UCSD MAE288A
Optimal Control

CLASS WILL START AT 5PM

Spring 2024

Lecture 18

Viscosity Solutions
and the Method of Characteristics

Recall Continuous-Time/Continuous-Space Deterministic Control Problem

- Problem definition:

$$\dot{\xi}_t = f(\xi_t, u_t), \qquad (D)$$
$$\xi_s = x \in \mathbf{R}^n, \qquad (IC)$$
$$U \subseteq \mathbf{R}^m, \quad \mathcal{U}_{s,T} \doteq L_2\big((s,T); U\big),$$
$$J(s,x,u_\cdot) \doteq \int_s^T L(\xi_t, u_t)\, dt + \psi(\xi_T), \qquad (P)$$
$$\bar{V}(s,x) \doteq \inf_{u \in \mathcal{U}_{s,T}} J(s,x,u) \;\; \forall (s,x) \in [0,T] \times \mathbf{R}^n, \qquad (V).$$

- Assumed f, L, Ψ continuous (stronger than necessary) and:

$$\exists K_f < \infty \text{ s.t. } |f(x,v) - f(y,v)| \le K_f |x-y| \;\; \forall x,y \in \mathbf{R}^n,\ v \in U, \qquad (A.1)$$
$$\exists C_f < \infty \text{ s.t. } |f(x,v)| \le C_f(1+|v|) \;\; \forall x \in \mathbf{R}^n,\ v \in U. \qquad (A.2)$$
$$0 \le L(x,v) \le C_L(1+|x|^2+|v|^2) \;\; \forall x \in \mathbf{R}^n,\ v \in U, \qquad (A.3)$$
$$0 \le \psi(x) \le C_\psi(1+|x|^2) \;\; \forall x \in \mathbf{R}^n. \qquad (A.4)$$

- Results can be obtained under weaker assumptions (with sufficient effort...).

HJ PDE Problem

- The associated Hamilton-Jacobi PDE (HJ PDE) problem is given by

$$
\begin{aligned}
0 &= -V_s + H(s, x, \nabla_x V) \\
&\quad - V_s - \inf_{v \in U} \left\{ L(x, v) + V_t(s, x) + \nabla_x V(s, x) \cdot f(x, v) \right\}, \qquad (DPE) \\
V(T, x) &= \Psi(x). \qquad (TC)
\end{aligned}
$$

- Solve this on $(0, T) \times \boldsymbol{R}^n$.
- If we solve this, then we expect to obtain the optimal control [as a feedback!] given by $\bar{u}(t, x) \in \mathrm{argmin}_{v \in U} \left\{ L(x, v) + \nabla_x V(s, x) \cdot f(x, v) \right\}$.

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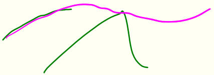
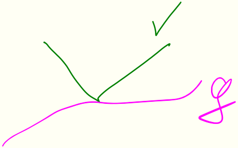
- Signs matter here! Write the HJ PDE as:

$$0 = -V_s + H(s, x, \nabla_x V) \qquad (HJPDE)$$

$$\text{where } H(s, x, p) \doteq -\inf_{v \in U} \{L(x, v) + p \cdot f(x, v)\}.$$





Definition:

Let $\mathcal{D} \doteq (0, T) \times \boldsymbol{R}^n$, and suppose $V \in C(\mathcal{D})$.

1) Suppose that for all $g \in C^1(\mathcal{D})$ and all $(\bar{s}, \bar{x}) \in \mathcal{D}$ s.t. $V - g$ has a local maximum at $(\bar{s}, \bar{x})$ with $V(\bar{s}, \bar{x}) = g(\bar{s}, \bar{x})$,

$$-g_s(\bar{s}, \bar{x}) + H(\bar{s}, \bar{x}, \nabla_x g(\bar{s}, \bar{x})) \leq 0.$$

Then V is a viscosity subsolution of $(HJPDE)$ on $\mathcal{D}$.

2) Suppose that for all $g \in C^1(\mathcal{D})$ and all $(\bar{s}, \bar{x}) \in \mathcal{D}$ s.t. $V - g$ has a local minimum at $(\bar{s}, \bar{x})$ with $V(\bar{s}, \bar{x}) = g(\bar{s}, \bar{x})$,

$$-g_s(\bar{s}, \bar{x}) + H(\bar{s}, \bar{x}, \nabla_x g(\bar{s}, \bar{x})) \geq 0.$$

Then V is a viscosity supersolution of $(HJPDE)$ on $\mathcal{D}$.

3) If V is both a viscosity subsolution and a viscosity supersolution on $\mathcal{D}$, then it is a viscosity solution on $\mathcal{D}$.

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Theory relating the Control problem and the HJ PDE problem

- Using the Gronwall inequality and other tools (and skipping quite a bit), we showed that $\bar{V}$ is Lipschitz continuous, and hence differentiable almost everywhere.
- We have a definition of *continuous* viscosity solutions of HJ PDEs.
- There are two methods for relating the HJ PDE problem viscosity solution to the corresponding control problem:

① a) SHOW $\bar{V}$ SAT'S (HJ PDE) (+)

b) FIND A RESULT SHOWING UNIQUENESS OF SOL'S

② a) SHOW ANY SOL. OF (HJPDE) IS $\bar{V}$ ($\Rightarrow$!) (VERIFICATION PROOF)

b) FIND RESULT SHOWING $\exists$ OF A SOL. OF (HJ PDE)

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Theorem

Value function, $\bar{V}$ is a viscosity solution of the HJ PDE problem.

- Partial proof:

(HJPDE) $\longleftrightarrow$ $0 = -V_t - \inf_{v \in U} \{ L(x,v) + \nabla_x V^T f(x,v) \}$

(TC) $V(T,x) = \psi(x)$

(TC) IS OBVIOUS.

PROVE $\bar{V}$ SAT'S (HJPDE) IN VISCOSITY SENSE.

SUPPOSE $\bar{V}$ NOT A VISC. SUBSOLUTION.

THEN $\exists\, g \in C^1$, $(\bar{t},\bar{x}) \in \Omega$ W/

$V - g$ LOC MAX AT $(\bar{t},\bar{x})$ S.T.

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$\exists\, \epsilon > 0$ s.t.

$$(1) \quad -g_2 - \inf_{v \in V} \{ L(\bar{x}, v) + \nabla_x g(\bar{t}, \bar{x})^T f(\bar{x}, v) \} \geq \epsilon$$

ASSUMED V COMPACT, AND (1) IS CONTAIN IN V.

$\Rightarrow \exists\, \bar{v} \in V$ s.t.

$$-g_t(\bar{t}, \bar{x}) - L(\bar{x}, \bar{v}) - \nabla_x g(\bar{t}, \bar{x})^T f(\bar{x}, \bar{v}) \geq \epsilon > 0$$

LET $u^0_t \equiv \bar{v} \quad \forall t$

$$\Rightarrow -g_t(\bar{t}, \bar{x}) - L(\bar{x}, u^0_t) - \nabla_x g(\bar{t}, \bar{x})^T f(\bar{x}, u^0_t) \geq \epsilon > 0 \quad \forall t$$

By $g \in C^1$; $L, f \in C$, $\exists\, \delta > 0$ s.t.

$$-g_t(t, x) - L(x, u^0_t) - \nabla_x g(t, x)^T f(x, u^0_t) \geq \frac{\epsilon}{2} > 0$$

$$\forall\, |t - \bar{t}| \leq \delta, \quad |x - \bar{x}| \leq \delta$$

$\exists\ \tilde{\delta} \in (0, \delta)$ S.T. ξ^0_t SAT'ING DYN'S W/ $\xi^0_{\bar{t}} = \bar{x}$ S.T.

$$|\xi^0_t - \bar{x}| \le \delta \quad \forall\ t \in [0, \bar{t} + \tilde{\delta})$$

$$(5) \quad -g_t(t, \xi^0_t) - L(\xi^0_t, u^0_t) - \nabla_x g(t, \xi^0_t)^T f(\xi^0_t, u^0_t) \ge \frac{\theta}{2} > 0$$

BY $\bar{V}$ BEING VALUE AND THE DPP

$$\bar{V}(\bar{t}, \bar{x}) = \inf_{u \in \mathcal{U}_{\bar{t},\hat{t}}} \left\{ \int_{\bar{t}}^{\hat{t}} L(\xi_s, u_s)\, ds + \bar{V}(\hat{t}, \xi_{\hat{t}}) \right\}$$

$$\Rightarrow \quad 0 \le \int_{\bar{t}}^{\hat{t}} L(\xi^0_s, u^0_s)\, ds + \bar{V}(\hat{t}, \xi^0_{\hat{t}}) - \bar{V}(\bar{t}, \bar{x}) \qquad (6)$$

BUT $\quad \bar{V}(\bar{t}, \bar{x}) - g(\bar{t}, \bar{x}) \ge \bar{V}(\hat{t}, \xi^0_{\hat{t}}) - g(\hat{t}, \xi^0_{\hat{t}})$ FOR $|\hat{t} - \bar{t}| \le \tilde{\delta}$

$$\Rightarrow \quad \bar{V}(\hat{t}, \xi^0_{\hat{t}}) - \bar{V}(\bar{t}, \bar{x}) \le g(\hat{t}, \xi^0_{\hat{t}}) - g(\bar{t}, \bar{x}) \qquad (7)$$

 SUB (7) → (6)

$$0 \le \int_s^{\hat{t}} L(\xi^0_r, u^0_r)\, dr + g(\hat{t}, \xi^0_{\hat{t}}) - g(\bar{s}, \bar{x})$$

SINCE $g \in C^1$

$$= \int_s^{\hat{t}} L(\xi^0_r, u^0_r) + g_s(r, \xi^0_r) + \nabla_x g(r, \xi^0_r)^T f(\xi^0_r, u^0_r)\, dr$$

BY (5)

$$\le \int_s^{\hat{t}} -\frac{\varepsilon}{2}\, dr = -\frac{\varepsilon}{2}(\hat{t} - \bar{s})$$

CONTRADICTION

∴ $\bar{V}$ IS A VISC. SUBSOL.

SUPERSOL IS EXERCISE!

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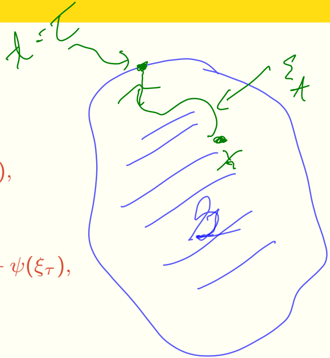
Exit Problem case

- Exit problem:

$$
\begin{aligned}
&\dot{\xi}_t = f(\xi_t, u_t),\\
&\xi_0 = x \in \mathcal{G}, \quad (\mathcal{G} \text{ open}),\\
&u \in \mathcal{U}_{0,\infty} \doteq L_2^{loc}((0,\infty); U),\\
&\tau \doteq \inf_{t \geq 0} \{\xi_t \notin \mathcal{G}\},\\
&J(x,u) \doteq \int_0^\tau L(\xi_t, u_t)\, dt + \psi(\xi_\tau),\\
&\bar{V}(x) \doteq \inf_{u \in L_2^{loc}} J(x,u).
\end{aligned}
$$



- Corresponding HJ PDE problem:

$$
\begin{aligned}
&0 = -\inf_{v \in R} \{L(x,v) + \nabla_x V^T f(x,v)\} \qquad \forall\, x \in \mathcal{G},\\
&V(x) = \psi(x) \qquad \forall\, x \in \partial\mathcal{G}.
\end{aligned}
$$

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Simplest viscosity-solution problem example

Example 1: EXIT PROBLEM, $G = (-1, 1)$

$$\dot{\xi}_t = u_t, \quad \xi_0 = x \in G$$

$$u_\cdot \in \mathcal{U}_{0,\infty} = L_2^{loc}((0,\infty); \mathbb{R}) \quad [L_2(0,T) \ \forall T]$$

$$J(x, u) = \int_0^\tau 1 + \frac{u_t^2}{2} \, dt$$

$$V(x) = \inf_{u \in \mathcal{U}_{0,\infty}} \{ J(x, u_\cdot) \}$$

$$\Longleftrightarrow \quad 0 = -\inf_{v \in \mathbb{R}} \left(1 + \frac{v^2}{2} + V_x^T v \right)$$

$$(TC0) \quad V(x) = 0 \quad \forall x \in \partial G = \{-1, 1\}$$

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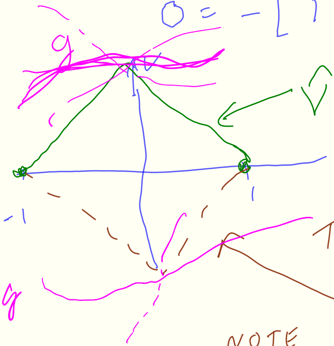
$0 = v + V_x \Rightarrow v^* = -V_x$



$0 = -\left[1 - \frac{V_x^2}{2}\right] = \frac{V_x^2}{2} - 1$ (*)

$\Rightarrow V_x^2 = 2$

$|V_x| = \sqrt{2}$

$V_x = \pm\sqrt{2}$

TRY $\tilde{V} = -\sqrt{2} + \sqrt{2}|x| = \sqrt{2}(|x| - 1)$

NOTE $\tilde{V} \in C^1((-1,0)\cup(0,1))$

$\Rightarrow$ ONLY CHECK CLASSIC THEORS

$\tilde{V}_x = \pm\sqrt{2} \Rightarrow$ O.K. AWAY FROM $x=0$.

AT $x=0$, CHECK V. SOL. COND'S

IF $V - g$ LOC MIN AT $x = 0$

$\Rightarrow |g_x(0)| \le \sqrt{2}$

IF $|g_x(0)| \le \sqrt{2}$

$$\Rightarrow \frac{|g_x(0)|^2}{2} - 1 \le \frac{(\sqrt{2})^2}{2} - 1 \le 0$$

IF $\check{g}(x) \le -\sqrt{2} \Rightarrow \check{g}_x(0) = 0$

$$\Rightarrow \frac{|g_x(0)|^2}{2} - 1 = -1 < 0$$

BUT $\Rightarrow \check{V}$ NOT A V. SUPERSOL

$\Rightarrow$ NOT A V. SOL

TRY $\hat{V}(x) = \sqrt{2}\,(1 - |x|)$

IF $\hat{V} - g$ LOCAL MAX AT 0, $g \in C^1$

$\Rightarrow |g_x(0)| \le \sqrt{2}$

$$\Rightarrow \frac{|g_x(0)|^2}{2} - 1 \le 0$$

The (nearly only) classical-solution example ⇒ $\hat{V}$ A V. SUB SOL, $\check{V}$ A V. SOL.

Example 2:

$$\dot{\xi}_t = A\xi_t + Bu_t, \quad \xi_s = x \in \mathbb{R}^n$$

$$J(s,x,u) = \int_s^T \frac{1}{2}\xi_t^T C \xi_t + \frac{1}{2} u_t^T D u_t \, dt + \frac{1}{2}\xi_T^T F \xi_T$$

$$\hat{V}(s,x) = \inf_{u \in L_2([s,T];\mathbb{R}^m)} J(s,x,u)$$

(PDE 1) $$0 = -V_s - \inf_{v \in \mathbb{R}^m}\left\{ \frac{1}{2}x^T C x + \frac{1}{2} v^T D v + \nabla_x V^T (Ax + Bv) \right\}$$

(TC) $$V(T,x) = \frac{1}{2}x^T F x$$

LOOK FOR $\tilde{V}(s,x) = \frac{1}{2}x^T P_s x + r_s$ (7)

NO NEED FOR V. SOL, ∃ ! CLASSICAL SOL OF

 FORM (7)

NOTE
$$\left.\begin{aligned}\tilde{V}_s(s,x) &= \tfrac{1}{2}x^T \mathring{P}_s x + \mathring{r}_s \\ D_x \tilde{V}(s,x) &= P_s x\end{aligned}\right\} \quad (8)$$

BY (TC1) $\quad P_T = F, \quad r_T = 0$

SUB (8) → (PDE 1) TO GET ODE'S FOR $P.$, $r.$

$$0 = -\tfrac{1}{2}x^T \dot{P}_s x - \dot{r}_s - \min_{v\in\mathbb{R}^m}\left\{\tfrac{1}{2}x^T C x + \tfrac{1}{2}v^T D v + (P_s x)^T[Ax + Bv]\right\}$$

$$= -\tfrac{1}{2}x^T \dot{P} x - \dot{r} - \tfrac{1}{2}x^T C x - \tfrac{1}{2}x^T P A x - \tfrac{1}{2}x^T A^T P x$$

$$- \min_v \left\{\tfrac{1}{2}v^T D v + v^T B^T P_s x\right\}$$

$$D v^* + B^T P x = 0$$

$$v^* = -D^{-1} B^T P_s x$$

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$$0 = -\tfrac{1}{2} x' \dot{P}_s x - \dot{r}_s - \tfrac{1}{2} x' C x - \tfrac{1}{2} x' \{ P_s A + A' P_s \} x + \tfrac{1}{2} x^T P_s B D^{-1} B' P_s x$$

COLLECTING QUADRATIC TERMS,

$$0 = -\dot{P}_s - C - (P_s A + A' P_s) + P_s B D^{-1} B' P_s$$

$$0 = -\dot{r}_s \implies r_s \equiv 0 \quad (\text{RECALL } r_T = 0)$$

$$\dot{P}_s = P_s B D^{-1} B' P_s - P_s A - A' P_s - C$$

w/ $P_T = F$ $\longrightarrow P_0.$

AND $V(s, x) = \tfrac{1}{2} x^T P_s x$

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$\bar{u}^*(t,x) = -D^{-1}B^{\top}P_t x \quad \forall t,x$

OPTIMAL CONTROL AS A F/B.

Generalization of simplest viscosity-solution problem example

Example 3:

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Some Solution Methods

- The method of [generalized] characteristics (partially motivational).

- Finite elements specifically designed for HJ PDE.

- Max-plus/curse-of-dimensionality-free methods.

The [Generalized] Method of Characteristics

- Might as well use the exit-problem case for development of the method; it works as well for other problem forms.

$$0 = -H(x, \nabla_x V(x)), \quad x \in \mathcal{G},$$
$$V(x) = g(x), \quad x \in \partial\mathcal{G}.$$

- Refs and relations: L.C. Evans, Fritz-John; Hamiltonian Mechanics; Schr odnger equation; etc.
- Very formal development
- Consider a (state) trajectory, ξ_t moving into the interior from the boundary.
- Let ϕ_t denote the gradient of the solution, $V(x)$, at ξ_t (i.e., the "co-state").
- Finish.

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Comments Regarding the [Generalized] Method of Characteristics

- The standard method of characteristics needs **LOTS** of assumptions to be satisfied for it to work.
- Problems analogous to shocks and rarefaction waves.
- Generalized characteristics (cf. A. Melikyan) address these at the cost of seriously problematic “bookkeeping” issues.

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