \psi(x) \le C_\psi(1+|x|^2) \;\; \forall x \in \boldsymbol{R}^n. \qquad (A.4)$$

- Results can be obtained under weaker assumptions (with sufficient effort...).

HJ PDE Problem

- The associated Hamilton-Jacobi PDE (HJ PDE) problem is given by

$$
\begin{aligned}
0 &= -V_s + H(s, x, \nabla_x V) \\
&\quad - V_s - \inf_{v \in U} \left\{ L(x, v) + V_t(s, x) + \nabla_x V(s, x) \cdot f(x, v) \right\}, \qquad (DPE) \\
V(T, x) &= \Psi(x). \qquad (TC)
\end{aligned}
$$

- Solve this on $(0, T) \times \boldsymbol{R}^n$.
- If we solve this, then we expect to obtain the optimal control [as a feedback!] given by $\bar{u}(t, x) \in \operatorname{argmin}_{v \in U} \left\{ L(x, v) + \nabla_x V(s, x) \cdot f(x, v) \right\}$.

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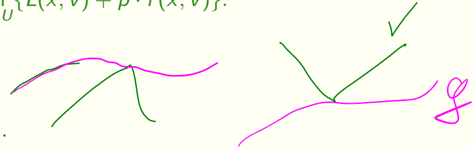
Viscosity Solution Definition

- Signs matter here! Write the HJ PDE as:

$$0 = -V_s + H(s, x, \nabla_x V) \qquad (HJPDE)$$

$$\text{where } H(s, x, p) \doteq -\inf_{v \in U} \{L(x, v) + p \cdot f(x, v)\}.$$



Definition:

Let $\mathcal{D} \doteq (0, T) \times \boldsymbol{R}^n$, and suppose $V \in C(\mathcal{D})$.

1) Suppose that for all $g \in C^1(\mathcal{D})$ and all $(\bar{s}, \bar{x}) \in \mathcal{D}$ s.t. $V - g$ has a local maximum at $(\bar{s}, \bar{x})$ with $V(\bar{s}, \bar{x}) = g(\bar{s}, \bar{x})$,

$$-g_s(\bar{s}, \bar{x}) + H(\bar{s}, \bar{x}, \nabla_x g(\bar{s}, \bar{x})) \leq 0.$$

Then V is a viscosity subsolution of $(HJPDE)$ on $\mathcal{D}$.

2) Suppose that for all $g \in C^1(\mathcal{D})$ and all $(\bar{s}, \bar{x}) \in \mathcal{D}$ s.t. $V - g$ has a local minimum at $(\bar{s}, \bar{x})$ with $V(\bar{s}, \bar{x}) = g(\bar{s}, \bar{x})$,

$$-g_s(\bar{s}, \bar{x}) + H(\bar{s}, \bar{x}, \nabla_x g(\bar{s}, \bar{x})) \geq 0.$$

Then V is a viscosity supersolution of $(HJPDE)$ on $\mathcal{D}$.

3) If V is both a viscosity subsolution and a viscosity supersolution on $\mathcal{D}$, then it is a viscosity solution on $\mathcal{D}$.

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Theory relating the Control problem and the HJ PDE problem

- Using the Gronwall inequality and other tools (and skipping quite a bit), we showed that $\bar{V}$ is Lipschitz continuous, and hence differentiable almost everywhere.
- We have a definition of *continuous* viscosity solutions of HJ PDEs.
- There are two methods for relating the HJ PDE problem viscosity solution to the corresponding control problem:

(1) a) SHOW $\bar{V}$ SAT'S (HJ PDE)
b) FIND A RESULT SHOWING UNIQUENESS OF SOL'S

(2) a) SHOW A

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Main Theorem of the Section

Theorem

Value function, $\bar{V}$ is a viscosity solution of the HJ PDE problem.

- Partial proof:

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Exit Problem case

- Exit problem:

$$\begin{aligned}
&\dot{\xi}_t = f(\xi_t, u_t),\\
&\xi_0 = x \in \mathcal{G}, \quad (\mathcal{G} \text{ open}),\\
&u \in \mathcal{U}_{0,\infty} \doteq L_2^{loc}((0,\infty); U),\\
&\tau \doteq \inf_{t \geq 0} \{\xi_t \notin \mathcal{G}\},\\
&J(x,u) \doteq \int_0^\tau L(\xi_t, u_t)\, dt + \psi(\xi_\tau),\\
&\bar{V}(x) \doteq \inf_{u \in L_2^{loc}} J(x,u).
\end{aligned}$$

- Corresponding HJ PDE problem:

$$\begin{aligned}
&0 = -\inf_{v \in R} \{L(x,v) + \nabla_x V^T f(x,v)\} \qquad \forall x \in \mathcal{G},\\
&V(x) = \psi(x) \qquad \forall x \in \partial\mathcal{G}.
\end{aligned}$$

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Simplest viscosity-solution problem example

Example 1:

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The (nearly only) classical-solution example

Example 2:

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Generalization of simplest viscosity-solution problem example

Example 3:

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Some Solution Methods

- The method of [generalized] characteristics (partially motivational).

- Finite elements specifically designed for HJ PDE.

- Max-plus/curse-of-dimensionality-free methods.

The [Generalized] Method of Characteristics

- Might as well use the exit-problem case for development of the method; it works as well for other problem forms.

$$0 = -H(x, \nabla_x V(x)), \quad x \in \mathcal{G},$$
$$V(x) = g(x), \quad x \in \partial\mathcal{G}.$$

- Refs and relations: L.C. Evans, Fritz-John; Hamiltonian Mechanics; Schr odnger equation; etc.
- Very formal development
- Consider a (state) trajectory, ξ_t moving into the interior from the boundary.
- Let ϕ_t denote the gradient of the solution, $V(x)$, at ξ_t (i.e., the "co-state").
- Finish.

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Comments Regarding the [Generalized] Method of Characteristics

- The standard method of characteristics needs **LOTS** of assumptions to be satisfied for it to work.
- Problems analogous to shocks and rarefaction waves.
- Generalized characteristics (cf. A. Melikyan) address these at the cost of seriously problematic "bookkeeping" issues.

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